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Growth of solutions to two systems of q-difference differential equations

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Abstract

This paper is devoted to studying the growth of entire or meromorphic solutions to two systems of q-difference differential equations. The estimates on the growth order of meromorphic solutions are obtained, which are extensions of previous results due to Xu et al . Examples are given to illustrate the existence of solutions of such systems.

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1 Introduction and main results

Let $f(z)$ be a non-constant meromorphic function in the complex plane $\mathbb{C}$. We use $\rho(f)$ and $\mu(f)$ to denote the order and the lower order of $f(z)$, and use $\lambda(\frac{1}{f})$ and $\overline{\lambda}(\frac{1}{f})$ to denote the exponent of convergence of poles and that of the distinct poles of $f(z)$, respectively. In addition, we say a meromorphic function $\alpha(z)$ ($\not\equiv 0, \infty$) is a small function of $f(z)$ provided that $T(r,\alpha) = S(r,f)$, where $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of r with finite logarithmic measure. Nevanlinna theory is an important tool in this paper, its standard symbols and fundamental results come mainly from [12, 21].

As we all know, it is an interesting problem to consider the Malmquist theorem [16] for differential equations. Laine [15] gave the following result.

Theorem A ([15]) *Let*

$$\bigl(w'(z)\bigr)^n = R(z,w), \tag{1}$$

where the right-hand side

$$R(z,w) = \frac{\sum_{i=0}^{k} a_i(z)w^i}{\sum_{j=0}^{l} b_j(z)w^j}$$

is rational in both arguments. If equation (1) *has a transcendental meromorphic solution, then* $l = 0$ *and* $k \leq 2n$.

With the establishment of the difference analog of Nevanlinna theory, many studies [1, 8, 13, 20] about the Malmquist-type theorem of complex difference equations or systems have appeared. Gundersen et al. [11] considered the growth of meromorphic solutions to a certain type of complex q-difference equation and proved the following result.

Theorem B ([11]) *Let $w(z)$ be a transcendental meromorphic solution of the equation*

$$w(qz) = R(z, w),$$

where $q \in \mathbb{C}$, $|q| > 1$, $R(z, w)$ is irreducible in w, which is defined as in Theorem A, and the coefficients $a_i(z)$ and $b_j(z)$ are small functions of w and $a_k(z)b_l(z) \not\equiv 0$. If $m := \max\{k, l\} \geq 1$, then $\rho(w) = \frac{\log m}{\log |q|}$.

After these results, many scholars studied a series of complex q-difference differential equations and systems about the Malmquist-type theorem [5, 6, 19]. Xu et al. [19] investigated the following system:

$$\begin{cases} [w_1'(q_1 z)]^{n_1} = R_2(z, w_2(z)), \\ [w_2'(q_2 z)]^{n_2} = R_1(z, w_1(z)), \end{cases} \tag{2}$$

where $q_1, q_2 \in \mathbb{C} \setminus \{0\}$, $n_1, n_2 \in \mathbb{Z}_+$, and

$$R_1\big(z, w_1(z)\big) = \frac{\sum_{i=0}^{k_1} a_i(z) w_1(z)^i}{\sum_{j=0}^{l_1} b_j(z) w_1(z)^j}, \qquad R_2\big(z, w_2(z)\big) = \frac{\sum_{i=0}^{k_2} c_i(z) w_2(z)^i}{\sum_{j=0}^{l_2} d_j(z) w_2(z)^j} \tag{3}$$

are irreducible rational functions, and $a_i(z)$, $b_j(z)$ are small functions with respect to w_1, and $c_i(z)$, $d_j(z)$ are small functions with respect to w_2. They obtained the estimates on the growth order for meromorphic solutions of system (2).

We consider the question of what happens if system (2) is more general, for example,

$$\begin{cases} \Omega_1(z, w_1^{(h_1)}(q_1 z)) = R_2(z, w_2(z)), \\ \Omega_2(z, w_2^{(h_2)}(q_2 z)) = R_1(z, w_1(z)), \end{cases} \tag{4}$$

where $q_1, q_2 \in \mathbb{C} \setminus \{0\}$, $h_1, h_2 \in \mathbb{Z}_+$, and $R_1(z, w_1(z))$, $R_2(z, w_2(z))$ are defined as in (3), and

$$\begin{aligned} \Omega_1\big(z, w_1^{(h_1)}(q_1 z)\big) &= \frac{\sum_{m_1=0}^{p_1} u_{m_1}^1(z) [w_1^{(h_1)}(q_1 z)]^{m_1}}{\sum_{n_1=0}^{s_1} v_{n_1}^1(z) [w_1^{(h_1)}(q_1 z)]^{n_1}}, \\ \Omega_2\big(z, w_2^{(h_2)}(q_2 z)\big) &= \frac{\sum_{m_2=0}^{p_2} u_{m_2}^2(z) [w_2^{(h_2)}(q_2 z)]^{m_2}}{\sum_{n_2=0}^{s_2} v_{n_2}^2(z) [w_2^{(h_2)}(q_2 z)]^{n_2}} \end{aligned} \tag{5}$$

are irreducible rational functions in $w_1^{(h_1)}(q_1 z)$, $w_2^{(h_2)}(q_2 z)$, respectively, and the meromorphic coefficients $u_{m_t}^t(z)$ ($m_t = 0, \ldots, p_t$), $v_{n_t}^t(z)$ ($n_t = 0, \ldots, s_t$) are of growth $S(r, w_t)$, $t = 1, 2$, and $u_{p_t}^t(z) v_{s_t}^t(z) \not\equiv 0$, $t = 1, 2$.

For the question above, we study the growth of solutions to the system of q-difference differential equations (4). Further, set

$$\tau_t = \max\{p_t, s_t\} \quad \text{and} \quad \sigma_t = \max\{k_t, l_t\}, \quad t = 1, 2.$$

Clearly, $\tau_t \geq 1$ and $\sigma_t \geq 1$. Also set

$$\tau = \tau_1\tau_2, \qquad \sigma = \sigma_1\sigma_2, \qquad q = q_1q_2,$$

and

$$\kappa_1 = \tau(h_1 + 1), \qquad \kappa_2 = \tau(h_2 + 1), \qquad \kappa = \tau(h_1 + 1)(h_2 + 1).$$

Now, we state the first result in this paper.

