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Stability of Ulam–Hyers and Ulam–Hyers–Rassias for a class of fractional differential equations

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Abstract

In this paper, we investigate a class of nonlinear fractional differential equations with integral boundary condition. By means of Krasnosel'skiĭ fixed point theorem and contraction mapping principle we prove the existence and uniqueness of solutions for a nonlinear system. By means of Bielecki-type metric and the Banach fixed point theorem we investigate the Ulam–Hyers and Ulam–Hyers–Rassias stability of nonlinear fractional differential equations. Besides, we discuss an example for illustration of the main work.

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Keywords: Caputo fractional derivative; Integral boundary condition; Ulam–Hyers stability; Ulam–Hyers–Rassias stability

1 Introduction

Fractional derivatives provide an effective instrument in the modeling of many physical phenomena. Fractional differential equations and fractional integral equations appeared in various fields such as polymer rheology, blood flow phenomena, electrodynamics of complex medium, modeling and control theory, signal processing, and so on; see [1, 2]. In recent years, many researchers proved the existence and uniqueness of solutions to fractional differential equations [3–8]. Moreover, integral boundary problems had a variety of applications in real-life problems such as blood flow, underground water flow, population dynamics, thermoplasticity, chemical engineering, and so on; see [9–11].

On the other hand, S.M. Ulam presented the stability problem of the solutions of functional equations (of group homomorphisms) in 1940 in a talk given at Wisconsin University [12]. In 1941, Hyers [13] gave the first answer to the question in Banach spaces. Since then, many researchers were interested in Ulam-type stability. With a wide expansion of the fractional calculus, the study of stability for fractional differential equations also attracted the attention of researchers [14, 15].

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In 2011, Wang et al. [16] investigated the Ulam stability and data dependence for fractional differential equations with Caputo derivative:

$$^cD^\alpha x(t) = f(t, x(t)), \quad t \in [a, b), b = +\infty,$$

where $^cD^\alpha(\cdot)$ is the Caputo fractional derivative, $\alpha \in (0, 1)$.

Abbas, Benchohra et al. [17] researched the existence and Ulam stability for the fractional differential equation

$$\begin{cases} {}^H D_{1+}^{\alpha,\beta} u(t) = f(t, u(t)), \quad t \in [1, T], \\ {}^H I_{1+}^{\alpha,1-\gamma} u(t)|_{t=1} = \phi, \end{cases}$$

where $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $\gamma = \alpha + \beta - \alpha\beta$, $T > 1$, $\phi \in \mathbb{R}$, $^H D_{1+}^{\alpha,\beta}(\cdot)$ is the Hilfer–Hadamard fractional derivative, and $^H I_{1+}^{\alpha,1-\gamma}(\cdot)$ is the Hadamard integral.

Chalishajar et al. [18] discussed the existence, uniqueness, and Ulam–Hyers stability of solutions for the following coupled system of fractional differential equations with integral boundary conditions:

$$\begin{cases} {}^c D_{0+}^{\alpha} x(t) = f(t, y(t)), \quad \alpha \in (1, 2], t \in [0, 1], \\ {}^c D_{0+}^{\beta} y(t) = g(t, x(t)), \quad \beta \in (1, 2], t \in [0, 1], \\ px(0) + qx'(0) = \int_0^1 a_1(x(s))\, ds, \quad px(1) + qx'(1) = \int_0^1 a_2(x(s))\, ds, \\ \widetilde{p}y(0) + \widetilde{q}y'(0) = \int_0^1 \widetilde{a}_1(y(s))\, ds, \quad \widetilde{p}y(1) + \widetilde{q}y'(1) = \int_0^1 \widetilde{a}_2(y(s))\, ds, \end{cases}$$

where $^c D_{0+}^{\alpha}(\cdot)$ and $^c D_{0+}^{\beta}(\cdot)$ are the Caputo fractional derivatives, $p, \widetilde{p} > 0$, $q, \widetilde{q} \geq 0$ are real numbers, and $a_1, a_2, \widetilde{a}_1, \widetilde{a}_2$ are continuous functions.

Vanterler da C. Sousa et al. [19–21] studied the ψ-Hilfer fractional derivative and the stability of Hyers–Ulam–Rassias and Hyers–Ulam of the following Volterra integro-differential equation [14]:

$$\begin{cases} {}^H D_{0+}^{\alpha,\beta;\psi} u(t) = f(t, u(t)) + \int_0^t K(t, s, u(t))\, ds, \quad t \in [0, T] = I, \\ I_{0+}^{1-\gamma} u(0) = \sigma, \end{cases}$$

where $f(t, u)$ is a continuous function with respect to the variables t and u on $I \times \mathbb{R}$, $K(t, s, u)$ is continuous with respect to t, s, and u on $I \times I \times \mathbb{R}$, σ is a given constant, $^H D_{0+}^{\alpha,\beta;\psi}(\cdot)$ is the right-sided ψ-Hilfer fractional derivative with $\alpha \in (0, 1)$ and $\beta \in [0, 1]$, and $I_{0+}^{1-\gamma}(\cdot)$ is the ψ-Riemann–Liouville fractional integral with $\gamma \in [0, 1)$.

