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Formal and analytic normal forms for non-autonomous difference systems with uniform dichotomy spectrum

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Abstract

In this paper, we extend formal and analytic normal forms from autonomous difference systems to non-autonomous ones based on the uniform dichotomy spectrum of their linear part.

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1 Introduction and main results

The normal form theory founded for dynamical systems by Poincaré is widely used to analyse local dynamical properties, whose main idea is to simplify systems to proper forms. Many researchers including Dulac, Sternberg, Chen, Birkhoff and Il'yashenko devoted their great efforts to the development of the theory. Nowadays, it plays an important role in the study of bifurcations and stabilities, which is also widely applied to celestial mechanics, biomathematics, the control theory and so on. More topics for the systematic study of normal forms can be found in many monographs such as [4, 6, 7] and a series of papers.

In the classical theory, transformations that we seek are established near fixed points or (quasi)-periodic orbits. Due to the talent idea of the concept exponential dichotomy by Sacker and Sell, the spectral theory can be established to characterise the exponential dynamical behaviour of general linear non-autonomous systems in [3, 11], which shall lead to the simplification of nonlinear parts in [8, 12, 13]. Additionally, normal forms of random dynamical systems and non-autonomous differential systems were archived in [9, 14] based on the Lyapunov exponent and the nonuniform dichotomy spectrum, respectively. Previously, we proved the analytic conjugacy of Poincaré type and Poincaré–Dulac type for analytic non-autonomous differential systems based on the dichotomy spectrum of their linear parts in [13] using the methods from [2, 5]. In this paper we persuade to modify the technique of [2] to handle the case of non-autonomous difference ones.

Firstly, we recall the uniform exponential dichotomy spectral theory in [3] of linear difference systems

$$x_{k+1} = A_k x_k, \quad x_k \in \mathbb{R}^n \tag{1}$$

with invertible matrices $A_k \in \mathbb{R}^{n\times n}$. Let $\Phi : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}^{n\times n}$, $(k,t) \to \Phi(k,t)$, and denote the evolution operator of system (1) by

$$\Phi(k,t) = \begin{cases} A_{k-1}\cdots A_t, & k > t \\ I, & k = t \\ A_k^{-1}\cdots A_{t-1}^{-1}, & k < t. \end{cases}$$

System (1) is said to be of *bounded growth*, provided that there exist constants K and $a \geq 1$ such that

$$\|\Phi(k,t)\| \leq Ka^{|k-t|} \quad \text{for } k,t \in \mathbb{Z},$$

which is equivalent to the fact that A_k and A_k^{-1} are bounded for $\forall k \in \mathbb{Z}$. Hence system (1) has a nonempty and compact dichotomy spectrum $\Sigma(A) = \bigcup_{i=1}^d [a_i, b_i]$ with $0 < a_1 \leq b_1 < \cdots < a_d \leq b_d$, where $1 \leq d \leq n$. Similar to the block diagonalization with respect to the Jordan normal form of the linear part matrix for autonomous systems, [11] gave the block diagonalization theorem for non-autonomous ones by the dichotomy spectrum. More precisely, it means that there exists a kinematic similarity matrix function $S : \mathbb{Z} \to \mathbb{R}^{n\times n}$ such that system (1) is kinematically similar to

$$y_{k+1} = B_k y_k, \tag{2}$$

where $B_k = \operatorname{diag}\{B_k^1, \ldots, B_k^d\}$ is in the d block diagonal form, and each block $B_k^i : \mathbb{Z} \to \mathbb{R}^{n\times n}$ corresponds to the spectral interval $[a_i, b_i]$ for $i = 1, \ldots, d$ and $n_1 + \cdots + n_d = n$.