Theorem 1.1 *Let* (w_1, w_2) *be a pair of transcendental solutions of system* (4). *Then one of the following cases holds.*

(i) *For* $|q_1| > 1$, $|q_2| > 1$, *if* w_1, w_2 *are meromorphic and* $\sigma > \kappa$, *then* $\mu(w_t) \geq \frac{\log\sigma - \log\kappa}{\log|q|}$, $t = 1, 2$; *if* w_t *is meromorphic and* $\sigma > \kappa_t$, $t = 1$ *or* $t = 2$, *and the other is entire, then* $\mu(w_t) \geq \frac{\log\sigma - \log\kappa_t}{\log|q|}$, $t = 1, 2$; *if* w_1, w_2 *are entire and* $\sigma > \tau$, *then* $\mu(w_t) \geq \frac{\log\sigma - \log\tau}{\log|q|}$, $t = 1, 2$.

(ii) *For* $|q_1| < 1$, $|q_2| < 1$, *if* w_1, w_2 *are meromorphic and* $\sigma \leq \kappa$, *then* $\rho(w_t) \leq \frac{\log\sigma - \log\kappa}{\log|q|}$, $t = 1, 2$; *if* w_t *is meromorphic and* $\sigma \leq \kappa_t$, $t = 1$ *or* $t = 2$, *and the other is entire, then* $\rho(w_t) \leq \frac{\log\sigma - \log\kappa_t}{\log|q|}$, $t = 1, 2$; *if* w_1, w_2 *are entire and* $\sigma \leq \tau$, *then* $\rho(w_t) \leq \frac{\log\sigma - \log\tau}{\log|q|}$, $t = 1, 2$.

(iii) *For* $|q_1| = |q_2| = 1$, *if* w_1, w_2 *are meromorphic, then* $\sigma \leq \kappa$, *furthermore, if* $\kappa_t < \sigma \leq \kappa$, $t = 1, 2$, *then* $\overline{\lambda}(\frac{1}{w_t}) = \lambda(\frac{1}{w_t}) = \rho(w_t)$, $t = 1, 2$; *if* w_t *is meromorphic*, $t = 1$ *or* $t = 2$, *and the other is entire, then* $\sigma \leq \kappa_t$, $t = 1$ *or* $t = 2$, *furthermore, if* $\tau < \sigma \leq \kappa_t$, $t = 1$ *or* $t = 2$, *then* $\overline{\lambda}(\frac{1}{w_t}) = \lambda(\frac{1}{w_t}) = \rho(w_t)$, $t = 1$ *or* $t = 2$; *if* w_1, w_2 *are entire, then* $\sigma \leq \tau$.

In the past few decades, meromorphic solutions of complex functional equations were studied by Bergweiler et al. [3, 4], Heittokangas et al. [14], and Rieppo [17]. Silvennoinen [18] investigated the existence and growth of solutions to an equation of the form $w(g(z)) = R(z, w)$ and proved the following result.

Theorem C ([18]) *Let*

$$w\big(g(z)\big) = R(z, w), \tag{6}$$

where the right-hand side $R(z, w)$ *is defined as in Theorem* A, $a_i(z)$, $b_j(z)$ *are of growth* $S(r, w)$, *and* $g(z)$ *is entire. If equation* (6) *has a non-constant meromorphic solution* w, *then* $g(z)$ *is a polynomial.*

Gao [7] considered the system of functional equations

$$\begin{cases} w_1(g(z)) = R_2(z, w_2(z)), \\ w_2(g(z)) = R_1(z, w_1(z)), \end{cases} \tag{7}$$

where $g(z)$ is an entire function, $R_1(z, w_1(z))$, $R_2(z, w_2(z))$ are defined as in (3), and obtained the following result.

Theorem D ([7]) *If system* (7) *has a pair of non-constant meromorphic solutions* (w_1, w_2), *then* $g(z)$ *is a polynomial.*

There are some results about the existence and growth of meromorphic solutions of several systems of complex functional equations [8, 19, 20]. Xu et al. [19] studied the problem when $R_1(z, w_1(z))$, $R_2(z, w_2(z))$ in (2) are replaced by $R_1(z, w_1(g_1(z)))$, $R_2(z, w_2(g_2(z)))$, respectively, and (2) is turned into the following system:

$$\begin{cases} [w_1'(q_1 z)]^{n_1} = R_2(z, w_2(g_2(z))), \\ [w_2'(q_2 z)]^{n_2} = R_1(z, w_1(g_1(z))), \end{cases} \tag{8}$$

where $g_1(z), g_2(z)$ are polynomials, and obtained the estimates of the growth order of meromorphic solutions of system (8).

A similar question to ask is what happens if system (8) is more general, for example,

$$\begin{cases} \Omega_1(z, w_1^{(h_1)}(q_1 z)) = R_2(z, w_2(g_2(z))), \\ \Omega_2(z, w_2^{(h_2)}(q_2 z)) = R_1(z, w_1(g_1(z))), \end{cases} \tag{9}$$

where $R_1(z, w_1(z))$, $R_2(z, w_2(z))$, $\Omega_1(z, w_1^{(h_1)}(q_1 z))$, and $\Omega_2(z, w_2^{(h_2)}(q_2 z))$ are defined as in (3), (5), respectively. Further, set

$$g_1(z) = \alpha_{\gamma_1} z^{\gamma_1} + \alpha_{\gamma_1 - 1} z^{\gamma_1 - 1} + \cdots + \alpha_0$$

and

$$g_2(z) = \beta_{\gamma_2} z^{\gamma_2} + \beta_{\gamma_2 - 1} z^{\gamma_2 - 1} + \cdots + \beta_0$$

be two polynomials, where $\alpha_{\gamma_1}, \alpha_{\gamma_1-1}, \ldots, \alpha_0$, $\beta_{\gamma_2}, \beta_{\gamma_2-1}, \ldots, \beta_0$ are complex constants, and $\gamma_t \geq 2$ $(t = 1, 2)$ are two positive integers.

