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New model of May cooperative system with strong and weak cooperative partners

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Abstract

In this paper, based on the model of Zhao, Qin, and Chen [Adv. Differ. Equ. 2018:172, 2018], we propose a new model of the May cooperative system with strong and weak cooperative partners. The model overcomes the drawback of the corresponding model of Zhao, Qin, and Chen. By using the differential inequality theory, a set of sufficient conditions that ensure the permanence of the system are obtained. By combining the differential inequality theory and the iterative method, a set of sufficient conditions that ensure the extinction of the weak partners and the attractivity of the strong partners and the other species is obtained. Numeric simulations show that too large transform rate will lead to more complicated fluctuation; however, the system is still permanent.

Keywords: Mutualism; Iterative method; Weak partners; Strong partners; Global stability

1 Introduction

During the last decade, many scholars [1–36] investigated the dynamic behaviors of the mutualism and commensalism models. Some substantial progress has been made on the stability, permanence, and extinction of the mutualism model. For example, under some very simple assumption, Xie, Chen, and Xue [9] showed that the unique positive equilibrium of a cooperative system incorporating harvesting is globally attractive; Xie et al. [11] showed that the unique positive equilibrium of an integrodifferential model of mutualism is globally attractive. Chen, Xie, and Chen [17] proposed a stage structured cooperative system, and they showed that the stage structure plays an important role in the persistence and extinction property of the system. Lei [21] proposed a stage structured commensalism model. By constructing some suitable Lyapunov function, he obtained the conditions that ensure global asymptotic stability of the positive equilibrium.

Recently, stimulated by the idea of Mohammadi and Mahzoon [37], Zhao, Qin, and Chen [36] proposed the following May cooperative system with strong and weak cooperative partners:

$$\frac{dH_1}{dt} = r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \frac{\alpha H_2}{r_1} \right),$$

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$$\begin{aligned} \frac{dH_2}{dt} &= H_2(\alpha H_1 + d - eH_2), \\ \frac{dP}{dt} &= r_2 P \left(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P \right), \end{aligned} \tag{1.1}$$

where $r_i, a_i, b_i, c_i, d, i = 1, 2$, are positive constants. The authors considered system (1.1) together with the initial condition $H_i(0) > 0, i = 1, 2, P(0) > 0$. Obviously, any solution of system (1.1) with positive initial conditions remains positive for all $t \geq 0$.

Concerned with the non-persistence property of the species, by using the differential inequality theory, the authors of [36] obtained the following result.

Theorem A *If assumption (B_3) holds, where*

$$(B_3) \quad M = 1 - \frac{\alpha d}{r_1 e} < 0,$$

then the weak partners H_2 and the second species P are permanent, while the stronger partners H_1 will be driven to extinction.

Though it seems to be correct in mathematical deduction of Theorem A, from an ecological point of view, it is unreasonable. It is hard to imagine in natural world that a species only leave the weak partners while the stronger partners die out. Generally speaking, the strong partners will have more chances to survive, let alone the fact that in system (1.1) the strong partners will obtain the help from the species P .

After carefully checking the deduction of [36], we think the reason for such an unreasonable phenomenon relies on the fact that the authors made the following assumption:

Without the strong partners, the weak partners in system (1.1) are always permanent.

Indeed, the authors assumed that the weak partners satisfy the equation

$$\frac{dH_2}{dt} = H_2(d - eH_2). \tag{1.2}$$

This is the famous logistic equation, and the system admits a unique positive equilibrium $H_2^* = \frac{d}{e}$, which is globally asymptotically stable. Such an assumption seems curious since, generally speaking, weak partners are more easily driven to extinction.

This motivated us to propose the following May cooperative system with strong and weak cooperative partners:

$$\begin{aligned} \frac{dH_1}{dt} &= r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \frac{\alpha H_2}{r_1} \right), \\ \frac{dH_2}{dt} &= H_2(\alpha H_1 - d - eH_2), \\ \frac{dP}{dt} &= r_2 P \left(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P \right). \end{aligned} \tag{1.3}$$

One may argue that system (1.3) is very similar to system (1.1), maybe they have similar dynamic behaviors. However, this is impossible, there are some essential differences between system (1.1) and (1.3). In system (1.3), as far as the weak partners are concerned,

without the transformation of strong partners, it satisfies the equation

$$\frac{dH_2}{dt} = H_2(-d - eH_2). \tag{1.4}$$

Hence,

$$\frac{dH_2}{dt} \leq -dH_2. \tag{1.5}$$

Consequently,

$$H_2(t) \leq H_2(0) \exp\{-dt\} \rightarrow 0 \text{ as } t \rightarrow +\infty. \tag{1.6}$$

That is to say, without the transformation of the strong partners, the weak partners will finally be driven to extinction.

Noting that in analyzing the persistence and extinction property of the species in system (1.1), the authors of [17] deeply relied on the fact that the weak partners have positive lower bound, while this could not hold for system (1.3), as was shown in (1.6), the weak partners will be driven to extinction.

Now, an interesting issue is proposed: Is it possible for us to investigate the persistence and extinction property of system (1.3)? The aim of this paper is to give a positive answer to this issue.

The rest of the paper is arranged as follows. We introduce a lemma in the next section, and we investigate the persistence property of system (1.3) in Sect. 3. In Sect. 4, by using the iterative method, we establish a set of sufficient conditions which ensure the global attractivity of the boundary equilibrium. Some numeric simulations are carried out in Sect. 5 to show the feasibility of the main results. We end this paper with a brief discussion.

2 Lemma

We need the following lemma to prove the main results.

Lemma 2.1 ([38]) *Let $a > 0, b > 0$.*

- (I) *If $\frac{dx}{dt} \geq x(b - ax)$, then $\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b}{a}$ for $t \geq 0$ and $x(0) > 0$;*
- (II) *If $\frac{dx}{dt} \leq x(b - ax)$, then $\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b}{a}$ for $t \geq 0$ and $x(0) > 0$.*

3 Permanence

The aim of this section is to obtain a set of sufficient conditions to ensure the permanence of system (1.3).

Definition If there exist positive constants $m_i, M_i, i = 1, 2, 3$, which are independent of the positive solution of system (1.3) such that

$$m_1 \leq \liminf_{t \rightarrow +\infty} H_1(t) \leq \limsup_{t \rightarrow +\infty} H_1(t) \leq M_1, \tag{3.1}$$

$$m_2 \leq \liminf_{t \rightarrow +\infty} H_2(t) \leq \limsup_{t \rightarrow +\infty} H_2(t) \leq M_2, \tag{3.2}$$

$$m_3 \leq \liminf_{t \rightarrow +\infty} P(t) \leq \limsup_{t \rightarrow +\infty} P(t) \leq M_3. \tag{3.3}$$

Obviously, by means of permanence means that the species could survive for a long time.