In this paper, we consider the following class of fractional differential equations with integral boundary condition:

$$u'(t) + {}^c D_{0+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0, 1], \tag{1.1}$$

$$u(1) = I_{0+}^{\beta} u(\eta) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} u(s)\, ds, \tag{1.2}$$

where $^c D_{0+}^{\alpha}(\cdot)$ is the Caputo derivative with $0 < \alpha < 1$, $I_{0+}^{\beta}(\cdot)$ is the Riemann–Liouville fractional integral with $\beta > 0$, $\eta \in (0, 1]$ is a fixed real number, $u \in C^1[0, 1]$, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Equation (1.1) represents a constitutive relation for viscoelastic model of fractional differential equation. Equation (1.1) [22] can also be used to describe macroscopic models II for electrodiffusion of ions in nerve cells when molecular diffusion is anomalous subdiffusion due to binding, crowding, or trapping.

This paper is organized as follows. In the second section, we recall some basic definitions of fractional calculus, the concepts of Ulam–Hyers and Ulam–Hyers–Rassias stability for Eq. (1.1) and fixed point theorems. In the third section, we investigate the existence and uniqueness of solutions for problem (1.1)–(1.2). Moreover, we discuss the Ulam–Hyers and Ulam–Hyers–Rassias stability for Eq. (1.1). In the last section, we provide an illustrative example.

2 Preliminaries

In this section, we recall some useful definitions, notations, and the fundamental results about fractional derivatives (refer to [23, 24] and [25]). Also, we present the concepts of the Ulam–Hyers and Ulam–Hyers–Rassias stability for Eq. (1.1).

Let $C^1[0,1] = \{u|u$ is a differentiable function on $[0,1]$ and its derivative is continuous$\}$ with the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Definition 2.1 ([23–25]) For a real-valued integrable function $f : (0,+\infty) \to \mathbb{R}$, the Riemann–Liouville fractional integral of order $0 < \alpha < 1$ is defined as

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > 0,$$

where Γ is the gamma function.

Definition 2.2 ([23–25]) The Caputo fractional derivative ${}^c D_{0+}^{\alpha}$ of an absolutely continuous (or differentiable) function $f(t)$ of order $0 < \alpha < 1$ is defined as

$$ {}^c D_{0+}^{\alpha} f(x) = I_{0+}^{1-\alpha} f'(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\,dt.$$

Definition 2.3 ([23–25]) The two-parametric Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0.$$

The Laplace transform of the Caputo derivative ${}^c D_{0+}^{\alpha} f(t)$ is

$$\mathcal{L}\{{}^c D_{0+}^{\alpha} f(t)\}(s) = s^{\alpha}\widetilde{f}(s) - s^{\alpha-1} f(0), \quad 0 < \alpha < 1. \tag{2.1}$$

The Laplace transform of the two-parametric Mittag-Leffler function is

$$\mathcal{L}\big\{t^{\beta-1}E_{\alpha,\beta}\big(\pm at^{\alpha}\big)\big\}(s) = \frac{s^{\alpha-\beta}}{(s^{\alpha}\mp a)},$$

$$\operatorname{Re}(s) > |a|^{\frac{1}{\alpha}}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \alpha, \beta \in \mathbb{C},$$

$$\mathcal{L}\big\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}\big(\pm at^{\alpha}\big)\big\}(s) = \frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}},$$

$$\operatorname{Re}(s) > |a|^{\frac{1}{\alpha}}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \alpha, \beta \in \mathbb{C},$$

(2.2)

where $E_{\alpha,\beta}^{(k)}(y) = \frac{d^{k}}{dy^{k}}E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty}\frac{(j+k)!y^{j}}{j!\Gamma(\alpha j+\alpha k+\beta)}$, $k = 0, 1, 2, \dots$.

Definition 2.4 ([26]) A function f is of exponential order λ if there exist constants $M > 0$ and λ such that for some $t_0 > 0$,

$$\big|f(t)\big| \le Me^{\lambda t} \quad \text{for } t \ge t_0.$$

Next, we present the concepts of the Ulam– and Ulam–Hyers–Rassias stability for Eq. (1.1). The following Definitions 2.5 and 2.6 are adapted from [14].

Definition 2.5 If $x(t)$ is a continuously differentiable function satisfying

$$\big|x'(t) +^{c} D_{0+}^{\alpha}x(t) - f\big(t, x(t)\big)\big| \le \theta, \quad t \in [0, 1],$$

where $\theta > 0$, and there are a solution $u(t)$ of Eq. (1.1) and a constant $C > 0$ independent of $x(t)$ and $u(t)$ such that

$$\big|x(t) - u(t)\big| \le C\theta, \quad t \in [0, 1],$$

then we say that the Eq. (1.1) has the Ulam–Hyers stability.

