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Fixed time synchronization of delayed quaternion-valued memristor-based neural networks

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Abstract

This paper investigates the fixed time synchronization issue for a class of quaternion-valued memristor-based neural networks (QVMNN) at the presence of time varying delays. Differential inclusion and fixed time stability theory are used, and new synchronization conditions are formulated to achieve the synchronization of delayed QVMNN within a fixed time based on a Lyapunov function and a suitable controller. The feasibility of the proposed method is shown through numerical simulations.

Keywords: Fixed time synchronization; Lyapunov function; Quaternion; Time varying delays; Memristor-based neural networks

1 Introduction

Synchronization control is the hot topic of research into nonlinear systems, which is widely applied in various areas. Different synchronization schemes have been presented, including exponential synchronization, complete synchronization, projective synchronization, lag synchronization, and adaptive synchronization [1–5]. It is worthy to point out that these kinds of synchronization strategies considered an infinite time. Therefore, the method which can realize the synchronization within a finite time has important practical application values. The finite time synchronization was first introduced by Kamenkov in [6].

As we all know, neural networks (NN) own numerous specialties, including powerful self-learning and tolerance ability, parallel computing, which have been applied in pattern recognition, signal processing, optimal, and so on. In recent years, the study of its dynamical characteristic, such as finite time stability and synchronization, has gained tremendous attention, and considerable works have been done [7–13].

Finite time synchronization relies on its setting time, which depends on its initial error. To resolve this issue, the concept of fixed time synchronization (FTS) is introduced in [14], implying that the setting time has a convergence upper bound which is independent of the initial values. Because of this merit, FTS has attracted great attention in lots of fields, including power system [15], traffic [16], and consensus [17]. So far, some inter-

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esting results on FTS of NN have been presented [18–24]. Based on Lyapunov functionals, FTS for recurrent memristive NN (MNN) with time delay was presented in [21]. In [22], authors studied the FTS of complex-valued NN (CVNN) with discontinuous activation functions and parameter uncertainties by decomposing CVNN into two real-valued NN (RVNN). By applying the Lyapunov stability theorem, Ref. [23] considered the FTS of uncertain Cohen–Grossberg NN with time varying delay. In [24], the FTS for a class of impulsive MNN with time varying delay was discussed through designing a suitable controller.

Recently, some scholars introduced quaternion algebra into NN to form quaternion-valued NN (QVNN). Compared with RVNN and CVNN, QVNN can be applied to deal with multidimensional problems, such as image processing, 3-D wind processing, color night vision [25–27]. Moreover, QVNN possess superiority in dealing with optimization and estimation problems and they have widely potential application in engineering field [28, 29]. Thereupon, it is necessary to consider QVNN. In recent years, a few results on dynamical properties of QVNN, including stability, periodicity, dissipativity, and passivity, have appeared [30–33]. Moreover, the memristor, as a fourth fundamental circuit element, was reported in Nature. The memristor exhibits numerous superiority such as high density, good scalability. Therefore, many scholars constructed the memristor-based neural networks (MNN) by replacing the traditional resistor with memristor. However, there are few results considering FTS for QVNN [34], not mentioning quaternion-valued MNN (QVMNN). On the other hand, due to the finite switching speed of amplifiers, time delays inevitable exist in NN, which may become a key source of instability, oscillation or chaos in the neural system. Furthermore, the delays are time varying in nature. Consequently, it is necessary to consider QVMNN at the presence of time varying delays in our study.

Motivated by the above discussions, this paper focuses on studying the FTS of QVMNN with time varying delays. The three distinctive advantages are listed below. First, taking time delays and memristor into consideration, the model of QVMNN with time varying delays is established, which is more complex and more general. Second, the issue of FTS in delayed QVMNN is discussed. By applying the fixed time control and inequalities skills, some novel conclusions are developed in this paper. Third, the obtained results in this paper can be applied to handle the RVNN, CVNN, and QVNN with or without delays.

Notations. In this paper, $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ refer to the real numbers, complex numbers, and quaternion numbers, respectively. $C([-\tau, 0], \mathbb{Q}^n)$ denotes the continuous functions from $[-\tau, 0]$ to \mathbb{Q}^n .

2 Preliminaries

A quaternion-valued y is described as $y = y^R + \iota y^I + J y^J + \kappa y^K$, where $y^R, y^I, y^J, y^K \in \mathbb{R}$, the imaginary units ι, J, κ can satisfy the Hamilton rules:

	1	ι	J	κ
1	1	ι	J	κ
ι	ι	-1	κ	- J
J	J	- κ	-1	ι
κ	κ	J	- ι	-1

Remark 1 Unlike real value and complex value, the commutative law is not true for quaternion according to Hamilton rules.

The model of delayed QVMNN is described as follows:

$$\begin{aligned} \dot{z}_p(t) = & -d_p z_p(t) + \sum_{q=1}^n a_{pq}(z_p(t)) f_q(z_q(t)) \\ & + \sum_{q=1}^n b_{pq}(z_p(t)) g_q(z_q(t - \tau(t))) \\ & + l_p(t), \quad t \geq 0, p = 1, 2, \dots, n, \end{aligned} \tag{1}$$

where $z_p(t) \in \mathbb{Q}$ denotes the state vector of the p th neuron. $d_p > 0$ stands for the self-feedback coefficient, $a_{pq}(z_p(t)), b_{pq}(z_p(t)) \in \mathbb{Q}$ are the memristive connection weight matrices, $f_q(\cdot) \in \mathbb{Q}$ stands for the neuron activation function. $\tau(t)$ represents time varying delays, $l_p \in \mathbb{Q}$ denotes the external input vector. The initial condition is given by $z(s) = \psi(s) \in C([-\tau, 0], \mathbb{Q}^n)$, $-\tau \leq s \leq 0$. System (1) is considered as a drive system.

The memristor connection weights $a_{pq}(z_p(t))$ and $b_{pq}(z_p(t))$ satisfy the following conditions:

$$\begin{aligned} a_{pq}(z_p(t)) = & \begin{cases} \acute{a}_{pq}, & |z_p(t)| \leq T_p; \\ \grave{a}_{pq}, & |z_p(t)| > T_p, \end{cases} \\ b_{pq}(z_p(t)) = & \begin{cases} \acute{b}_{pq}, & |z_p(t)| \leq T_p; \\ \grave{b}_{pq}, & |z_p(t)| > T_p, \end{cases} \end{aligned} \tag{2}$$

where $T_p > 0$ is the switching jumps, $\acute{a}_{pq}, \grave{a}_{pq}, \acute{b}_{pq}, \grave{b}_{pq} \in \mathbb{Q}$ are constants.

Remark 2 The considered system (1) takes into account quaternion values, time-varying delays, and memristors. Compared with [33, 34], the system in [33, 34] is just a particular case of this paper.

To carry forward the main results, some necessary assumptions are presented.

Assumption 1 Suppose $z_p = z_p^R + \iota z_p^I + J z_p^J + \kappa z_p^K$, the activation function

$$f_q(z_q) = f_q^R(z_q^R) + \iota f_q^I(z_q^I) + J f_q^J(z_q^J) + \kappa f_q^K(z_q^K).$$

Assumption 2 For $p = 1, 2, \dots, n, \forall z_p^\mu, \tilde{z}_p^\mu \in \mathbb{R} (\mu = R, I, J, K)$, there exist $l_p^\mu, h_p^\mu > 0$ such that

$$\begin{aligned} |f_p^\mu(z_p^\mu) - f_p^\mu(\tilde{z}_p^\mu)| & \leq l_p^\mu |z_p^\mu - \tilde{z}_p^\mu|, \\ |g_p^\mu(z_p^\mu) - g_p^\mu(\tilde{z}_p^\mu)| & \leq h_p^\mu |z_p^\mu - \tilde{z}_p^\mu|. \end{aligned}$$

Assumption 3 $\tau(t)$ is differentiable and satisfies $0 \leq \tau(t) \leq \tau$ and $\dot{\tau}(t) \leq \sigma < 1$, where σ and τ are constants.