Concerned with the persistence property of system (1.3), we have the following result.

Theorem 3.1 *Assume that*

(A₁)

$$\frac{\alpha}{c_1} > d; \tag{3.4}$$

(A₂)

$$r_1 > \alpha M_2 \tag{3.5}$$

and

(A₃)

$$\alpha m_1 > d \tag{3.6}$$

hold, where M_2 is defined by (3.13) and m_1 is defined by (3.20). Then system (1.3) is permanent.

Proof Noting that m_1, M_2 are fixed positive constants, it follows from conditions (A₂), (A₃) that, for small enough positive constant $\varepsilon > 0$, the inequalities

$$r_1 > \alpha(M_2 + \varepsilon), \tag{3.7}$$

$$\alpha(m_1 - \varepsilon) > d \tag{3.8}$$

hold. Indeed, we could choose ε that satisfies the inequality $\varepsilon < \min\{\frac{r_1}{\alpha} - M_2, m_1 - \frac{d}{\alpha}\}$.

From the first equation of system (1.3) we have

$$\begin{aligned} \frac{dH_1}{dt} &= r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \alpha \frac{H_2}{r_1} \right) \\ &\leq r_1 H_1 (1 - c_1 H_1). \end{aligned} \tag{3.9}$$

Applying Lemma 2.1 to the above inequality leads to

$$\limsup_{t \rightarrow +\infty} H_1(t) \leq \frac{1}{c_1} \stackrel{\text{def}}{=} M_1. \tag{3.10}$$

For above $\varepsilon > 0$, from (3.10), there exists $T_1 > 0$ such that, for all $t > T_1$,

$$H_1(t) < \frac{1}{c_1} + \varepsilon. \tag{3.11}$$

For $t > T_1$, it follows from (3.11) and the second equation of system (1.3) that

$$\frac{dH_2}{dt} = H_2(\alpha H_1 - d - e H_2)$$

$$\leq H_2 \left(\alpha \left(\frac{1}{c_1} + \varepsilon \right) - d - eH_2 \right).$$

Applying Lemma 2.1 to the above inequality leads to

$$\limsup_{t \rightarrow +\infty} H_2(t) \leq \frac{\alpha \left(\frac{1}{c_1} + \varepsilon \right) - d}{e}. \tag{3.12}$$

Setting $\varepsilon \rightarrow 0$ in (3.12) leads to

$$\limsup_{t \rightarrow +\infty} H_2(t) \leq \frac{\alpha}{c_1} - d \stackrel{\text{def}}{=} M_2. \tag{3.13}$$

From (3.13), for any $\varepsilon > 0$ that satisfies (3.7) and (3.8), there exists $T_2 > T_1$ such that

$$H_2(t) < M_2 + \varepsilon \quad \text{for all } t \geq T_2. \tag{3.14}$$

From the third equation of system (1.3), one has

$$\begin{aligned} \frac{dP}{dt} &= r_2 P \left(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P \right) \\ &\leq r_2 P (1 - c_2 P). \end{aligned} \tag{3.15}$$

Applying Lemma 2.1 to the above inequality leads to

$$\limsup_{t \rightarrow +\infty} P(t) \leq \frac{1}{c_2} \stackrel{\text{def}}{=} M_3. \tag{3.16}$$

For any small enough positive constant $\varepsilon > 0$, it follows from (3.16) that there exists $T_3 > 0$,

$$P(t) < \frac{1}{c_2} + \varepsilon \quad \text{for all } t \geq T_3. \tag{3.17}$$

For $t \geq T_3$, from (3.14) and the first equation of system (1.3), we have

$$\begin{aligned} \frac{dH_1}{dt} &= r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \alpha \frac{H_2}{r_1} \right) \\ &\geq r_1 H_1 \left(1 - \frac{H_1}{a_1} - c_1 H_1 - \alpha \frac{M_2 + \varepsilon}{r_1} \right). \end{aligned} \tag{3.18}$$

Applying Lemma 2.1 to (3.18) leads to

$$\liminf_{t \rightarrow +\infty} H_1(t) \geq \frac{1 - \alpha \frac{M_2 + \varepsilon}{r_1}}{c_1 + \frac{1}{a_1}}. \tag{3.19}$$

Setting $\varepsilon \rightarrow 0$ in (3.19), we have

$$\liminf_{t \rightarrow +\infty} H_1(t) \geq \frac{1 - \alpha \frac{M_2}{r_1}}{c_1 + \frac{1}{a_1}} \stackrel{\text{def}}{=} m_1. \tag{3.20}$$

For $\varepsilon > 0$ that satisfies (3.7) and (3.8), it follows from (3.20) that there exists $T_4 > T_3$ such that

$$H_1(t) > m_1 - \varepsilon \quad \text{for all } t \geq T_4. \tag{3.21}$$

For $t > T_4$, from (3.21) and the second equation of system (1.3), we have

$$\begin{aligned} \frac{dH_2}{dt} &= H_2(\alpha H_1 - d - eH_2) \\ &\geq H_2(\alpha(m_1 - \varepsilon) - d - eH_2). \end{aligned}$$

Applying Lemma 2.1 to the above inequality leads to

$$\liminf_{t \rightarrow +\infty} H_2(t) \geq \frac{\alpha(m_1 - \varepsilon) - d}{e}. \tag{3.22}$$

Setting $\varepsilon \rightarrow 0$ in the above inequality, we have

$$\liminf_{t \rightarrow +\infty} H_2(t) \geq \frac{\alpha m_1 - d}{e} \stackrel{\text{def}}{=} m_2. \tag{3.23}$$

From the third equation of system (1.3), one has

$$\begin{aligned} \frac{dP}{dt} &= r_2 P \left(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P \right) \\ &\geq r_2 P \left(1 - \frac{P}{a_2} - c_2 P \right). \end{aligned} \tag{3.24}$$

Applying Lemma 2.1 to the above inequality leads to

$$\liminf_{t \rightarrow +\infty} P(t) \geq \frac{1}{\frac{1}{a_2} + c_2} \stackrel{\text{def}}{=} m_3. \tag{3.25}$$

(3.10), (3.13), (3.16), (3.20), (3.23), and (3.25) show that under the assumption of Theorem 3.1, system (1.3) is permanent. This ends the proof of Theorem 3.1. \square

Remark 3.1 Condition (A_1) shows that the transform rate of strong partners to weak partners should be large enough, while condition (A_2) requires the transform rate of strong partners to weak partners to be restricted to some area. Combining these two inequalities ensures the permanence of system (1.3), the transform rate (represent by α) should be restricted to some interval.

