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Matrix transformations of Norlund–Orlicz difference sequence spaces of nonabsolute type and their Toeplitz duals

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Abstract

In this paper, the Nörlund–Orlicz difference sequence space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, u, q)$ of nonabsolute type is introduced as a domain of Nörlund means which is isomorphic to the space $\ell(p)$ and the basis of the space is constructed. Additionally, the α–, β–, and γ-duals of the spaces are computed and their matrix transformations are given. Finally, the properties like rotundity, modularity of the newly formed spaces are established.

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1 Introduction and preliminaries

Summability is a wide field of mathematics in functional analysis and has many applications, for instance, in numerical analysis to speed up the rate of convergence, in operator theory, the theory of orthogonal series, approximation theory, etc. Toeplitz [22] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices. By w, we mean the space of all complex sequences. Any vector subspace of w is called a sequence space. The spaces of all bounded, convergent, and null sequences are denoted respectively by ℓ_∞, c, and c_0. We indicate the set of natural numbers including 0 by $\mathbb{N}$, and $\mathcal{G}$ denotes the collection of all finite subsets of $\mathbb{N}$. Let λ and η be two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then the matrix A defines the A-transformation from λ into η if, for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in η; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}.$$

For example, if $A = I$, the unit matrix for all n, then $x_k \to \ell(I)$ means precisely that $x_k \to \ell$ as $k \to \infty$. By $(\lambda : \eta)$, we denote the class of all matrices A such that $A : \lambda \to \eta$. For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined as

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}. \tag{1}$$

Also, we write $A_n = (a_{nk})_{k\in\mathbb{N}}$ for the sequence in the nth row of A.

A sequence (b_n) in a normed space X is called a Schauder basis for X if, for every $x \in X$, there is one kind of sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, that is,

$$\lim_m \left\| x - \sum_{n=0}^{m} \alpha_n b_n \right\| = 0.$$

In [10] Lindenstrauss and Tzafriri utilized the idea of Orlicz function to define the Orlicz space of sequences. A sequence $\mathcal{F} = (F_k)$ of Orlicz functions is called a *Musielak–Orlicz function* (see [13, 15]). For detailed definition of Orlicz sequence spaces and paranormed spaces, see [1, 2, 18–21, 23, 25] and the references therein.

Now, we define the sequence spaces $\ell(q, \Delta_n^m)$ and $\ell_\infty(q, \Delta_n^m)$ as follows:

$$\ell(q, \Delta_n^m) = \left\{ x = (x_k) \in \omega : \sum_k \left|\Delta_n^m x_k\right|^{q_k} < \infty \right\},$$

$$\ell_\infty(q, \Delta_n^m) = \left\{ x = (x_k) \in \omega : \sup_k \left|\Delta_n^m x_k\right|^{q_k} < \infty \right\},$$

which are the complete spaces (see [5, 27]).

Kızmaz [8] gave the concept of the spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ by using difference operator, and it was additionally summed up by Et and Çolak [6]. Let n, m be nonnegative integers, then for a given sequence space Z, we have

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\}$$

for $Z = c$, c_0 and ℓ_∞, where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k, n \in \mathbb{N}$, which is equal to the accompanying binomial representation

$$\Delta_n^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k+n\nu}.$$

Taking $n = 1$, we get the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$, and $c_0(\Delta^m)$ studied by Et and Çolak [6]. Taking $m = n = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ introduced and studied by Kızmaz [8].

Let $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$, where (t_k) is a sequence of nonnegative real numbers with $t_0 > 0$. Then the Nörlund means $\mathcal{N}^t = (c_{nk}^t)$ is defined by

$$c_{nk}^t = \begin{cases} \frac{t_{n-k}}{T_n}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. For more details about Nörlund spaces, one can refer to [14, 17, 24]. Let $t_0 = D_0 = 1$ and define D_n for $n \in \{1, 2, 3, \dots\}$ by

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 0 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \ddots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \ddots & t_1 \end{vmatrix}.$$

The inverse matrix $V^t = (v^t_{nk})$ of the matrix $N^t = (c^t_{nk})$ (see [14]) is as follows:

$$v^t_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_k, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Also, for $k \in \{1, 2, 3, \dots\}$, we have

$$D_k = \sum_{j=1}^{k-1} (-1)^{j-1} D_{k-j} + (-1)^{k-1} t_k.$$

In [26] Yeşilkayagil introduced the Nörlund sequence space $\mathcal{N}^t(q)$ defined by

$$\mathcal{N}^t(q) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \right|^{q_k} < \infty \right\},$$

where $0 < q_k \le D < \infty$. Throughout the paper we shall assume that $q_k^{-1} + (q'_k)^{-1} = 1$ provided $1 < \inf q_k \le D < \infty$. By bs, cs, ℓ_1, and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series respectively.

The main purpose of this paper is to introduce some difference sequence spaces generated by Nörlund matrix and Musielak–Orlicz function. We show that these spaces are complete paranormed spaces. Section three is devoted to determining the α-, β-, and γ-duals of these spaces, and in the fourth section, we discuss the matrix transformations on these spaces. Finally, the rotundity of the Nörlund–Orlicz sequence spaces $\mathcal{N}^t(\mathcal{F}, \Delta^m_n, \mu, q)$ is characterized, and some properties of these spaces are given.