By employing quaternion multiplication with the non-commutativity, system (1) can be represented as the following four parts:

$$\left. \begin{aligned}
 \dot{z}_p^R(t) &= -d_p z_p^R(t) + \sum_{q=1}^n [a_{pq}^R(z_p^R(t)) f_q^R(z_q^R(t)) \\
 &\quad - a_{pq}^I(z_p^I(t)) f_q^I(z_q^I(t)) \\
 &\quad - a_{pq}^J(z_p^J(t)) f_q^J(z_q^J(t)) - a_{pq}^K(z_p^K(t)) f_q^K(z_q^K(t))] \\
 &\quad + \sum_{q=1}^n [b_{pq}^R(z_p^R(t)) g_q^R(z_q^R(t - \tau(t))) \\
 &\quad - b_{pq}^I(z_p^I(t)) g_q^I(z_q^I(t - \tau(t))) \\
 &\quad - b_{pq}^J(z_p^J(t)) g_q^J(z_q^J(t - \tau(t))) \\
 &\quad - b_{pq}^K(z_p^K(t)) g_q^K(z_q^K(t - \tau(t)))] , \\
 \dot{z}_p^I(t) &= -d_p z_p^I(t) + \sum_{q=1}^n [a_{pq}^I(z_p^I(t)) f_q^R(z_q^R(t)) \\
 &\quad + a_{pq}^R(z_p^R(t)) f_q^I(z_q^I(t)) \\
 &\quad + a_{pq}^K(z_p^K(t)) f_q^J(z_q^J(t)) - a_{pq}^J(z_p^J(t)) f_q^K(z_q^K(t))] \\
 &\quad + \sum_{q=1}^n [b_{pq}^I(z_p^I(t)) g_q^R(z_q^R(t - \tau(t))) \\
 &\quad + b_{pq}^R(z_p^R(t)) g_q^I(z_q^I(t - \tau(t))) \\
 &\quad + b_{pq}^K(z_p^K(t)) g_q^J(z_q^J(t - \tau(t))) \\
 &\quad - b_{pq}^J(z_p^J(t)) g_q^K(z_q^K(t - \tau(t)))] , \\
 \dot{z}_p^J(t) &= -d_p z_p^J(t) + \sum_{q=1}^n [a_{pq}^J(z_p^J(t)) f_q^R(z_q^R(t)) \\
 &\quad + a_{pq}^K(z_p^K(t)) f_q^I(z_q^I(t)) \\
 &\quad + a_{pq}^R(z_p^R(t)) f_q^J(z_q^J(t)) - a_{pq}^I(z_p^I(t)) f_q^K(z_q^K(t))] \\
 &\quad + \sum_{q=1}^n [b_{pq}^J(z_p^J(t)) g_q^R(z_q^R(t - \tau(t))) \\
 &\quad + b_{pq}^K(z_p^K(t)) g_q^I(z_q^I(t - \tau(t))) \\
 &\quad + b_{pq}^R(z_p^R(t)) g_q^J(z_q^J(t - \tau(t))) \\
 &\quad - b_{pq}^I(z_p^I(t)) g_q^K(z_q^K(t - \tau(t)))] , \\
 \dot{z}_p^K(t) &= -d_p z_p^K(t) + \sum_{q=1}^n [a_{pq}^K(z_p^K(t)) f_q^R(z_q^R(t)) \\
 &\quad - a_{pq}^I(z_p^I(t)) f_q^I(z_q^I(t)) \\
 &\quad + a_{pq}^J(z_p^J(t)) f_q^J(z_q^J(t)) + a_{pq}^R(z_p^R(t)) f_q^K(z_q^K(t))] \\
 &\quad + \sum_{q=1}^n [b_{pq}^K(z_p^K(t)) g_q^R(z_q^R(t - \tau(t))) \\
 &\quad - b_{pq}^J(z_p^J(t)) g_q^I(z_q^I(t - \tau(t))) \\
 &\quad + b_{pq}^I(z_p^I(t)) g_q^J(z_q^J(t - \tau(t))) \\
 &\quad - b_{pq}^R(z_p^R(t)) g_q^K(z_q^K(t - \tau(t)))] .
 \end{aligned} \right\} \tag{3}$$

According to the characteristics of memristor and current voltage, we have

$$\begin{aligned}
 a_{pq}^\mu(z_p^\mu(t)) &= \begin{cases} \acute{a}_{pq}^\mu, & |z_p^\mu(t)| \leq T_p; \\ \grave{a}_{pq}^\mu, & |z_p^\mu(t)| > T_p, \end{cases} \\
 b_{pq}^\mu(z_p^\mu(t)) &= \begin{cases} \acute{b}_{pq}^\mu, & |z_p^\mu(t)| \leq T_p; \\ \grave{b}_{pq}^\mu, & |z_p^\mu(t)| > T_p, \end{cases}
 \end{aligned} \tag{4}$$

where $T_p > 0$ is the switching jumps, $\acute{a}_{pq}^\mu, \grave{a}_{pq}^\mu, \acute{b}_{pq}^\mu, \grave{b}_{pq}^\mu$ are constants, $\mu = R, I, J, K$.

Applying the theory of differential inclusion and the definition of Filippov solution, system (3) can be rewritten as follows:

$$\left\{ \begin{aligned} \dot{z}_p^R(t) &\in -d_p z_p^R(t) + \sum_{q=1}^n [co\{\bar{a}_{pq}^R, \underline{a}_{pq}^R\} f_q^R(z_q^R(t)) - co\{\bar{a}_{pq}^I, \underline{a}_{pq}^I\} f_q^I(z_q^I(t)) \\ &\quad - co\{\bar{a}_{pq}^J, \underline{a}_{pq}^J\} f_q^J(z_q^J(t)) - co\{\bar{a}_{pq}^K, \underline{a}_{pq}^K\} f_q^K(z_q^K(t))] \\ &\quad + \sum_{q=1}^n [co\{\bar{b}_{pq}^R, \underline{b}_{pq}^R\} g_q^R(z_q^R(t - \tau(t))) - co\{\bar{b}_{pq}^I, \underline{b}_{pq}^I\} g_q^I(z_q^I(t - \tau(t))) \\ &\quad - co\{\bar{b}_{pq}^J, \underline{b}_{pq}^J\} g_q^J(z_q^J(t - \tau(t))) - co\{\bar{b}_{pq}^K, \underline{b}_{pq}^K\} g_q^K(z_q^K(t - \tau(t)))]], \\ \dot{z}_p^I(t) &\in -d_p z_p^I(t) + \sum_{q=1}^n [co\{\bar{a}_{pq}^I, \underline{a}_{pq}^I\} f_q^R(z_q^R(t)) + co\{\bar{a}_{pq}^R, \underline{a}_{pq}^R\} f_q^I(z_q^I(t)) \\ &\quad + co\{\bar{a}_{pq}^K, \underline{a}_{pq}^K\} f_q^J(z_q^J(t)) - co\{\bar{a}_{pq}^J, \underline{a}_{pq}^J\} f_q^K(z_q^K(t))] \\ &\quad + \sum_{q=1}^n [co\{\bar{b}_{pq}^I, \underline{b}_{pq}^I\} g_q^R(z_q^R(t - \tau(t))) + co\{\bar{b}_{pq}^R, \underline{b}_{pq}^R\} g_q^I(z_q^I(t - \tau(t))) \\ &\quad + co\{\bar{b}_{pq}^K, \underline{b}_{pq}^K\} g_q^J(z_q^J(t - \tau(t))) - co\{\bar{b}_{pq}^J, \underline{b}_{pq}^J\} g_q^K(z_q^K(t - \tau(t)))]], \\ \dot{z}_p^J(t) &\in -d_p z_p^J(t) + \sum_{q=1}^n [co\{\bar{a}_{pq}^J, \underline{a}_{pq}^J\} f_q^R(z_q^R(t)) + co\{\bar{a}_{pq}^K, \underline{a}_{pq}^K\} f_q^I(z_q^I(t)) \\ &\quad + co\{\bar{a}_{pq}^R, \underline{a}_{pq}^R\} f_q^J(z_q^J(t)) - co\{\bar{a}_{pq}^I, \underline{a}_{pq}^I\} f_q^K(z_q^K(t))] \\ &\quad + \sum_{q=1}^n [co\{\bar{b}_{pq}^J, \underline{b}_{pq}^J\} g_q^R(z_q^R(t - \tau(t))) + co\{\bar{b}_{pq}^K, \underline{b}_{pq}^K\} g_q^I(z_q^I(t - \tau(t))) \\ &\quad + co\{\bar{b}_{pq}^R, \underline{b}_{pq}^R\} g_q^J(z_q^J(t - \tau(t))) - co\{\bar{b}_{pq}^I, \underline{b}_{pq}^I\} g_q^K(z_q^K(t - \tau(t)))]], \\ \dot{z}_p^K(t) &\in -d_p z_p^K(t) + \sum_{q=1}^n [co\{\bar{a}_{pq}^K, \underline{a}_{pq}^K\} f_q^R(z_q^R(t)) - co\{\bar{a}_{pq}^I, \underline{a}_{pq}^I\} f_q^I(z_q^I(t)) \\ &\quad + co\{\bar{a}_{pq}^J, \underline{a}_{pq}^J\} f_q^J(z_q^J(t)) + co\{\bar{a}_{pq}^R, \underline{a}_{pq}^R\} f_q^K(z_q^K(t))] \\ &\quad + \sum_{q=1}^n [co\{\bar{b}_{pq}^K, \underline{b}_{pq}^K\} g_q^R(z_q^R(t - \tau(t))) - co\{\bar{b}_{pq}^I, \underline{b}_{pq}^I\} g_q^I(z_q^I(t - \tau(t))) \\ &\quad + co\{\bar{b}_{pq}^J, \underline{b}_{pq}^J\} g_q^J(z_q^J(t - \tau(t))) - co\{\bar{b}_{pq}^R, \underline{b}_{pq}^R\} g_q^K(z_q^K(t - \tau(t)))]]. \end{aligned} \right. \tag{5}$$