Remark 3.2 Conditions (A_1) – (A_3) could be combined with the following inequality:

$$\alpha \cdot \frac{1 - \frac{\alpha(c_1 - d)}{r_1 e}}{c_1 + \frac{1}{a_1}} > d, \tag{3.26}$$

which is equivalent to

$$1 > \frac{\alpha(\alpha - c_1 d)}{r_1 e c_1} + \frac{d}{\alpha} \left(c_1 + \frac{1}{a_1} \right). \tag{3.27}$$

4 Stability of the boundary equilibrium

As was shown in the [Introduction](#) section, without the transformation of the strong partners to the weak partners, the weak partners will be driven to extinction. Also, we have shown in [Sect. 3](#) that to ensure the permanence of the system, the transform rate should be restricted to a certain range. A natural issue is proposed: What would happen if the transform rate is small? This section will give the answer to this question.

Theorem 4.1 *Assume that*

$$(A_4)$$

$$\frac{\alpha}{c_1} < d \tag{4.1}$$

holds, then the weak partners H_2 in system (1.3) will be driven to extinction, while the strong partners and the species P will be finally attracted to the unique positive equilibrium of the following system:

$$\begin{aligned} \frac{dH_1}{dt} &= r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 \right), \\ \frac{dP}{dt} &= r_2 P \left(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P \right). \end{aligned} \tag{4.2}$$

Noting that the expression of the positive equilibrium of (4.2) is not easy to be expressed, here we give the following lemma.

Lemma 4.1 *System (4.2) admits a unique positive equilibrium (H_1^*, P^*) .*

Proof The positive equilibrium of system (4.2) satisfies the equations

$$\begin{aligned} 1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 &= 0, \\ 1 - \frac{P}{a_2 + b_2 H_1} - c_2 P &= 0. \end{aligned} \tag{4.3}$$

From the second equation of system (4.3), we have

$$P = \frac{b_2 H_1 + a_2}{H_1 b_2 c_2 + a_2 c_2 + 1}. \tag{4.4}$$

Substituting it into the first equation of system (4.3) and simplifying it, we have

$$A_1 H_1^2 + A_2 H_1 + A_3 = 0, \tag{4.5}$$

here

$$\begin{aligned} A_1 &= a_1 b_2 c_1 c_2 + b_1 b_2 c_1 + b_2 c_2 > 0, \\ A_2 &= a_1 a_2 c_1 c_2 - a_1 b_2 c_2 + a_2 b_1 c_1 + a_1 c_1 + a_2 c_2 - b_1 b_2 + 1, \\ A_3 &= -a_1 a_2 c_2 - a_2 b_1 - a_1 < 0. \end{aligned}$$

Obviously, system (4.5) has the unique positive solution

$$H_1^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}. \tag{4.6}$$

Thus, system (4.3) has a unique positive solution (H_1^*, P^*) , where H_1^* is defined by (4.6) and

$$P^* = \frac{b_2H_1^* + a_2}{H_1^*b_2c_2 + a_2c_2 + 1}. \tag{4.7}$$

This ends the proof of Lemma 4.1. □

Proof of Theorem 4.1 For any small enough positive constant $\varepsilon > 0$, without loss of generality, we may assume that $\varepsilon < \min\{\frac{1}{\frac{\alpha}{r_1} + c_1 + \frac{1}{a_1}}, \frac{1}{2} \frac{1}{c_2 + \frac{1}{a_2}}\}$, it follows from (4.1) that the inequality

$$\alpha \left(\frac{1}{c_1} + \varepsilon \right) < d \tag{4.8}$$

holds. For above $\varepsilon > 0$, similar to the analysis of (3.9)–(3.11), we have

$$\limsup_{t \rightarrow +\infty} H_1(t) \leq \frac{1}{c_1}. \tag{4.9}$$

For above $\varepsilon > 0$, it follows from (4.9) that there exists $T_{11} > 0$ such that

$$H_1(t) < \frac{1}{c_1} + \varepsilon \stackrel{\text{def}}{=} M_1^{(1)} \quad \text{for all } t > T_{11}. \tag{4.10}$$

For $t > T_{11}$, from (4.10) and the second equation of system (1.3), we have

$$\begin{aligned} \frac{dH_2}{dt} &= H_2(\alpha H_1 - d - eH_2) \\ &\leq H_2 \left(\alpha \left(\frac{1}{c_1} + \varepsilon \right) - d \right). \end{aligned}$$

Hence

$$H_2(t) \leq H_2(T_{11}) \exp \left\{ \left(\alpha \left(\frac{1}{c_1} + \varepsilon \right) - d \right) (t - T_{11}) \right\}. \tag{4.11}$$

Therefore,

$$\lim_{t \rightarrow +\infty} H_2(t) = 0. \tag{4.12}$$

It follows from (4.12) that the weak parters will be driven to extinction. Also, for any small enough positive constant $\varepsilon > 0$, it follows from (4.12) that there exists $T_{12} > T_{11} > 0$ such that

$$H_2(t) < \varepsilon \quad \text{for all } t > T_{12}. \tag{4.13}$$

From the third equation of system (1.3), similar to the analysis of (3.15)–(3.16), we have

$$\limsup_{t \rightarrow +\infty} P(t) \leq \frac{1}{c_2}. \tag{4.14}$$

For any small enough positive constant $\varepsilon > 0$, it follows from (4.14) that there exists $T_{13} > 0$ such that, for all $t > T_{13}$,

$$P(t) < \frac{1}{c_2} + \varepsilon \stackrel{\text{def}}{=} M_2^{(1)}. \tag{4.15}$$

For $t > T_{13}$, from (4.15) and the first equation of system (1.3) we have

$$\begin{aligned} \frac{dH_1}{dt} &= r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \alpha \frac{H_2}{r_1} \right) \\ &\leq r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 M_2^{(1)}} - c_1 H_1 \right). \end{aligned} \tag{4.16}$$

Applying Lemma 2.1 to (4.16) leads to

$$\limsup_{t \rightarrow +\infty} H_1(t) \leq \frac{1}{c_1 + \frac{1}{a_1 + b_1 M_2^{(1)}}}. \tag{4.17}$$

Hence, for $\varepsilon > 0$ that satisfies (4.8), there exists $T_{21} > T_{13}$ such that, for all $t > T_{21}$, one has

$$H_1(t) < \frac{1}{c_1 + \frac{1}{a_1 + b_1 M_2^{(1)}}} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_1^{(2)}. \tag{4.18}$$

For $t > T_{21}$, from (4.18) and the third equation of system (1.3), we have

$$\begin{aligned} \frac{dP}{dt} &= r_2 P \left(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P \right) \\ &\leq r_2 P \left(1 - \frac{P}{a_2 + b_2 M_1^{(2)}} - c_2 P \right). \end{aligned} \tag{4.19}$$

Applying Lemma 2.1 to (4.19) leads to

$$\limsup_{t \rightarrow +\infty} P(t) \leq \frac{1}{c_2 + \frac{1}{a_2 + b_2 M_1^{(2)}}}. \tag{4.20}$$