2 Nörlund–Orlicz sequence space $\mathcal{N}^t(\mathcal{F}, \Delta^m_n, \mu, q)$ and its properties

The current section contains completeness and introduction of Nörlund–Orlicz sequence space $\mathcal{N}^t(\mathcal{F}, \Delta^m_n, \mu, q)$. We also show that the Nörlund–Orlicz sequence space and $\ell(q, \Delta^m_n)$ are linearly isomorphic and determine the basis for the space.

Let $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function, $q = (q_k)$ be a bounded sequence of positive real numbers, and $\mu = (\mu_j)$ be a sequence of positive real numbers. Then we define new difference sequence space $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ as follows:

$$\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)=\left\{x=(x_k)\in w:\sum_k\left|\frac{1}{T_k}\sum_{j=0}^{k}F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k}<\infty,\right.$$
$$\left.\text{for some } \rho>0\right\}$$

with $0<q_k\leq D<\infty$, $k\in\mathbb{N}$. With the definition of matrix domain (1), the sequence space $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ may be redefined as

$$\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)=\{\ell(q,\Delta_n^m)\}_{\mathcal{N}^t(\mathcal{F},\mu)},$$

where $\mathcal{N}^t(\mathcal{F},\mu)$ denotes the matrix $\mathcal{N}^t(\mathcal{F},\mu)=a_{nk}^t(\mathcal{F},\mu)$ defined by

$$a_{nk}^t(\mathcal{F},\mu)=\begin{cases}\frac{1}{T_n}F_n(\frac{|\mu_n t_{n-k}|}{\rho}), & \text{if } 0\leq k\leq n,\\ 0, & \text{if } k>n.\end{cases}$$

Define $y=(y_k)=(\Delta_n^m y_k)$ to be a sequence used as the $\mathcal{N}^t(\mathcal{F},\mu)$-transform of sequence $x=(x_k)=(\Delta_n^m x_k)$, so we have

$$y=(y_k)=\frac{1}{T_k}\sum_{j=0}^{k}F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right). \tag{2}$$

Theorem 1 *For Musielak–Orlicz function $\mathcal{F}=(F_j)$ and let $\mu=(\mu_j)$ be a sequence of positive real numbers. Then $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ is a complete paranormed linear metric space given by*

$$g(x)=\left(\sum_k\left|\frac{1}{T_k}\sum_{j=0}^{k}F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k}\right)^{\frac{1}{H}}$$

with $0\leq q_k\leq D<\infty$ and $H=\max\{1,D\}$.

Proof The linearity of $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ follows from the following inequality. For $x=(x_j)$, $y=(y_j)\in\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ (see [12], p. 30),

$$\begin{aligned}&\left(\sum_k\left|\frac{1}{T_k}\sum_{j=0}^{k}F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m(x_j+y_j)|}{\rho}\right)\right|^{q_k}\right)^{\frac{1}{H}}\\ &\quad\leq\left(\sum_k\left|\frac{1}{T_k}\sum_{j=0}^{k}F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k}\right)^{\frac{1}{H}}\\ &\qquad+\left(\sum_k\left|\frac{1}{T_k}\sum_{j=0}^{k}F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m y_j|}{\rho}\right)\right|^{q_k}\right)^{\frac{1}{H}}\end{aligned} \tag{3}$$

and

$$|\beta|^{q_k} \leq \max\bigl(1, |\beta|^H\bigr), \quad \forall \beta \in \mathbb{R} \text{ (see [11])}. \tag{4}$$

Clearly $g(x) \geq 0$ for $x = (x_k) \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, p)$. Since $M_k(0) = 0$, we get $g(0) = 0$ and $g(x) = g(-x)$. Therefore, inequalities (3) and (4) give the subadditivity of g and

$$g(\beta x) \leq \max\bigl(1, |\beta|\bigr)g(x).$$

Let $\{x^n\} \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ be any sequence, then

$$g\bigl(x^n - x\bigr) \to 0,$$

and let (β^n) be a sequence of scalars such that $\beta^n \to \beta$. Thus

$$\begin{aligned} g\bigl(\beta_n x^n - \beta x\bigr) &= \left(\sum_k \left| \frac{1}{T_k} \sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j \Delta_n^m(\beta_n x_j^n - \beta x_j)|}{\rho}\right)\right|^{q_k}\right)^{\frac{1}{H}} \\ &\leq |\beta_n - \beta|^{\frac{1}{H}} g\bigl(x^n\bigr) + |\beta|^{\frac{1}{H}} g\bigl(x^n - x\bigr) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence g is paranorm.