The second result in this paper concerns the growth of solutions to the system of functional equations (9).

Theorem 1.2 *Let* (w_1, w_2) *be a pair of transcendental solutions of system* (9). *Then one of the following cases holds.*

(i) *If* w_1, w_2 *are meromorphic and* $\sigma \leq \kappa$, *then*

$$T(r, w_t(z)) = O\big((\log r)^{\alpha_1}\big), \quad t = 1, 2,$$

where $\alpha_1 = \frac{\log \kappa - \log \sigma}{\log(\gamma_1 \gamma_2)}$.

(ii) *If* w_t *is meromorphic and* $\sigma \leq \kappa_t$, $t = 1$ *or* $t = 2$, *and the other is entire, then*

$$T(r, w_t(z)) = O\big((\log r)^{\alpha_2}\big), \quad t = 1 \text{ or } t = 2,$$

where $\alpha_2 = \frac{\log \kappa_t - \log \sigma}{\log(\gamma_1 \gamma_2)}$, $t = 1$ *or* $t = 2$.

(iii) *If w_1, w_2 are entire and $\sigma \le \tau$, then*

$$T\big(r, w_t(z)\big) = O\big((\log r)^{\alpha_3}\big), \quad t = 1, 2,$$

where $\alpha_3 = \frac{\log\tau - \log\sigma}{\log(\gamma_1\gamma_2)}$.

2 Examples

In this section, we give examples to illustrate that the cases can occur in Theorem 1.1.

The following Examples 2.1–2.4 are about case (i) of Theorem 1.1.

Example 2.1 Let $q_1 = 2$, $q_2 = 3$. Then $(w_1, w_2) = (\frac{e^z}{z}, \frac{e^z}{z^2})$ satisfies the system

$$\begin{cases} \frac{[w_1'(2z)]^2 + w_1'(2z) + 1}{[w_1'(2z)]^3 + w_1'(2z) + 2} = \frac{2z^4(2z-1)^2 w_2(z)^4 + 4z^2(2z-1)w_2(z)^2 + 8}{z^6(2z-1)^3 w_2(z)^6 + 4z^2(2z-1)w_2(z)^2 + 16}, \\ \frac{[w_2'(3z)]^2 + w_2'(3z) + 2}{w_2'(3z) + 1} = \frac{(3z-2)^2 w_1(z)^6 + 9(3z-2)w_1(z)^3 + 162}{9(3z-2)w_1(z)^3 + 81}, \end{cases}$$

where $h_1 = h_2 = 1$, $\tau_1 = 3$, $\tau_2 = 2$, and $\sigma_1 = \sigma_2 = 6$. Here $\sigma = 36 > \kappa = 24$ and $\mu(w_t) = 1 \ge \frac{\log 36 - \log 24}{\log 6} = \frac{\log 36 - \log 24}{\log 36 - \log 6}$, $t = 1, 2$.

Example 2.2 Let $q_1 = 2$, $q_2 = 3$. Then $(w_1, w_2) = (e^z, \frac{e^z}{z})$ satisfies the system

$$\begin{cases} \frac{z[w_1''(2z)]^3 + 2w_1''(2z) + z^2}{z^2[w_1''(2z)]^2 + w_1''(2z) + z} = \frac{64z^6 w_2(z)^6 + 8zw_2(z)^2 + z}{16z^5 w_2(z)^4 + 4zw_2(z)^2 + 1}, \\ \frac{9z^4[w_2'(3z)]^2 + 3z^2 w_2'(3z)}{w_2'(3z) + 2} = \frac{3z^2(3z-1)^2 w_1(z)^6 + 3z^2(3z-1)w_1(z)^3}{(3z-1)w_1(z)^3 + 6z^2}, \end{cases}$$

where $h_1 = 2$, $h_2 = 1$, $\tau_1 = 3$, $\tau_2 = 2$, and $\sigma_1 = \sigma_2 = 6$. Then we have $\sigma = 36 > \kappa_2 = 12$ and $\mu(w_t) = 1 \ge \frac{\log 36 - \log 12}{\log 6} = \frac{\log 3}{\log 6}$, $t = 1, 2$.

Example 2.3 Let $q_1 = 3$, $q_2 = 2$. Then $(w_1, w_2) = (\frac{e^z}{z^2}, e^z)$ satisfies the system

$$\begin{cases} \frac{[w_1'(3z)]^3 + z[w_1'(3z)]^2}{[w_1'(3z)]^2 + w_1'(3z)} = \frac{(3z-2)^2 w_2(z)^6 + 9z^4(3z-2)w_2(z)^3}{9z^3(3z-2)w_2(z)^3 + 81z^6}, \\ \frac{(z+1)w_2''(2z) + 1}{[w_2''(2z)]^2 + w_2''(2z) + z} = \frac{4z^4(z+1)w_1(z)^2 + 1}{16z^8 w_1(z)^4 + 4z^4 w_1(z)^2 + z}, \end{cases}$$

where $h_1 = 1$, $h_2 = 2$, $\tau_1 = 3$, $\tau_2 = 2$, $\sigma_1 = 4$, and $\sigma_2 = 6$. It is known that $\sigma = 24 > \kappa_1 = 12$ and $\mu(w_t) = 1 \ge \frac{\log 24 - \log 12}{\log 6} = \frac{\log 2}{\log 6}$, $t = 1, 2$.

Example 2.4 Let $q_1 = 2$, $q_2 = 3$. Then $(w_1, w_2) = (e^z, ze^z)$ satisfies the system

$$\begin{cases} \frac{w_1''(2z) + z}{[w_1''(2z)]^3 + z^2 w_1''(2z) + 1} = \frac{4z^4 w_2(z)^2 + z^7}{64w_2(z)^6 + 4z^6 w_2(z)^2 + z^6}, \\ \frac{[w_2''(3z)]^2 + zw_2''(3z) + 1}{w_2''(3z) + z} = \frac{81(3z+2)^2 w_1(z)^6 + 9z(3z+2)w_1(z)^3 + 1}{9(3z+2)w_1(z)^3 + z}, \end{cases}$$

where $h_1 = h_2 = 2$, $\tau_1 = 3$, $\tau_2 = 2$, and $\sigma_1 = \sigma_2 = 6$. Thus, $\sigma = 36 > \tau = 6$ and $\mu(w_t) = 1 \ge \frac{\log 36 - \log 6}{\log 6} = 1$, $t = 1, 2$.

