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Quantum Hermite–Hadamard inequality by means of a Green function

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Abstract

The purpose of this work is to present the quantum Hermite–Hadamard inequality through the Green function approach. While doing this, we deduce some novel quantum identities. Using these identities, we establish some new inequalities in this direction. We contemplate the possibility of expanding the method, outlined herein, to recast the proofs of some known inequalities in the literature.

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1 Introduction

Let $\psi : [b_1, b_2] \to \mathbb{R}$ be a convex function. Then the following double inequality holds:

$$\psi\left(\frac{b_1+b_2}{2}\right) \le \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \psi(x)\,dx \le \frac{\psi(b_1)+\psi(b_2)}{2}. \tag{1.1}$$

It is known in the literature as the Hermite–Hadamard inequality. This inequality has instigated pletora of papers. Results concerning generalization, refinement, and extension of (1.1) are also found; see [1–9, 12, 14, 15, 17–20, 23, 27–30] and the references therein.

In the early 16th century, the concept of q-calculus was introduced. Ever since, integral inequalities of the trapeziod, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Grüss, Hölder, Grüss–Čebyšev, and other types have been established in the q-calculus sense. In 2014, Tariboon and Ntouyas [33] obtained the following q-calculus version of (1.1).

Theorem 1.1 *Let $\psi : [b_1, b_2] \to \mathbb{R}$ be a convex continuous function on (b_1, b_2), and let $0 < q < 1$. Then*

$$\psi\left(\frac{b_1+b_2}{2}\right) \le \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \psi(x)_{b_1}\,d_q x \le \frac{q\psi(b_1)+\psi(b_2)}{1+q}. \tag{1.2}$$

In 2016, Kunt and Işçan [21] observed, via a counterexample, that the left-hand side of inequality (1.2) is not necessarily true. Subsequently, Alp et al. [11] proved the following correct version of (1.2).

Theorem 1.2 ([11]) *Let $\psi : [b_1, b_2] \to \mathbb{R}$ be a convex differentiable function on (b_1, b_2), and let $0 < q < 1$. Then*

$$\psi\left(\frac{qb_1 + b_2}{1+q}\right) \leq \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \psi(x)_{b_1}\, d_q x \leq \frac{q\psi(b_1) + \psi(b_2)}{1+q}. \tag{1.3}$$

Remark 1.3 It is important to note that the inequality in Theorem 1.2 was first established by Marinković et al. [24, Theorem 5.3].

The aim of this work is to recast inequality (1.3) in Theorem 1.2 via another approach different from that presented in [11]. Specifically, we do this using a Green function. In the process, we establish some identities that are also used to obtain more results in this direction.

We organize this paper as follows. Section 2 contains a brief introduction of the quantum calculus. Our main results are then framed and proved in Sect. 3.

2 Preliminaries

Quantum calculus is known as the calculus without limits. In this section, we present a quick overview of the theory of q-calculus. The interested reader is invited to the book [16] for an in-depth study of this subject. We begin with these basic definitions.

Definition 2.1 ([32]) Let $\psi : [b_1, b_2] \to \mathbb{R}$ be a continuous function, and let $w \in [b_1, b_2]$. Then the expression

$${}_{b_1}D_q\psi(w) = \frac{\psi(w) - \psi(qw + (1-q)b_1)}{(1-q)(w - b_1)}, \quad w \neq b_1, {}_{b_1}D_q\psi(b_1) = \lim_{w \to b_1} {}_{b_1}D_q\psi(w) \tag{2.1}$$

is called the q-derivative on $[b_1, b_2]$ of the function at w.

We call ψ q-differentiable on $[b_1, b_2]$ if ${}_{b_1}D_q\psi(w)$ exists for all $w \in [b_1, b_2]$.

Definition 2.2 ([32]) Let $\psi : [b_1, b_2] \to \mathbb{R}$ be a continuous function. Then the q-integral on $[b_1, b_2]$ is defined as

$$\int_{b_1}^{w} \psi(x)_{b_1}\, d_q x = (1-q)(w - b_1)\sum_{k=0}^{\infty} q^k \psi\left(q^k w + \left(1 - q^k\right)b_1\right) \tag{2.2}$$

for $w \in [b_1, b_2]$. Moreover, if $c \in (b_1, w)$, then the q-integral on $[b_1, b_2]$ is defined as

$$\int_{c}^{w} \psi(x)_{b_1}\, d_q x = \int_{b_1}^{w} \psi(x)_{b_1}\, d_q x - \int_{b_1}^{c} \psi(x)_{b_1}\, d_q x. \tag{2.3}$$

Remark 2.3 In light of Definitions 2.1 and 2.2, we make the following remarks:

1. By taking $b_1 = 0$ expression (2.1) boils down to the well-known q-derivative $D_q\psi(w)$ of the function $\psi(w)$ defined by

$$D_q\psi(w) = \frac{\psi(w) - \psi(qw)}{(1-q)w}.$$

2. Also, if $b_1 = 0$, then (2.2) reduces to the classical q-integral of a function $\psi : [0, \infty) \to \mathbb{R}$ defined by

$$\int_0^w \psi(x)_0 \, d_q x = (1-q)w \sum_{k=0}^{\infty} q^k \psi\left(q^k w\right).$$

Some known results in continuous calculus have been extended to the q-calculus framework as follows.

Theorem 2.4 ([32]) *Let $\psi : [b_1, b_2] \to \mathbb{R}$ be a continuous function. Then we have*

$$\int_\delta^w {}_{b_1}D_q \psi(w)_{b_1} \, d_q x = \psi(w) - \psi(\delta) \quad \text{for } \delta \in (b_1, w).$$

Theorem 2.5 ([13]) *Let $\psi, \phi : [b_1, b_2] \to \mathbb{R}$ be two continuous functions and suppose $\psi(x) \le \phi(x)$ for all $x \in [b_1, b_2]$. Then*

$$\int_{b_1}^w \psi(x)_{b_1} \, d_q x \le \int_{b_1}^w \phi(x)_{b_1} \, d_q x.$$

Theorem 2.6 ([32]) *Let $\psi : [b_1, b_2] \to \mathbb{R}$ be a continuous function. Then*

$${}_{b_1}D_q \int_{b_1}^w \psi(x)_{b_1} \, d_q x = \psi(w);$$

$$\int_c^w {}_{b_1}D_q \psi(x)_{b_1} \, d_q x = \psi(w) - \psi(c), \quad \text{for } c \in (b_1, w).$$

Theorem 2.7 ([32]) *Let $\psi, \phi : [b_1, b_2] \to \mathbb{R}$ be continuous functions, and let $\alpha \in \mathbb{R}$. Then, for $w \in [b_1, b_2]$ and $c \in (b_1, w)$, we have*

$$\int_{b_1}^w \left[\psi(x) + \phi(x)\right]_{b_1} d_q x = \int_{b_1}^w \psi(x)_{b_1} \, d_q x + \int_{b_1}^w \phi(x)_{b_1} \, d_q x;$$

$$\int_{b_1}^w \alpha\psi(x)_{b_1} \, d_q x = \alpha \int_{b_1}^w \psi(x)_{b_1} \, d_q x;$$

$$\int_c^w \psi(x)_{b_1} D_q \phi(x)_{b_1} \, d_q x$$

$$= \psi(w)\phi(w) - \psi(c)\phi(c) - \int_c^w \phi\left(qx + (1-q)b_1\right)_{b_1} D_q \psi(x)_{b_1} \, d_q x.$$

3 Main results

We will prove our fundamental results with the help of the following lemma.

Lemma 3.1 ([10, 25]) *Let G be the Green function defined on $[b_1, b_2] \times [b_1, b_2]$ by*

$$G(x,u) = \begin{cases} b_1 - u, & b_1 \le u \le x; \\ b_1 - x, & x \le u \le b_2. \end{cases}$$

Then any $\psi \in C^2([b_1, b_2])$ *can be expressed as*

$$\psi(x) = \psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, \mu)\psi''(\mu)\, d\mu. \tag{3.1}$$

We now state and justify our main results.