Definition 2.6 If $x(t)$ is a continuously differentiable function satisfying

$$\big|x'(t) +^{c} D_{0+}^{\alpha}x(t) - f\big(t, x(t)\big)\big| \le \sigma(t),$$

where $\sigma : [0, 1] \to [0, +\infty)$ is a continuous function, and there exist a solution $u(t)$ of Eq. (1.1) and a constant $C > 0$ independent of $x(t)$ and $u(t)$ such that

$$\big|x(t) - u(t)\big| \le C\sigma(t), \quad t \in [0, 1],$$

then we say that the Eq. (1.1) has the Ulam–Hyers–Rassias stability.

Theorem 2.1 ([27] (Krasnosel'skiĭ fixed point theorem)) *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be operators such that*

  (i) *$Ax + By \in M$ whenever $x, y \in M$,*

 (ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2.2 ([28] (Banach)) *Let* (X, d) *be a generalized complete metric space. Assume that* $\Lambda : X \to X$ *is a strictly contractive operator with Lipschitz constant* $K < 1$. *If there exists a nonnegative integer* k *such that* $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ *for some* $x \in X$, *then the following three propositions hold*:

(1) *The sequence* $\{\Lambda^n x\}$ *converges to a fixed point* x^* *of* Λ;

(2) x^* *is a unique fixed point of* Λ *in* $X^* = \{y \in X : d(\Lambda^k x, y) < \infty\}$;

(3) *If* $y \in X^*$, *then* $d(y, x^*) \le \frac{1}{1-K} d(\Lambda y, y)$.

Theorem 2.3 ([26]) *If* f *is piecewise continuous function on* $[0, \infty)$ *of exponential order* λ, *then the Laplace transform* $\mathcal{L}(f(t))$ *exists for* $\mathrm{Re}(s) > \lambda$ *and converges absolutely*.

3 Main results

In this section, we derive the existence and uniqueness of solutions for the integral boundary problem (1.1)–(1.2). Moreover, we study the Ulam–Hyers and Ulam–Hyers–Rassias stability for Eq. (1.1).

3.1 Existence and uniqueness results

In this subsection, by means of the Krasnosel'skiĭ fixed point theorem and contraction mapping principle, we investigate the existence and uniqueness of solutions for problem (1.1)–(1.2) in $C^1[0, 1]$.

Lemma 3.1 *Let* $u(t) \in C^1[0, 1]$, $0 < \alpha < 1$, $\beta > 0$. *For any* $g \in C[0, 1]$ *and* $\eta \in (0, 1]$, *the solution of the boundary value problem*

$$u'(t) +^c D_{0+}^\alpha u(t) = g(t), \quad t \in [0, 1], \tag{3.1}$$

$$u(1) = I_{0+}^\beta u(\eta). \tag{3.2}$$

is given by

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta - 1} u(s) \, ds + \int_0^1 G(t, s) g(s) \, ds,$$

where $G(t, s)$ *is called the Green's function of problem* (3.1)–(3.2)*and is given by*

$$G(t, s) = \begin{cases} E_{1-\alpha,1}(-(t-s)^{1-\alpha}) - E_{1-\alpha,1}(-(1-s)^{1-\alpha}), & 0 \le s \le t, \\ -E_{1-\alpha,1}(-(1-s)^{1-\alpha}), & t \le s \le 1. \end{cases} \tag{3.3}$$

Proof Since $u(t) \in C^1[0, 1]$, $u(t)$ and $^c D_{0+}^\alpha u(t)$ are bounded. For any $t \in [0, 1]$, we have that u' and $^c D_{0+}^\alpha$ are of exponential order. By Theorem 2.3 and Definition 2.4 we have that the Laplace transform of both $u'(t)$ and $^c D_{0+}^\alpha u(t)$ exist for $u(t) \in C^1[0, 1]$.

Taking the Laplace transform on both sides of Eq. (3.1), by Eq. (2.1) we obtain

$$s\widetilde{u}(s) - u(0) + s^\alpha \widetilde{u}(s) - s^{\alpha-1} u(0) = \widetilde{g}(s).$$

$$\widetilde{u}(s) = \frac{1}{s} u(0) + \frac{s^{-\alpha}}{1 + s^{1-\alpha}} \widetilde{g}(s).$$

Using the inverse Laplace transform, by Eq. (2.2) we have

$$u(t) = u(0) + \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)g(s)\,ds. \tag{3.4}$$

Equation (3.4) is the equivalent fractional integral equation of Eq. (3.1), so we have

$$u(1) = u(0) + \int_0^1 E_{1-\alpha,1}\left(-(1-s)^{1-\alpha}\right)g(s)\,ds. \tag{3.5}$$

From Eqs. (3.2) and (3.5) we have

$$u(0) = I_{0+}^{\beta}u(\eta) - \int_0^1 E_{1-\alpha,1}\left(-(1-s)^{1-\alpha}\right)g(s)\,ds.$$