For $\varepsilon > 0$ that satisfies (4.8), it follows from (4.20) that there exists $T_{22} > T_{21}$ such that, for all $t > T_{22}$, one has

$$P(t) < \frac{1}{c_2 + \frac{1}{a_2 + b_2 M_1^{(2)}}} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_2^{(2)}. \tag{4.21}$$

From (4.21) and the first equation of system (1.3), we have

$$\frac{dH_1}{dt} = r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \alpha \frac{H_2}{r_1} \right)$$

$$\geq r_1 H_1 \left(1 - \frac{H_1}{a_1} - c_1 H_1 - \alpha \frac{\varepsilon}{r_1} \right). \tag{4.22}$$

Applying Lemma 2.1 to (4.22) leads to

$$\liminf_{t \rightarrow +\infty} H_1(t) \geq \frac{1 - \alpha \frac{\varepsilon}{r_1}}{c_1 + \frac{1}{a_1}}. \tag{4.23}$$

For $\varepsilon > 0$ that satisfies (4.8), it follows from (4.23) that there exists $T'_{11} > T_{22}$ such that, for all $t > T'_{11}$, one has

$$H_1(t) > \frac{1 - \alpha \frac{\varepsilon}{r_1}}{c_1 + \frac{1}{a_1}} - \varepsilon \stackrel{\text{def}}{=} m_1^{(1)}. \tag{4.24}$$

From the third equation of system (1.3), we have

$$\begin{aligned} \frac{dP}{dt} &= r_2 P \left(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P \right) \\ &\geq r_2 P \left(1 - \frac{P}{a_2} - c_2 P \right). \end{aligned} \tag{4.25}$$

Applying Lemma 2.1 to (4.25) leads to

$$\liminf_{t \rightarrow +\infty} P(t) \geq \frac{1}{c_2 + \frac{1}{a_2}}. \tag{4.26}$$

For $\varepsilon > 0$ that satisfies (4.8), it follows from (4.26) that there exists $T'_{12} > T'_{11}$ such that, for all $t > T'_{12}$, one has

$$P(t) > \frac{1}{c_2 + \frac{1}{a_2}} - \varepsilon \stackrel{\text{def}}{=} m_2^{(1)}. \tag{4.27}$$

From (4.27) and the first equation of system (1.3), for $t > T'_{12}$, one has

$$\begin{aligned} \frac{dH_1}{dt} &= r_1 H_1 \left(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \alpha \frac{H_2}{r_1} \right) \\ &\geq r_1 H_1 \left(1 - \frac{H_1}{a_1 + m_2^{(1)}} - c_1 H_1 - \alpha \frac{\varepsilon}{r_1} \right). \end{aligned} \tag{4.28}$$

Applying Lemma 2.1 to (4.28) leads to

$$\liminf_{t \rightarrow +\infty} H_1(t) \geq \frac{1 - \alpha \frac{\varepsilon}{r_1}}{c_1 + \frac{1}{a_1 + m_2^{(1)}}}. \tag{4.29}$$

For $\varepsilon > 0$ that satisfies (4.8), it follows from (4.29) that there exists $T'_{21} > T'_{12}$ such that, for all $t > T'_{21}$, one has

$$H_1(t) > \frac{1 - \alpha \frac{\varepsilon}{r_1}}{c_1 + \frac{1}{a_1 + m_2^{(1)}}} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_1^{(2)}. \tag{4.30}$$

From (4.30) and the third equation of system (1.3), we have

$$\begin{aligned} \frac{dP}{dt} &= r_2P \left(1 - \frac{P}{a_2 + b_2H_1} - c_2P \right) \\ &\geq r_2P \left(1 - \frac{P}{a_2 + b_2m_1^{(1)}} - c_2P \right). \end{aligned} \tag{4.31}$$

Applying Lemma 2.1 to (4.31) leads to

$$\liminf_{t \rightarrow +\infty} P(t) \geq \frac{1}{c_2 + \frac{1}{a_2 + b_2m_1^{(1)}}}. \tag{4.32}$$

For $\varepsilon > 0$ that satisfies (4.8), it follows from (4.32) that there exists $T'_{22} > T'_{21}$ such that, for all $t > T'_{22}$, one has

$$P(t) > \frac{1}{c_2 + \frac{1}{a_2 + b_2m_1^{(1)}}} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_2^{(2)}. \tag{4.33}$$

Noting that $\frac{1}{a_2 + b_2M_1^{(1)}} > 0, \frac{1}{a_1 + b_1M_2^{(1)}} > 0$, from (4.10), (4.15), (4.18), and (4.21), one has

$$M_1^{(2)} < M_1^{(1)}, \quad M_2^{(2)} < M_2^{(1)}. \tag{4.34}$$

Also, from $m_1^{(1)} > 0, m_2^{(1)} > 0$ we have

$$\frac{1}{a_1} > \frac{1}{a_1 + b_1m_2^{(1)}}, \quad \frac{1}{a_2} > \frac{1}{a_2 + b_2m_1^{(1)}}.$$

Hence, from (4.24), (4.27), (4.30), and (4.33), we have

$$m_1^{(2)} > m_1^{(1)}, \quad m_2^{(2)} > m_2^{(1)}. \tag{4.35}$$

Repeating the above procedure, we get four sequences $m_i^{(n)}, M_i^{(n)}, i = 1, 2, n = 1, 2, \dots$ such that

$$\begin{aligned} M_1^{(n)} &= \frac{1}{c_1 + \frac{1}{a_1 + b_1M_2^{(n-1)}}} + \frac{\varepsilon}{n}, \\ M_2^{(n)} &= \frac{1}{c_2 + \frac{1}{a_2 + b_2M_1^{(n)}}} + \frac{\varepsilon}{n}, \\ m_1^{(n)} &= \frac{1 - \alpha \frac{\varepsilon}{r_1}}{c_1 + \frac{1}{a_1 + b_1m_2^{(n-1)}}} - \frac{\varepsilon}{n}, \\ m_2^{(n)} &= \frac{1}{c_2 + \frac{1}{a_2 + b_2m_1^{(n-1)}}} - \frac{\varepsilon}{n}. \end{aligned} \tag{4.36}$$

From the deduction process, for $t > \max\{T_{2n}, T'_{2n}\}$, we have

$$m_1^{(n)} < H_1(t) < M_1^{(n)}, \quad m_2^{(n)} < P(t) < M_2^{(n)}. \tag{4.37}$$