Theorem 3.2 *Let* $\psi : [b_1, b_2] \to \mathbb{R}$ *be a convex twice differentiable function on* (b_1, b_2). *If* $0 < q < 1$, *then*

$$\psi\left(\frac{qb_1 + b_2}{q+1}\right) \le \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \psi(x)_{b_1}\, d_q x \le \frac{q\psi(b_1) + \psi(b_2)}{q+1}. \tag{3.2}$$

Proof If we set $x = \frac{qb_1+b_2}{q+1}$ in (3.1), then we get

$$\begin{aligned}\psi\left(\frac{qb_1 + b_2}{q+1}\right) &= \psi(b_1) + \left(\frac{qb_1 + b_2}{q+1} - b_1\right)\psi'(b_2) + \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q+1}, u\right)\psi''(u)\, du \\ &= \psi(b_1) + \frac{b_2 - b_1}{q+1}\psi'(b_2) + \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q+1}, u\right)\psi''(u)\, du. \end{aligned} \tag{3.3}$$

By computing we obtain that

$$\begin{aligned}&\frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \psi(x)_{b_1}\, d_q x \\ &\quad= \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \left\{\psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, u)\psi''(u)\, du\right\}_{b_1} d_q x \\ &\quad= \psi(b_1) + \frac{b_2 - b_1}{q+1}\psi'(b_2) + \frac{1}{b_2 - b_1}\int_{b_1}^{b_2}\int_{b_1}^{b_2} G(x, u)\psi''(u)\, du_{b_1}\, d_q x. \end{aligned} \tag{3.4}$$

Subtracting (3.4) from (3.3), we get:

$$\begin{aligned}&\psi\left(\frac{qb_1 + b_2}{q+1}\right) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \psi(x)_{b_1}\, d_q x \\ &\quad= \psi(b_1) + \frac{b_2 - b_1}{q+1}\psi'(b_2) + \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q+1}, u\right)\psi''(u)\, du \\ &\qquad- \psi(b_1) - \frac{b_2 - b_1}{q+1}\psi'(b_2) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2}\int_{b_1}^{b_2} G(x, u)\psi''(u)\, du_{b_1}\, d_q x \\ &\quad= \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q+1}, u\right)\psi''(u)\, du - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2}\int_{b_1}^{b_2} G(x, u)\psi''(u)\, du_{b_1}\, d_q x \\ &\quad= \int_{b_1}^{b_2} \left\{G\left(\frac{qb_1 + b_2}{q+1}, u\right) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} G(x, u)_{b_1}\, d_q x\right\}\psi''(u)\, du \\ &\quad= \int_{b_1}^{b_2} \left[G\left(\frac{qb_1 + b_2}{q+1}, u\right) + \frac{1}{b_2 - b_1}\left\{\frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1)\right\}\right]\psi''(u)\, du. \end{aligned} \tag{3.5}$$

Next, we consider the function

$$f(u) = G\left(\frac{qb_1 + b_2}{q+1}, u\right) + \frac{1}{b_2 - b_1}\left\{\frac{(u-b_1)^2}{q+1} + (b_2 - u)(u - b_1)\right\}. \tag{3.6}$$

For this, the following cases are possible.

Case 1. If $b_1 \le u \le \frac{qb_1+b_2}{q+1}$, then

$$f(u) = b_1 - u + \frac{1}{b_2 - b_1}\left\{\frac{(u-b_1)^2}{q+1} + (b_2 - u)(u - b_1)\right\}.$$

Therefore

$$f'(u) = -1 + \frac{1}{b_2 - b_1}\left\{\frac{2(u-b_1)}{q+1} + (b_2 - u) - (u - b_1)\right\};$$
$$f''(u) = \frac{1}{b_2 - b_1}\left\{\frac{2}{q+1} - 2\right\} = \frac{-2q}{(1+q)(b_2 - b_1)} < 0.$$

This implies that f' is decreasing and $f'(b_1) = 0$, which shows that $f'(u) \le 0$. Thus f is also decreasing, and $f(b_1) = 0$, that is, $f(u) \le 0$ for all $u \in [b_1, \frac{qb_1+b_2}{q+1}]$.

Case 2. If $\frac{qb_1+b_2}{q+1} \le u \le b_2$, then

$$f(u) = \frac{b_1 - b_2}{q+1} + \frac{1}{b_2 - b_1}\left\{\frac{(u-b_1)^2}{q+1} + (b_2 - u)(u - b_1)\right\};$$
$$f'(u) = \frac{1}{b_2 - b_1}\left\{\frac{u - b_1}{q+1}(1-q) + (b_2 - u)\right\} > 0.$$

Hence f is increasing and $f(b_2) = 0$. So, $f(u) \le 0$ for all $u \in [\frac{qb_1+b_2}{q+1}, b_2]$.

Now, using (3.5) and the fact that $\psi''(u) \ge 0$ for all $u \in [b_1, b_2]$, since ψ is convex, we obtain the first inequality:

$$\psi\left(\frac{qb_1 + b_2}{q+1}\right) \le \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \psi(x)_{b_1}\, d_q x.$$

For the right-hand side inequality, we recall that

$$\psi(x) = \psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x,u)\psi''(u)\, du;$$
$$\psi(b_2) = \psi(b_1) + (b_2 - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(b_2,u)\psi''(u)\, du; \tag{3.7}$$
$$\frac{q\psi(b_1) + \psi(b_2)}{q+1} = \psi(b_1) + \frac{b_2 - b_1}{q+1}\psi'(b_2) + \frac{1}{q+1}\int_{b_1}^{b_2} G(b_2,u)\psi''(u)\, du.$$

Subtracting (3.4) from (3.7), we get

$$\frac{q\psi(b_1) + \psi(b_2)}{q+1} - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \psi(x)_{b_1}\, d_q x$$

$$= \int_{b_1}^{b_2} \left[\frac{G(b_2,u)}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^2}{q+1} + (b_2-u)(u-b_1) \right\} \right] \psi''(u)\, du. \tag{3.8}$$

Let

$$F(u) = \frac{G(b_2,u)}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^2}{q+1} + (b_2-u)(u-b_1) \right\}.$$

Then

$$F(u) = \frac{b_1-u}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^2}{q+1} + (b_2-u)(u-b_1) \right\};$$

$$F'(u) = \frac{-1}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{2(u-b_1)}{q+1} + (b_2-u) - (u-b_1) \right\};$$

$$F''(u) = \frac{1}{b_2-b_1} \left\{ \frac{2}{q+1} - 2 \right\} = \frac{-2q}{(1+q)(b_2-b_1)} < 0.$$

Here we also observe two cases.

Case 3. If $b_1 \le u \le \frac{b_1+b_2}{2}$, then $F''(u) < 0$. Therefore F' is decreasing, and also $F'(\frac{b_1+b_2}{2}) = 0$, which shows that $F'(u) \ge 0$. Moreover, F is increasing, and $F(b_1) = 0$. Hence $F(u) \ge 0$ for all $u \in [b_1, \frac{b_1+b_2}{2}]$.

Case 4. Also, if $\frac{b_1+b_2}{2} \le u \le b_2$, then $F''(u) < 0$. So, F' is decreasing, and $F'(\frac{b_1+b_2}{2}) = 0$, which implies that $F'(u) \le 0$. Hence F is decreasing, and $F(b_2) = 0$, and then $F(u) \ge 0$ for all $u \in [\frac{b_1+b_2}{2}, b_2]$.

Combining these two cases, we conclude that $F(u) \ge 0$ for all $u \in [b_1, b_2]$. Applying (3.8) and the convexity of ψ, we establish the right-hand side of the desired inequality. The proof is complete. □

Next, we prove new quantum Hermite–Hadamard inequalities for the class of monotone and convex functions.

Theorem 3.3 *Let $\psi \in C^2([b_1, b_2])$ and $0 < q < 1$. Then*:

(i). *If $|\psi''|$ is an increasing function, then*

$$\left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x)\, {}_{b_1}d_q x \right| \le \left|\psi''(b_2)\right| \left[\frac{q(b_2-b_1)^2}{6(1+q)} \right].$$

(ii). *If $|\psi''|$ is a decreasing function, then*

$$\left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x)\, {}_{b_1}d_q x \right| \le \left|\psi''(b_1)\right| \left[\frac{q(b_2-b_1)^2}{6(1+q)} \right].$$

(iii). *If $|\psi''|$ is a convex function, then*

$$\left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x)\, {}_{b_1}d_q x \right| \le \max\left\{ \left|\psi''(b_1)\right|, \left|\psi''(b_2)\right| \right\} \left[\frac{q(b_2-b_1)^2}{6(1+q)} \right].$$

Proof To prove (i), by (3.8) we get:

$$
\begin{aligned}
&\left|\frac{q\psi(b_1)+\psi(b_2)}{1+q}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad=\left|\int_{b_1}^{b_2}\left[\frac{G(b_2,u)}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right]\psi''(u)\,du\right| \\
&\quad=\left|\int_{b_1}^{b_2}\left[\frac{b_1-u}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-b_1b_2+(b_1+b_2)u-u^2\right\}\right]\psi''(u)\,du\right| \\
&\quad\le\left|\psi''(b_2)\right|\left[\frac{b_1}{1+q}\int_{b_1}^{b_2}1\,du-\frac{1}{1+q}\int_{b_1}^{b_2}u\,du+\frac{1}{b_2-b_1}\left\{\int_{b_1}^{b_2}\frac{(u-b_1)^2}{1+q}\,du\right.\right. \\
&\qquad\left.\left.-b_1b_2\int_{b_1}^{b_2}1\,du+(b_1+b_2)\int_{b_1}^{b_2}u\,du-\int_{b_1}^{b_2}u^2\,du\right\}\right] \\
&\quad=\left|\psi''(b_2)\right|\left[\frac{b_1(b_2-b_1)}{1+q}-\frac{u^2}{2(1+q)}\Big|_{b_1}^{b_2}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^3}{3(1+q)}\Big|_{b_1}^{b_2}-b_1b_2(b_2-b_1)\right.\right. \\
&\qquad\left.\left.+(b_1+b_2)\frac{u^2}{2}\Big|_{b_1}^{b_2}-\frac{u^3}{3}\Big|_{b_1}^{b_2}\right\}\right] \\
&\quad=\left|\psi''(b_2)\right|\left[\frac{b_1(b_2-b_1)}{1+q}-\frac{b_2^2-b_1^2}{2(1+q)}+\frac{1}{b_2-b_1}\left\{\frac{(b_2-b_1)^3}{3(1+q)}-b_1b_2(b_2-b_1)\right.\right. \\
&\qquad\left.\left.+(b_1+b_2)\frac{b_2^2-b_1^2}{2}-\frac{b_2^3-b_1^3}{3}\right\}\right] \\
&\quad=\left|\psi''(b_2)\right|\left[\frac{q(b_2-b_1)^2}{6(1+q)}\right],
\end{aligned}
$$

which proves the inequality in (i).