Now we consider the reducibility of the following non-autonomous nonlinear difference system:

$$x_{k+1} = F_k(x_k) = A_k x_k + f_k(x_k), \quad x_k \in \mathbb{R}^n, \tag{3}$$

where F_k is analytic in a neighbourhood of the origin, the linear part is of bounded growth and $f_k(x_k) = o(\|x_k\|)$ as $x_k \to 0$ is analytic uniformly for all $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Then the uniform dichotomy spectrum of the linear part is $\Sigma(A) = \bigcup_{i=1}^d [a_i, b_i]$, where $a_1 \leq b_1 < \cdots < a_d \leq b_d$. Moreover, the linear part is kinematically similar to a block diagonal one of form (2). So in the rest of the paper we always assume that A_k in (3) is of the block diagonal form and each block A_k^i corresponds to one spectral interval $[a_i, b_i]$.

In order to do the cancellation of nonlinear terms, based on the block diagonal form of the linear part, the map on the index set is denoted by

$$\begin{aligned} \Gamma : \mathbb{N}_l^n &\to \mathbb{N}_l^d \\ \nu &\mapsto \tau \end{aligned}$$

for

$$\tau = (\nu_1 + \cdots + \nu_{n_1}, \nu_{n_1+1} + \cdots + \nu_{n_1+n_2}, \dots, \nu_{n-n_d+1} + \cdots + \nu_n)$$

with $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}_+^n$, $|\nu| = \nu_1 + \cdots + \nu_n$, $\mathbb{N}_l^d = \{\tau = (\tau_1, \dots, \tau_d) \in \mathbb{Z}_+^d : |\tau| = l\}$ and $|\tau| = |\nu| = l$. Rewrite the nonlinearity $f_k(x_k)$ as its Taylor series:

$$f_k(x_k) = \sum_{|\nu|\geq 2}^{\infty} f_{k,\nu} x_k^\nu, \quad x_k^\nu = x_{k,1}^{\nu_1} \cdots x_{k,n}^{\nu_n}.$$

As usual, e_j denotes the unit vector, whose jth component is 1. Our definition of the resonant term can be given as follows, which accords with Siegmund's in [11].

Definition 1.1 The dichotomy spectrum $\Sigma(A) = \bigcup_{i=1}^d [a_i, b_i]$ is called resonant, provided that there exists a vector $\tau = (\tau_1, \dots, \tau_d) \in \mathbb{Z}_+^d$ fulfilling $|\tau| \geq 2$ and

$$1 \in \left[a_{j'} \prod_{i=1}^d b_i^{-\tau_i}, b_{j'} \prod_{i=1}^d a_i^{-\tau_i} \right]$$

for some $j' \in \{1, \dots, d\}$. Then the monomial $x_k^\nu e_j$ is called a resonant term for $e_{j'} = \Gamma(e_j)$ and $\tau = \Gamma(\nu)$.

Finally, our main results can be summarised as follows.

Theorem 1.2 *Assume that the exponential dichotomy spectrum of system* (1) *is* $\Sigma(A) = \bigcup_{i=1}^d [a_i, b_i]$ *for* $a_1 \leq b_1 < \cdots < a_d \leq b_d$. *Then the following statements hold.*

(i) (*Formal conjugacy*) *If the linear part of system* (3) *is in the block diagonal form corresponding to the spectrum* $\Sigma(A)$, *then there exists a formal coordinates substitution, which turns system* (3) *into*

$$y_{k+1} = A_k y_k + g_k(y_k),$$

where $g_k(y_k)$ *consists of resonant terms only.*

(ii) (*Analytic conjugacy*) *If the spectrum* $\Sigma(A)$ *is in the Poincaré domain, i.e.* $a_1 > 1$ *or* $b_d < 1$, *then there exists a coordinates substitution* $x_k = y_k + h_k(y_k)$, *which is analytic with respect to the variable* y_k *in the uniform neighbourhood of the origin and turns system* (3) *into a polynomial one with respect to the variable* y_k *of degree no more than* $\max\{\ln a_1/\ln b_d, \ln b_d/\ln a_1\}$. *Moreover, if* $\Sigma(A)$ *is non-resonant, then system* (3) *is locally analytically conjugated to its linear part.*

The rest of the article is organised as follows. In Sect. 2 basic definitions and lemmas are provided, which are key to our main arguments. Then comes the proof of Theorem 1.2, while two examples are well illustrated as the applications in Sect. 3.