Let $\{x^i\} \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ be any Cauchy sequence, where $x^i = \{x_0^i, x_1^i, \ldots\}$. Given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$g\bigl(x^i - x^j\bigr) < \epsilon \quad \forall i, j \geq n_0(\epsilon). \tag{5}$$

For each fixed $k \in \mathbb{N}$,

$$\begin{aligned} &\bigl|\bigl(\mathcal{N}^t(\mathcal{F}, \mu)x^i\bigr)_k - \bigl(\mathcal{N}^t(\mathcal{F}, \mu)x^j\bigr)_k\bigr| \\ &\quad \leq \left(\sum_k \bigl|\bigl(\mathcal{N}^t(\mathcal{F}, \mu)x^i\bigr)_k - \bigl(\mathcal{N}^t(\mathcal{F}, \mu)x^j\bigr)_k\bigr|^{q_k}\right)^{\frac{1}{H}} < \epsilon \quad \text{for all } i, j \geq n_0(\epsilon), \end{aligned}$$

which yields a Cauchy sequence of real numbers $\{(\mathcal{N}^t(\mathcal{F}, \mu)x^0)_k, (\mathcal{N}^t(\mathcal{F}, \mu)x^1)_k, \ldots\}$ for each fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete so that

$$\bigl(\mathcal{N}^t(\mathcal{F}, \mu)x^i\bigr)_k \to \bigl(\mathcal{N}^t(\mathcal{F}, \mu)x\bigr)_k \quad \text{as } i \to \infty.$$

By using $(\mathcal{N}^t(\mathcal{F}, \mu)x)_0, (\mathcal{N}^t(\mathcal{F}, \mu)x)_1, \ldots$, infinitely many limits, we define $\{(\mathcal{N}^t(\mathcal{F}, \mu)x)_0, (\mathcal{N}^t(\mathcal{F}, \mu)x)_1, \ldots\}$. For each $t \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$, from (5)

$$\sum_{k=0}^{t} \bigl|\bigl(\mathcal{N}^t(\mathcal{F}, \mu)x^i\bigr)_k - \bigl(\mathcal{N}^t(\mathcal{F}, \mu)x^j\bigr)_k\bigr|^{q_k} \leq g\bigl(x^i - x^j\bigr)^H < \epsilon^H. \tag{6}$$

Taking $j \to \infty$ in (6) and then $t \to \infty$, we obtain $g(x^i - x) \leq \epsilon$.

Taking $\epsilon = 1$ in (6) with $i \geq n_0(1)$, we have

$$\left[\sum_{k=0}^{t} \left|\left(\mathcal{N}^t(\mathcal{F},\mu)x\right)_k\right|^{q_k}\right]^{\frac{1}{H}} \leq g\left(x^i - x\right) + g\left(x^i\right) \\ \leq 1 + g\left(x^i\right)$$

gives $x \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$. We know $g(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, therefore $x^i \to x$ as $i \to \infty$. Hence, the space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ is complete. □

Theorem 2 *Let $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function and $\mu = (\mu_j)$ be a sequence of positive real numbers. Then the sequence space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ of non-absolute type is linearly isomorphic to $\ell(q, \Delta_n^m)$, where $0 < q_k \leq H < \infty$.*

Proof To demonstrate that the spaces $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ and $\ell(q, \Delta_n^m)$ are linearly isomorphic, we have to prove that there exists a linear bijection between these spaces. Define a linear transformation $T : \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q) \to \ell(q, \Delta_n^m)$ by $x \to y = Tx = \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)x$ by using equation (2). So, linearity of T is trivial. Clearly, $x = \theta$ whenever $Tx = \theta$ and therefore T is injective.

Suppose any sequence $y \in \ell(q, \Delta_n^m)$ and define the sequence $x = (x_k) = (\Delta_n^m x_k)$ by

$$x = (x_k) = \sum_{i=0}^{k} \frac{1}{F_j}\left(\frac{1}{\mu_j}(-1)^{k-i} D_{k-i}\rho T_i \Delta_n^m y_i\right) \quad \text{for } k \in \mathbb{N}.$$

Thus,

$$\begin{aligned} g(x) &= \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j \Delta_n^m x_j|}{\rho}\right)\right|^{q_k}\right)^{\frac{1}{H}} \\ &= \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j(\sum_{i=0}^{k}\frac{1}{F_j}(\frac{1}{\mu_j}(-1)^{k-i}D_{k-i}\rho T_i \Delta_n^m y_i)|}{\rho}\right)\right|^{q_k}\right)^{\frac{1}{H}} \\ &= \left(\sum_k |y_k|^{q_k}\right)^{\frac{1}{H}} \\ &< \infty. \end{aligned}$$

This means that $x \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$. Hence, the proof is completed. □

Theorem 3 *Define sequence $b^{(k)}(t) = \{b_n^{(k)}(t)\}_{n\in\mathbb{N}}$ of the elements of $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(t) = \begin{cases} \frac{1}{F_k}(\frac{1}{\mu_k}(-1)^{n-k} D_{n-k}\rho T_k), & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Then the sequence $\{b^{(k)}(t)\}_{k\in\mathbb{N}}$ *is a basis for* $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ *and any* $x\in\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ *has a unique representation of the form*

$$x=\sum_k \lambda_k(t)b^{(k)}(t), \tag{7}$$

where $\lambda_k(t)=(\mathcal{N}^t(\mathcal{F},\mu)x)_k$, $\forall k\in\mathbb{N}$ *and* $0<q_k\leq D<\infty$.