The following Examples 2.5–2.8 are about case (ii) of Theorem 1.1.

Example 2.5 Let $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{3}$. Then $(w_1, w_2) = (\frac{e^z}{z}, \frac{e^z}{z-1})$ satisfies the system

$$\begin{cases} \frac{[w_1'(\frac{1}{2}z)]^2+1}{[w_1'(\frac{1}{2}z)]^4+1} = \frac{z^4(z-1)(z-2)^2w_2(z)+z^8}{(z-1)^2(z-2)^4w_2(z)^2+z^8}, \\ \frac{[w_2'(\frac{1}{3}z)]^3+1}{[w_2'(\frac{1}{3}z)]^6+1} = \frac{z(z-6)^3(z-3)^6w_1(z)+(z-3)^{12}}{z^2(z-6)^6w_1(z)^2+(z-3)^{12}}, \end{cases}$$

where $h_1 = h_2 = 1$, $\tau_1 = 4$, $\tau_2 = 6$, and $\sigma_1 = \sigma_2 = 2$. Clearly, $\sigma = 4 \le \kappa = 96$ and $\rho(w_t) = 1 \le \frac{\log 4 - \log 96}{\log \frac{1}{6}} = \frac{\log 24}{\log 6}$, $t = 1, 2$.

Example 2.6 Let $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{2}$. Then $(w_1, w_2) = (e^z, \frac{e^z}{z-1})$ satisfies the system

$$\begin{cases} \frac{[w_1'(\frac{1}{3}z)]^6+1}{[w_1'(\frac{1}{3}z)]^3+1} = \frac{(z-1)^2w_2(z)^2+729}{27(z-1)^2w_2(z)^2+729}, \\ \frac{[w_2'(\frac{1}{2}z)]^4+1}{[w_2'(\frac{1}{2}z)]^2+1} = \frac{(z-4)^4w_1(z)^2+(z-2)^8}{(z-4)^2(z-2)^4w_1(z)+(z-2)^8}, \end{cases}$$

where $h_1 = h_2 = 1$, $\tau_1 = 6$, $\tau_2 = 4$, and $\sigma_1 = \sigma_2 = 2$. Then we have $\sigma = 4 \le \kappa_2 = 48$ and $\rho(w_t) = 1 \le \frac{\log 4 - \log 48}{\log \frac{1}{6}} = \frac{\log 12}{\log 6}$, $t = 1, 2$.

Example 2.7 Let $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{4}$. Then $(w_1, w_2) = (\frac{ze^z}{z-1}, ze^z)$ satisfies the system

$$\begin{cases} [w_1''(\frac{z}{2})]^2 = \frac{(z^3-4z^2-4z+32)^2w_2(z)}{16z(z-2)^6}, \\ [w_2'(\frac{z}{4})]^8 = \frac{(z-1)^2(z+4)^8w_1(z)^2}{16^8z^2}, \end{cases}$$

where $h_1 = 2$, $h_2 = 1$, $\tau_1 = 2$, $\tau_2 = 8$, $\sigma_1 = 2$, and $\sigma_2 = 1$. It is known that $\sigma = 2 \le \kappa_1 = 48$ and $\rho(w_t) = 1 \le \frac{\log 2 - \log 48}{\log \frac{1}{8}} = \frac{\log 24}{\log 8}$, $t = 1, 2$.

Example 2.8 Let $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{4}$. Then $(w_1, w_2) = (ze^z, e^{2z})$ satisfies the system

$$\begin{cases} 4096[w_1''(\frac{z}{2})]^4 = (z+4)^4w_2(z), \\ 4z[w_2'(\frac{z}{4})]^2 = w_1(z), \end{cases}$$

where $h_1 = 2$, $h_2 = 1$, $\tau_1 = 4$, $\tau_2 = 2$, and $\sigma_1 = \sigma_2 = 1$. Here $\sigma = 1 \le \tau = 8$ and $\rho(w_t) = 1 \le \frac{\log 1 - \log 8}{\log \frac{1}{8}} = 1$, $t = 1, 2$.

The following Examples 2.9–2.12 are about case (iii) of Theorem 1.1.

Example 2.9 Let $q_1 = q_2 = 1$. Then $(w_1, w_2) = (\frac{e^z}{e^z-1}, \frac{ze^z}{e^z-1})$ satisfies the system

$$\begin{cases} z^2[w_1'(z)]^2 = w_2(z)^2, \\ w_2'(z) = -zw_1(z)^2 + (1+z)w_1(z), \end{cases}$$

where $h_1 = h_2 = 1$, $\tau_1 = 2$, $\tau_2 = 1$, and $\sigma_1 = \sigma_2 = 2$. Clearly, $\sigma = 4 \le \kappa = 8$.

Example 2.10 Let $q_1 = 1$, $q_2 = -1$. Then $(w_1, w_2) = (\frac{1}{e^z-1}, \frac{1}{1-e^z})$ satisfies the system

$$\begin{cases} w_1''(z) = -2w_2(z)^3 + 3w_2(z)^2 - w_2(z), \\ w_2'(-z) = -w_1(z)^2 - w_1(z), \end{cases}$$

where $h_1 = 2$, $h_2 = 1$, $\tau_1 = \tau_2 = 1$, $\sigma_1 = 2$, and $\sigma_2 = 3$. Then we have $\kappa_t < \sigma = 6 \le \kappa = 6$, $t = 1, 2$, and $\overline{\lambda}(\frac{1}{w_t}) = \lambda(\frac{1}{w_t}) = \rho(w_t) = 1$, $t = 1, 2$.