Therefore the unique solution of problem (3.1)–(3.2) is

$$u(t) = I_{0+}^{\beta}u(\eta) + \int_0^1 G(t,s)g(s)\,ds,$$

where $G(t,s)$ is given by (3.3). This completes the proof. $\qquad\square$

Remark 3.1 By the definition of the two-parameter Mittag-Leffler function we get

$$\int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)ds = tE_{1-\alpha,2}\left(-t^{1-\alpha}\right), \quad t \in [0,1], 0 < \alpha < 1,$$

which is a convergent series of real numbers. Therefore there exists a constant $E_{1-\alpha,2} > 0$ such that

$$\left|\int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)ds\right| \le \left|E_{1-\alpha,2}\left(-t^{1-\alpha}\right)\right| \le E_{1-\alpha,2}.$$

Moreover, by Eq. (3.3) and the continuity of the two-parameter Mittag-Leffler function there exists a constant $M > 0$ such that

$$\int_0^t \left|G(t,s)\right|ds \le M, \quad t \in [0,1].$$

Remark 3.2 For a continuous function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists a constant $N > 0$ such that

$$N = \max_{t\in[0,1],s\in\mathbb{R}}\left|f(t,s)\right|.$$

Theorem 3.1 *Let $0 < \alpha < 1$, $\beta > 0$, and $\eta \in [0,1]$ be fixed real numbers. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition with respect to second argument, that is,*

$$\left|f(t,h_1) - f(t,h_2)\right| \le L|h_1 - h_2|$$

for all $t \in [0,1]$ and $h_1, h_2 \in \mathbb{R}$, where $L > 0$ is a Lipschitz constant. Then problem (1.1)–(1.2) has unique solution in $C^1[0,1]$, provided that

$$\frac{\eta^\beta}{\Gamma(\beta+1)} + LM < 1.$$

Proof By Lemma 3.1 the equivalent fractional integral of problem (1.1)–(1.2) is given by

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} u(s)\,ds + \int_0^1 G(t,s)f(s,u(s))\,ds,$$

where $G(t,s)$ is given by Eq. (3.3).

Consider the operator T defined on $C^1[0,1]$ by

$$(Tu)(t) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} u(s)\,ds + \int_0^1 G(t,s)f(s,u(s))\,ds. \tag{3.6}$$

By the continuity of the functions $G(t,s)$ and $f(t,u(t))$ we have $Tu \in C^1[0,1]$ for any $u \in C^1[0,1]$. This proves that T maps $C^1[0,1]$ into itself.

We define the set $\mathcal{B} = \{u \in C^1[0,1] : \|u\| < \delta\}$ and choose $\delta > \frac{MN\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta^\beta}$.

From Eq. (3.6), for $u \in \mathcal{B}$, we obtain

$$\left|(Tu)(t)\right| \leq \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1}\left|u(s)\right|\,ds + \int_0^1 \left|G(t,s)\right|\left|f(s,u(s))\right|\,ds$$

$$\leq \frac{\eta^\beta}{\Gamma(\beta+1)}\delta + MN \leq \delta$$

$$\implies \quad \|Tu\| \leq \delta, \quad u \in \mathcal{B}.$$

Hence $T\mathcal{B} \subseteq \mathcal{B}$.

Next, we show that T is a contraction operator. For $u_1, u_2 \in C^1[0,1]$ and $t \in [0,1]$, from Eq. (3.6), using the Lipschitz condition on f, we have

$$\left|(Tu_1)(t) - (Tu_2)(t)\right| \leq \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1}\left|u_1(s) - u_2(s)\right|\,ds$$

$$+ \int_0^1 \left|G(t,s)\right|\left|f(s,u_1(s)) - f(s,u_2(s))\right|\,ds$$

$$\leq \frac{\eta^\beta}{\Gamma(\beta+1)}\|u_1 - u_2\| + LM\|u_1 - u_2\|$$

$$= \left(\frac{\eta^\beta}{\Gamma(\beta+1)} + LM\right)\|u_1 - u_2\|.$$

As $\frac{\eta^\beta}{\Gamma(\beta+1)} + LM < 1$, T is a contraction mapping. By the contraction mapping principle it has a unique fixed point, which is the unique solution of problem (1.1)–(1.2).   □

Theorem 3.2 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and let f satisfy the Lipschitz condition with respect to second argument*:

$$\left|f(t,h_1) - f(t,h_2)\right| \leq L|h_1 - h_2|$$

for all $t \in [0, 1]$ and $h_1, h_2 \in \mathbb{R}$, where $L > 0$ is a Lipschitz constant, and

$$\frac{\eta^\beta}{\Gamma(\beta + 1)} + LM < 1.$$

Then problem (1.1)–(1.2) *has at least one solution in $C^1[0, 1]$.*

Proof We consider the operators A and B on $C^1[0, 1]$ defined by

$$(Au)(t) = -\int_t^1 E_{1-\alpha,1}\left(-(1-s)^{1-\alpha}\right) f\left(s, u(s)\right) ds,$$

$$(Bu)(t) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta - 1} u(s) \, ds$$

$$+ \int_0^t \left[E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right) - E_{1-\alpha,1}\left(-(1-s)^{1-\alpha}\right) \right] f\left(s, u(s)\right) ds.$$