Or equivalently, there exist $\hat{a}_{pq}^\mu \in co\{\bar{a}_{pq}^\mu, \underline{a}_{pq}^\mu\}$, $\hat{b}_{pq}^\mu \in co\{\bar{b}_{pq}^\mu, \underline{b}_{pq}^\mu\}$ such that

$$\left\{ \begin{aligned} \dot{z}_p^R(t) &= -d_p z_p^R(t) + \sum_{q=1}^n [\hat{a}_{pq}^R f_q^R(z_q^R(t)) - \hat{a}_{pq}^I f_q^I(z_q^I(t)) - \hat{a}_{pq}^J f_q^J(z_q^J(t)) \\ &\quad - \hat{a}_{pq}^K f_q^K(z_q^K(t))] + \sum_{q=1}^n [\hat{b}_{pq}^R g_q^R(z_q^R(t - \tau(t))) - \hat{b}_{pq}^I g_q^I(z_q^I(t - \tau(t))) \\ &\quad - \hat{b}_{pq}^J g_q^J(z_q^J(t - \tau(t))) - \hat{b}_{pq}^K g_q^K(z_q^K(t - \tau(t)))]], \\ \dot{z}_p^I(t) &= -d_p z_p^I(t) + \sum_{q=1}^n [\hat{a}_{pq}^I f_q^R(z_q^R(t)) + \hat{a}_{pq}^R f_q^I(z_q^I(t)) + \hat{a}_{pq}^K f_q^J(z_q^J(t)) \\ &\quad - \hat{a}_{pq}^J f_q^K(z_q^K(t))] + \sum_{q=1}^n [\hat{b}_{pq}^I g_q^R(z_q^R(t - \tau(t))) + \hat{b}_{pq}^R g_q^I(z_q^I(t - \tau(t))) \\ &\quad + \hat{b}_{pq}^K g_q^J(z_q^J(t - \tau(t))) - \hat{b}_{pq}^J g_q^K(z_q^K(t - \tau(t)))]], \\ \dot{z}_p^J(t) &= -d_p z_p^J(t) + \sum_{q=1}^n [\hat{a}_{pq}^J f_q^R(z_q^R(t)) + \hat{a}_{pq}^K f_q^I(z_q^I(t)) + \hat{a}_{pq}^R f_q^J(z_q^J(t)) \\ &\quad - \hat{a}_{pq}^I f_q^K(z_q^K(t))] + \sum_{q=1}^n [\hat{b}_{pq}^J g_q^R(z_q^R(t - \tau(t))) + \hat{b}_{pq}^K g_q^I(z_q^I(t - \tau(t))) \\ &\quad + \hat{b}_{pq}^R g_q^J(z_q^J(t - \tau(t))) - \hat{b}_{pq}^I g_q^K(z_q^K(t - \tau(t)))]], \\ \dot{z}_p^K(t) &= -d_p z_p^K(t) + \sum_{q=1}^n [\hat{a}_{pq}^K f_q^R(z_q^R(t)) - \hat{a}_{pq}^I f_q^I(z_q^I(t)) + \hat{a}_{pq}^J f_q^J(z_q^J(t)) \\ &\quad + \hat{a}_{pq}^R f_q^K(z_q^K(t))] + \sum_{q=1}^n [\hat{b}_{pq}^K g_q^R(z_q^R(t - \tau(t))) - \hat{b}_{pq}^I g_q^I(z_q^I(t - \tau(t))) \\ &\quad + \hat{b}_{pq}^J g_q^J(z_q^J(t - \tau(t))) - \hat{b}_{pq}^R g_q^K(z_q^K(t - \tau(t)))]]. \end{aligned} \right. \tag{6}$$

Make the definition that $\bar{a}_{pq}^\mu = \max\{\hat{a}_{pq}^\mu, \check{a}_{pq}^\mu\}$, $\underline{a}_{pq}^\mu = \min\{\hat{a}_{pq}^\mu, \check{a}_{pq}^\mu\}$, $\bar{b}_{pq}^\mu = \max\{\hat{b}_{pq}^\mu, \check{b}_{pq}^\mu\}$, $\underline{b}_{pq}^\mu = \min\{\hat{b}_{pq}^\mu, \check{b}_{pq}^\mu\}$, $\mu = R, I, J, K$.

The response system is given by

$$\begin{aligned} \dot{\tilde{z}}_p(t) = & -d_p \tilde{z}_p(t) + \sum_{q=1}^n a_{pq}(\tilde{z}_p(t)) f_q(\tilde{z}_q(t)) \\ & + \sum_{q=1}^n b_{pq}(\tilde{z}_p(t)) g_q(\tilde{z}_q(t - \tau(t))) \\ & + l_p(t) + u_p(t), \quad t \geq 0, p = 1, 2, \dots, n, \end{aligned} \tag{7}$$

where $u_p(t) \in \mathbb{Q}$ is an appropriate controller. The initial condition is $\tilde{z}(s) = \varphi(s) \in C([- \tau, 0], \mathbb{Q}^n)$, $-\tau \leq s \leq 0$.

Similarly, system (7) can be written as follows:

$$\left\{ \begin{aligned} \dot{\tilde{z}}_p^R(t) = & -d_p \tilde{z}_p^R(t) + \sum_{q=1}^n [\check{a}_{pq}^R f_q^R(\tilde{z}_q^R(t)) - \check{a}_{pq}^I f_q^I(\tilde{z}_q^I(t)) - \check{a}_{pq}^J f_q^J(\tilde{z}_q^J(t)) \\ & - \check{a}_{pq}^K f_q^K(\tilde{z}_q^K(t))] + \sum_{q=1}^n [\check{b}_{pq}^R g_q^R(\tilde{z}_q^R(t - \tau(t))) - \check{b}_{pq}^I g_q^I(\tilde{z}_q^I(t - \tau(t))) \\ & - \check{b}_{pq}^J g_q^J(\tilde{z}_q^J(t - \tau(t))) - \check{b}_{pq}^K g_q^K(\tilde{z}_q^K(t - \tau(t)))] + u_p^R(t), \\ \dot{\tilde{z}}_p^I(t) = & -d_p \tilde{z}_p^I(t) + \sum_{q=1}^n [\check{a}_{pq}^I f_q^R(\tilde{z}_q^R(t)) + \check{a}_{pq}^R f_q^I(\tilde{z}_q^I(t)) + \check{a}_{pq}^K f_q^J(\tilde{z}_q^J(t)) \\ & - \check{a}_{pq}^J f_q^K(\tilde{z}_q^K(t))] + \sum_{q=1}^n [\check{b}_{pq}^I g_q^R(\tilde{z}_q^R(t - \tau(t))) + \check{b}_{pq}^R g_q^I(\tilde{z}_q^I(t - \tau(t))) \\ & + \check{b}_{pq}^K g_q^J(\tilde{z}_q^J(t - \tau(t))) - \check{b}_{pq}^J g_q^K(\tilde{z}_q^K(t - \tau(t)))] + u_p^I(t), \\ \dot{\tilde{z}}_p^J(t) = & -d_p \tilde{z}_p^J(t) + \sum_{q=1}^n [\check{a}_{pq}^J f_q^R(\tilde{z}_q^R(t)) + \check{a}_{pq}^K f_q^I(\tilde{z}_q^I(t)) + \check{a}_{pq}^I f_q^J(\tilde{z}_q^J(t)) \\ & - \check{a}_{pq}^I f_q^K(\tilde{z}_q^K(t))] + \sum_{q=1}^n [\check{b}_{pq}^J g_q^R(\tilde{z}_q^R(t - \tau(t))) + \check{b}_{pq}^K g_q^I(\tilde{z}_q^I(t - \tau(t))) + u_p^J(t) \\ & + \check{b}_{pq}^R g_q^J(\tilde{z}_q^J(t - \tau(t))) - \check{b}_{pq}^I g_q^K(\tilde{z}_q^K(t - \tau(t)))] + u_p^J(t), \\ \dot{\tilde{z}}_p^K(t) = & -d_p \tilde{z}_p^K(t) + \sum_{q=1}^n [\check{a}_{pq}^K f_q^R(\tilde{z}_q^R(t)) - \check{a}_{pq}^I f_q^I(\tilde{z}_q^I(t)) + \check{a}_{pq}^J f_q^J(\tilde{z}_q^J(t)) \\ & + \check{a}_{pq}^R f_q^K(\tilde{z}_q^K(t))] + \sum_{q=1}^n [\check{b}_{pq}^K g_q^R(\tilde{z}_q^R(t - \tau(t))) - \check{b}_{pq}^I g_q^I(\tilde{z}_q^I(t - \tau(t))) \\ & + \check{b}_{pq}^J g_q^J(\tilde{z}_q^J(t - \tau(t))) - \check{b}_{pq}^R g_q^K(\tilde{z}_q^K(t - \tau(t)))] + u_p^K(t), \end{aligned} \right. \tag{8}$$