Example 2.11 Let $q_1 = q_2 = 1$. Then $(w_1, w_2) = (e^z, \frac{1}{e^z-1})$ satisfies the system

$$\begin{cases} w_1''(z)^2 = \frac{[w_2(z)+1]^2}{w_2(z)^2}, \\ w_2'(z)^4 = \frac{w_1(z)^4}{[w_1(z)-1]^8}, \end{cases}$$

where $h_1 = 2$, $h_2 = 1$, $\tau_1 = 2$, $\tau_2 = 4$, $\sigma_1 = 8$, and $\sigma_2 = 2$. It is known that $\tau = 8 < \sigma = 16 \le \kappa_2 = 16$ and $\overline{\lambda}(\frac{1}{w_2}) = \lambda(\frac{1}{w_2}) = \rho(w_2) = 1$.

Example 2.12 Let $q_1 = q_2 = 1$. Then $(w_1, w_2) = (e^z, ze^{2z})$ satisfies the system

$$\begin{cases} \frac{[w_1''(z)]^2+1}{[w_1''(z)]^4+1} = \frac{zw_2(z)+z^2}{w_2(z)+z^2}, \\ \frac{[w_2'(z)]^2+1}{w_2'(z)+1} = \frac{(2z+1)^2 w_1(z)^4+1}{(2z+1)w_1(z)^2+1}, \end{cases}$$

where $h_1 = 2$, $h_2 = 1$, $\tau_1 = 4$, $\tau_2 = 2$, $\sigma_1 = 4$, and $\sigma_2 = 1$. Then we have $\sigma = 4 \le \tau = 8$.

3 Lemmas

To prove Theorems 1.1 and 1.2, we need the following lemmas. Yang and Yi [21] showed the value distribution of a meromorphic function and its derivative.

Lemma 3.1 ([21])

$$N\bigl(r,f^{(k)}\bigr) = N(r,f) + k\overline{N}(r,f), \qquad T\bigl(r,f^{(k)}\bigr) \le T(r,f) + k\overline{N}(r,f) + S(r,f).$$

The following lemma is to compare the Nevanlinna functions of $f(z)$ and $f(cz)$.

Lemma 3.2 ([3])

$$\overline{N}\bigl(r,f(cz)\bigr) = \overline{N}\bigl(|c|r,f(z)\bigr) + O(1), \qquad T\bigl(r,f(cz)\bigr) = T\bigl(|c|r,f(z)\bigr) + O(1).$$

In 1972, Bank [2] established the following lemma.

Lemma 3.3 ([2]) *Let $g(r)$ and $h(r)$ be monotone non-decreasing functions on $(0, +\infty)$ such that $g(r) \le h(r)$, possibly outside a set of r with finite logarithmic measure. Then, for any real number $a > 1$, there exists $r_0 > 0$ such that $g(r) \le h(ar)$ for all $r > r_0$.*

Gundersen et al. [11] showed a method to obtain an upper bound for the growth order of a meromorphic function.

Lemma 3.4 ([11]) *Let $f(z)$ be a non-constant meromorphic function, and let $\Psi : (1,\infty) \to (0,\infty)$ be a monotone non-decreasing function. If for some real number $a \in (0,1)$, there exist real numbers $K_1 > 0$ and $K_2 \geq 1$ such that*

$$T(r,f) \leq K_1\Psi(ar) + K_2T(ar,f) + S(ar,f),$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log a} + \limsup_{r\to\infty} \frac{\log \Psi(r)}{\log r}.$$

The following lemma gives us a method to have a lower bound for the lower order of a meromorphic function.

Lemma 3.5 ([17]) *Let $\Psi : (r_0,\infty) \to (1,\infty)$ be a monotone non-decreasing function, where $r_0 \geq 1$. If for some real number $a > 1$, there exists a real number $b > 1$ such that $\Psi(ar) \geq b\Psi(r)$, then*

$$\liminf_{r\to\infty} \frac{\log \Psi(r)}{\log r} \geq \frac{\log b}{\log a}.$$

The following result about estimate of the Nevanlinna characteristic function of a meromorphic function composed with polynomials is given by Goldstein.

Lemma 3.6 ([9]) *Let $f(z)$ be a transcendental meromorphic function and $g(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_0$ be a polynomial with degree m (≥ 1). For given $\delta \in (0,|a_m|)$, let $\lambda = |a_m| + \delta$, $\mu = |a_m| - \delta$, then*

$$(1-\varepsilon)T\bigl(\mu r^m,f\bigr) \leq T(r,f\circ g) \leq (1+\varepsilon)T\bigl(\lambda r^m,f\bigr)$$

for any given $\varepsilon > 0$ and sufficiently large r.

Goldstein [10] showed the following lemma.

Lemma 3.7 ([10]) *Let $\phi(r)$ be a positive function defined on $[r_0,\infty)$ and bounded in every finite interval. Assume that $\phi(\mu r^k) \leq a\phi(r) + b$ $(r \geq r_0)$, where μ (> 0), k (> 1), a (≥ 1), and b are constants. Then $\phi(r) = O((\log r)^\alpha)$ with $\alpha = \frac{\log a}{\log k}$, unless $a = 1$ and $b > 0$; and if $a = 1$ and $b > 0$, then for any $\varepsilon > 0$, $\phi(r) = O((\log r)^\varepsilon)$.*

4 Proofs of the results

Proof of Theorem 1.1 Suppose first that (w_1,w_2) is a pair of transcendental solutions of system (4). In the following, we consider three cases.

Case (i): $|q_1| > 1$ and $|q_2| > 1$. Suppose that both w_1 and w_2 are meromorphic. It follows from Valiron–Mohon'ko theorem [15, Theorem 2.2.5], Lemma 3.1, and Lemma 3.2 that

$$\begin{aligned} T\bigl(r, R_2(z, w_2)\bigr) = \sigma_2 T(r, w_2) + S(r, w_2) \\ &= T\bigl(r, \Omega_1\bigl(z, w_1^{(h_1)}(q_1 z)\bigr)\bigr) \\ &= \tau_1 T\bigl(r, w_1^{(h_1)}(q_1 z)\bigr) + S\bigl(r, w_1^{(h_1)}(q_1 z)\bigr) \\ &\le \tau_1\bigl[T\bigl(r, w_1(q_1 z)\bigr) + h_1 \overline{N}\bigl(r, w_1(q_1 z)\bigr) + S\bigl(r, w_1(q_1 z)\bigr)\bigr] \\ &\quad + S\bigl(r, w_1^{(h_1)}(q_1 z)\bigr) \\ &\le \tau_1(h_1 + 1) T\bigl(|q_1| r, w_1\bigr) + S\bigl(|q_1| r, w_1\bigr), \end{aligned}$$

that is,

$$\sigma_2 T(r, w_2) + S(r, w_2) \le \tau_1(h_1 + 1) T\bigl(|q_1| r, w_1\bigr) + S\bigl(|q_1| r, w_1\bigr). \tag{10}$$