Part (ii) can be proved in a similar fashion. For part (iii), using (3.8) and the fact that $|\psi''|$ is bounded above, on the interval $[b_1,b_2]$, by $\max\{|\psi''(b_1)|,|\psi''(b_2)|\}$ as a convex function, we obtain:

$$
\begin{aligned}
&\left|\frac{q\psi(b_1)+\psi(b_2)}{1+q}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad\le\max\left\{\left|\psi''(b_1)\right|,\left|\psi''(b_2)\right|\right\}\left[\frac{q(b_2-b_1)^2}{6(1+q)}\right].
\end{aligned}
$$

□

Theorem 3.4 *Let $\psi\in C^2([b_1,b_2])$, and let $|\psi''|$ be a concave function. Then, for $0<q<1$,*

$$
\begin{aligned}
&\left|\frac{q\psi(b_1)+\psi(b_2)}{1+q}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad\le(b_2-b_1)^2\left[\frac{1}{2(q+1)}\left|\psi''\left(\frac{b_1+2b_2}{3}\right)\right|\right. \\
&\qquad\left.+\frac{1}{3(q+1)}\left|\psi''\left(\frac{b_1+3b_2}{4}\right)\right|+\frac{1}{6}\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\right].
\end{aligned}
$$

Proof Employing identity (3.8), we have:

$$\begin{aligned}
&\left|\frac{q\psi(b_1)+\psi(b_2)}{1+q}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad=\left|\int_{b_1}^{b_2}\left[\frac{G(b_2,u)}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right]\psi''(u)\,du\right| \\
&\quad=\left|\int_{b_1}^{b_2}\left[\frac{b_1-u}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right]\psi''(u)\,du\right|.
\end{aligned}$$

Suppose $u=(1-t)b_1+tb_2$ with $t\in[0,1]$. Then

$$\begin{aligned}
&=\left|\int_0^1\left[\frac{b_1-(1-t)b_1-tb_2}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{((1-t)b_1+tb_2-b_1)^2}{q+1}\right.\right.\right. \\
&\quad\left.\left.\left.+\big(b_2-(1-t)b_1-tb_2\big)(-tb_1+tb_2)\right\}\right]\psi''\big((1-t)b_1+tb_2\big)(b_2-b_1)\,dt\right| \\
&=\left|\int_0^1\left[\frac{-t(b_2-b_1)}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{t^2(b_2-b_1)^2}{q+1}\right.\right.\right. \\
&\quad\left.\left.\left.+t(1-t)(b_2-b_1)(b_2-b_1)\right\}\right]\psi''\big((1-t)b_1+tb_2\big)(b_2-b_1)\,dt\right| \\
&\le(b_2-b_1)^2\Big[\frac{1}{q+1}\left|\int_0^1 t\psi''\big((1-t)b_1+tb_2\big)\,dt\right| \\
&\quad+\frac{1}{q+1}\left|\int_0^1 t^2\psi''\big((1-t)b_1+tb_2\big)\,dt\right|+\left|\int_0^1 t(1-t)\psi''\big((1-t)b_1+tb_2\big)\,dt\right|\Big]. \qquad (3.9)
\end{aligned}$$

Now, using the Jensen integral inequality, we get the following estimates:

$$\begin{aligned}
&\left|\int_0^1 t\psi''\big((1-t)b_1+tb_2\big)\,dt\right| \\
&\quad\le\int_0^1 t\,dt\left|\psi''\left(\frac{\int_0^1 t((1-t)b_1+tb_2)\,dt}{\int_0^1 t\,dt}\right)\right| \\
&\quad=\frac{1}{2}\left|\psi''\left(\frac{b_1\int_o^1(t-t^2)\,dt+b_2\int_0^1 t^2\,dt}{\frac{1}{2}}\right)\right| \\
&\quad=\frac{1}{2}\left|\psi''\left(\frac{b_1+2b_2}{3}\right)\right|, \qquad (3.10) \\
&\left|\int_0^1 t^2\psi''\big((1-t)b_1+tb_2\big)\,dt\right| \\
&\quad\le\int_0^1 t^2\,dt\left|\psi''\left(\frac{\int_0^1 t^2((1-t)b_1+tb_2)\,dt}{\int_0^1 t^2\,dt}\right)\right| \\
&\quad=\frac{1}{3}\left|\psi''\left(\frac{b_1\int_0^1(t^2-t^3)\,dt+b_2\int_0^1 t^3\,dt}{\frac{1}{3}}\right)\right| \\
&\quad=\frac{1}{3}\left|\psi''\left(\frac{b_1+3b_2}{4}\right)\right|, \qquad (3.11)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^1 \left(t-t^2\right)\psi''\left((1-t)b_1+tb_2\right)dt\right| \\
&\quad\le \int_0^1 \left(t-t^2\right)dt\left|\psi''\left(\frac{\int_0^1 (t-t^2)((1-t)b_1+tb_2)\,dt}{\int_0^1 (t-t^2)\,dt}\right)\right| \\
&\quad= \frac{1}{6}\left|\psi''\left(\frac{b_1\int_0^1 (t-t^2)(1-t)\,dt+b_2\int_0^1 (t^2-t^3)\,dt}{\frac{1}{6}}\right)\right| \\
&\quad= \frac{1}{6}\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|. \qquad (3.12)
\end{aligned}$$

Putting (3.10), (3.11), and (3.12) into (3.9), we get

$$\begin{aligned}
&\left|\frac{q\psi(b_1)+\psi(b_2)}{1+q}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad\le (b_2-b_1)^2\left[\frac{1}{2(q+1)}\left|\psi''\left(\frac{b_1+2b_2}{3}\right)\right|\right. \\
&\qquad\left.+\frac{1}{3(q+1)}\left|\psi''\left(\frac{b_1+3b_2}{4}\right)\right|+\frac{1}{6}\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\right].
\end{aligned}$$

□

Theorem 3.5 *Suppose* $\psi\in C^2([b_1,b_2])$ *and* $q\in(0,1)$. *Then*:

(i). *If* $|\psi''|$ *is an increasing function, then*

$$\begin{aligned}
&\left|\psi''\left(\frac{qb_1+b_2}{1+q}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad\le \left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_1\frac{qb_1+b_2}{1+q}+\frac{q(b_2-b_1)^2}{(1+q)^2}-\frac{(b_2-b_1)^2}{3(q+1)}\right] \\
&\qquad-\frac{|\psi''(b_2)|}{6(b_2-b_1)}\left[2b_1^3+b_2^3-3b_1b_2^2\right].
\end{aligned}$$

(ii). *If* $|\psi''|$ *is a decreasing function, then*

$$\begin{aligned}
&\left|\psi''\left(\frac{qb_1+b_2}{1+q}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad\le \left|\psi''(b_1)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_1\frac{qb_1+b_2}{1+q}+\frac{q(b_2-b_1)^2}{(1+q)^2}-\frac{(b_2-b_1)^2}{3(q+1)}\right] \\
&\qquad-\frac{|\psi''(b_2)|}{6(b_2-b_1)}\left[2b_1^3+b_2^3-3b_1b_2^2\right].
\end{aligned}$$

(iii). *If* $|\psi''|$ *is a convex function, then*

$$\begin{aligned}
&\left|\psi''\left(\frac{qb_1+b_2}{1+q}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad\le \max\left\{\left|\psi''(b_2)\right|,\left|\psi''(b_1)\right|\right\}\left[\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_1\frac{qb_1+b_2}{1+q}+\frac{q(b_2-b_1)^2}{(1+q)^2}\right. \\
&\qquad\left.-\frac{(b_2-b_1)^2}{3(q+1)}\right]-\min\left\{\left|\psi''(b_1)\right|,\left|\psi''(b_2)\right|\right\}\left[\frac{2b_1^3+b_2^3-3b_1b_2^2}{6(b_2-b_1)}\right].
\end{aligned}$$