Proof Clearly, $\{b^{(k)}(t)\}\subset\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$, also

$$\mathcal{N}^t(\mathcal{F},\mu)b^{(k)}(t)=e^{(t)}\in\ell\big(q,\Delta_n^m\big)\quad\text{for all } k\in\mathbb{N}, \tag{8}$$

where $e^{(t)}$ is the sequence whose only nonzero term is 1 in the kth place for each $k\in\mathbb{N}$ and $0<q_k\leq D<\infty$. Let $x\in\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$. For every nonnegative integer m, we take

$$x^{[m]}=\sum_{k=0}^{m}\lambda_k(t)b^{(k)}(t). \tag{9}$$

Then, by applying $\mathcal{N}^t(\mathcal{F},\mu)$ to (9) with (8), we have

$$\begin{aligned}\mathcal{N}^t(\mathcal{F},\mu)x^{[m]}&=\sum_{k=0}^{m}\lambda_k(t)\mathcal{N}^t(\mathcal{F},\mu)b^{(k)}(t)\\&=\sum_{k=0}^{m}\big(\mathcal{N}^t(\mathcal{F},\mu)x\big)_k e^{(k)}.\end{aligned}$$

Now, for $i,m\in\mathbb{N}$,

$$\big\{\mathcal{N}^t(\mathcal{F},\mu)\big(x-x^{[m]}\big)\big\}_i=\begin{cases}0, & 0\leq i\leq m,\\ (\mathcal{N}^t(\mathcal{F},\mu)x)_i, & i>m.\end{cases}$$

For $\epsilon>0$ given, there is an integer m_0 such that

$$\left[\sum_{i=m+1}\big|\big(\mathcal{N}^t(\mathcal{F},\mu)x\big)_i\big|^{q_k}\right]^{1/H}<\epsilon,\qquad\forall(m+1)\geq m_0.$$

Therefore,

$$\begin{aligned}g\big[\mathcal{N}^t(\mathcal{F},\mu)\big(x-x^{[m]}\big)\big]&=\left[\sum_{i=m+1}^{\infty}\big|\big(\mathcal{N}^t(\mathcal{F},\mu)x\big)_i\big|^{q_k}\right]^{1/H}\\&\leq\left[\sum_{i=m_0}^{\infty}\big|\big(\mathcal{N}^t(\mathcal{F},\mu)x\big)_i\big|^{q_k}\right]^{1/H}\\&<\epsilon,\end{aligned}$$

for all $(m+1)\leq m_0$. To show the unique representation for $x\in\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$, suppose that there exists a representation $x=\sum_k\eta_k(t)b^{(k)}(t)$. Since T is continuous from Theorem 2, we have

$$\begin{aligned}\big(\mathcal{N}^t(\mathcal{F},\mu)x\big)_n &= \sum_k \eta_k(t)\big\{\mathcal{N}^t(\mathcal{F},\mu)b^{(k)}(t)\big\}_n \\ &= \sum_k \eta_k(t)e_n^{(k)} = \eta_n(t)\end{aligned}$$

for every natural number n which contradicts that $(\mathcal{N}^t(\mathcal{F},\mu)x)_n = \lambda_n(t)$, $\forall n \in \mathbb{N}$. Hence, the result. □

3 Toeplitz duals of the space $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$

For the sequence spaces X and Y, define the set

$$S(X:Y) = \big\{z = (z_k) : xz = (x_k z_k) \in Y \text{ for all } x = (x_k) \in X\big\}.$$

The α-, β-, and γ-duals of a sequence space X, respectively denoted by X^α, X^β, and X^γ, are defined by

$$X^\alpha = S(X:\ell_1), \qquad X^\beta = S(X:cs) \quad \text{and} \quad X^\gamma = S(X:bs).$$

Firstly, we state some lemmas which are required in this section.

Lemma 3.1 (see [7], Theorem 5.1.0)

(i) *Suppose that $1 < q_k \le D < \infty$ for all k. Then $A = (a_{nk}) \in (\ell(q):\ell_1)$ iff there exists an integer $B > 1$ such that*

$$\sup_{K\in\mathcal{G}} \sum_k \Big|\sum_{n\in K} a_{nk}B^{-1}\Big|^{q_k'} < \infty. \tag{10}$$

(ii) *Let $0 < q_k \le 1$. Then $A = (a_{nk}) \in (\ell(q):\ell_1)$ iff*

$$\sup_{K\in\mathcal{G}} \sup_k \Big|\sum_{n\in K} a_{nk}\Big|^{q_k} < \infty. \tag{11}$$

Lemma 3.2 (see [9], Theorem 1) *The following statements hold*:

(i) *Let $1 < q_k \le D < \infty$ for all k. Then $A = (a_{nk}) \in (\ell(q):\ell_\infty)$ iff there exists an integer $B > 1$ such that*

$$\sup_n \sum_k \big|a_{nk}B^{-1}\big|^{q_k'} < \infty. \tag{12}$$

(ii) *Let $0 < q_k \le 1$ for every $k \in \mathbb{N}$. Then $A = (a_{nk}) \in (\ell(q):\ell_\infty)$ iff*

$$\sup_{n,k} |a_{nk}|^{q_k} < \infty. \tag{13}$$

Lemma 3.3 (see [9], Theorem 1) *Let $0 < q_k \le D < \infty$ for every $k \in \mathbb{N}$. Then $A = (a_{nk}) \in (\ell(q):c)$ iff (12) and (13) hold along with there is $\beta_k \in \mathbb{C}$ such that $\lim_n a_{nk} = \beta_k$ for every natural number k.*