where $\check{a}_{pq}^\mu \in co\{\bar{a}_{pq}^\mu, \underline{a}_{pq}^\mu\}$, $\check{b}_{pq}^\mu \in co\{\bar{b}_{pq}^\mu, \underline{b}_{pq}^\mu\}$.

Let $e_p(t) = \tilde{z}_p(t) - z_p(t)$ be the error of synchronization, giving that the system error can be modeled as follows:

$$\begin{aligned} \dot{e}_p(t) \in & -d_p e_p(t) + \sum_{q=1}^n co\{\bar{a}_{pq}, \underline{a}_{pq}\} F_q(e_q(t)) \\ & + \sum_{q=1}^n co\{\bar{b}_{pq}, \underline{b}_{pq}\} G_q(e_q(t - \tau(t))) \\ & + u_p(t), \quad t \geq 0, p = 1, 2, \dots, n, \end{aligned} \tag{9}$$

or equivalently, there exist $\bar{a}_{pq} \in co\{\bar{a}_{pq}, \underline{a}_{pq}\}$, $\bar{b}_{pq} \in co\{\bar{b}_{pq}, \underline{b}_{pq}\}$ such that

$$\begin{aligned} \dot{e}_p(t) = & -d_p e_p(t) + \sum_{q=1}^n \bar{a}_{pq} F_q(e_q(t)) + \sum_{q=1}^n \bar{b}_{pq} G_q(e_q(t - \tau(t))) \\ & + u_p(t), \quad t \geq 0, p = 1, 2, \dots, n, \end{aligned} \tag{10}$$

where $F_q(e_q(t)) = f_q(\tilde{z}_q(t)) - f_q(z_q(t))$, $G_q(e_q(t - \tau_q)) = g_q(\tilde{z}_q(t - \tau_q)) - g_q(z_q(t - \tau_q))$. The initial condition is $\phi(s) = \psi(s) - \varphi(s)$.

Based on the above analysis, the error system $e_p(t) \triangleq e_p^R(t) + I e_p^I(t) + J e_p^J(t) + \kappa e_p^K(t)$ can be represented as the following four parts:

$$\left\{ \begin{aligned}
 \dot{e}_p^R(t) &= -d_p e_p^R(t) + \sum_{q=1}^n [\bar{a}_{pq}^R F_q^R(e_q^R(t)) - \bar{a}_{pq}^I F_q^I(e_q^I(t)) - \bar{a}_{pq}^J F_q^J(e_q^J(t)) \\
 &\quad - \bar{a}_{pq}^K F_q^K(e_q^K(t))] \\
 &\quad + \sum_{q=1}^n [\bar{b}_{pq}^R G_q^R(e_q^R(t - \tau(t))) - \bar{b}_{pq}^I G_q^I(e_q^I(t - \tau(t))) - \bar{b}_{pq}^J G_q^J(e_q^J(t - \tau(t))) \\
 &\quad - \bar{b}_{pq}^K G_q^K(e_q^K(t - \tau(t)))] + u_p^R(t), \\
 \dot{e}_p^I(t) &= -d_p e_p^I(t) + \sum_{q=1}^n [\bar{a}_{pq}^I F_q^R(e_q^R(t)) + \bar{a}_{pq}^R F_q^I(e_q^I(t)) + \bar{a}_{pq}^K F_q^J(e_q^J(t)) \\
 &\quad - \bar{a}_{pq}^J F_q^K(e_q^K(t))] \\
 &\quad + \sum_{q=1}^n [\bar{b}_{pq}^I G_q^R(e_q^R(t - \tau(t))) + \bar{b}_{pq}^R G_q^I(e_q^I(t - \tau(t))) + \bar{b}_{pq}^K G_q^J(e_q^J(t - \tau(t))) \\
 &\quad - \bar{b}_{pq}^J G_q^K(e_q^K(t - \tau(t)))] + u_p^I(t), \\
 \dot{e}_p^J(t) &= -d_p e_p^J(t) + \sum_{q=1}^n [\bar{a}_{pq}^J F_q^R(e_q^R(t)) + \bar{a}_{pq}^K F_q^I(e_q^I(t)) + \bar{a}_{pq}^R F_q^J(e_q^J(t)) \\
 &\quad - \bar{a}_{pq}^I F_q^K(e_q^K(t))] \\
 &\quad + \sum_{q=1}^n [\bar{b}_{pq}^J G_q^R(e_q^R(t - \tau(t))) + \bar{b}_{pq}^K G_q^I(e_q^I(t - \tau(t))) + \bar{b}_{pq}^R G_q^J(e_q^J(t - \tau(t))) \\
 &\quad - \bar{b}_{pq}^I G_q^K(e_q^K(t - \tau(t)))] + u_p^J(t), \\
 \dot{e}_p^K(t) &= -d_p e_p^K(t) + \sum_{q=1}^n [\bar{a}_{pq}^K F_q^R(e_q^R(t)) - \bar{a}_{pq}^I F_q^I(e_q^I(t)) + \bar{a}_{pq}^J F_q^J(e_q^J(t)) \\
 &\quad + \bar{a}_{pq}^R F_q^K(e_q^K(t))] \\
 &\quad + \sum_{q=1}^n [\bar{b}_{pq}^K G_q^R(e_q^R(t - \tau(t))) - \bar{b}_{pq}^I G_q^I(e_q^I(t - \tau(t))) + \bar{b}_{pq}^J G_q^J(e_q^J(t - \tau(t))) \\
 &\quad - \bar{b}_{pq}^R G_q^K(e_q^K(t - \tau(t)))] + u_p^K(t),
 \end{aligned} \right. \tag{11}$$

where $F_q^\mu(e_q^\mu(t)) = f_q^\mu(\tilde{z}_q^\mu(t)) - f_q^\mu(z_q^\mu(t))$, $G_q^\mu(e_q^\mu(t - \tau_q)) = g_q^\mu(\tilde{z}_q^\mu(t - \tau_q)) - g_q^\mu(z_q^\mu(t - \tau_q))$, $\bar{a}_{pq}^\mu \in co\{\bar{a}_{pq}^\mu, \underline{a}_{pq}^\mu\}$, $\bar{b}_{pq}^\mu \in co\{\bar{b}_{pq}^\mu, \underline{b}_{pq}^\mu\}$, $\mu = R, I, J, K$.

For the subsequent discussion, the definitions and lemmas are given.

Definition 1 ([6]) For any initial values ϕ , at an existing time of $0 < T(\phi) < \infty$ such that

- (1) $\lim_{t \rightarrow T(\phi)} \|e(t, \phi)\| = 0$,
- (2) $\|e(t, \phi)\| = 0, t > T(\phi)$,

drive-response systems (1) and (7) can obtain the finite time synchronization.

Definition 2 Drive-response systems (1) and (7) are said to reach the FTS if the conditions of finite time synchronization hold and the setting time $T(\phi)$ is upper bounded by T_{\max} , i.e., $T(\phi) \leq T_{\max}$.