We claim that sequences $M_i^{(n)}, i = 1, 2$, are strictly decreasing, and sequences $m_i^{(n)}, i = 1, 2$, are strictly increasing. To prove this claim, we carry on by induction. (4.34) and (4.35) show that the conclusion holds for $n = 2$. Let us assume now that our claim is true for $n = k$, that is,

$$M_i^{(k)} < M_i^{(k-1)}, \quad m_i^{(k)} > m_i^{(k-1)}, \quad i = 1, 2.$$

Then

$$\frac{1}{a_1 + b_1 M_2^{(k-1)}} < \frac{1}{a_1 + b_1 M_2^{(k)}}, \quad \frac{1}{a_2 + b_2 M_1^{(k-1)}} < \frac{1}{a_2 + b_2 M_1^{(k)}}, \tag{4.38}$$

$$\frac{1}{a_1 + b_1 m_2^{(k-1)}} < \frac{1}{a_1 + b_1 m_2^{(k)}}, \quad \frac{1}{a_2 + b_2 m_1^{(k-1)}} > \frac{1}{a_2 + b_2 m_1^{(k)}}. \tag{4.39}$$

And so

$$\begin{aligned} M_1^{(k+1)} &= \frac{1}{c_1 + \frac{1}{a_1 + b_1 M_2^{(k)}}} + \frac{\varepsilon}{k+1} \\ &< \frac{1}{c_1 + \frac{1}{a_1 + b_1 M_2^{(k-1)}}} + \frac{\varepsilon}{k} = M_1^{(k)}, \\ M_2^{(k+1)} &= \frac{1}{c_2 + \frac{1}{a_2 + b_2 M_1^{(k)}}} + \frac{\varepsilon}{k+1} \\ &< \frac{1}{c_2 + \frac{1}{a_2 + b_2 M_1^{(k-1)}}} + \frac{\varepsilon}{k} = M_2^{(k)}, \\ m_1^{(k+1)} &= \frac{1 - \alpha \frac{\varepsilon}{r_1}}{c_1 + \frac{1}{a_1 + b_1 m_2^{(k)}}} - \frac{\varepsilon}{k+1} \\ &> \frac{1 - \alpha \frac{\varepsilon}{r_1}}{c_1 + \frac{1}{a_1 + b_1 m_2^{(k-1)}}} - \frac{\varepsilon}{k} = m_1^{(k)}, \\ m_2^{(k+1)} &= \frac{1}{c_2 + \frac{1}{a_2 + b_2 m_1^{(k)}}} - \frac{\varepsilon}{k+1} \\ &> \frac{1}{c_2 + \frac{1}{a_2 + b_2 m_1^{(k-1)}}} - \frac{\varepsilon}{k} = m_2^{(k)}. \end{aligned} \tag{4.40}$$

The above analysis shows that $M_i^{(n)}$ is a strictly decreasing sequence, $m_i^{(n)}$ is a strictly increasing sequence. Set

$$\begin{aligned} \lim_{n \rightarrow +\infty} M_1(n) &= \overline{H}_1, \\ \lim_{n \rightarrow +\infty} M_2(n) &= \overline{P}, \\ \lim_{n \rightarrow +\infty} m_1(n) &= \underline{H}_1, \\ \lim_{n \rightarrow +\infty} m_2(n) &= \underline{P}. \end{aligned} \tag{4.41}$$

Setting $n \rightarrow +\infty$ in (4.40) leads to

$$\begin{aligned}
 \overline{H}_1 &= \frac{1}{c_1 + \frac{1}{a_1 + b_1 \overline{P}}}, \\
 \overline{P} &= \frac{1}{c_2 + \frac{1}{a_2 + b_2 \overline{H}_1}}, \\
 \underline{H}_1 &= \frac{1}{c_1 + \frac{1}{a_1 + b_1 \underline{P}}}, \\
 \underline{P} &= \frac{1}{c_2 + \frac{1}{a_2 + b_2 \underline{H}_1}}.
 \end{aligned}
 \tag{4.42}$$

(4.42) shows that $(\overline{H}_1, \overline{P}), (\underline{H}_1, \underline{P})$ are all the solutions of (4.3). By Lemma 4.1, (4.3) has a unique positive solution (H_1^*, P^*) . Hence, we conclude that $\overline{H}_1 = \underline{H}_1 = H_1^*, \overline{P} = \underline{P} = P^*$, that is,

$$\lim_{t \rightarrow +\infty} H_1(t) = H_1^*, \quad \lim_{t \rightarrow +\infty} P(t) = P^*.
 \tag{4.43}$$

Thus, the unique interior equilibrium (H_1^*, P^*) is globally attractive. This completes the proof of Theorem 4.1. □

5 Numeric simulations

Now let us consider the following three examples.

Example 5.1

$$\begin{aligned}
 \frac{dH_1}{dt} &= 3H_1 \left(1 - \frac{H_1}{2 + 2P} - H_1 - \frac{2H_2}{3} \right), \\
 \frac{dH_2}{dt} &= H_2(2H_1 - 0.5 - 2H_2), \\
 \frac{dP}{dt} &= 2P \left(1 - \frac{P}{2 + 0.8H_1} - 1.5P \right).
 \end{aligned}
 \tag{5.1}$$