Proof To prove item (i), we use (3.5) and the fact that $|\psi''|$ is an increasing function to obtain:

$$
\begin{aligned}
&\left|\psi''\left(\frac{qb_1+b_2}{1+q}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad=\left|\int_{b_1}^{b_2}\left[G\left(\frac{qb_1+b_2}{q+1},u\right)+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right]\psi''(u)\,du\right| \\
&\quad\le\left|\psi''(b_2)\right|\int_{b_1}^{b_2}\left|G\left(\frac{qb_1+b_2}{q+1},u\right)+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right|du \\
&\quad\le\left|\psi''(b_2)\right|\left[\int_{b_1}^{\frac{qb_1+b_2}{1+q}}\left|b_1-u+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-u^2+(b_1+b_2)u-b_1b_2\right\}\right|du\right. \\
&\qquad\left.+\int_{\frac{qb_1+b_2}{1+q}}^{b_2}\left|\frac{b_1-b_2}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-u^2+(b_1+b_2)u-b_1b_2\right\}\right|du\right] \\
&\quad=\left|\psi''(b_2)\right|\left[\int_{b_1}^{\frac{qb_1+b_2}{1+q}}\left\{u-b_1-\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-u^2+(b_1+b_2)u-b_1b_2\right\}\right\}du\right. \\
&\qquad\left.+\int_{\frac{qb_1+b_2}{1+q}}^{b_2}\left\{\frac{b_2-b_1}{q+1}-\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-u^2+(b_1+b_2)u-b_1b_2\right\}\right\}du\right] \\
&\quad=\left|\psi''(b_2)\right|\left[\left(\frac{u^2}{2}-b_1u\right)\Big|_{b_1}^{\frac{qb_1+b_2}{1+q}}-\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^3}{3(q+1)}\Big|_{b_1}^{\frac{qb_1+b_2}{1+q}}-\frac{u^3}{3}\Big|_{b_1}^{\frac{qb_1+b_2}{1+q}}\right.\right. \\
&\qquad\left.+(b_1+b_2)\frac{u^2}{2}\Big|_{b_1}^{\frac{qb_1+b_2}{1+q}}-b_2b_1u\Big|_{b_1}^{\frac{qb_1+b_2}{1+q}}\right\}+\frac{b_2-b_1}{1+q}\left(b_2-\frac{qb_1+b_2}{1+q}\right)-\frac{1}{b_2-b_1} \\
&\qquad\left.\times\left\{\frac{(u-b_1)^3}{3(q+1)}|_{\frac{qb_1+b_2}{1+q}}^{b_2}-\frac{u^3}{3}|_{\frac{qb_1+b_2}{1+q}}^{b_2}+(b_1+b_2)\frac{u^2}{2}|_{\frac{qb_1+b_2}{1+q}}^{b_2}-b_2b_1u|_{\frac{qb_1+b_2}{1+q}}^{b_2}\right\}\right] \\
&\quad=\left|\psi''(b_2)\right|\left[\left(\frac{(\frac{qb_1+b_2}{1+q})^2}{2}-b_1\left(\frac{qb_1+b_2}{1+q}\right)-\frac{b_1^2}{2}+b_1^2\right)-\frac{1}{b_2-b_1}\left\{\frac{(\frac{qb_1+b_2}{1+q}-b_1)^3}{3(q+1)}\right.\right. \\
&\qquad-\frac{(qb_1+b_2)^3}{3(1+q)^3}+\frac{b_1^3}{3}+(b_1+b_2)\frac{(qb_1+b_2)^2}{2(1+q)^2}-(b_1+b_2)\frac{b_1^2}{2}-b_2b_1\left(\frac{qb_1+b_2}{1+q}-b_1\right)\Big\} \\
&\qquad+\frac{b_2-b_1}{1+q}\left(b_2-\frac{qb_1+b_2}{1+q}\right)-\frac{1}{b_2-b_1}\left\{\frac{(b_2-b_1)^3}{3(q+1)}-\frac{(\frac{qb_1+b_2}{1+q}-b_1)^3}{3(q+1)}-\frac{b_2^3}{3}\right. \\
&\qquad\left.\left.+\frac{(qb_1+b_2)^3}{3(1+q)^3}+(b_1+b_2)\frac{b_2^2}{2}-(b_1+b_2)\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_2b_1\left(b_2-\frac{qb_1+b_2}{1+q}\right)\right\}\right] \\
&\quad=\left|\psi''(b_2)\right|\left[\left(\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_1\left(\frac{qb_1+b_2}{1+q}\right)+\frac{b_1^2}{2}\right)-\frac{1}{b_2-b_1}\left\{\frac{(b_2-b_1)^3}{3(q+1)^4}\right.\right. \\
&\qquad-\frac{(qb_1+b_2)^3}{3(1+q)^3}+\frac{b_1^3}{3}+(b_1+b_2)\frac{(qb_1+b_2)^2}{2(1+q)^2}-(b_1+b_2)\frac{b_1^2}{2}-b_2b_1\left(\frac{b_2-b_1}{1+q}\right)\Big\} \\
&\qquad+\frac{q(b_2-b_1)^2}{(1+q)^2}-\frac{1}{b_2-b_1}\left\{\frac{(b_2-b_1)^3}{3(q+1)}-\frac{(b_2-b_1)^3}{3(q+1)^4}-\frac{b_2^3}{3}+\frac{(qb_1+b_2)^3}{3(1+q)^3}\right. \\
&\qquad\left.\left.+(b_1+b_2)\frac{b_2^2}{2}-(b_1+b_2)\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_2b_1\frac{q(b_2-b_1)}{1+q}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\left(\frac{qb_1+b_2}{1+q}\right) + \frac{b_1^2}{2} - \frac{1}{b_2-b_1}\left\{\frac{b_1^3}{3} - \frac{(b_1+b_2)b_1^2}{2}\right.\right.\\
&\quad \left. - b_2b_1\left(\frac{b_2-b_1}{1+q}\right)\right\} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{1}{b_2-b_1}\left\{-\frac{b_2^3}{3} + \frac{(b_1+b_2)b_2^2}{2} + \frac{(b_2-b_1)^3}{3(q+1)}\right.\\
&\quad \left.\left. - b_2b_1\frac{q(b_2-b_1)}{1+q}\right\}\right]\\
&= \left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\frac{qb_1+b_2}{1+q} + \frac{b_1^2}{2} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{1}{b_2-b_1}\left\{\frac{b_1^3}{3} - \frac{b_2^3}{3}\right.\right.\\
&\quad \left.\left. - \frac{(b_1+b_2)b_1^2}{2} - b_2b_1\left(\frac{b_2-b_1}{1+q}\right) + \frac{(b_1+b_2)b_2^2}{2} + \frac{(b_2-b_1)^3}{3(q+1)} - b_2b_1\frac{q(b_2-b_1)}{1+q}\right\}\right]\\
&= \left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\frac{qb_1+b_2}{1+q} + \frac{b_1^2}{2} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{1}{b_2-b_1}\left\{\frac{b_1^3}{3}\right.\right.\\
&\quad \left.\left. - \frac{(b_1+b_2)b_1^2}{2} - \frac{b_2^3}{3} + \frac{(b_1+b_2)b_2^2}{2} + \frac{(b_2-b_1)^3}{3(q+1)} - b_2b_1(b_2-b_1)\right\}\right]\\
&= \frac{|\psi''(b_2)|}{b_2-b_1}\left[\frac{(qb_1+b_2)^2(b_2-b_1)}{2(1+q)^2} - b_1\frac{(qb_1+b_2)(b_2-b_1)}{1+q} + \frac{b_1^2(b_2-b_1)}{2} - \frac{b_1^3}{3}\right.\\
&\quad \left. + \frac{q(b_2-b_1)^3}{(1+q)^2} + \frac{(b_1+b_2)b_1^2}{2} + \frac{b_2^3}{3} - \frac{(b_1+b_2)b_2^2}{2} - \frac{(b_2-b_1)^3}{3(q+1)} + b_2^2b_1 - b_2b_1^2\right]\\
&= \left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\frac{qb_1+b_2}{1+q} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{(b_2-b_1)^2}{3(q+1)}\right]\\
&\quad + \frac{|\psi''(b_2)|}{b_2-b_1}\\
&\quad \times \left[\frac{3b_1^2b_2 - 3b_1^3 - 2b_1^3 + 3b_1^3 + 3b_1^2b_2 + 2b_2^3 - 3b_1b_2^2 - 3b_2^3 + 6b_2^2b_1 - 6b_2b_1^2}{6}\right]\\
&= \left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\frac{qb_1+b_2}{1+q} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{(b_2-b_1)^2}{3(q+1)}\right]\\
&\quad - \frac{|\psi''(b_2)|}{6(b_2-b_1)}\left[2b_1^3 + b_2^3 - 3b_1b_2^2\right].
\end{aligned}
$$

Part (ii) can be proved in a similar way. For part (iii), using (3.5) and the fact that $|\psi''|$ is bounded above, on the interval $[b_1, b_2]$, by $\max\{|\psi''(b_1)|, |\psi''(b_2)|\}$ as a convex function, we obtain:

$$
\begin{aligned}
&\left|\psi''\left(\frac{qb_1+b_2}{1+q}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right|\\
&\quad \le \max\left\{\left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\frac{qb_1+b_2}{1+q} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{(b_2-b_1)^2}{3(q+1)}\right]\right.\\
&\quad\quad - \frac{|\psi''(b_2)|}{6(b_2-b_1)}\left[2b_1^3 + b_2^3 - 3b_1b_2^2\right], \left|\psi''(b_1)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\frac{qb_1+b_2}{1+q}\right.\\
&\quad\quad \left.\left. + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{(b_2-b_1)^2}{3(q+1)}\right] - \frac{|\psi''(b_1)|}{6(b_2-b_1)}\left[2b_1^3 + b_2^3 - 3b_1b_2^2\right]\right\}\\
&\quad = \max\left\{\left|\psi''(b_2)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2} - b_1\frac{qb_1+b_2}{1+q} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{(b_2-b_1)^2}{3(q+1)}\right],\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left|\psi''(b_1)\right|\left[\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_1\frac{qb_1+b_2}{1+q}+\frac{q(b_2-b_1)^2}{(1+q)^2}-\frac{(b_2-b_1)^2}{3(q+1)}\right]\Bigg\}\\
&\quad+\max\left\{-\left|\psi''(b_1)\right|\left[\frac{2b_1^3+b_2^3-3b_1b_2^2}{6(b_2-b_1)}\right],\right.\\
&\quad\left.-\left|\psi''(b_2)\right|\left[\frac{2b_1^3+b_2^3-3b_1b_2^2}{6(b_2-b_1)}\right]\right\}\\
&=\max\left\{\left|\psi''(b_2)\right|,\left|\psi''(b_1)\right|\right\}\left[\frac{(qb_1+b_2)^2}{2(1+q)^2}-b_1\frac{qb_1+b_2}{1+q}+\frac{q(b_2-b_1)^2}{(1+q)^2}-\frac{(b_2-b_1)^2}{3(q+1)}\right]\\
&\quad-\min\left\{\left|\psi''(b_1)\right|,\left|\psi''(b_2)\right|\right\}\left[\frac{2b_1^3+b_2^3-3b_1b_2^2}{6(b_2-b_1)}\right],
\end{aligned}
$$