Lemma 1 ([35]) If $V : R^n \rightarrow R_+ \cup \{0\}$ is a continuous radially unbounded function satisfying:

- (1) $V(y) = 0 \Leftrightarrow y = 0$;
- (2) For $\xi, \eta > 0, \delta > 1, 0 < \theta < 1$, the following inequality holds:

$$D^+ V(y(t)) \leq -\xi V^\delta(y(t)) - \eta V^\theta(y(t)), \quad \forall y(t),$$

then the origin is fixed time stable; moreover,

$$V(y(t)) \equiv 0, \quad t \geq T(y_0),$$

with the setting time bounded by

$$T(y_0) \leq T_{\max} := \frac{1}{\eta} \left(\frac{\eta}{\xi} \right)^{\frac{1-\theta}{\delta-\theta}} \left(\frac{1}{1-\theta} + \frac{1}{\delta-1} \right).$$

Lemma 2 ([36]) *Assume $y_1, y_2, \dots, y_n \geq 0, 0 < \varepsilon \leq 1, \zeta > 1$, then the following inequalities hold:*

$$\sum_{q=1}^n y_p^\varepsilon \geq \left(\sum_{q=1}^n y_p \right)^\varepsilon, \quad \sum_{q=1}^n y_p^\zeta \geq n^{1-\zeta} \left(\sum_{q=1}^n y_p \right)^\zeta.$$

3 Fixed time synchronization

In order to achieve the FTS of delayed QVMNN, the controller is designed as follows:

$$\begin{cases} u_p^R(t) = -\lambda_{1p}^R e_p^R(t) - \text{sign}(e_p^R(t)) [\lambda_{2p}^R |e_p^R(t)|^\delta + \lambda_{3p}^R |e_p^R(t)|^\theta + \lambda_{4p}^R |e_p^R(t - \tau(t))|], \\ u_p^I(t) = -\lambda_{1p}^I e_p^I(t) - \text{sign}(e_p^I(t)) [\lambda_{2p}^I |e_p^I(t)|^\delta + \lambda_{3p}^I |e_p^I(t)|^\theta + \lambda_{4p}^I |e_p^I(t - \tau(t))|], \\ u_p^J(t) = -\lambda_{1p}^J e_p^J(t) - \text{sign}(e_p^J(t)) [\lambda_{2p}^J |e_p^J(t)|^\delta + \lambda_{3p}^J |e_p^J(t)|^\theta + \lambda_{4p}^J |e_p^J(t - \tau(t))|], \\ u_p^K(t) = -\lambda_{1p}^K e_p^K(t) - \text{sign}(e_p^K(t)) [\lambda_{2p}^K |e_p^K(t)|^\delta + \lambda_{3p}^K |e_p^K(t)|^\theta + \lambda_{4p}^K |e_p^K(t - \tau(t))|], \end{cases} \quad (12)$$

where $\lambda_{1p}^\mu, \lambda_{2p}^\mu, \lambda_{3p}^\mu, \lambda_{4p}^\mu$ ($\mu = R, I, J, K, p = 1, 2, \dots, n$) are control gains, $\delta > 1, 0 < \theta < 1$.

Theorem 1 *Let Assumptions 1–3 hold, when the control gains satisfy the following conditions:*

$$\begin{cases} \lambda_{1p}^\mu \geq \sum_{q=1}^n [|\bar{a}_{qp}^R| + |\bar{a}_{qp}^I| + |\bar{a}_{qp}^J| + |\bar{a}_{qp}^K|] l_p^\mu - d_p, \\ \lambda_{4p}^\mu \geq \sum_{q=1}^n [|\bar{b}_{qp}^R| + |\bar{b}_{qp}^I| + |\bar{b}_{qp}^J| + |\bar{b}_{qp}^K|] h_p^\mu, \\ \lambda_{2p}^\mu \geq 0, \quad \lambda_{3p}^\mu \geq 0, \quad p = 1, 2, \dots, n, \mu = R, I, J, K, \end{cases} \quad (13)$$

where $\delta > 1, 0 < \theta < 1$, then the response system (7) can synchronize the drive system (1) in a fixed time under controller (12). The fixed time T is estimated as follows:

$$T \leq \frac{1}{\lambda_3} \left(\frac{\lambda_3}{\lambda_2(4n)^{1-\delta}} \right)^{\frac{1-\theta}{\delta-\theta}} \left(\frac{1}{1-\theta} + \frac{1}{\delta-1} \right), \quad (14)$$

where $\lambda_2 = \min_p \{\lambda_{2p}^\mu\}, \lambda_3 = \min_p \{\lambda_{3p}^\mu\}$.

Proof Constructing the Lyapunov function

$$\begin{aligned} V(t) &= V_1(t) + V_2 + V_3(t) + V_4(t) \\ &\triangleq \sum_{p=1}^n |e_p^R(t)| + \sum_{p=1}^n |e_p^I(t)| + \sum_{p=1}^n |e_p^J(t)| + \sum_{p=1}^n |e_p^K(t)|, \end{aligned} \quad (15)$$

taking the time derivative of $V_1(t)$, one gets

$$\begin{aligned} \dot{V}_1(t) &= \sum_{p=1}^n \text{sign}(e_p^R(t)) \left\{ -d_p e_p^R(t) + \sum_{q=1}^n [\bar{a}_{pq}^R F_q^R(e_q^R(t)) - \bar{a}_{pq}^I F_q^I(e_q^I(t)) - \bar{a}_{pq}^J F_q^J(e_q^J(t)) \right. \\ &\quad - \bar{a}_{pq}^K F_q^K(e_q^K(t))] + \sum_{q=1}^n [\bar{b}_{pq}^R G_q^R(e_q^R(t - \tau(t))) - \bar{b}_{pq}^I G_q^I(e_q^I(t - \tau(t))) \\ &\quad - \bar{b}_{pq}^J G_q^J(e_q^J(t - \tau(t))) - \bar{b}_{pq}^K G_q^K(e_q^K(t - \tau(t)))] \\ &\quad \left. - \lambda_{1p}^R e_p^R(t) - \text{sign}(e_p^R(t)) [\lambda_{2p}^R |e_p^R(t)|^\delta + \lambda_{3p}^R |e_p^R(t)|^\theta + \lambda_{4p}^R |e_p^R(t - \tau(t))|] \right\} \\ &\leq \sum_{p=1}^n \left\{ -(d_p + \lambda_{1p}^R) |e_p^R(t)| + \sum_{q=1}^n [|\bar{a}_{pq}^R| |l_q^R| |e_q^R(t)| + |\bar{a}_{pq}^I| |l_q^I| |e_q^I(t)| + |\bar{a}_{pq}^J| |l_q^J| |e_q^J(t)| \right. \\ &\quad + |\bar{a}_{pq}^K| |l_q^K| |e_q^K(t)|] + \sum_{q=1}^n [|\bar{b}_{pq}^R| |h_q^R| |e_q^R(t - \tau(t))| \\ &\quad + |\bar{b}_{pq}^I| |h_q^I| |e_q^I(t - \tau(t))| + |\bar{b}_{pq}^J| |h_q^J| |e_q^J(t - \tau(t))| + |\bar{b}_{pq}^K| |h_q^K| |e_q^K(t - \tau(t))|] \\ &\quad \left. - \lambda_{2p}^R |e_p^R(t)|^\delta - \lambda_{3p}^R |e_p^R(t)|^\theta - \lambda_{4p}^R |e_p^R(t - \tau(t))| \right\} \\ &= \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^R + \sum_{q=1}^n |\bar{a}_{qp}^R| |l_p^R| \right\} |e_p^R(t)| + \sum_{p=1}^n \sum_{q=1}^n [|\bar{a}_{pq}^I| |l_q^I| |e_q^I(t)| + |\bar{a}_{pq}^J| |l_q^J| |e_q^J(t)| \\ &\quad + |\bar{a}_{pq}^K| |l_q^K| |e_q^K(t)|] + \sum_{p=1}^n \left\{ -\lambda_{4p}^R + \sum_{q=1}^n |\bar{b}_{qp}^R| |h_p^R| \right\} |e_p^R(t - \tau(t))| \\ &\quad + \sum_{p=1}^n \sum_{q=1}^n [|\bar{b}_{pq}^I| |h_q^I| |e_q^I(t - \tau(t))| + |\bar{b}_{pq}^J| |h_q^J| |e_q^J(t - \tau(t))| + |\bar{b}_{pq}^K| |h_q^K| |e_q^K(t - \tau(t))|] \\ &\quad - \sum_{p=1}^n [\lambda_{2p}^R |e_p^R(t)|^\delta + \lambda_{3p}^R |e_p^R(t)|^\theta]. \end{aligned}$$