Similarly, we have

$$\sigma_1 T(r, w_1) + S(r, w_1) \le \tau_2(h_2 + 1) T\bigl(|q_2| r, w_2\bigr) + S\bigl(|q_2| r, w_2\bigr). \tag{11}$$

Thus, from (10) and (11), we obtain

$$\sigma T(r, w_t) + S(r, w_t) \le \kappa T\bigl(|q| r, w_t\bigr) + S\bigl(|q| r, w_t\bigr), \quad t = 1, 2. \tag{12}$$

Now $\sigma > \kappa$, and for any given $\varepsilon > 0$,

$$\sigma(1 - \varepsilon) T(r, w_t) \le \kappa(1 + \varepsilon) T\bigl(|q| r, w_t\bigr), \quad t = 1, 2, \tag{13}$$

for sufficiently large r, possibly outside a set of r with finite logarithmic measure. By Lemma 3.3, with $a > 1$ and (13) we have

$$\sigma(1 - \varepsilon) T(r, w_t) \le \kappa(1 + \varepsilon) T\bigl(a|q| r, w_t\bigr), \quad t = 1, 2, \tag{14}$$

for all $r \ge r_0$. It follows from Lemma 3.5 and (14) that

$$\mu(w_t) \ge \frac{\log[\sigma(1 - \varepsilon)] - \log[\kappa(1 + \varepsilon)]}{\log(a|q|)}, \quad t = 1, 2.$$

As $\varepsilon \to 0^+$ and $a \to 1^+$, we get

$$\mu(w_t) \ge \frac{\log \sigma - \log \kappa}{\log |q|}, \quad t = 1, 2.$$

Suppose that only one between w_1 and w_2 is meromorphic, without loss of generality, we assume that w_1 is meromorphic and w_2 is entire. Then, similar to (11), we have

$$\sigma_1 T(r, w_1) + S(r, w_1) \le \tau_2 T\bigl(|q_2| r, w_2\bigr) + S\bigl(|q_2| r, w_2\bigr). \tag{15}$$

Thus, it follows from (10) and (15) that

$$\sigma T(r,w_t)+S(r,w_t)\leq \kappa_1 T\big(|q|r,w_t\big)+S\big(|q|r,w_t\big),\quad t=1,2. \tag{16}$$

Similar to the above argument, since $\sigma>\kappa_1$ and for any small $\varepsilon>0$, we know that there exists $a>1$ such that

$$\sigma(1-\varepsilon)T(r,w_t)\leq \kappa_1(1+\varepsilon)T\big(a|q|r,w_t\big),\quad t=1,2, \tag{17}$$

for all $r\geq r_0$. Applying Lemma 3.5 to (17) yields that

$$\mu(w_t)\geq \frac{\log[\sigma(1-\varepsilon)]-\log[\kappa_1(1+\varepsilon)]}{\log(a|q|)},\quad t=1,2.$$

By letting $\varepsilon\to 0^+$ and $a\to 1^+$, we obtain

$$\mu(w_t)\geq \frac{\log\sigma-\log\kappa_1}{\log|q|},\quad t=1,2.$$

Suppose that both w_1 and w_2 are entire. Then, similar to (10), we have

$$\sigma_2 T(r,w_2)+S(r,w_2)\leq \tau_1 T\big(|q_1|r,w_1\big)+S\big(|q_1|r,w_1\big). \tag{18}$$

Thus, it follows from (15) and (18) that

$$\sigma T(r,w_t)+S(r,w_t)\leq \tau T\big(|q|r,w_t\big)+S\big(|q|r,w_t\big),\quad t=1,2.$$

Now, $\sigma>\tau$, we know that for $\varepsilon>0$ there exists $a>1$ such that

$$\sigma(1-\varepsilon)T(r,w_t)\leq \tau(1+\varepsilon)T\big(a|q|r,w_t\big),\quad t=1,2, \tag{19}$$

for all $r\geq r_0$. Recalling Lemma 3.5 and letting $\varepsilon\to 0^+$ and $a\to 1^+$, we conclude that

$$\mu(w_t)\geq \frac{\log\sigma-\log\tau}{\log|q|},\quad t=1,2.$$

Case (ii): $|q_1|<1$ and $|q_2|<1$. Suppose that both w_1 and w_2 are meromorphic. Then, similar to the previous argument, we have that for $\varepsilon>0$ there exists $a>1$ such that $a|q|<1$, (12) and (14) hold for all $r\geq r_0$. Since $\sigma\leq\kappa$, then $\frac{\kappa(1+\varepsilon)}{\sigma(1-\varepsilon)}>1$. Hence, applying Lemma 3.4 to (14) yields that

$$\rho(w_t)\leq \frac{\log[\kappa(1+\varepsilon)]-\log[\sigma(1-\varepsilon)]}{-\log(a|q|)},\quad t=1,2,$$

which implies

$$\rho(w_t)\leq \frac{\log\sigma-\log\kappa}{\log|q|},\quad t=1,2,$$

as $\varepsilon\to 0^+$ and $a\to 1^+$.