Consider $W_r = \{u \in C^1[0, 1] : \|u\| \le r\}$ and choose

$$0 < \frac{MN}{1 - \frac{\eta^\beta}{\Gamma(\beta+1)}} \le r.$$

For any $u, v \in W_r$, having in mind Remark 3.1, Remark 3.2, and the definitions of the operators A and B, we conclude that

$$\|Au + Bv\| \le N \int_0^1 \left| G(t, s) \right| ds + \|v\| \frac{\eta^\beta}{\Gamma(\beta + 1)}$$

$$\le NM + \frac{r\eta^\beta}{\Gamma(\beta + 1)}.$$

Therefore we obtain $Au + Bv \in W_r$.

By Theorem 3.1 the operator B is a contraction mapping if

$$\frac{\eta^\beta}{\Gamma(\beta + 1)} + LM < 1.$$

It follows from the proof of the operator T.

By the continuity of the two-parameter Mittag-Leffler function and $f(t.u(t))$, for any continuous function $u \in W_r$, the operator A is continuous.

For any $u \in W_r$, from Remarks 3.1 and 3.2 we have

$$\left| Au(t) \right| \le MN.$$

Hence A is uniformly bounded on W_r.

For any $u \in W_r$ and $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$,

$$\left\| Au(t_1) - Au(t_2) \right\| = \left\| \int_{t_1}^1 E_{1-\alpha,1}\left(-(1-s)^{1-\alpha}\right) f\left(s, u(s)\right) ds \right.$$

$$\left. - \int_{t_2}^1 E_{1-\alpha,1}\left(-(1-s)^{1-\alpha}\right) f\left(s, u(s)\right) ds \right\|$$

$$= \left\| \int_{t_1}^{t_2} E_{1-\alpha,1}\big(-(1-s)^{1-\alpha}\big)f\big(s,u(s)\big)\,ds \right\|$$

$$\leq NE_{1-\alpha,1}\big(-(1-\theta)^{1-\alpha}\big)|t_2 - t_1|, \quad t_1 < \theta < t_2.$$

The constant $NE_{1-\alpha,1}(-(1-\theta)^{1-\alpha})$ is independent of u, so A is relatively compact on W_r. Therefore by the Arzelà–Ascoli theorem the operator A is compact on W_r. By Theorem 2.1 problem (1.1)–(1.2) has at least one solution on $[0,1]$. □

3.2 Ulam–Hyers–Rassias and Ulam–Hyers stability

In this subsection, by means of the Bielecki metric and Banach fixed-point theorem we investigate the Ulam–Hyers–Rassias and Ulam–Hyers stability in $C^1[0,1]$ for Eq. (1.1).

Consider the space $C^1[0,1]$ endowed with the Bielecki-type metric

$$d(w,v) = \sup_{t \in [0,1]} \frac{|w(t) - v(t)|}{\sigma(t)}, \quad w, v \in C^1[0,1],$$

where $\sigma : [0,1] \to (0,\infty)$ is a nondecreasing continuous function. Obviously, $(C^1[0,1], d)$ is a complete metric space.

Theorem 3.3 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition*

$$\big|f(t,h_1) - f(t,h_2)\big| \leq L|h_1 - h_2|, \quad t \in [0,1], h_1, h_2 \in \mathbb{R},$$

with $L > 0$. Moreover, let $\sigma : [0,1] \to (0,\infty)$ be a nondecreasing continuous function, and suppose that there exists a constant $\xi \in [0,1)$ such that

$$\int_0^t E_{1-\alpha,1}\big(-(t-s)^{1-\alpha}\big)\sigma(s)\,ds \leq \xi\sigma(t). \tag{3.7}$$

If $x \in C^1[0,1]$ satisfies

$$\big|x'(t) + {}^cD_{0+}^\alpha x(t) - f\big(t,x(t)\big)\big| \leq \sigma(t), \quad t \in [0,1] \tag{3.8}$$

and $L\xi < 1$, then there exists a solution $u(t)$ of Eq. (1.1) in $C^1[0,1]$ such that

$$\big|x(t) - u(t)\big| \leq \frac{\xi}{1-L\xi}\sigma(t), \quad t \in [0,1]. \tag{3.9}$$

This means that under the above conditions, the fractional differential Eq. (1.1) has the Ulam–Hyers–Rassias stability.

Proof By Lemma 3.1 the equivalent fractional integral equation of Eq. (1.1) is given by

$$u(t) = u(0) + \int_0^t E_{1-\alpha,1}\big(-(t-s)^{1-\alpha}\big)f\big(s,u(s)\big)\,ds, \tag{3.10}$$

which follows from the proof of Eq. (3.4).

We conclude that $u(t)$ satisfies Eq. (1.1) if and only if $u(t)$ satisfies the integral Eq. (3.10).