Similarly:

$$\begin{aligned} \dot{V}_2(t) &\leq \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^I + \sum_{q=1}^n |\bar{a}_{qp}^R| |l_p^I| \right\} |e_p^I(t)| + \sum_{p=1}^n \sum_{q=1}^n [|\bar{a}_{pq}^I| |l_q^R| |e_q^R(t)| + |\bar{a}_{pq}^K| |l_q^I| |e_q^I(t)| \\ &\quad + |\bar{a}_{pq}^J| |l_q^K| |e_q^K(t)|] + \sum_{p=1}^n \left\{ -\lambda_{4p}^I + \sum_{q=1}^n |\bar{b}_{qp}^R| |h_p^I| \right\} |e_p^I(t - \tau(t))| \\ &\quad + \sum_{p=1}^n \sum_{q=1}^n [|\bar{b}_{pq}^I| |h_q^R| |e_q^R(t - \tau(t))| + |\bar{b}_{pq}^K| |h_q^I| |e_q^I(t - \tau(t))| + |\bar{b}_{pq}^J| |h_q^K| |e_q^K(t - \tau(t))|] \\ &\quad - \sum_{p=1}^n [\lambda_{2p}^I |e_p^I(t)|^\delta + \lambda_{3p}^I |e_p^I(t)|^\theta], \\ \dot{V}_3(t) &\leq \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^J + \sum_{q=1}^n |\bar{a}_{qp}^R| |l_p^J| \right\} |e_p^J(t)| + \sum_{p=1}^n \sum_{q=1}^n [|\bar{a}_{pq}^J| |l_q^R| |e_q^R(t)| + |\bar{a}_{pq}^K| |l_q^I| |e_q^I(t)| \end{aligned}$$

$$\begin{aligned}
 & + |\bar{a}_{pq}^I |l_q^K |e_q^K(t)| + \sum_{p=1}^n \left\{ -\lambda_{4p}^J + \sum_{q=1}^n |\bar{b}_{qp}^R |h_p^I \right\} |e_p^J(t - \tau(t))| \\
 & + \sum_{p=1}^n \sum_{q=1}^n [|\bar{b}_{pq}^J |h_q^R |e_q^R(t - \tau(t))| + |\bar{b}_{pq}^K |h_q^I |e_q^I(t - \tau(t))| + |\bar{b}_{pq}^I |h_q^K |e_q^K(t - \tau(t))|] \\
 & - \sum_{p=1}^n [\lambda_{2p}^J |e_p^J(t)|^\delta + \lambda_{3p}^J |e_p^J(t)|^\theta], \\
 \dot{V}_4(t) \leq & \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^K + \sum_{q=1}^n |\bar{a}_{qp}^R |l_p^K \right\} |e_p^K(t)| + \sum_{p=1}^n \sum_{q=1}^n [|\bar{a}_{pq}^K |l_q^R |e_q^R(t)| + |\bar{a}_{pq}^J |l_q^I |e_q^I(t)| \\
 & + |\bar{a}_{pq}^I |l_q^J |e_q^J(t)|] + \sum_{p=1}^n \left\{ -\lambda_{4p}^K + \sum_{q=1}^n |\bar{b}_{qp}^R |h_p^K \right\} |e_p^K(t - \tau(t))| \\
 & + \sum_{p=1}^n \sum_{q=1}^n [|\bar{b}_{pq}^K |h_q^R |e_q^R(t - \tau(t))| + |\bar{b}_{pq}^J |h_q^I |e_q^I(t - \tau(t))| + |\bar{b}_{pq}^I |h_q^J |e_q^J(t - \tau(t))|] \\
 & - \sum_{p=1}^n [\lambda_{2p}^K |e_p^K(t)|^\delta + \lambda_{3p}^K |e_p^K(t)|^\theta].
 \end{aligned}$$

By combining the above inequalities and according to Lemma 2, one gets the following:

$$\begin{aligned}
 \dot{V}(t) \leq & \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^R + \sum_{q=1}^n [|\bar{a}_{qp}^R | + |\bar{a}_{qp}^I | + |\bar{a}_{qp}^J | + |\bar{a}_{qp}^K |] l_p^R \right\} |e_p^R(t)| \\
 & + \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^I + \sum_{q=1}^n [|\bar{a}_{qp}^R | + |\bar{a}_{qp}^I | + |\bar{a}_{qp}^J | + |\bar{a}_{qp}^K |] l_p^I \right\} |e_p^I(t)| \\
 & + \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^J + \sum_{q=1}^n [|\bar{a}_{qp}^R | + |\bar{a}_{qp}^I | + |\bar{a}_{qp}^J | + |\bar{a}_{qp}^K |] l_p^J \right\} |e_p^J(t)| \\
 & + \sum_{p=1}^n \left\{ -d_p - \lambda_{1p}^K + \sum_{q=1}^n [|\bar{a}_{qp}^R | + |\bar{a}_{qp}^I | + |\bar{a}_{qp}^J | + |\bar{a}_{qp}^K |] l_p^K \right\} |e_p^K(t)| \\
 & + \sum_{p=1}^n \left\{ -\lambda_{4p}^R + \sum_{q=1}^n [|\bar{b}_{qp}^R | + |\bar{b}_{qp}^I | + |\bar{b}_{qp}^J | + |\bar{b}_{qp}^K |] h_p^R \right\} |e_p^R(t - \tau(t))| \\
 & + \sum_{p=1}^n \left\{ -\lambda_{4p}^I + \sum_{q=1}^n [|\bar{b}_{qp}^R | + |\bar{b}_{qp}^I | + |\bar{b}_{qp}^J | + |\bar{b}_{qp}^K |] h_p^I \right\} |e_p^I(t - \tau(t))| \\
 & + \sum_{p=1}^n \left\{ -\lambda_{4p}^J + \sum_{q=1}^n [|\bar{b}_{qp}^R | + |\bar{b}_{qp}^I | + |\bar{b}_{qp}^J | + |\bar{b}_{qp}^K |] h_p^J \right\} |e_p^J(t - \tau(t))| \\
 & + \sum_{p=1}^n \left\{ -\lambda_{4p}^K + \sum_{q=1}^n [|\bar{b}_{qp}^R | + |\bar{b}_{qp}^I | + |\bar{b}_{qp}^J | + |\bar{b}_{qp}^K |] h_p^K \right\} |e_p^K(t - \tau(t))| \\
 & - \sum_{p=1}^n [\lambda_{2p}^R |e_p^R(t)|^\delta + \lambda_{3p}^R |e_p^R(t)|^\theta] - \sum_{p=1}^n [\lambda_{2p}^I |e_p^I(t)|^\delta + \lambda_{3p}^I |e_p^I(t)|^\theta]
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{p=1}^n [\lambda_{2p}^J |e_p^J(t)|^\delta + \lambda_{3p}^J |e_p^J(t)|^\theta] - \sum_{p=1}^n [\lambda_{2p}^K |e_p^K(t)|^\delta + \lambda_{3p}^K |e_p^K(t)|^\theta] \\
 \leq & - \sum_{p=1}^n [\lambda_{2p}^R |e_p^R(t)|^\delta + \lambda_{3p}^R |e_p^R(t)|^\theta] - \sum_{p=1}^n [\lambda_{2p}^I |e_p^I(t)|^\delta + \lambda_{3p}^I |e_p^I(t)|^\theta] \\
 & - \sum_{p=1}^n [\lambda_{2p}^J |e_p^J(t)|^\delta + \lambda_{3p}^J |e_p^J(t)|^\theta] - \sum_{p=1}^n [\lambda_{2p}^K |e_p^K(t)|^\delta + \lambda_{3p}^K |e_p^K(t)|^\theta] \\
 \leq & -\lambda_2 \sum_{p=1}^n [|e_p^R(t)|^\delta + |e_p^I(t)|^\delta + |e_p^J(t)|^\delta + |e_p^K(t)|^\delta] \\
 & - \lambda_3 \sum_{p=1}^n [|e_p^R(t)|^\theta + |e_p^I(t)|^\theta + |e_p^J(t)|^\theta + |e_p^K(t)|^\theta] \\
 \leq & -\lambda_2 (4n)^{1-\delta} \left[\sum_{p=1}^n |e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)| \right]^\delta \\
 & - \lambda_3 \left[\sum_{p=1}^n |e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)| \right]^\theta \\
 = & -\lambda_2 (4n)^{1-\delta} V^\delta(t) - \lambda_3 V^\theta(t).
 \end{aligned}$$

Therefore, following from Lemma 1, the FTS between drive-response systems (1) and (7) can be achieved. Moreover, the upper bound of setting time is estimated by $T \leq \frac{1}{\lambda_3} \left(\frac{\lambda_3}{\lambda_2 (4n)^{1-\delta}} \right)^{\frac{1-\theta}{\delta-\theta}} \left(\frac{1}{1-\theta} + \frac{1}{\delta-1} \right)$. □

Remark 3 Ref. [34] investigated the FTS of QVNN without considering time-delay, and some conditions of FTS and bound of the settling time were obtained. In our paper, the FTS of more complex QVMNN at the presence of time varying delays is considered. Moreover, the design of controllers and the conclusions are more complex and more general than those in Ref. [34].