Suppose that only one between w_1 and w_2 is meromorphic. Without loss of generality, we assume that w_1 is meromorphic and w_2 is entire. Then we similarly obtain that, for $\varepsilon > 0$, there exists $a > 1$ such that $a|q| < 1$, (16) and (17) hold for all $r \geq r_0$. Since $\sigma \leq \kappa_1$, then $\frac{\kappa_1(1+\varepsilon)}{\sigma(1-\varepsilon)} > 1$. Thus, we conclude by Lemma 3.4 and (17) that

$$\rho(w_t) \leq \frac{\log[\kappa_1(1+\varepsilon)] - \log[\sigma(1-\varepsilon)]}{-\log(a|q|)}, \quad t = 1, 2,$$

and let $\varepsilon \to 0^+$ and $\alpha \to 1^+$, it yields

$$\rho(w_t) \leq \frac{\log\sigma - \log\kappa_1}{\log|q|}, \quad t = 1, 2.$$

Suppose that both w_1 and w_2 are entire. Similarly, for $\varepsilon > 0$, there exists $a > 1$ such that $a|q| < 1$, (15), (18), and (19) hold for all $r \geq r_0$. Since $\sigma \leq \tau$, then $\frac{\tau(1+\varepsilon)}{\sigma(1-\varepsilon)} > 1$. Therefore, recalling Lemma 3.4, we have

$$\rho(w_t) \leq \frac{\log[\tau(1+\varepsilon)] - \log[\sigma(1-\varepsilon)]}{-\log(a|q|)}, \quad t = 1, 2,$$

which deduces

$$\rho(w_t) \leq \frac{\log\sigma - \log\tau}{\log|q|}, \quad t = 1, 2,$$

as $\varepsilon \to 0^+$ and $a \to 1^+$.

Case (iii): $|q_1| = |q_2| = 1$. Suppose that both w_1 and w_2 are meromorphic. Then, from Valiron–Mohon'ko theorem [15, Theorem 2.2.5] and Lemma 3.1, we conclude that

$$\begin{aligned} \sigma_2 T(r, w_2) + S(r, w_2) &\leq \tau_1\bigl[T(r, w_1) + h_1\overline{N}(r, w_1) + S(r, w_1)\bigr] + S\bigl(r, w_1^{(h_1)}\bigr) \\ &\leq \tau_1(h_1 + 1)T(r, w_1) + S(r, w_1), \end{aligned} \tag{20}$$

and

$$\begin{aligned} \sigma_1 T(r, w_1) + S(r, w_1) &\leq \tau_2\bigl[T(r, w_2) + h_2\overline{N}(r, w_2) + S(r, w_2)\bigr] + S\bigl(r, w_2^{(h_2)}\bigr) \\ &\leq \tau_2(h_2 + 1)T(r, w_2) + S(r, w_2). \end{aligned} \tag{21}$$

From (20) and (21), we have $\sigma \leq \kappa$. Furthermore, if $\kappa_t < \sigma \leq \kappa$, $t = 1, 2$, then

$$\frac{\sigma - \kappa_2}{h_1\kappa_2} T(r, w_1) + S(r, w_1) \leq \overline{N}(r, w_1) + S(r, w_1) \leq T(r, w_1) + S(r, w_1)$$

and

$$\frac{\sigma - \kappa_1}{h_1\kappa_1} T(r, w_2) + S(r, w_2) \leq \overline{N}(r, w_2) + S(r, w_2) \leq T(r, w_2) + S(r, w_2),$$

which imply that $\overline{\lambda}(\frac{1}{w_t}) = \lambda(\frac{1}{w_t}) = \rho(w_t)$, $t = 1, 2$.

Suppose that only one between w_1 and w_2 is meromorphic. Without loss of generality, we assume that w_1 is meromorphic and w_2 is entire. Then we get (20) and

$$\sigma_1 T(r,w_1) + S(r,w_1) \leq \tau_2 T(r,w_2) + S(r,w_2). \tag{22}$$

Hence, it follows from (20) and (22) that $\sigma \leq \kappa_1$. Furthermore, if $\tau < \sigma \leq \kappa_1$, it yields

$$\frac{\sigma - \tau}{\tau h_1} T(r,w_1) + S(r,w_1) \leq \overline{N}(r,w_1) + S(r,w_1) \leq T(r,w_1) + S(r,w_1),$$

which implies $\overline{\lambda}(\frac{1}{w_1}) = \lambda(\frac{1}{w_1}) = \rho(w_1)$. Similarly, if w_2 is meromorphic and w_1 is entire, we obtain that $\overline{\lambda}(\frac{1}{w_2}) = \lambda(\frac{1}{w_2}) = \rho(w_2)$ when $\tau < \sigma \leq \kappa_2$.

Suppose that both w_1 and w_2 are entire. Then, similar to the above argument, we can get (22) and

$$\sigma_2 T(r,w_2) + S(r,w_2) \leq \tau_1 T(r,w_1) + S(r,w_1). \tag{23}$$

Thus, it follows from (22) and (23) that $\sigma \leq \tau$.

From Cases (i)–(iii), the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2 Suppose first that (w_1, w_2) is a pair of transcendental solutions of system (9). In what follows, we consider three cases.

Case (i): Suppose that both w_1 and w_2 are meromorphic. Then, by Valiron–Mohon'ko theorem [15, Theorem 2.2.5], Lemma 3.1, and Lemma 3.2, we get

$$\sigma_1 T\bigl(r, w_1\bigl(g_1(z)\bigr)\bigr) + S\bigl(r, w_1\bigl(g_1(z)\bigr)\bigr) \leq \tau_2(h_2+1) T\bigl(|q_2|r, w_2\bigr) + S\bigl(|q_2|r, w_2\bigr) \tag{24}$$

and

$$\sigma_2 T\bigl(r, w_2\bigl(g_2(z)\bigr)\bigr) + S\bigl(r, w_2\bigl(g_2(z)\bigr)\bigr) \leq \tau_1(h_1+1) T\bigl(|q_1|r, w_1\bigr) + S\bigl(|q_1|r, w_1\bigr). \tag{25}$$

By Lemma 3.6, for given $0 < \delta_1 < |\alpha_{\gamma_1}|$, $0 < \delta_2 < |\beta_{\gamma_2}|$, and $\mu_1 = |\alpha_{\gamma_1}| - \delta_1$, $\mu_2 = |\beta_{\gamma_2}| - \delta_2$, we know that for any small $\varepsilon > 0$ there exists two sets E_1, E_2 of finite logarithmic measure such that

$$\sigma_1(1-\varepsilon) T\bigl(\mu_1 r^{\gamma_1}, w_1\bigr) \leq \tau_2(h_2+1)(1+\varepsilon) T\bigl(|q_2|r, w_2\bigr), \quad r \notin E_1, \tag{26}$$

and

$$\sigma_2(1-\varepsilon) T\bigl(\mu_2 r^{\gamma_2}, w_2\bigr) \leq \tau_1(h_1+1)(1+\varepsilon) T\bigl(|q_1|r, w_1\bigr), \quad r \notin E_2. \tag{27}$$