Theorem 4 *Let $1 < q_k \leq D < \infty$ and $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function. Define the sets $D_1(\mathcal{F}, \Delta_n^m, \mu, q)$ and $D_2(\mathcal{F}, \Delta_n^m, \mu, q)$ as follows*:

$$D_1(\mathcal{F}, \Delta_n^m, \mu, q) = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{G}} \sum_{k \in \mathbb{N}} \left| \sum_{n \in K} \frac{1}{F_k} \left(\frac{1}{\mu_k} (-1)^{n-k} a_n D_{n-k} \rho T_k B^{-1} \right) \right|^{q'_k} < \infty \right\}$$

and

$$D_2(\mathcal{F}, \Delta_n^m, \mu, q) = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k} \left| \sum_{i=k}^{n} \frac{1}{F_k} \left(\frac{1}{\mu_k} (-1)^{i-k} a_i D_{i-k} \rho T_k B^{-1} \right) \right|^{q'_k} < \infty \right\}.$$

Then

(i) $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\alpha = D_1(\mathcal{F}, \Delta_n^m, \mu, q)$;

(ii) $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\gamma = D_2(\mathcal{F}, \Delta_n^m, \mu, q)$;

(iii) $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\beta = D_2(\mathcal{F}, \Delta_n^m, \mu, q) \cap cs$.

Proof Suppose $a = (a_k) \in w$. Therefore, by using (1) we have

$$a_n x_n = \sum_{k=0}^{n} \frac{1}{F_k} \left(\frac{1}{\mu_k} (-1)^{n-k} D_{n-k} \rho T_k \Delta_n^m a_n y_k \right) = (Fy)_n, \tag{14}$$

where $F = (f_{nk})$ is defined as follows:

$$f_{nk} = \begin{cases} \frac{1}{F_k}(\frac{1}{\mu_k}(-1)^{n-k} D_{n-k} \rho T_k a_n), & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus, by combining equation (14) with part (i) of Lemma 3.1, we have $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ iff $Fy \in \ell_1$ whenever $y \in \ell(q, \Delta_n^m)$. This gives the result $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\alpha = D_1(\mathcal{F}, \Delta_n^m, \mu, q)$.

Further take

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n-1} \sum_{i=k}^{n} \frac{1}{F_k} \left(\frac{1}{\mu_k} (-1)^{i-k} D_{i-k} \rho T_k \Delta_n^m a_i y_k \right) + \frac{1}{F_k} \left(\frac{1}{\mu_k} T_n \Delta_n^m a_n y_n \right) \\ &= (Ey)_n \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{15}$$

here $E = (e_{nk})$ with

$$e_{nk} = \begin{cases} \sum_{i=k}^{n} \frac{1}{F_k}(\frac{1}{\mu_k}(-1)^{i-k} D_{i-k} \rho T_k a_i), & \text{if } 0 \leq k \leq n-1, \\ \frac{1}{F_k}(\frac{1}{\mu_k} T_n a_n), & \text{if } k = n, \\ 0, & \text{if } k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus, from Lemma 3.2 with equality (15) we have $ax = (a_n x_n) \in bs$ whenever $x = (x_k) \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ iff $Ey \in \ell_\infty$ whenever $y \in \ell(q, \Delta_n^m)$. Hence, from Lemma 3.2 we have $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\gamma = D_2(\mathcal{F}, \Delta_n^m, \mu, q)$.

It is seen immediately that $ax = (a_n x_n) \in cs$ whenever $x = (x_k) \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ iff $Ey \in c$ whenever $y = (y_k) \in \ell(q, \Delta_n^m)$. Using by Lemma 3.3, the proof of the theorem is completed. □

Theorem 5 *Let $0 < q_k \leq 1$ and let $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function. Define the sets $D_3(\mathcal{F}, \Delta_n^m, \mu, q)$ and $D_4(\mathcal{F}, \Delta_n^m, \mu, q)$ by*

$$\begin{aligned} &D_3(\mathcal{F}, \Delta_n^m, \mu, q) \\ &\quad = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{G}} \sum_{k \in \mathbb{N}} \left| \sum_{n \in K} \frac{1}{F_k} \left(\frac{1}{\mu_k} (-1)^{n-k} a_n D_{n-k} \rho T_k \right) \right|^{q_k} < \infty \right\} \end{aligned}$$

and

$$\begin{aligned} &D_4(\mathcal{F}, \Delta_n^m, \mu, q) \\ &\quad = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k} \left| \sum_{i=k}^{n} \frac{1}{F_k} \left(\frac{1}{\mu_k} (-1)^{i-k} a_i D_{i-k} \rho T_k \right) \right|^{q_k} < \infty \right\}. \end{aligned}$$

Then

(i) $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\alpha = D_3(\mathcal{F}, \Delta_n^m, \mu, q)$;

(ii) $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\gamma = D_4(\mathcal{F}, \Delta_n^m, \mu, q)$;

(iii) $\{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\beta = cs \cap D_4(\mathcal{F}, \Delta_n^m, \mu, q)$.