Remark 4 Recently, many excellent results on FTS of RVNN and CVNN [18–24] have been presented. Compared with previous works, QVNN possess superiority in dealing with multidimensional problems, and they have widely potential application in engineering field. Hence, our result is more general and meaningful.

Remark 5 It is worthy of pointing out that quaternion multiplication does not satisfy the commutative rule. Therefore, traditional methods and techniques for solving problems of CVNN or RVNN cannot be directly employed to study QVNN. To avoid the non-commutativity of quaternion multiplication, a feasible method is to decompose the quaternion-valued systems into real-valued systems. Choosing different approaches on investigating the dynamics characteristics of QVNN is still an open and challenging task. In the future, we are focusing on studying the synchronization of QVMNN via direct method.

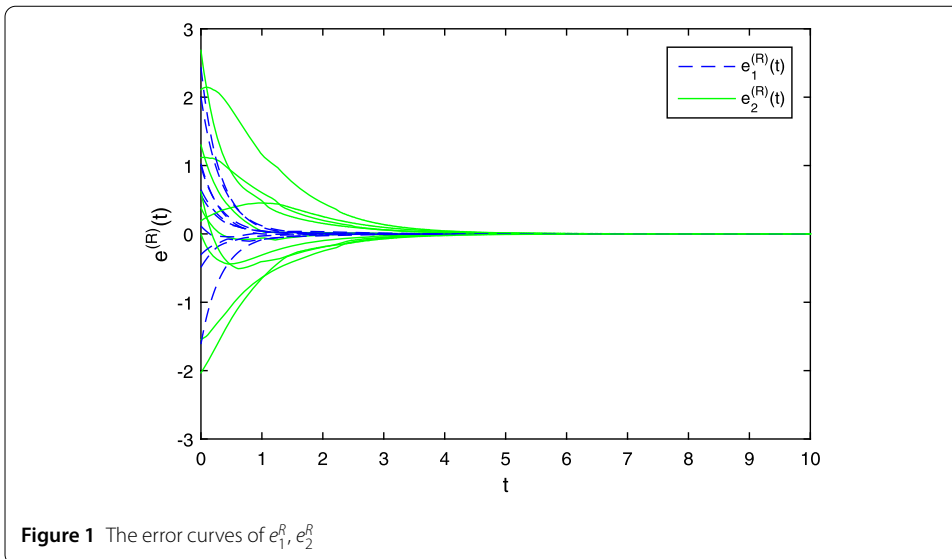
4 Numerical example

The following QVMNN with time delay is taken as a drive system:

$$\begin{aligned} \dot{z}_p(t) = & -d_p z_p(t) + \sum_{q=1}^2 a_{pq}(z_p(t)) f_q(z_q(t)) \\ & + \sum_{q=1}^2 b_{pq}(z_p(t)) g_q(z_q(t - \tau(t))) \\ & + l_p(t), \quad t \geq 0, p = 1, 2, \end{aligned} \quad (16)$$

where $z_p(t) = z_p^R(t) + \iota z_p^I(t) + j z_p^J(t) + \kappa z_p^K(t)$, $\tau(t) = 0.2 + 0.78 \sin(t)$, $l_1(t) = 0.5 + 0.4\iota - 0.5j - 0.3\kappa$, $l_2(t) = -0.3 + 0.3\iota + 2j - 1\kappa$, $d_1 = d_2 = 1$.

$$\begin{aligned} a_{11}^R(x_1^R) &= \begin{cases} -0.4, & |x_1^R| \leq 1, \\ 0.1, & |x_1^R| > 1, \end{cases} & a_{12}^R(x_1^R) &= \begin{cases} 0.1, & |x_1^R| \leq 1, \\ -0.1, & |x_1^R| > 1, \end{cases} \\ a_{21}^R(x_2^R) &= \begin{cases} -0.1, & |x_2^R| \leq 1, \\ 0.1, & |x_2^R| > 1, \end{cases} & a_{22}^R(x_2^R) &= \begin{cases} 0.1, & |x_2^R| \leq 1, \\ -0.1, & |x_2^R| > 1, \end{cases} \\ a_{11}^I(x_1^I) &= \begin{cases} 0.2, & |x_1^I| \leq 1, \\ 0.1, & |x_1^I| > 1, \end{cases} & a_{12}^I(x_1^I) &= \begin{cases} 0.1, & |x_1^I| \leq 1, \\ -0.1, & |x_1^I| > 1, \end{cases} \\ a_{21}^I(x_2^I) &= \begin{cases} 0.2, & |x_2^I| \leq 1, \\ 0.1, & |x_2^I| > 1, \end{cases} & a_{22}^I(x_2^I) &= \begin{cases} 0.1, & |x_2^I| \leq 1, \\ -0.1, & |x_2^I| > 1, \end{cases} \\ a_{11}^J(x_1^J) &= \begin{cases} -0.1, & |x_1^J| \leq 1, \\ 0.1, & |x_1^J| > 1, \end{cases} & a_{12}^J(x_1^J) &= \begin{cases} 0.1, & |x_1^J| \leq 1, \\ -0.1, & |x_1^J| > 1, \end{cases} \\ a_{21}^J(x_2^J) &= \begin{cases} -0.1, & |x_2^J| \leq 1, \\ 0.2, & |x_2^J| > 1, \end{cases} & a_{22}^J(x_2^J) &= \begin{cases} 0.1, & |x_2^J| \leq 1, \\ -0.1, & |x_2^J| > 1, \end{cases} \\ a_{11}^K(x_1^K) &= \begin{cases} -0.1, & |x_1^K| \leq 1, \\ 0.2, & |x_1^K| > 1, \end{cases} & a_{12}^K(x_1^K) &= \begin{cases} 0.1, & |x_1^K| \leq 1, \\ -0.1, & |x_1^K| > 1, \end{cases} \\ a_{21}^K(x_2^K) &= \begin{cases} -0.1, & |x_2^K| \leq 1, \\ 0.1, & |x_2^K| > 1, \end{cases} & a_{22}^K(x_2^K) &= \begin{cases} 0.1, & |x_2^K| \leq 1, \\ 0.2, & |x_2^K| > 1, \end{cases} \\ b_{11}^R(x_1^R) &= \begin{cases} 0.2, & |x_1^R| \leq 1, \\ 0.1, & |x_1^R| > 1, \end{cases} & b_{12}^R(x_1^R) &= \begin{cases} 0.1, & |x_1^R| \leq 1, \\ -0.1, & |x_1^R| > 1, \end{cases} \\ b_{21}^R(x_2^R) &= \begin{cases} 0.2, & |x_2^R| \leq 1, \\ 0.1, & |x_2^R| > 1, \end{cases} & b_{22}^R(x_2^R) &= \begin{cases} 0.1, & |x_2^R| \leq 1, \\ -0.1, & |x_2^R| > 1, \end{cases} \\ b_{11}^I(x_1^I) &= \begin{cases} -0.1, & |x_1^I| \leq 1, \\ 0.1, & |x_1^I| > 1, \end{cases} & b_{12}^I(x_1^I) &= \begin{cases} 0.1, & |x_1^I| \leq 1, \\ 0.2, & |x_1^I| > 1, \end{cases} \end{aligned}$$



$$\begin{aligned}
 b_{21}^I(x_2^I) &= \begin{cases} -0.1, & |x_2^I| \leq 1, \\ 0.4, & |x_2^I| > 1, \end{cases} & b_{22}^I(x_2^I) &= \begin{cases} 0.1, & |x_2^I| \leq 1, \\ 0.2, & |x_2^I| > 1, \end{cases} \\
 b_{11}^J(x_1^J) &= \begin{cases} -0.1, & |x_1^J| \leq 1, \\ 0.2, & |x_1^J| > 1, \end{cases} & b_{12}^J(x_1^J) &= \begin{cases} 0.1, & |x_1^J| \leq 1, \\ -0.1, & |x_1^J| > 1, \end{cases} \\
 b_{21}^J(x_2^J) &= \begin{cases} -0.1, & |x_2^J| \leq 1, \\ 0.2, & |x_2^J| > 1, \end{cases} & b_{22}^J(x_2^J) &= \begin{cases} 0.1, & |x_2^J| \leq 1, \\ -0.1, & |x_2^J| > 1, \end{cases} \\
 b_{11}^K(x_1^K) &= \begin{cases} -0.1, & |x_1^K| \leq 1, \\ 0.1, & |x_1^K| > 1, \end{cases} & b_{12}^K(x_1^K) &= \begin{cases} 0.1, & |x_1^K| \leq 1, \\ 0.2, & |x_1^K| > 1, \end{cases} \\
 b_{21}^K(x_2^K) &= \begin{cases} -0.1, & |x_2^K| \leq 1, \\ 0.1, & |x_2^K| > 1, \end{cases} & b_{22}^K(x_2^K) &= \begin{cases} 0.1, & |x_2^K| \leq 1, \\ 0.2, & |x_2^K| > 1. \end{cases}
 \end{aligned}$$