which gives the inequality in item (iii). □

Theorem 3.6 *Let $\psi\in C^2([b_1,b_2])$, and let $|\psi''|$ be a convex function. Then for $q\in(0,1)$, the following inequality holds*:

$$
\begin{aligned}
&\left|\frac{q\psi(b_1)+\psi(b_2)}{1+q}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right|\\
&\quad\le\left|\psi''(b_1)\right|\left[\frac{4b_1b_2+q(b_1+b_2)^2}{12(q+1)}\right]+\left|\psi''(b_2)\right|\left[\frac{q(b_2-b_1)^2-2(q+1)b_1b_2}{12(q+1)}\right].
\end{aligned}
$$

Proof Employing (3.8), we get:

$$
\begin{aligned}
&\left|\frac{q\psi(b_1)+\psi(b_2)}{1+q}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right|\\
&\quad=\left|\int_{b_1}^{b_2}\left[\frac{G(b_2,u)}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right]\psi''(u)\,du\right|\\
&\quad=\left|\int_{b_1}^{b_2}\left[\frac{b_1-u}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right]\psi''(u)\,du\right|.\\
&\quad\le\int_{b_1}^{b_2}\left[\frac{b_1-u}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-u^2+(b_1+b_2)u-b_1b_2\right\}\right]\left|\psi''(u)\right|du.
\end{aligned}
$$

Putting $u=(1-t)b_1+tb_2$ with $t\in[0,1]$, we get

$$
\begin{aligned}
&=\int_0^1\left[\frac{b_1-(1-t)b_1-tb_2}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{((1-t)b_1+tb_2-b_1)^2}{q+1}-\big((1-t)b_1+tb_2\big)^2\right.\right.\\
&\quad\left.\left.+(b_1+b_2)\big((1-t)b_1+tb_2\big)-b_1b_2\right\}\right]\left|\psi''\big((1-t)b_1+tb_2\big)\right|(b_2-b_1)\,dt\\
&\le\int_0^1\left[\frac{t(b_1-b_2)}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{t^2(b_2-b_1)^2}{q+1}-(1-t)^2b_1^2-t^2b_2^2-2(1-t)tb_1b_2\right.\right.\\
&\quad\left.\left.+(b_1+b_2)b_1(1-t)+(b_1+b_2)b_2t-b_1b_2\right\}\right]\left[(1-t)\left|\psi''(b_1)\right|+t\left|\psi''(b_2)\right|\right](b_2-b_1)\,dt\\
&=(b_2-b_1)\left|\psi''(b_1)\right|\int_0^1\left[\frac{t(1-t)(b_1-b_2)}{q+1}+\frac{1}{b_2-b_1}\left\{\frac{t^2(1-t)(b_2-b_1)^2}{q+1}\right.\right.\\
&\quad\left.\left.-(1-t)^3b_1^2-t^2(1-t)b_2^2-2(1-t)^2tb_1b_2+(b_1+b_2)b_1(1-t)^2\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ (b_1 + b_2)b_2t(1-t) - b_1b_2(1-t)\Big\}\Big]\,dt + (b_2-b_1)\big|\psi''(b_2)\big| \int_0^1 \Big[\frac{t^2(b_1-b_2)}{q+1} \\
&+ \frac{1}{b_2-b_1}\Big\{\frac{t^3(b_2-b_1)^2}{q+1} - t(1-t)^2b_1^2 + t^3b_2^2 + 2(1-t)t^2b_1b_2 \\
&+ (b_1+b_2)b_1t(1-t) + (b_1+b_2)b_2t^2 - b_1b_2t\Big\}\Big]\,dt \\
&= \big|\psi''(b_1)\big| \int_0^1 \Big[\frac{(t^2-t)(b_2-b_1)^2}{q+1} + \frac{(t^2-t^3)(b_2-b_1)^2}{q+1} - (1-t)^3b_1^2 - \big(t^2-t^3\big)b_2^2 \\
&- 2(1-t)^2tb_1b_2 + (b_1+b_2)b_1(1-t)^2 + (b_1+b_2)b_2\big(t-t^2\big) - b_1b_2(1-t)\Big]\,dt \\
&+ \big|\psi''(b_2)\big| \int_0^1 \Big[\frac{-t^2(b_2-b_1)^2}{q+1} + \frac{t^3(b_2-b_1)^2}{q+1} - t(1-t)^2b_1^2 - t^3b_2^2 - 2\big(t^2-t^3\big)b_1b_2 \\
&+ (b_1+b_2)b_1\big(t-t^2\big) + (b_1+b_2)b_2t^2 - b_1b_2t\}\Big]\,dt \\
&= \big|\psi''(b_1)\big|\Big[\frac{(\frac{t^3}{3}-\frac{t^2}{2})(b_2-b_1)^2}{q+1} + \frac{(\frac{t^3}{3}-\frac{t^4}{4})(b_2-b_1)^2}{q+1} + \frac{(1-t)^4}{4}b_1^2 - \Big(\frac{t^3}{3}-\frac{t^4}{4}\Big)b_2^2 \\
&+ b_1b_2\Big(t^2+\frac{t^4}{2}-\frac{4t^3}{3}\Big) - (b_1+b_2)b_1\frac{(1-t)^3}{3} + (b_1+b_2)b_2\Big(\frac{t^2}{2}-\frac{t^3}{3}\Big) \\
&- b_1b_2\Big(t-\frac{t^2}{2}\Big)\Big]_0^1 + \big|\psi''(b_2)\big|\Big[\frac{-t^3(b_2-b_1)^2}{3(q+1)} + \frac{t^4(b_2-b_1)^2}{4(q+1)} - \Big(\frac{t^2}{2}+\frac{t^4}{4}-\frac{2t^3}{3}\Big)b_1^2 \\
&- \frac{t^4}{4}b_2^2 - 2\Big(\frac{t^3}{3}-\frac{t^4}{4}\Big)b_1b_2 + (b_1+b_2)b_1\Big(\frac{t^2}{2}-\frac{t^3}{3}\Big) + (b_1+b_2)b_2\frac{t^3}{3} - b_1b_2\frac{t^2}{2}\Big]_0^1 \\
&= \big|\psi''(b_1)\big|\Big[\frac{-(b_2-b_1)^2}{6(q+1)} + \frac{(b_2-b_1)^2}{12(q+1)} - \frac{b_1^2}{4} - \frac{b_2^2}{12} + b_1b_2\Big(1+\frac{1}{2}-\frac{4}{3}\Big) \\
&+ \frac{(b_1+b_2)b_1}{3} + \frac{(b_1+b_2)b_2}{6} - \frac{b_1b_2}{2}\Big] + \big|\psi''(b_2)\big|\Big[-\frac{(b_2-b_1)^2}{3(q+1)} + \frac{(b_2-b_1)^2}{4(q+1)} \\
&- \Big(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\Big)b_1^2 - \frac{b_2^2}{4} - \frac{b_1b_2}{3} + \frac{(b_1+b_2)b_1}{6} + \frac{(b_1+b_2)b_2}{3} - \frac{b_1b_2}{2}\Big] \\
&= \big|\psi''(b_1)\big|\Big[\frac{4b_1b_2+q(b_1+b_2)^2}{12(q+1)}\Big] + \big|\psi''(b_2)\big|\Big[\frac{q(b_2-b_1)^2-2(q+1)b_1b_2}{12(q+1)}\Big]. \qquad \square
\end{aligned}
$$

We now present our last result.