Thus, for sufficiently large r and $r \notin E_1 \cup E_2$, we can deduce from (26) and (27) that

$$\sigma(1-\varepsilon)^2 T\biggl(\frac{\mu_1 \mu_2^{\gamma_1}}{|q_2|^{\gamma_1}} r^{\gamma_1 \gamma_2}, w_1\biggr) \leq \kappa(1+\varepsilon)^2 T\bigl(|q_1|r, w_1\bigr) \tag{28}$$

and

$$\sigma(1-\varepsilon)^2 T\biggl(\frac{\mu_2 \mu_1^{\gamma_2}}{|q_1|^{\gamma_2}} r^{\gamma_1 \gamma_2}, w_2\biggr) \leq \kappa(1+\varepsilon)^2 T\bigl(|q_2|r, w_2\bigr). \tag{29}$$

By Lemma 3.3, with $a > 1$ and (28), we have

$$\sigma(1-\varepsilon)^2 T\left(\frac{\mu_1\mu_2^{\gamma_1}}{|q_2|^{\gamma_1}} r^{\gamma_1\gamma_2}, w_1\right) \le \kappa(1+\varepsilon)^2 T\bigl(a|q_1|r, w_1\bigr) \tag{30}$$

for all $r \ge r_0$. Set $R = a|q_1|r$. Then (30) can be rewritten as

$$T\left(\frac{\mu_1\mu_2^{\gamma_1}}{|q_2|^{\gamma_1}|aq_1|^{\gamma_1\gamma_2}} R^{\gamma_1\gamma_2}, w_1\right) \le \frac{\kappa(1+\varepsilon)^2}{\sigma(1-\varepsilon)^2} T(R, w_1). \tag{31}$$

If $\sigma \le \kappa$, then $\frac{\kappa(1+\varepsilon)^2}{\sigma(1-\varepsilon)^2} \ge 1$. Since $\frac{\mu_1\mu_2^{\gamma_1}}{|q_2|^{\gamma_1}|aq_1|^{\gamma_1\gamma_2}} > 0$, $\gamma_t \ge 2$ $(t = 1, 2)$, applying Lemma 3.7 to (31) yields that

$$T(r, w_1) = O\bigl((\log r)^{\alpha_1}\bigr),$$

where

$$\alpha_1 = \frac{\log[\kappa(1+\varepsilon)^2] - \log[\sigma(1-\varepsilon)^2]}{\log(\gamma_1\gamma_2)},$$

which deduces

$$\alpha_1 = \frac{\log\kappa - \log\sigma}{\log(\gamma_1\gamma_2)},$$

as $\varepsilon \to 0^+$. Similarly, from (29), we conclude that

$$T(r, w_2) = O\bigl((\log r)^{\alpha_1}\bigr),$$

where

$$\alpha_1 = \frac{\log\kappa - \log\sigma}{\log(\gamma_1\gamma_2)}.$$

Case (ii): Suppose that only one between w_1 and w_2 is meromorphic. Without loss of generality, we assume that w_2 is meromorphic and w_1 is entire. By Valiron–Mohon'ko theorem [15, Theorem 2.2.5], Lemma 3.1, and Lemma 3.2, we get (24) and

$$\sigma_2 T\bigl(r, w_2\bigl(g_2(z)\bigr)\bigr) + S\bigl(r, w_2\bigl(g_2(z)\bigr)\bigr) \le \tau_1 T\bigl(|q_1|r, w_1\bigr) + S\bigl(|q_1|r, w_1\bigr). \tag{32}$$

Thus, by an argument similar to the proof of Case (i) of Theorem 1.2, we can deduce

$$T\left(\frac{\mu_1\mu_2^{\gamma_1}}{|q_2|^{\gamma_1}|aq_1|^{\gamma_1\gamma_2}} R^{\gamma_1\gamma_2}, w_1\right) \le \frac{\kappa_2(1+\varepsilon)^2}{\sigma(1-\varepsilon)^2} T(R, w_1).$$

If $\sigma \le \kappa_2$, then $\frac{\kappa_2(1+\varepsilon)^2}{\sigma(1-\varepsilon)^2} \ge 1$. Since $\frac{\mu_1\mu_2^{\gamma_1}}{|q_2|^{\gamma_1}|aq_1|^{\gamma_1\gamma_2}} > 0$, $\gamma_t \ge 2$ $(t = 1, 2)$, it follows from Lemma 3.7 that

$$T(r, w_1) = O\bigl((\log r)^{\alpha_2}\bigr),$$

where

$$\alpha_2 = \frac{\log \kappa_2 - \log \sigma}{\log(\gamma_1 \gamma_2)}.$$

Similarly, we have

$$T(r, w_2) = O\bigl((\log r)^{\alpha_2}\bigr),$$

where

$$\alpha_2 = \frac{\log \kappa_1 - \log \sigma}{\log(\gamma_1 \gamma_2)}.$$

Case (iii): Suppose that both w_1 and w_2 are entire. Then, similar to the above argument, we can get (32) and

$$\sigma_1 T\bigl(r, w_1\bigl(g_1(z)\bigr)\bigr) + S\bigl(r, w_1\bigl(g_1(z)\bigr)\bigr) \leq \tau_2 T\bigl(|q_2|r, w_2\bigr) + S\bigl(|q_2|r, w_2\bigr).$$

Hence, by an argument similar to the proof of Case (ii) of Theorem 1.2, if $\sigma \leq \tau$, then we can obtain

$$T(r, w_t) = O\bigl((\log r)^{\alpha_3}\bigr), \quad t = 1, 2,$$

where

$$\alpha_3 = \frac{\log \tau - \log \sigma}{\log(\gamma_1 \gamma_2)}.$$

From Cases (i)–(iii), the proof of Theorem 1.2 is completed. □

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