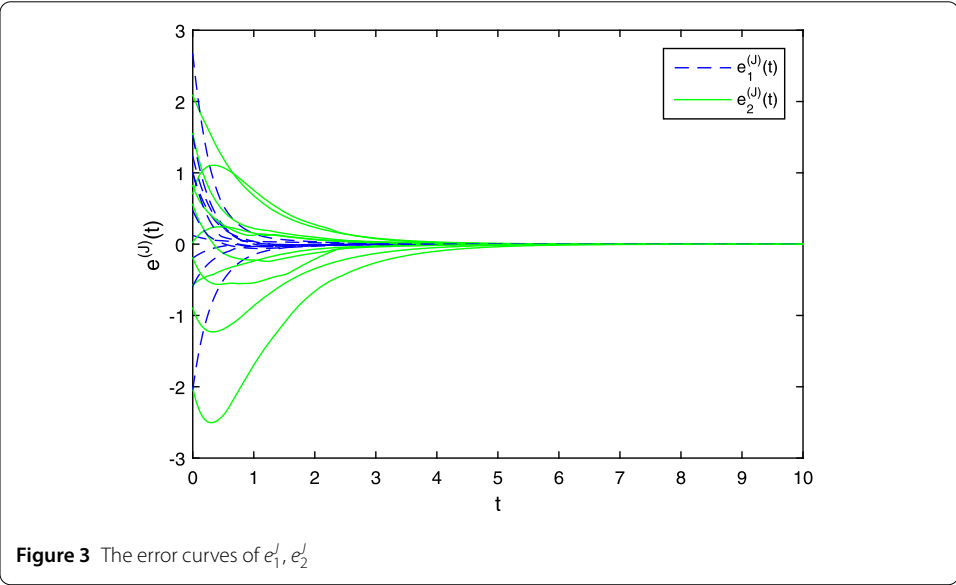
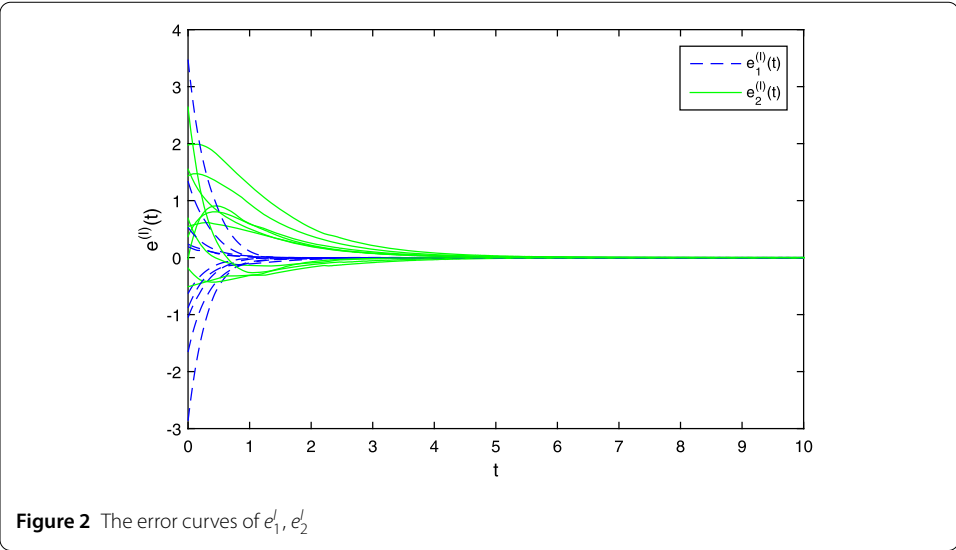
The neuron activation function is

$$f_p(z_p(t)) = g_p(z_p(t)) \frac{1}{1 + e^{z_p^R(t)}} + \frac{1}{1 + e^{z_p^I(t)}} I + \frac{1}{1 + e^{z_p^J(t)}} J + \frac{1}{1 + e^{z_p^K(t)}} K$$

for $p = 1, 2$, which implies $l_p^\mu = h_p^\mu = 1, \mu = R, I, J, K$.

The response system is described as follows:

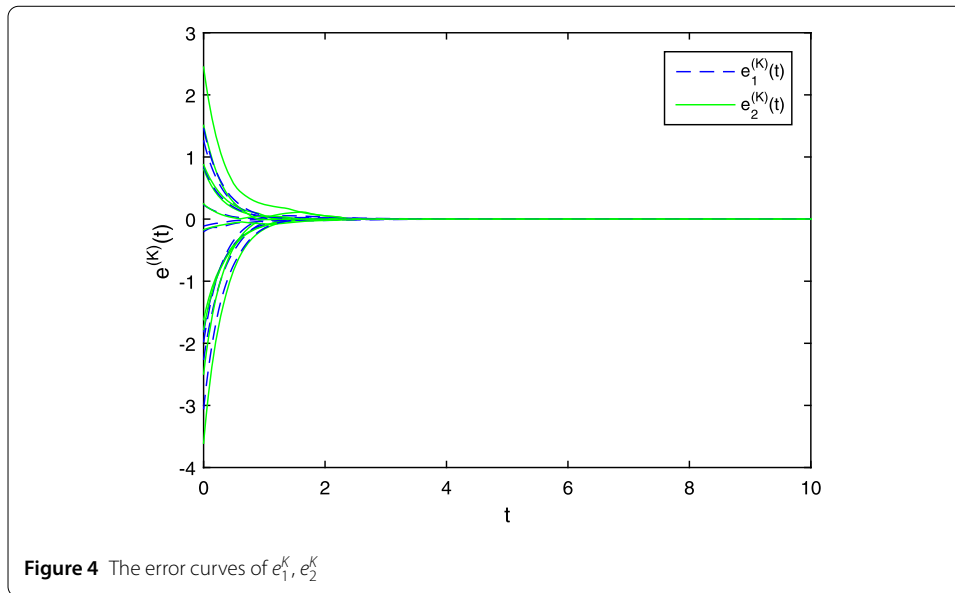
$$\begin{aligned}
 \dot{\tilde{z}}_p(t) &= -d_p \tilde{z}_p(t) + \sum_{q=1}^n a_{pq}(\tilde{z}_p(t)) f_q(\tilde{z}_q(t)) \\
 &\quad + \sum_{q=1}^n b_{pq}(\tilde{z}_p(t)) g_q(\tilde{z}_q(t - \tau_q)) \\
 &\quad + l_p(t) + u_p(t), \quad t \geq 0, p = 1, 2,
 \end{aligned} \tag{17}$$



where $\tilde{z}_p(t) = \tilde{z}_p^R(t) + \iota \tilde{z}_p^I(t) + j \tilde{z}_p^J(t) + \kappa \tilde{z}_p^K(t)$. $u_p(t) = u_p^R(t) + \iota u_p^I(t) + j u_p^J(t) + \kappa u_p^K(t)$ is the controller. The parameters used here are similar to those used in (16). Choosing $\lambda_{1p}^\mu = 4$, $\lambda_{2p}^\mu = 5$, $\lambda_{3p}^\mu = 10$, $\lambda_{4p}^\mu = 2$ ($\mu = R, I, J, K$, $p = 1, 2, \dots, n$), $\delta = 1.5$, $\theta = 0.5$, the conditions in Theorem 1 hold. According to Theorem 1, drive-response systems (16) and (17) can synchronize with the setting time $T_{\max} = 0.6$. Figures 1–4 show the synchronization trajectories with ten random initial values.

5 Conclusions

The FTS issue for a class of QVMNN at the presence of time varying delays is investigated based on fixed time stability theory. With the help of a Lyapunov function and a nonlinear controller, some sufficient conditions are established to implement the FTS of delayed QVMNN. Finally, a numerical example is used to present the effectiveness of the proposed



method. The synchronization of delayed QVMNN with parameter uncertainties warrants further investigation.

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Availability of data and materials

The data used to support the findings of this study are available from the corresponding author upon request.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript. All the authors read and approved the final manuscript.

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