We consider the operator $\Lambda : C^1[0,1] \to C^1[0,1]$ defined by

$$(\Lambda v)(t) = v(0) + \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)f\left(s,v(s)\right)ds, \quad t \in [0,1], v \in C^1[0,1].$$

By the continuity of the two-parameter Mittag-Leffler function and f the operator Λ is continuous.

First, we prove that the operator Λ is strictly contractive in $(C^1[0,1], d)$. From Eq. (3.7), for any $v, w \in C^1[0,1]$, we obtain

$$
\begin{aligned}
d(\Lambda w, \Lambda v) &= \sup_{t\in[0,1]} \frac{|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})[f(s,w(s)) - f(s,v(s))]\,ds|}{\sigma(t)} \\
&\le L \sup_{t\in[0,1]} \frac{|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})|w(s)-v(s)|\,ds|}{\sigma(t)} \\
&= L \sup_{t\in[0,1]} \frac{|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\sigma(s)\frac{|w(s)-v(s)|}{\sigma(s)}\,ds|}{\sigma(t)} \\
&\le L\xi d(w,v).
\end{aligned}
$$

Since $L\xi < 1$, the operator Λ is strictly contractive.

On the other hand, let $x \in C^1[0,1]$ satisfy Eq. (3.8). By the Laplace transform and the inverse Laplace transform we obtain that x satisfies

$$
\begin{aligned}
&\left| x(t) - x(0) - \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)f\left(s,x(s)\right)ds \right| \\
&\le \left| \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)\sigma(s)\,ds \right|,
\end{aligned}
\tag{3.11}
$$

which follows from the proof of Lemma 3.1.

By Eq. (3.7), Eq. (3.11), and the definition of the operator Λ we get

$$
\begin{aligned}
\left| (\Lambda x)(t) - x(t) \right| &= \left| x(0) + \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)f\left(s,x(s)\right)ds - x(t) \right| \\
&\le \left| \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)\sigma(s)\,ds \right| \le \xi\sigma(t).
\end{aligned}
$$

Therefore we conclude that

$$d(\Lambda x, x) \le \xi < \infty, \quad \xi \in [0,1). \tag{3.12}$$

By means of item 2 of Theorem 2.2 there exists a unique element

$$u \in C^*[0,1] = \left\{ y \in C^1[0,1] : d(\Lambda x, y) < \infty \right\}$$

such that $\Lambda u = u$ or, equivalently,

$$u(t) = u(0) + \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)f\left(s,u(s)\right)ds.$$

Since Eq. (3.10) is the equivalent integral equation of Eq. (1.1), we conclude that $u(t)$ is a solution of Eq. (1.1). Also, from item 3 of Theorem 2.2 and Eq. (3.12) we have

$$d(x,u) \leq \frac{1}{1-L\xi} d(\Lambda x, x) \leq \frac{\xi}{1-L\xi}.$$

By the definition of d we obtain that inequality (3.9) holds. □

Theorem 3.4 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition*

$$\left| f(t,h_1) - f(t,h_2) \right| \leq L|h_1 - h_2|, \quad t \in [0,1], h_1, h_2 \in \mathbb{R},$$

with $L > 0$. Moreover, let $\sigma : [0,1] \to (0,\infty)$ be a nondecreasing continuous function, and suppose that there exists a constant $\xi \in [0,1)$ such that

$$\int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)\sigma(s)\,ds \leq \xi\sigma(t) \tag{3.13}$$

and $L\xi < 1$. If $x \in C^1[0,1]$ satisfies

$$\left| x'(t) + {}^c D_{0+}^\alpha x(t) - f\left(t, x(t)\right) \right| \leq \theta, \quad t \in [0,1], \tag{3.14}$$

where $\theta > 0$, then there exists a solution $u(t)$ of Eq. (1.1) in $C^1[0,1]$ such that

$$\left| x(t) - u(t) \right| \leq \frac{\sigma(1)E_{1-\alpha,2}}{(1-L\xi)\sigma(0)}\theta, \quad t \in [0,1]. \tag{3.15}$$

This means that under the above conditions, the fractional differential Eq. (1.1) has the Ulam–Hyers stability.

Proof: The first part of the proof follows the same steps as in the proof of Theorem 3.3. Consider the operator $\Lambda : C^1[0,1] \to C^1[0,1]$ defined by

$$(\Lambda v)(t) = v(0) + \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)f\left(s, v(s)\right)\,ds, \quad t \in [0,1], v \in C^1[0,1].$$

For any $v, w \in C^1[0,1]$, we have

$$d(\Lambda w, \Lambda v) \leq L\xi\, d(w,v).$$

Since $L\xi < 1$, we conclude that the operator Λ is strictly contractive in $(C^1[0,1], d)$, which follows from the proof of Theorem 3.3.