Theorem 3.7 *Let $\psi \in ([b_1, b_2])$ be such that $|\psi''|$ is a convex function. Then for any $q \in (0,1)$, the following inequality holds*:

$$
\begin{aligned}
&\left|\psi''\Big(\frac{qb_1+b_2}{1+q}\Big) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad\le \big|\psi''(b_1)\big|\Big[\frac{(b_2-b_1)^2(5q^2+4q+1)-4(b_1+b_2)b_1}{12(1+q)^3} + \frac{3b_1^2-b_2^2+6b_1b_2}{12}\Big] \\
&\qquad + \big|\psi''(b_2)\big|\Big[\frac{(b_2-b_1)^2(3(1+q)^2+10)}{12(1+q)^3} - \frac{(b_2-b_1)^2}{12}\Big].
\end{aligned}
$$

Proof Using (3.5), we get:

$$
\begin{aligned}
&\left|\psi''\left(\frac{qb_1+b_2}{1+q}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)\,{}_{b_1}d_qx\right| \\
&\quad=\left|\int_{b_1}^{b_2}\left[G\left(\frac{qb_1+b_2}{q+1},u\right)+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right]\psi''(u)\,du\right| \\
&\quad\le\int_{b_1}^{\frac{qb_1+b_2}{1+q}}\left|b_1-u+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right|\left|\psi''(u)\right|du \\
&\qquad+\int_{\frac{qb_1+b_2}{1+q}}^{b_2}\left|\frac{b_1-b_2}{1+q}+\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}+(b_2-u)(u-b_1)\right\}\right|\left|\psi''(u)\right|du \\
&\quad=\int_{b_1}^{\frac{qb_1+b_2}{1+q}}\left[u-b_1-\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-u^2+(b_1+b_2)u-b_1b_2\right\}\right]\left|\psi''(u)\right|du \\
&\qquad+\int_{\frac{qb_1+b_2}{1+q}}^{b_2}\left[\frac{b_2-b_1}{1+q}-\frac{1}{b_2-b_1}\left\{\frac{(u-b_1)^2}{q+1}-u^2+(b_1+b_2)u-b_1b_2\right\}\right]\left|\psi''(u)\right|du \\
&\quad=(b_2-b_1)\int_0^{\frac{1}{1+q}}\left[(1-t)b_1+tb_2-b_1-\frac{1}{b_2-b_1}\left\{\frac{((1-t)b_1+tb_2-b_1)^2}{q+1}\right.\right. \\
&\qquad\left.\left.-\big((1-t)b_1+tb_2\big)^2+(b_1+b_2)\big((1-t)b_1+tb_2\big)-b_1b_2\right\}\right]\left|\psi''\big((1-t)b_1+tb_2\big)\right|dt \\
&\qquad+\int_{\frac{1}{1+q}}^1\left[\frac{b_2-b_1}{1+q}-\frac{1}{b_2-b_1}\left\{\frac{((1-t)b_1+tb_2-b_1)^2}{q+1}-\big((1-t)b_1+tb_2\big)^2\right.\right. \\
&\qquad\left.\left.+(b_1+b_2)\big((1-t)b_1+tb_2\big)-b_1b_2\right\}\right]\left|\psi''\big((1-t)b_1+tb_2\big)\right|(b_2-b_1)\,dt \\
&\quad\le(b_2-b_1)\int_0^{\frac{1}{1+q}}\left[t(b_2-b_1)-\frac{1}{b_2-b_1}\left\{\frac{t^2(b_2-b_1)^2}{q+1}-(1-t)^2b_1^2-t^2b_2^2\right.\right. \\
&\qquad\left.\left.-2t(1-t)b_1b_2+(b_1+b_2)b_1(1-t)+t(b_1+b_2)b_2-b_1b_2\right\}\right]\left[(1-t)\left|\psi''(b_1)\right|\right. \\
&\qquad\left.+t\left|\psi''(b_2)\right|\right]dt+\int_{\frac{1}{1+q}}^1\left[\frac{b_2-b_1}{1+q}-\frac{1}{b_2-b_1}\left\{\frac{t^2(b_2-b_1)^2}{q+1}-(1-t)^2b_1^2\right.\right. \\
&\qquad\left.\left.-t^2b_2^2-2t(1-t)b_1b_2+(b_1+b_2)b_1(1-t)+t(b_1+b_2)b_2-b_1b_2\right\}\right] \\
&\qquad\times\left[(1-t)\left|\psi''(b_1)\right|+t\left|\psi''(b_2)\right|\right](b_2-b_1)\,dt \\
&\quad=(b_2-b_1)\left|\psi''(b_1)\right|\int_0^{\frac{1}{1+q}}\left[t(1-t)(b_2-b_1)-\frac{1}{b_2-b_1}\left\{\frac{t^2(1-t)(b_2-b_1)^2}{q+1}\right.\right. \\
&\qquad\left.\left.-(1-t)^3b_1^2-t^2(1-t)b_2^2-2t(1-t)^2b_1b_2+(b_1+b_2)b_1(1-t)^2\right.\right. \\
&\qquad\left.\left.+t(1-t)(b_1+b_2)b_2-b_1b_2(1-t)\right\}\right]+(b_2-b_1)\left|\psi''(b_2)\right|\int_0^{\frac{1}{1+q}}\left[t^2(b_2-b_1)\right. \\
&\qquad\left.-\frac{1}{b_2-b_1}\left\{\frac{t^3(b_2-b_1)^2}{q+1}-t(1-t)^2b_1^2-t^3b_2^2-2t^2(1-t)b_1b_2\right.\right. \\
&\qquad\left.\left.+(b_1+b_2)b_1t(1-t)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ t^2(b_1+b_2)b_2 - tb_1b_2\Bigg\}\Bigg]dt + (b_2-b_1)\big|\psi''(b_1)\big|\int_{\frac{1}{1+q}}^{1}\Bigg[\frac{b_2-b_1}{1+q}(1-t)\\
&- \frac{1}{b_2-b_1}\Bigg\{\frac{t^2(1-t)(b_2-b_1)^2}{q+1} - (1-t)^3b_1^2 - t^2(1-t)b_2^2 - 2t(1-t)^2b_1b_2\\
&+ (b_1+b_2)b_1(1-t)^2 + t(1-t)(b_1+b_2)b_2 - b_1b_2(1-t)\Bigg\}\Bigg]dt + (b_2-b_1)\big|\psi''(b_2)\big|\\
&\times\int_{\frac{1}{1+q}}^{1}\Bigg[\frac{b_2-b_1}{1+q}t - \frac{1}{b_2-b_1}\Bigg\{\frac{t^3(b_2-b_1)^2}{q+1} - t(1-t)^2b_1^2 - t^3b_2^2 - 2t^2(1-t)b_1b_2\\
&+ (b_1+b_2)b_1t(1-t) + t^2(b_1+b_2)b_2 - tb_1b_2\Bigg\}\Bigg]dt\\
&= (b_2-b_1)\big|\psi''(b_1)\big|\int_0^{\frac{1}{1+q}}\Bigg[\big(t-t^2\big)(b_2-b_1) - \frac{1}{b_2-b_1}\Bigg\{\frac{(t^2-t^3)(b_2-b_1)^2}{q+1}\\
&- (1-t)^3b_1^2 - \big(t^2-t^3\big)b_2^2 - 2\big(t+t^3-2t^2\big)b_1b_2 + (b_1+b_2)b_1(1-t)^2\\
&+ \big(t-t^2\big)(b_1+b_2)b_2 - b_1b_2(1-t)\Bigg\}\Bigg] + (b_2-b_1)\big|\psi''(b_2)\big|\int_0^{\frac{1}{1+q}}\Bigg[t^2(b_2-b_1)\\
&- \frac{1}{b_2-b_1}\Bigg\{\frac{t^3(b_2-b_1)^2}{q+1} - \big(t+t^3-2t^2\big)b_1^2 - t^3b_2^2 - 2\big(t^2-t^3\big)b_1b_2\\
&+ (b_1+b_2)b_1\big(t-t^2\big)\\
&+ t^2(b_1+b_2)b_2 - tb_1b_2\Bigg\}\Bigg]dt + (b_2-b_1)\big|\psi''(b_1)\big|\int_{\frac{1}{1+q}}^{1}\Bigg[\frac{b_2-b_1}{1+q}(1-t)\\
&- \frac{1}{b_2-b_1}\Bigg\{\frac{(t^2-t^3)(b_2-b_1)^2}{q+1} - (1-t)^3b_1^2 - \big(t^2-t^3\big)b_2^2 - 2\big(t+t^3-t^2\big)b_1b_2\\
&+ (b_1+b_2)b_1(1-t)^2 + \big(t-t^2\big)(b_1+b_2)b_2 - b_1b_2(1-t)\Bigg\}\Bigg]dt + (b_2-b_1)\big|\psi''(b_2)\big|\\
&\times\int_{\frac{1}{1+q}}^{1}\Bigg[\frac{b_2-b_1}{1+q}t - \frac{1}{b_2-b_1}\Bigg\{\frac{t^3(b_2-b_1)^2}{q+1} - \big(t+t^3-2t^2\big)b_1^2 - t^3b_2^2 - 2\big(t^2-t^3\big)b_1b_2\\
&+ (b_1+b_2)b_1\big(t-t^2\big) + t^2(b_1+b_2)b_2 - tb_1b_2\Bigg\}\Bigg]dt\\
&= (b_2-b_1)\big|\psi''(b_1)\big|\Bigg[\left(\frac{t^2}{2}-\frac{t^3}{3}\right)(b_2-b_1) - \frac{1}{b_2-b_1}\Bigg\{\frac{(\frac{t^3}{3}-\frac{t^4}{4})(b_2-b_1)^2}{q+1}\\
&+ \frac{(1-t)^4}{4}b_1^2 - \left(\frac{t^3}{3}-\frac{t^4}{4}\right)b_2^2 - 2\left(\frac{t^2}{2}+\frac{t^4}{4}-2\frac{t^3}{3}\right)b_1b_2 - (b_1+b_2)b_1\frac{(1-t)^3}{3}\\
&+ \left(\frac{t^2}{2}-\frac{t^3}{3}\right)(b_1+b_2)b_2 - b_1b_2\left(t-\frac{t^2}{2}\right)\Bigg\}\Bigg]_0^{\frac{1}{1+q}} + (b_2-b_1)\big|\psi''(b_2)\big|\\
&\times\Bigg[\frac{t^3}{3}(b_2-b_1) - \frac{1}{b_2-b_1}\Bigg\{\frac{t^4(b_2-b_1)^2}{4(q+1)} - \left(\frac{t^2}{2}+\frac{t^4}{4}-\frac{2t^3}{3}\right)b_1^2 - 2\left(\frac{t^3}{3}-\frac{t^4}{4}\right)b_1b_2\\
&- \frac{t^4}{4}b_2^2 + (b_1+b_2)b_1\left(\frac{t^2}{2}-\frac{t^3}{3}\right) + \frac{t^3}{3}(b_1+b_2)b_2 - \frac{t^2}{2}b_1b_2\Bigg\}\Bigg]_0^{\frac{1}{1+q}}\\