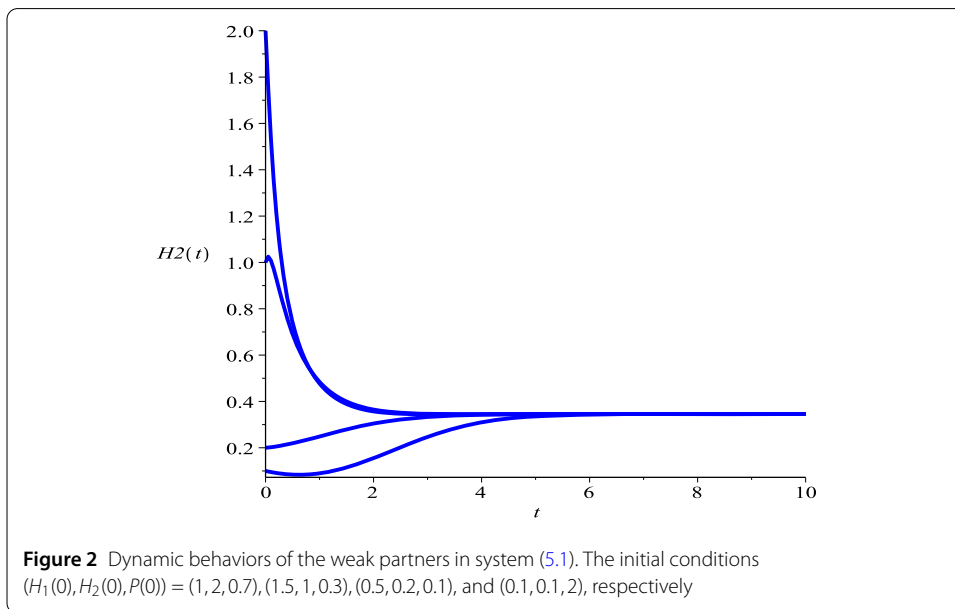
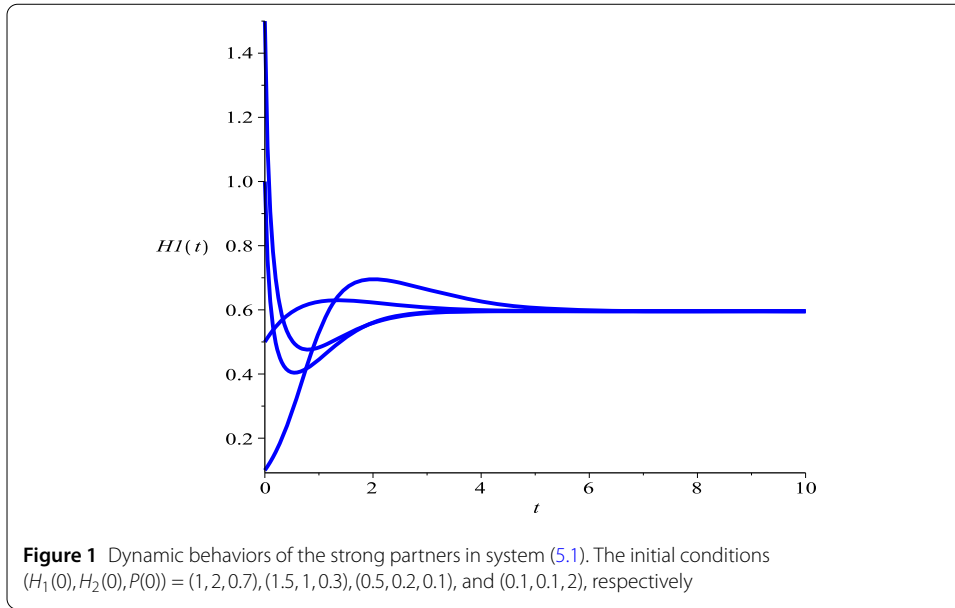
Here, corresponding to system (1.3), we take $r_1 = 3, a_1 = 2, b_1 = 2, c_1 = 1, \alpha = 2, r_2 = 2, d = 0.5, e = 2, a_2 = 2, b_2 = 0.8, c_2 = 1.5$. By simple computation, we have

$$\frac{\alpha(\alpha - c_1 d)}{r_1 e c_1} + \frac{d}{\alpha} \left(c_1 + \frac{1}{a_1} \right) = \frac{7}{8} < 1.
 \tag{5.2}$$

That is, inequality (3.27) holds, it follows from Remark 3.2 and Theorem 3.1 that system (1.3) is permanent. Figures 1–3 support this assertion.

Example 5.2

$$\begin{aligned}
 \frac{dH_1}{dt} &= 3H_1 \left(1 - \frac{H_1}{2 + 2P} - H_1 - \frac{0.5H_2}{3} \right), \\
 \frac{dH_2}{dt} &= H_2(0.5H_1 - 2 - 2H_2),
 \end{aligned}
 \tag{5.3}$$

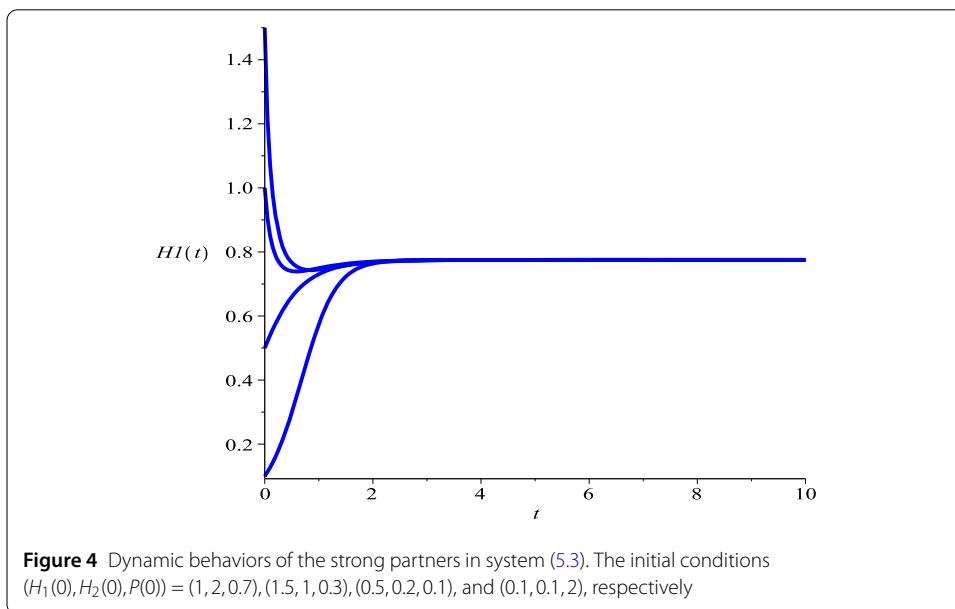
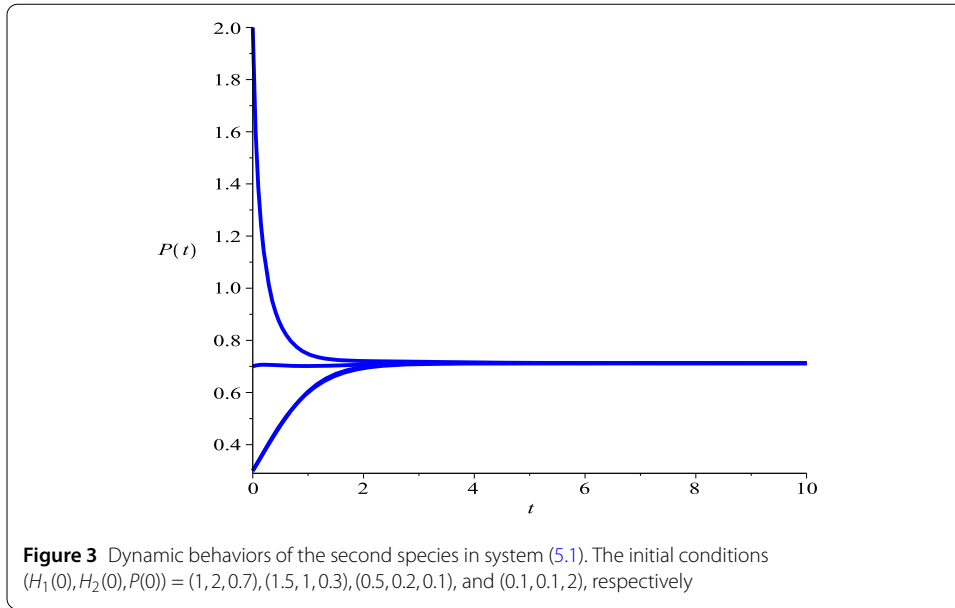


$$\frac{dP}{dt} = 2P \left(1 - \frac{P}{2 + 0.8H_1} - 1.5P \right).$$

Here, corresponding to system (1.3), we take $r_1 = 3, a_1 = 2, b_1 = 2, c_1 = 1, \alpha = 0.5, r_2 = 2, d = 2, e = 2, a_2 = 2, b_2 = 0.8, c_2 = 1.5.$ By simple computation, we have

$$\frac{\alpha}{c_1} = 0.5 < 2 = d. \tag{5.4}$$

That is, inequality (4.1) holds. It follows from Theorem 4.1 that the weak partners H_2 in system (5.3) will be driven to extinction, while the strong partners and the species P will