Proof We can find easily the proof of the theorem as in the proof of Theorem 4 through Lemma 3.1, Lemma 3.2, and Lemma 3.3. □

4 Characterizations of matrix transformations on the space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$

This segment deals with portrayal of the matrix mappings from the space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ into any specified space η and from a given sequence space η.

Theorem 6 *Let $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function. Let the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ be connected with*

$$b_{nk} = \sum_{j=k}^{\infty} \frac{1}{F_j} \left(\frac{1}{\mu_j} (-1)^{j-k} D_{j-k} \rho T_k a_{nk} \right) \tag{16}$$

for all $n, k \in \mathbb{N}$ and sequence space η be given. Thus $A \in (\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q) : \eta)$ iff $A_n \in \{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\beta$ $\forall n, k \in \mathbb{N}$ and $B \in (\ell(q, \Delta_n^m) : \eta)$.

Proof Let η be any sequence space, relation (16) holds between the elements of the matrices $A = (a_{nk})$ and $B = (b_{nk})$ since the space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ and $\ell(q, \Delta_n^m)$ are linearly isomorphic.

Suppose $A \in (\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q) : \eta)$ and choose any $y \in \ell(q, \Delta_n^m)$. Then

$$\begin{aligned}\big(B\mathcal{N}^t(\mathcal{F},\mu)\big)_{nk} &= \sum_{j=k}^{\infty} b_{nj} a_{nk}^t(\mathcal{F},\mu) \\ &= \sum_{j=k}^{\infty} \frac{1}{F_j}\left(\frac{1}{\mu_j}(-1)^{j-k} D_{j-k}\rho T_k a_{nk}\right)\frac{1}{T_j} F_j\left(\frac{|\mu_j t_{j-k}|}{\rho}\right) \\ &= a_{nk}.\end{aligned}$$

Therefore, $B\mathcal{N}^t(\mathcal{F},\mu)$ exists and $A_n \in \{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\beta$, which gives that $B_n \in \ell_1$ for each $n \in \mathbb{N}$. Thus, By exists and hence

$$\begin{aligned}\sum_k^{\infty} b_{nk} y_k &= \sum_{j=k}^{\infty} \frac{1}{F_j}\left(\frac{1}{\mu_j}(-1)^{j-k} D_{j-k}\rho T_k a_{nk}\right) \times \frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j \Delta_n^m x_j|}{\rho}\right) \\ &= \sum_k a_{nk} x_k\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $By = Ax$, which leads to the consequence $B \in (\ell(q, \Delta_n^m) : \eta)$.

On the contrary, let $A_n \in \{\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)\}^\beta$ for every natural number n and $B \in (\ell(q, \Delta_n^m) : \eta)$, let us choose $x = (x_k) \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$. Then Ax exists. Thus, we have

$$\begin{aligned}\sum_k a_{nk} x_k &= \sum_k a_{nk}\left[\frac{1}{F_j}\left(\frac{1}{\mu_j}(-1)^{k-i} D_{k-i}\rho T_i \Delta_n^m y_i\right)\right] \\ &= \sum_k^{\infty} b_{nk} y_k \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

which gives $Ax = By$ and gives $A \in (\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q) : \eta)$. □

5 The rotundity of the space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$

In this section we use the concept of rotundity and give some conditions to prove the rotundity of the space $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$. For details about rotundity, Opial property, modularity, see [3, 4, 13, 26].

Definition 5.1 Let $S(X)$ be the unit sphere of a Banach space X. Then a point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space X is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point.

Let $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function, $\mu = (\mu_j)$ be a sequence of positive real numbers, and $q = (q_k)$ be a bounded sequence of positive real numbers. We portray $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}$ on $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ by

$$\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) = \sum_k \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j \Delta_n^m x_j|}{\rho}\right)\right|^{q_k}.$$

If $q_k \geq 1$ for all $k \in \mathbb{N}_1 = \{1,2,\ldots\}$, by the convexity of the function $t \to |t|^{q_k}$ for each $k \in \mathbb{N}$, $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}$ is a convex modular on $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$. We consider $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ furnished with Luxemburg norm

$$\|x\| = \inf\left\{\gamma > 0 : \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}\left(\frac{x}{\gamma}\right) \leq 1\right\}. \tag{17}$$

The space $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ is a complete normed space with above norm. This can be proved in a similar manner as in the proof of Theorem 7 in [16].