&+ (b_2-b_1)\big|\psi''(b_1)\big|
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\frac{b_2-b_1}{1+q}\left(t-\frac{t^2}{2}\right) - \frac{1}{b_2-b_1}\left\{ \frac{(\frac{t^3}{3}-\frac{t^4}{4})(b_2-b_1)^2}{q+1} + \frac{(1-t)^4}{4}b_1^2 - \left(\frac{t^3}{3}-\frac{t^4}{4}\right)b_2^2 \right.\right.\\
&- 2\left(\frac{t^2}{2}+\frac{t^4}{4}-\frac{2t^3}{3}\right)b_1b_2 - (b_1+b_2)b_1\frac{(1-t)^3}{3} + \left(\frac{t^2}{2}-\frac{t^3}{3}\right)(b_1+b_2)b_2\\
&\left.\left. - b_1b_2\left(t-\frac{t^2}{2}\right)\right\}\right]_{\frac{1}{1+q}}^{1} + (b_2-b_1)\left|\psi''(b_2)\right|\left[\frac{b_2-b_1}{2(1+q)}t^2 - \frac{1}{b_2-b_1}\left\{ \frac{t^4(b_2-b_1)^2}{4(q+1)} \right.\right.\\
&- \left(\frac{t^2}{2}+\frac{t^4}{4}-\frac{2t^3}{3}\right)b_1^2 - \frac{t^4}{4}b_2^2 - 2\left(\frac{t^3}{3}-\frac{t^4}{4}\right)b_1b_2 + (b_1+b_2)b_1\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\\
&\left.\left. + \frac{t^3}{3}(b_1+b_2)b_2 - \frac{t^2}{2}b_1b_2\right\}\right]_{\frac{1}{1+q}}^{1}\\
&= (b_2-b_1)\left|\psi''(b_1)\right|\left[\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)(b_2-b_1) - \frac{1}{b_2-b_1}\left\{ \frac{q^4b_1^2}{4(1+q)^4} \right.\right.\\
&- \frac{b_1^2}{4} + \frac{(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4})(b_2-b_1)^2}{q+1} - \left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_2^2\\
&- 2\left(\frac{1}{2(1+q)^2}+\frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1b_2 - \frac{q^3(b_1+b_2)b_1}{3(1+q)^3} + \frac{(b_1+b_2)b_1}{3(1+q)^3}\\
&\left.\left. + \left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)(b_1+b_2)b_2 - \frac{(1+2q)b_1b_2}{2(1+q)^2}\right\}\right]\\
&+ (b_2-b_1)\left|\psi''(b_2)\right|\left[\frac{b_2-b_1}{3(1+q)^3} - \frac{1}{b_2-b_1}\left\{ \frac{(b_2-b_1)^2}{4(q+1)^5} - \left(\frac{1}{2(1+q)^2}\right.\right.\right.\\
&\left.\left.\left. + \frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1^2 - \frac{1}{4(1+q)^4}b_2^2 - 2\left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_1b_2\right.\right.\\
&\left.\left. + (b_1+b_2)b_1\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right) + \frac{(b_1+b_2)b_2}{3(1+q)^3} - \frac{b_1b_2}{2(1+q)^2}\right\}\right]\\
&+ (b_2-b_1)\left|\psi''(b_1)\right|\left[\frac{b_2-b_1}{2(1+q)} - \frac{(b_2-b_1)(1+2q)}{2(1+q)^3} - \frac{1}{b_2-b_1}\left\{ \frac{(b_2-b_1)^2}{12(q+1)} \right.\right.\\
&- \frac{(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4})(b_2-b_1)^2}{q+1} - \frac{q^4b_1^2}{4(1+q)^4} - \left(\frac{1}{3}-\frac{1}{4}\right)b_2^2\\
&+ \left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_2^2 - 2\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)b_1b_2 + 2\left(\frac{1}{2(1+q)^2}\right.\\
&\left. + \frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1b_2 + \frac{q^3(b_1+b_2)b_1}{3(1+q)^3} + \left(\frac{1}{2}-\frac{1}{3}\right)(b_1+b_2)b_2\\
&\left.\left. - \left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)(b_1+b_2)b_2 - \frac{b_1b_2}{2} + \frac{b_1b_2(1+2q)}{2(1+q)^2}\right\}\right]\\
&+ (b_2-b_1)\left|\psi''(b_2)\right|\left[\frac{b_2-b_1}{2(1+q)} - \frac{b_2-b_1}{2(1+q)^3} - \frac{1}{b_2-b_1}\left\{ \frac{(b_2-b_1)^2}{4(q+1)} - \frac{(b_2-b_1)^2}{4(q+1)^5} \right.\right.\\
&- \left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)b_1^2 + \left(\frac{1}{2(1+q)^2}+\frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1^2 - \frac{b_2^2}{4} + \frac{b_2^2}{4(1+q)^4}\\
&\left.\left. - 2\left(\frac{1}{3}-\frac{1}{4}\right)b_1b_2 + 2\left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_1b_2 + (b_1+b_2)b_1\left(\frac{1}{2}-\frac{1}{3}\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&- (b_1+b_2)b_1\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)+\frac{(b_1+b_2)b_2}{3}-\frac{(b_1+b_2)b_2}{3(1+q)^3}-\frac{b_1b_2}{2}\\
&+\frac{b_1b_2}{2(1+q)^2}\Bigg\}\Bigg]\\
&=(b_2-b_1)\left|\psi''(b_1)\right|\Bigg[\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)(b_2-b_1)-\frac{1}{b_2-b_1}\Bigg\{\frac{q^4b_1^2}{4(1+q)^4}\\
&-\frac{b_1^2}{4}+\frac{(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4})(b_2-b_1)^2}{q+1}-\left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_2^2\\
&-2\left(\frac{1}{2(1+q)^2}+\frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1b_2-\frac{q^3(b_1+b_2)b_1}{3(1+q)^3}+\frac{(b_1+b_2)b_1}{3(1+q)^3}\\
&+\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)(b_1+b_2)b_2-\frac{(1+2q)b_1b_2}{2(1+q)^2}\Bigg\}+\frac{b_2-b_1}{2(1+q)}\\
&-\frac{(b_2-b_1)(1+2q)}{2(1+q)^3}-\frac{1}{b_2-b_1}\Bigg\{\frac{(b_2-b_1)^2}{12(q+1)}-\frac{(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4})(b_2-b_1)^2}{q+1}\\
&-\frac{q^4b_1^2}{4(1+q)^4}-\left(\frac{1}{3}-\frac{1}{4}\right)b_2^2+\left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_2^2-2\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)b_1b_2\\
&+2\left(\frac{1}{2(1+q)^2}+\frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1b_2+\frac{q^3(b_1+b_2)b_1}{3(1+q)^3}\\
&+\left(\frac{1}{2}-\frac{1}{3}\right)(b_1+b_2)b_2\\
&-\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)(b_1+b_2)b_2-\frac{b_1b_2}{2}+\frac{b_1b_2(1+2q)}{2(1+q)^2}\Bigg\}\Bigg]\\
&+(b_2-b_1)\left|\psi''(b_2)\right|\Bigg[\frac{b_2-b_1}{3(1+q)^3}-\frac{1}{b_2-b_1}\Bigg\{\frac{(b_2-b_1)^2}{4(q+1)^5}-\Bigg(\frac{1}{2(1+q)^2}\\
&+\frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\Bigg)b_1^2-\frac{1}{4(1+q)^4}b_2^2-2\left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_1b_2\\
&+(b_1+b_2)b_1\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)+\frac{(b_1+b_2)b_2}{3(1+q)^3}-\frac{b_1b_2}{2(1+q)^2}\Bigg\}\\
&+\frac{b_2-b_1}{2(1+q)}-\frac{b_2-b_1}{2(1+q)^3}-\frac{1}{b_2-b_1}\Bigg\{\frac{(b_2-b_1)^2}{4(q+1)}-\frac{(b_2-b_1)^2}{4(q+1)^5}-\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)b_1^2\\
&+\left(\frac{1}{2(1+q)^2}+\frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1^2-\frac{b_2^2}{4}+\frac{b_2^2}{4(1+q)^4}-2\left(\frac{1}{3}-\frac{1}{4}\right)b_1b_2\\