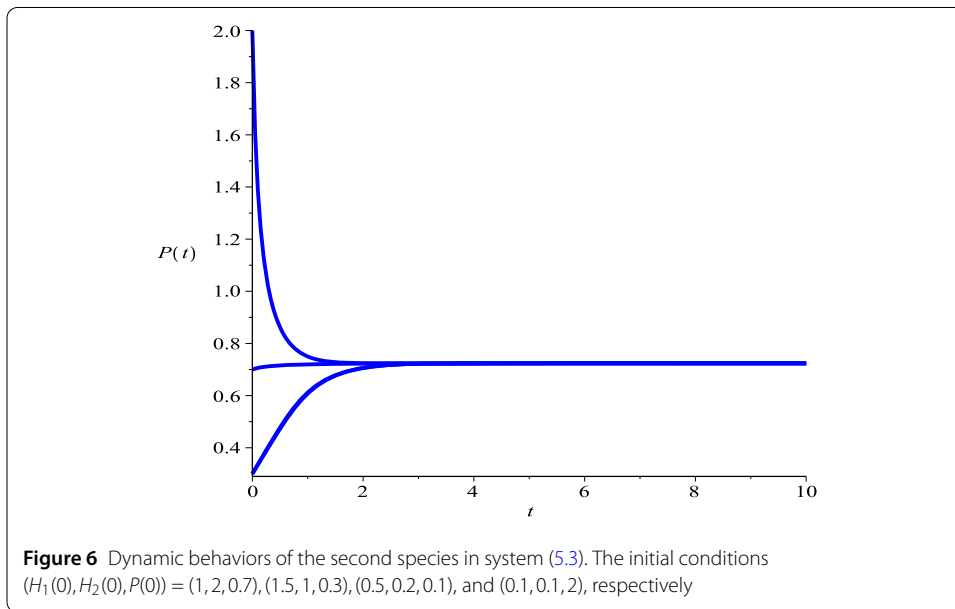
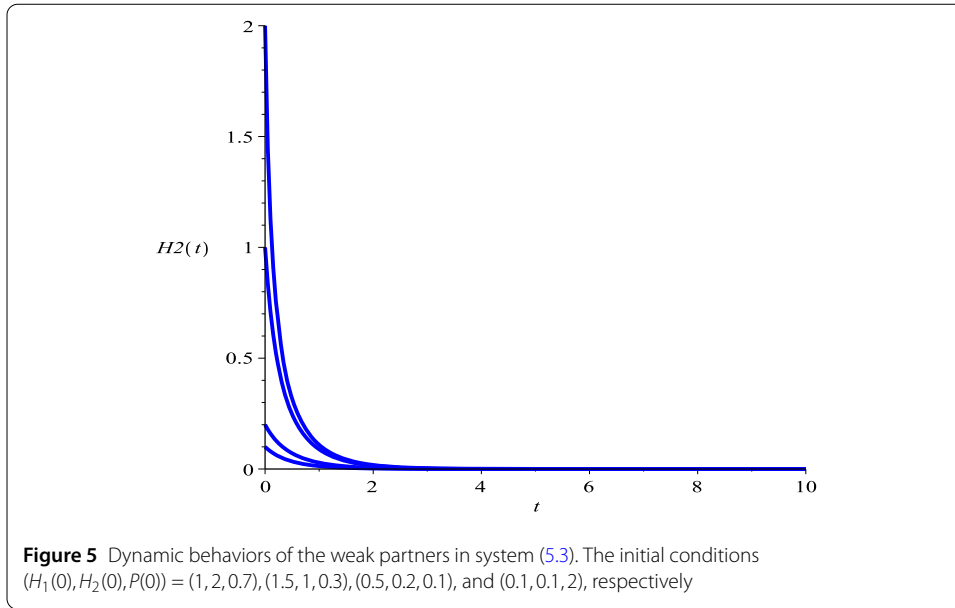


be finally attracted to the unique positive equilibrium of the following system:

$$\begin{aligned} \frac{dH_1}{dt} &= 3H_1 \left(1 - \frac{H_1}{2 + 2P} - H_1 \right), \\ \frac{dP}{dt} &= 2P \left(1 - \frac{P}{2 + 0.8H_1} - 1.5P \right). \end{aligned} \tag{5.5}$$

Figures 4–6 support this assertion.

In Theorems 3.1 and 4.1, we made the assumption that the transform rate of the strong partners to the weak partners is limited to an interval, one may argue what would happen if the transform rate is large enough. Now let us consider the following example.



Example 5.3

$$\begin{aligned} \frac{dH_1}{dt} &= 3H_1 \left(1 - \frac{H_1}{2 + 2P} - H_1 - \frac{10H_2}{3} \right), \\ \frac{dH_2}{dt} &= H_2(10H_1 - 2 - 2H_2), \\ \frac{dP}{dt} &= 2P \left(1 - \frac{P}{2 + 0.8H_1} - 1.5P \right). \end{aligned} \tag{5.6}$$

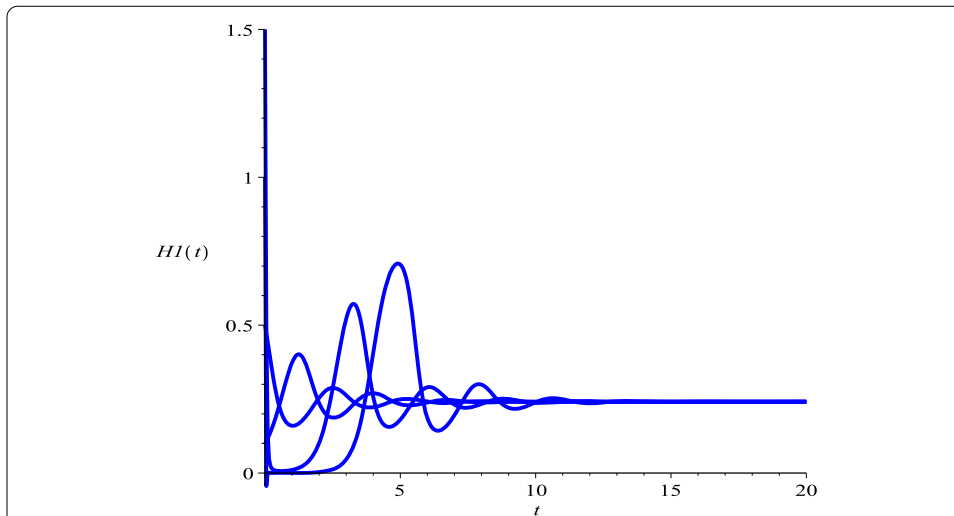


Figure 7 Dynamic behaviors of the strong partners in system (5.6). The initial conditions $(H_1(0), H_2(0), P(0)) = (1, 2, 0.7), (1.5, 1, 0.3), (0.5, 0.2, 0.1)$, and $(0.1, 0.1, 2)$, respectively