Suppose that $x \in C^1[0,1]$ satisfies Eq. (3.14). By means of the Laplace transform, the inverse Laplace transform, and Remark 3.1 we obtain

$$\left| x(t) - x(0) - \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)f\left(s, x(s)\right)\,ds \right| \leq \theta \left| \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)\,ds \right|$$

$$\leq \theta E_{1-\alpha,2}, \quad t \in [0,1].$$

Now by the definition of the operator Λ we get

$$
\begin{aligned}
\left|(\Lambda x)(t) - x(t)\right| &= \left| x(0) + \int_0^t E_{1-\alpha,1}\left(-(t-s)^{1-\alpha}\right)f\left(s, x(s)\right) ds - x(t) \right| \\
&\le \theta E_{1-\alpha,2}, \quad t \in [0,1].
\end{aligned}
$$

Since σ is a positive nondecreasing function, we have

$$
d(\Lambda x, x) = \sup_{t \in [0,1]} \frac{|(\Lambda x)(t) - x(t)|}{\sigma(t)} \le \frac{E_{1-\alpha,2}}{\sigma(0)}\theta < \infty. \tag{3.16}
$$

Having in mind item 2 of Theorem 2.2, there exists a unique element

$$
u \in C^*[0,1] = \left\{ y \in C^1[0,1] : d(\Lambda x, y) < \infty \right\}
$$

such that $\Lambda u = u$, which means that $u(t)$ is a solution of Eq. (1.1).

Thus from item 3 of Theorem 2.2 and Eq. (3.16) it follows that

$$
d(x, u) \le \frac{1}{1 - L\xi} d(\Lambda x, x) = \frac{1}{1 - L\xi} \sup_{t \in [0,1]} \frac{|(\Lambda x)(t) - x(t)|}{\sigma(t)} \le \frac{1}{1 - L\xi} \frac{E_{1-\alpha,2}}{\sigma(0)}\theta.
$$

By the definition of the Bielecki-type metric d we obtain

$$
\left| x(t) - u(t) \right| \le \frac{\sigma(t)E_{1-\alpha,2}}{(1 - L\xi)\sigma(0)}\theta, \quad t \in [0,1]. \tag{3.17}
$$

Therefore Eq. (3.15) follows directly from Eq. (3.17).

4 Example

Example 4.1 Consider the following fractional differential equation with integral boundary condition:

$$
u'(t) + {}^cD_{0+}^{\frac{1}{2}} u(t) = \frac{1}{t^2 + 8} \frac{1}{1 + |u(t)|}, \quad t \in [0,1], \tag{4.1}
$$

$$
u(1) = I_{0+}^{\frac{1}{2}} u\left(\frac{1}{4}\right), \tag{4.2}
$$

where $u(t) \in C^1[0,1]$.

Comparing with problem (1.1)–(1.2), we have

$$
\alpha = \frac{1}{2}, \qquad \beta = \frac{1}{2}, \qquad \eta = \frac{1}{4}, \qquad f\left(t, u(t)\right) = \frac{1}{t^2 + 8} \frac{1}{1 + |u(t)|}.
$$

Clearly, we obtain

$$
\left| f(t, h_1) - f(t, h_2) \right| \le \frac{1}{8}|h_1 - h_2|, \quad h_1, h_2 \in \mathbb{R}, t \in [0,1],
$$

$$
\left| f\left(t, u(t)\right) \right| \le \frac{1}{8}, \quad t \in [0,1], u \in C^1[0,1].
$$

Here we get $L = N = \frac{1}{8}$.

Further, by Remark 3.1 we have

$$\frac{\eta^\beta}{\Gamma(\beta+1)} + LM = \frac{\frac{1}{4}^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{1}{4} = \frac{1}{\sqrt{\pi}} + \frac{1}{4} < 1.$$

Therefore by Theorems 3.1 and 3.2 problem (4.1)–(4.2) has a unique solution.

Next, we investigate the Ulam–Hyers and Ulam–Hyers–Rassias stability for Eq. (4.1).

Letting $\sigma(t) = e^t$, by Remark 3.1 we obtain

$$\int_0^t E_{\frac{1}{2},1}\left(-(t-s)^{\frac{1}{2}}\right)e^s \, ds < e^t - 1 < \frac{3}{4}e^t, \quad t \in [0,1].$$

Thus $\sigma(t) = e^t$ satisfies Eq. (3.7) with $\xi = \frac{3}{4}$, and $L\xi = \frac{3}{32} < 1$.

Hence Theorem 3.3 guarantees that Eq. (4.1) has the Ulam–Hyers–Rassias stability. Further, Theorem 3.4 guarantees that Eq. (4.1) has the Ulam–Hyers stability.

The Ulam–Hyers and Ulam–Hyers–Rassias stability for Eq. (4.1) is independent of the initial value condition. Using MATLAB, the solution $u(t)$ of Eq. (4.1) with initial value condition $u(0) = 0$ is computed and depicted in Fig. 1.