&+2\left(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4}\right)b_1b_2+(b_1+b_2)b_1\left(\frac{1}{2}-\frac{1}{3}\right)-(b_1+b_2)b_1\Bigg(\frac{1}{2(1+q)^2}\\
&-\frac{1}{3(1+q)^3}\Bigg)+\frac{(b_1+b_2)b_2}{3}-\frac{(b_1+b_2)b_2}{3(1+q)^3}-\frac{b_1b_2}{2}+\frac{b_1b_2}{2(1+q)^2}\Bigg\}\Bigg]\\
&=\left|\psi''(b_1)\right|\Bigg[\left(\frac{1}{2(1+q)^2}-\frac{1}{3(1+q)^3}\right)(b_2-b_1)^2-\frac{q^4b_1^2}{4(1+q)^4}\\
&+\frac{b_1^2}{4}-\frac{(\frac{1}{3(1+q)^3}-\frac{1}{4(1+q)^4})(b_2-b_1)^2}{q+1}+\left(\frac{1}{3(1+q)^3}+\frac{1}{4(1+q)^4}\right)b_2^2\\
&+2\left(\frac{1}{2(1+q)^2}+\frac{1}{4(1+q)^4}-\frac{2}{3(1+q)^3}\right)b_1b_2+\frac{q^3(b_1+b_2)b_1}{3(1+q)^3}-\frac{(b_1+b_2)b_1}{3(1+q)^3}
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3}\right)(b_1+b_2)b_2 + \frac{(1+2q)b_1b_2}{2(1+q)^2} + \frac{(b_2-b_1)^2}{2(1+q)} \\
&- \frac{(b_2-b_1)^2(1+2q)}{2(1+q)^3} - \frac{(b_2-b_1)^2}{12(q+1)} + \frac{(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4})(b_2-b_1)^2}{q+1} \\
&+ \frac{q^4b_1^2}{4(1+q)^4} + \left(\frac{1}{3} - \frac{1}{4}\right)b_2^2 - \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4}\right)b_2^2 + 2\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)b_1b_2 \\
&- 2\left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3}\right)b_1b_2 - \frac{q^3(b_1+b_2)b_1}{3(1+q)^3} \\
&- \left(\frac{1}{2} - \frac{1}{3}\right)(b_1+b_2)b_2 \\
&+ \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3}\right)(b_1+b_2)b_2 + \frac{b_1b_2}{2} - \frac{b_1b_2(1+2q)}{2(1+q)^2}\Bigg] \\
&+ \left|\psi''(b_2)\right|\Bigg[\frac{(b_2-b_1)^2}{3(1+q)^3} - \frac{(b_2-b_1)^2}{4(q+1)^5} + \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3}\right)b_1^2 \\
&+ \frac{1}{4(1+q)^4}b_2^2 + 2\left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4}\right)b_1b_2 - (b_1+b_2)b_1\left(\frac{1}{2(1+q)^2}\right. \\
&\left. - \frac{1}{3(1+q)^3}\right) - \frac{(b_1+b_2)b_2}{3(1+q)^3} + \frac{b_1b_2}{2(1+q)^2} + \frac{(b_2-b_1)^2}{2(1+q)} + \frac{(b_2-b_1)^2}{2(1+q)^3} \\
&- \frac{(b_2-b_1)^2}{4(q+1)} + \frac{(b_2-b_1)^2}{4(q+1)^5} + \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)b_1^2 - \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4}\right. \\
&\left. - \frac{2}{3(1+q)^3}\right)b_1^2 + \frac{b_2^2}{4} - \frac{b_2^2}{4(1+q)^4} + 2\left(\frac{1}{3} - \frac{1}{4}\right)b_1b_2 - 2\left(\frac{1}{3(1+q)^3}\right. \\
&\left. - \frac{1}{4(1+q)^4}\right)b_1b_2 - (b_1+b_2)b_1\left(\frac{1}{2} - \frac{1}{3}\right) + (b_1+b_2)b_1\left(\frac{1}{2(1+q)^2}\right. \\
&\left. - \frac{1}{3(1+q)^3}\right) - \frac{(b_1+b_2)b_2}{3} + \frac{(b_1+b_2)b_2}{3(1+q)^3} + \frac{b_1b_2}{2} - \frac{b_1b_2}{2(1+q)^2}\Bigg] \\
&= \left|\psi''(b_1)\right|\Bigg[\left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3}\right)(b_2-b_1)^2 + \frac{b_1^2}{4} - \frac{(b_1+b_2)b_1}{3(1+q)^3} + \frac{(b_2-b_1)^2}{2(1+q)} \\
&- \frac{(b_2-b_1)^2(1+2q)}{2(1+q)^3} - \frac{(b_2-b_1)^2}{12(q+1)} + \left(\frac{1}{3} - \frac{1}{4}\right)b_2^2 + 2\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)b_1b_2 \\
&- \left(\frac{1}{2} - \frac{1}{3}\right)(b_1+b_2)b_2 + \frac{b_1b_2}{2}\Bigg] + \left|\psi''(b_2)\right|\Bigg[\frac{(b_2-b_1)^2}{3(1+q)^3} + \frac{(b_2-b_1)^2}{2(1+q)} \\
&+ \frac{(b_2-b_1)^2}{2(1+q)^3} - \frac{(b_2-b_1)^2}{4(q+1)} + \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)b_1^2 + \frac{b_2^2}{4} + 2\left(\frac{1}{3} - \frac{1}{4}\right)b_1b_2 \\
&- (b_1+b_2)b_1\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{(b_1+b_2)b_2}{3} + \frac{b_1b_2}{2}\Bigg] \\
&= \left|\psi''(b_1)\right|\Bigg[\frac{(1+3q)(b_2-b_1)^2}{6(1+q)^3} + \frac{b_1^2}{4} - \frac{(b_1+b_2)b_1}{3(1+q)^3} + \frac{(b_2-b_1)^2}{2(1+q)} \\
&- \frac{(b_2-b_1)^2(1+2q)}{2(1+q)^3} - \frac{(b_2-b_1)^2}{12(q+1)} + \frac{b_2^2}{12} + \frac{b_1b_2}{6} - \frac{(b_1+b_2)b_2}{6} + \frac{b_1b_2}{2}\Bigg] \\
&+ \left|\psi''(b_2)\right|\Bigg[\frac{(b_2-b_1)^2}{3(1+q)^3} + \frac{(b_2-b_1)^2}{2(1+q)} + \frac{(b_2-b_1)^2}{2(1+q)^3} - \frac{(b_2-b_1)^2}{4(q+1)} + \frac{b_1^2}{12} + \frac{b_2^2}{4}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{b_1b_2}{6}-\frac{(b_1+b_2)b_1}{6}-\frac{(b_1+b_2)b_2}{3}+\frac{b_1b_2}{2}\Bigg]\\
&=\left|\psi''(b_1)\right|\Bigg[\frac{(1+3q)(b_2-b_1)^2}{6(1+q)^3}-\frac{(b_1+b_2)b_1}{3(1+q)^3}+\frac{(b_2-b_1)^2}{2(1+q)}-\frac{(b_2-b_1)^2(1+2q)}{2(1+q)^3}\\
&\quad-\frac{(b_2-b_1)^2}{12(q+1)}+\frac{b_1^2}{4}+\frac{b_2^2}{12}+\frac{b_1b_2}{6}-\frac{(b_1+b_2)b_2}{6}+\frac{b_1b_2}{2}\Bigg]\\
&\quad+\left|\psi''(b_2)\right|\Bigg[\frac{(b_2-b_1)^2}{3(1+q)^3}+\frac{(b_2-b_1)^2}{2(1+q)}+\frac{(b_2-b_1)^2}{2(1+q)^3}-\frac{(b_2-b_1)^2}{4(q+1)}+\frac{b_1^2}{12}+\frac{b_2^2}{4}\\
&\quad+\frac{b_1b_2}{6}-\frac{(b_1+b_2)b_1}{6}-\frac{(b_1+b_2)b_2}{3}+\frac{b_1b_2}{2}\Bigg]\\
&=\left|\psi''(b_1)\right|\Bigg[\frac{(b_2-b_1)^2(5q^2+4q+1)-4(b_1+b_2)b_1}{12(1+q)^3}+\frac{3b_1^2-b_2^2+6b_1b_2}{12}\Bigg]\\
&\quad+\left|\psi''(b_2)\right|\Bigg[\frac{(b_2-b_1)^2(3(1+q)^2+10)}{12(1+q)^3}-\frac{(b_2-b_1)^2}{12}\Bigg].
\end{aligned}
$$

□

4 Conclusion

We revisited the Hermite–Hadamard inequality in quantum calculus. We deduced some new identities in the way. Using these identities, we obtained new estimates in this regard. Employing the method outlined in this paper, we anticipate that some other inequalities may be reestablished. More results on the Hermite–Hadamard inequality in quantum calculus can be found in [22, 26, 31, 34].

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Authors' contributions
MAK provided the main idea and carried out the proof of Theorem 3.2. NM carried out the proof of Theorems 3.4 and 3.5. ERN carried out the proof of Theorems 3.6 and 3.7. YMC carried out the proof of Theorem 3.3, drafted the first version of the manuscript, and completed the revision of the manuscript. All authors agreed to the authorship and their order in the final manuscript, and all authors read and approved the final manuscript.

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