Here, corresponding to system (1.3), we take $r_1 = 3, a_1 = 2, b_1 = 2, c_1 = 1, \alpha = 10, r_2 = 2, d = 2, e = 2, a_2 = 2, b_2 = 0.8, c_2 = 1.5$. By simple computation, we have

$$\frac{\alpha(\alpha - c_1 d)}{r_1 e c_1} + \frac{d}{\alpha} \left(c_1 + \frac{1}{a_1} \right) = \frac{409}{30} > 1 \tag{5.7}$$

and

$$\frac{\alpha}{c_1} = 10 > 2 = d. \tag{5.8}$$

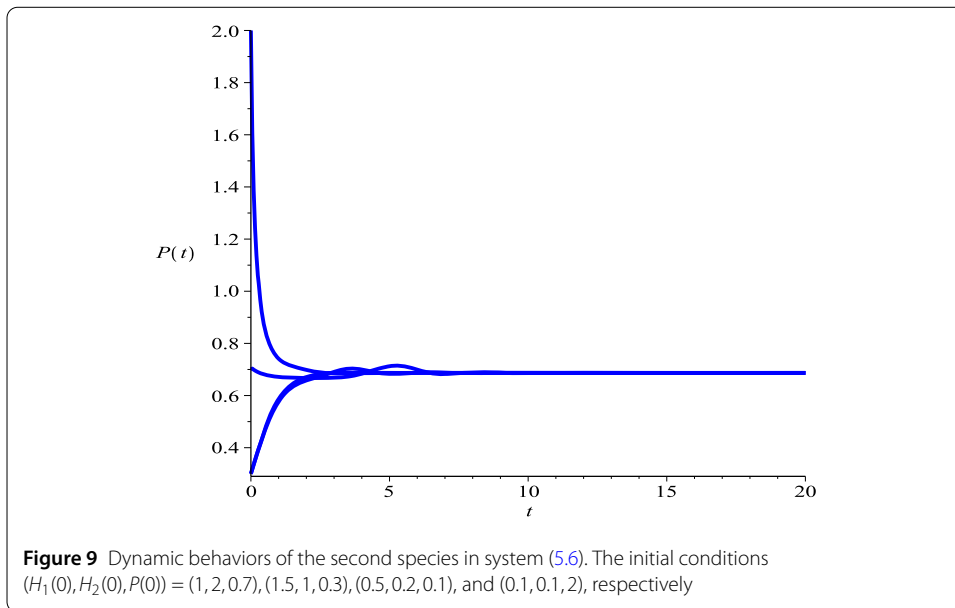
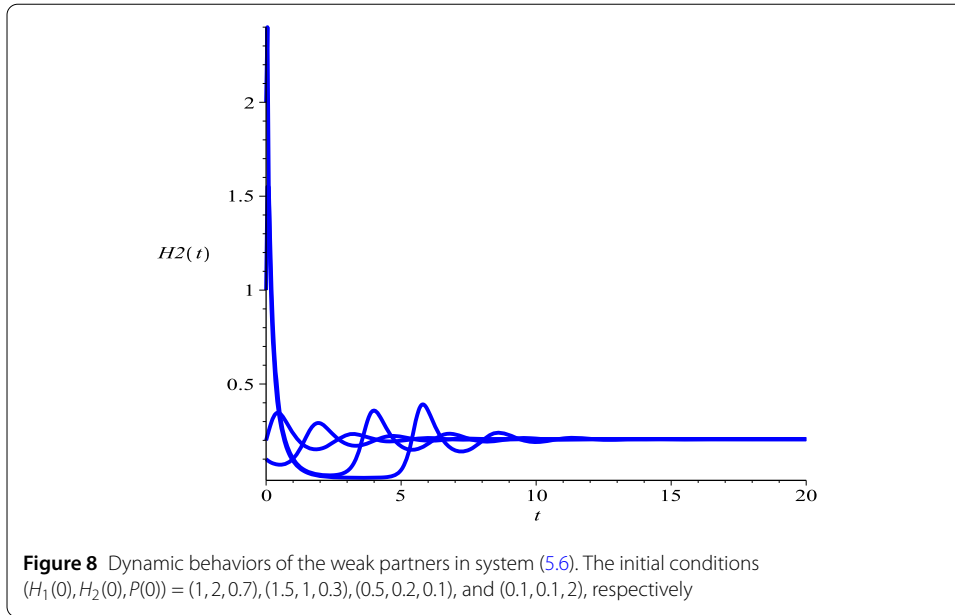
Thus, neither (3.27) nor (4.1) holds. We could not give any persistence or non-persistence property from Theorem 3.1 and 4.1. However, numeric simulations (Figs. 7–9) show that in this case the system is still permanent.

6 Discussion

Based on the traditional May cooperative system, Zhao, Qin, and Chen [36] proposed a May cooperative system with strong and weak cooperative partners (system (1.1)). The authors investigated the persistence and non-persistence property of system (1.1). By constructing a suitable Lyapunov function, they also obtained the conditions that ensure the global asymptotic stability of the positive equilibrium and boundary equilibrium. As was shown in the Introduction section, their result (Theorem A) from the mathematic point of view is correct; however, from the biological background, it is unreasonable. This motivated us to propose system (1.3), which overcomes the drawback of model (1.1).

By applying the differential inequality theory, we first obtain a set of conditions that ensure the permanence of system (1.3). Then, by using the iterative method and the differential inequality theory, we also obtain a set of conditions that ensure the extinction of the weak partners and the attractivity of the strong partners and the other species.

Since our results require the restriction of the transform rate, it is natural to ask what would happen if the transform rate is large enough. Zhao, Qin, and Chen [36] showed



that in system (1.1), if the transform rate is too large, the strong partners will be driven to extinction; however, for model (1.3), numeric simulations (Figs. 7–9) show that in this case the model is still permanent. From numeric simulations one could also see that the species need more time to attract to their final density. It is in this sense that the large transform rate makes the system unstable, since from Figs. 7–9 we could see that sometimes the density of the species may be very near to zero, which means the lack of the species, and could increase the chance of extinction of the species.

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The authors declare that there is no conflict of interests.

Consent for publication

Not applicable.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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