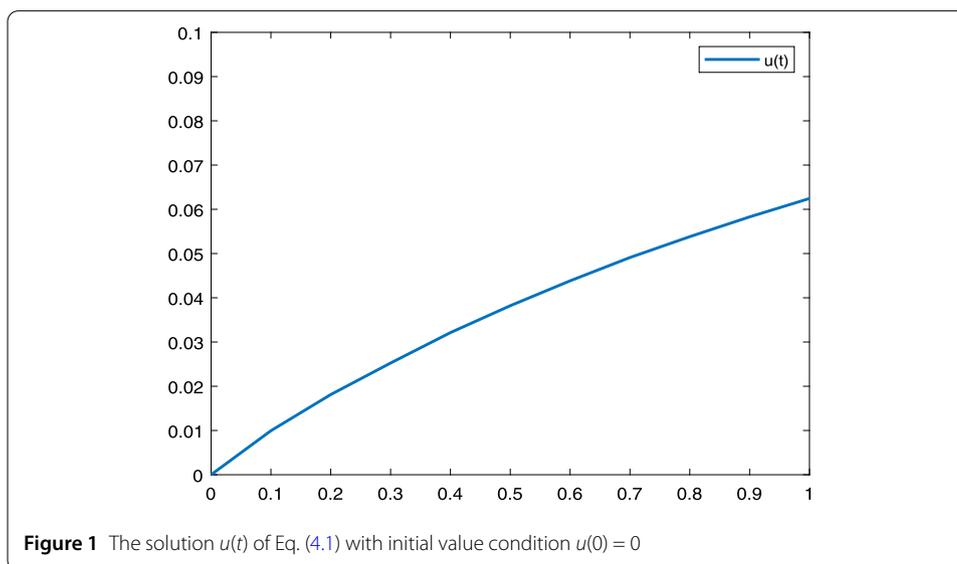Now consider $x \in C^1[0,1]$, the solution of the following fractional differential equation:

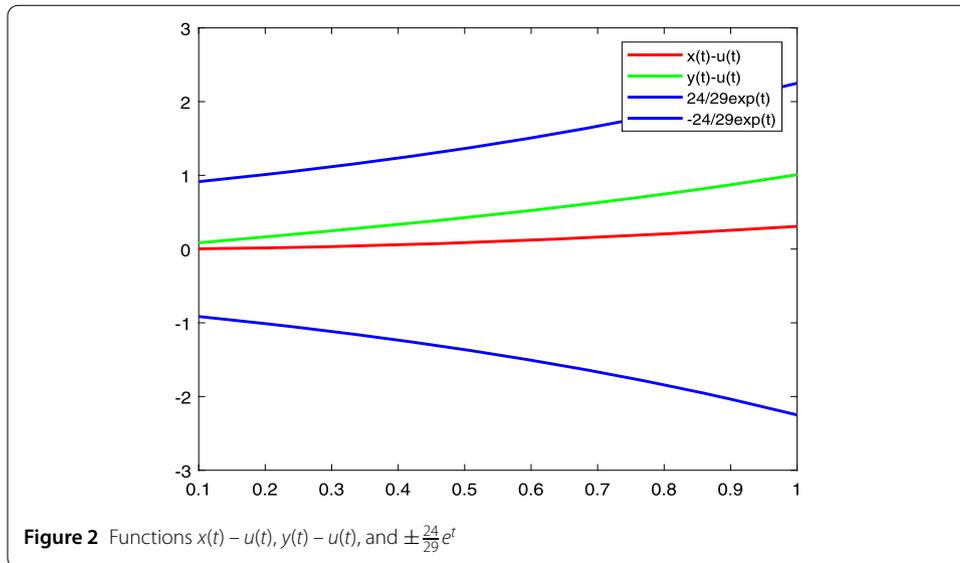$$x'(t) + {}^cD_{0+}^{\frac{1}{2}}x(t) = \frac{1}{t^2+8}\frac{1}{1+|x(t)|} + t, \quad t \in [0,1],$$

$$x(0) = 0.$$

We conclude that x satisfies Eq. (3.8). Therefore we have

$$\left|x(t) - u(t)\right| \le \frac{\xi}{1-L\xi}e^t = \frac{24}{29}e^t, \quad t \in [0,1];$$

see Fig. 2.

Figure 1 The solution $u(t)$ of Eq. (4.1) with initial value condition $u(0) = 0$

Figure 2 Functions $x(t) - u(t)$, $y(t) - u(t)$, and $\pm\frac{24}{29}e^t$

On the other hand, consider $y \in C^1[0, 1]$, the solution of the following fractional differential equation:

$$y'(t) + {}^cD_{0+}^{\frac{1}{2}}y(t) = \frac{1}{t^2 + 8}\frac{1}{1 + |y(t)|} + e^t, \quad t \in [0, 1],$$

$$y(0) = 0,$$

Then y satisfies Eq. (3.8), and we have

$$\left|y(t) - u(t)\right| \leq \frac{\xi}{1 - L\xi}e^t = \frac{24}{29}e^t, \quad t \in [0, 1];$$

see Fig. 2.

Availability of data and materials
Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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