Theorem 7 *For all $k \in \mathbb{N}$ and $q_k \geq 1$, the modular $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}$ on $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ satisfies the following properties*:

(i) *If $0 < \gamma \leq 1$, then $\gamma^K\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x/\gamma) \leq \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x)$ and $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(\gamma x) \leq \gamma\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x)$.*

(ii) *If $\gamma \geq 1$, then $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \leq \gamma^K\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x/\gamma)$.*

(iii) *If $0 < \gamma \leq 1$, then $\gamma\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x/\gamma) \leq \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x)$.*

(iv) *The modular $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}$ is continuous.*

Proof (i) Let $0 < \gamma \leq 1$. Then $\gamma^K/\gamma^{q_k} \leq 1$ for all $q_k \geq 1$. Therefore, we have

$$\begin{aligned}
\gamma^K\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}\left(\frac{x}{\gamma}\right) &= \sum_k \frac{\gamma^K}{\gamma^{q_k}}\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\
&\leq \sum_k \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\
&= \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x), \\
\sigma_{(\mathcal{F},\Delta_n^m,\mu,p)}(\gamma x) &= \sum_k \gamma^{q_k}\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\
&\leq \gamma\sum_k \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\
&= \gamma\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x).
\end{aligned}$$

(ii) Let $\gamma \geq 1$. Then $1 \leq \gamma^K/\gamma^{q_k}$ for all $q_k \geq 1$. So, we have

$$\sigma_{(\mathcal{F},\Delta_n^m,\mu,p)}(x) \leq \frac{\gamma^K}{\gamma^{p_k}}\sigma_{(\mathcal{F},\Delta_n^m,\mu,p)}(x) = \gamma^K\sigma_{(\mathcal{F},\Delta_n^m,\mu,p)}\left(\frac{x}{\gamma}\right). \tag{18}$$

(iii) Let $\gamma \geq 1$. Then $\gamma/\gamma^{p_k} \leq 1$ for all $q_k \geq 1$. Therefore, we have

$$\begin{aligned}
\gamma\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}\left(\frac{x}{\gamma}\right) &= \sum_k \frac{\gamma}{\gamma^{q_k}}\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\
&\leq \sum_k \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\
&= \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x).
\end{aligned}$$

(iv) If $\gamma > 1$, then we have

$$\begin{aligned}\sum_k \gamma \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} &= \sum_k \gamma^{p_k}\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\ &\leq \sum_k \gamma^{K}\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\ &= \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x).\end{aligned}$$

Therefore,

$$\gamma\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \leq \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(\gamma x) \leq \gamma^K\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x). \tag{19}$$

Taking γ as 1^+ in (19), we find $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(\gamma x) \to \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x)$.

If we consider $0 < \gamma < 1$, we find that

$$\begin{aligned}\sum_k \gamma^K \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} &= \sum_k \gamma^{p_k}\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\ &\leq \sum_k \gamma\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k} \\ &= \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x),\end{aligned}$$

that is,

$$\gamma^K\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \leq \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(\gamma x) \leq \gamma\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x). \tag{20}$$

Take γ as 1^- in (20), we get $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(\gamma x) \to \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x)$. Hence, $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}$ is continuous. □

Theorem 8 *Let $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function, $\mu = (\mu_j)$ be a sequence of positive real numbers, and $q = (q_k)$ be a bounded sequence of positive real numbers. For any $x \in \mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$, the following statements hold*:

(i) *If* $\|x\| < 1$, *then* $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \leq \|x\|$.

(ii) *If* $\|x\| > 1$, *then* $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \geq \|x\|$.

(iii) $\|x\| = 1$ *iff* $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) = 1$.

(iv) $\|x\| < 1$ *iff* $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) < 1$.

(v) $\|x\| > 1$ *iff* $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) > 1$.

(vi) *If* $0 < \gamma < 1$ *and* $\|x\| > \gamma$, *then* $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) > \gamma^K$.

(vii) *If* $\gamma \geq 1$ *and* $\|x\| < \gamma$, *then* $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) < \gamma^K$.

Proof Let $x \in \mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$.

(i) Let us take $\epsilon > 0$ such that $0 < \epsilon < 1 - \|x\|$. Using (20), there exists $\gamma > 0$ such that $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(\frac{x}{\gamma}) \leq 1$ and $\|x\| + \epsilon > \gamma$. Therefore, we have

$$\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \leq \sum_k \left(\frac{\|x\|+\epsilon}{\alpha}\right)^{q_k} \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m x_j|}{\rho}\right)\right|^{q_k}$$
$$\leq \left(\|x\|+\epsilon\right)\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}\left(\frac{x}{\gamma}\right) \leq \|x\| + \epsilon. \tag{21}$$

Since ϵ is arbitrary, we have $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \leq \|x\|$ from (21).

(ii) Let $\epsilon > 0$ such that $0 < \epsilon < 1 - \frac{1}{\|x\|}$, then $1 < (1-\epsilon)\|x\| < \|x\|$. Using (20) and part (iii) of Theorem 7, we have

$$1 < \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}\left[\frac{x}{(1-\epsilon)\|x\|}\right] \leq \frac{1}{(1-\epsilon)\|x\|}\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x).$$

Therefore, $(1-\epsilon)\|x\| < \|x\| \forall \epsilon \in (0, 1-(1/\|x\|))$. Thus, $\|x\| < \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x)$.

(iii) This can be done by the similar way used in the proof of Theorem 4 of [13] and continuity of $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}$. Similarly, we can find the others. □

Theorem 9 *Let $\mathcal{F} = (F_j)$ be a Musielak–Orlicz function, $\mu = (\mu_j)$ be a sequence of positive real numbers, and $q = (q_k)$ be a bounded sequence of positive real numbers. The space $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ is rotund iff $q_k > 1$ for every natural number k.*

Proof Let $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ be rotund and take a natural number k such that $q_k > 1$ for every $k < 3$. Now, we contemplate the sequences given by

$$x = (1, -X_1, X_2, -X_3, X_4, \ldots),$$
$$y = (0, Y_1, -Y_2X_1, Y_1X_2, -Y_1X_3, \ldots).$$

Clearly, $x \neq y$ and $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) = \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(y) = \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(\frac{x+y}{2}) = 1$.

By using (iii) of Theorem 5, $x, y, (x+y)/2 \in S[\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)]$, which contradicts that the sequence space $\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)$ is not rotund. Therefore, $q_k > 1$ for every natural number k.

On the contrary, suppose $x \in S[\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)]$ and $r, s \in S[\mathcal{N}^t(\mathcal{F},\Delta_n^m,\mu,q)]$, where $x = (r+s)/2$. By the convexity of $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}$ and Theorem 8, we have

$$1 = \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) \leq \frac{\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(r) + \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(s)}{2} = 1,$$

which gives

$$\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x) = \frac{\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(r) + \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(s)}{2}. \tag{22}$$

Since $x = (r+s)/2$, we obtain from (22) that

$$\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m(r_j+s_j)/2|}{\rho}\right)\right|^{q_k} = \frac{1}{2}\left(\left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m r_j|}{\rho}\right)\right|^{q_k} + \left|\frac{1}{T_k}\sum_{j=0}^{k} F_j\left(\frac{|t_{k-j}\mu_j\Delta_n^m s_j|}{\rho}\right)\right|^{q_k}\right).$$

Therefore,

$$\left|\frac{r_j+s_j}{2}\right|^{q_k} = \frac{|r_j|^{q_k}+|s_j|^{q_k}}{2} \tag{23}$$

for every natural number k. Since $t \to |t|^{q_k}$ is strictly convex for all $k \in \mathbb{N}$, it follows by (23) that $r_j = s_j$ for all $k \in \mathbb{N}$. Thus, $r = s$ and hence $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ is rotund. □

Theorem 10 *Suppose that $\mathcal{F} = (F_j)$ is a Musielak–Orlicz function, $\mu = (\mu_j)$ is a sequence of positive real numbers, and $q = (q_k)$ is a bounded sequence of positive real numbers. Let (x_n) be a sequence in $\mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$. Then the following statements hold*:

(i) $\lim_{n\to\infty}\|x_n\| = 1$ *implies* $\lim_{n\to\infty}\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x_n) = 1$;

(ii) $\lim_{n\to\infty}\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x_n) = 0$ *implies* $\lim_{n\to\infty}\|x_n\| = 0$.

Proof This can be proved by the similar way used in the proof of Theorem 10 in [16]. So, we omit it. □

Theorem 11 *Suppose that $\mathcal{F} = (F_j)$ is a Musielak–Orlicz function, $\mu = (\mu_j)$ is a sequence of positive real numbers, and $q = (q_k)$ is a bounded sequence of positive real numbers. Let $x \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ and $(x^{(n)}) \subset \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$. If $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x^{(n)}) \to \sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x)$ as $n \to \infty$ and $(x_k^{(n)}) \to x_k$ as $n \to \infty$ for all $k \in \mathbb{N}$, then $x^{(n)} \to x$ as $n \to \infty$.*

Proof Let $\epsilon > 0$. Since $x \in \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$ and $(x^{(n)}) \subset \mathcal{N}^t(\mathcal{F}, \Delta_n^m, \mu, q)$, we have

$$\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}\big(x^{(n)} - x\big) = \sum_k \big|\big\{\mathcal{N}^t(\mathcal{F},\mu)\big(x^{(n)} - x\big)\big\}_k\big| < \infty.$$

Then, we can find a natural number k_0 such that

$$\sum_{k=k_0+1}^{\infty} \big|\big\{\mathcal{N}^t(\mathcal{F},u)\big(x^{(n)} - x\big)\big\}_k\big| = \frac{\epsilon}{2}. \tag{24}$$

Since $x_k^{(n)} \to x_k$ as $n \to \infty$, we have

$$\sum_{k=1}^{k_0} \big|\big\{\mathcal{N}^t(\mathcal{F},\mu)\big(x^{(n)} - x\big)\big\}_k\big| = \frac{\epsilon}{2}. \tag{25}$$

From (24) and (25), we obtain $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x^{(n)} - x) < \epsilon$. Therefore, $\sigma_{(\mathcal{F},\Delta_n^m,\mu,q)}(x^{(n)} - x) \to 0$ as $n \to \infty$. This implies $\|x^n - x\| \to 0$ as $n \to \infty$ from (ii) of Theorem 7. Hence, the result. □

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Authors' contributions
Both authors jointly worked on deriving the results and writing the manuscript and approved the final version of manuscript.

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