2 Preliminaries

In order to state the proof of our results clearly, in this section we introduce some necessary contents. Let us begin with the precise definition of the exponential dichotomy spectrum.

Definition 2.1 ([3, 11]) We say that system (1) admits an exponential dichotomy if there exists an invariant projector P_k, which means

$$P_{k+1}A_k = A_kP_k, \quad \forall k \in \mathbb{Z}$$

and constants $K \geq 1$, $\alpha > 1$ such that

$$\begin{aligned}
&\|\Phi(k,t)P_t\| \leq K\left(\frac{1}{\alpha}\right)^{(k-t)}, \quad k \geq t;\\
&\|\Phi(k,t)(I-P_t)\| \leq K\alpha^{(k-t)}, \quad k \leq t.
\end{aligned}$$

Then the exponential dichotomy spectrum of system (1) is the set

$$\Sigma(A) = \left\{\gamma \in \mathbb{R}^+ : x_{k+1} = \frac{1}{\gamma}A_kx_k \text{ admits no exponential dichotomy}\right\}$$

and the resolvent set $\rho(A) = \mathbb{R}^+ \setminus \Sigma(A)$ is its complement.

Notice that the above definitions also appear in [10, 12].

Next comes the study of a linear system having tensor structures. Denote by $V_1, \ldots, V_k$ the finite-dimensional real vector spaces of dimensions $\dim V_i = n_i$ for $i = 1, \ldots, k$. Then let $V = V_1 \otimes \cdots \otimes V_k$ be their tensor product, a vector space of dimension $n = n_1n_2 \cdots n_k$, which is defined to be the vector space $L(V_1^* \times \cdots \times V_k^*, \mathbb{R})$ of k-linear forms on $V_1^* \times \cdots \times V_k^*$. As usual, V^* denotes the dual of V. From [12] we know that linear cocycles Φ_i on V_i deduce a linear cocycle $\Phi_1 \otimes \cdots \otimes \Phi_l$ on V, where V_i $(i = 1, \ldots, l)$ are finite dimensional real vector fields and $V = V_1 \otimes \cdots \otimes V_l$. Restricted to our attention, we have the following for $l = 2$, then the extension to general k is obvious.

Lemma 2.2 *If linear systems*

$$x_{k+1} = A_k^i x_k, \quad i = 1, 2,$$

have evolution operators $\Phi^i(k,t)$ with the corresponding exponential dichotomy spectrum $\Sigma(A_i) = \bigcup_{i=1}^{d_i}[a_1^{(i)}, b_1^{(i)}]$, then $\Phi^1(k,t) \otimes \Phi^2(k,t)$ is the evolution operator of the system $x_{k+1} = (A_k^1 \otimes A_k^2)x_k$ with the exponential dichotomy spectrum estimation

$$\bigcup_{i=1}^{d_1}\bigcup_{j=1}^{d_2}[a_i^{(1)}a_j^{(2)}, b_i^{(1)}b_j^{(2)}].$$

Proof Since $\Phi^i(k,t)$ has the following form:

$$\Phi^i(k,t) = \begin{cases} A_{k-1}^i \cdots A_t^i, & k > t;\\ I_i, & k = t;\\ (A_k^i)^{-1} \cdots (A_{t-1}^i)^{-1}, & k < t, \end{cases}$$

we have that

$$\begin{aligned}\Phi^1(k,t)\otimes\Phi^2(k,t) &= \left(A^1_{k-1}\cdots A^1_t\right)\otimes\left(A^2_{k-1}\cdots A^2_t\right)\\ &= \left(A^1_{k-1}\otimes A^2_{k-1}\right)\cdots\left(A^1_t\otimes A^2_t\right)\end{aligned}$$

for $k>t$, then $\Phi^1(k,t)\otimes\Phi^2(k,t)=I_1\otimes I_2=\overline{I}$ for $k=t$ and similarly $\Phi^1(k,t)\otimes\Phi^2(k,t)=((A^1_k)^{-1}\otimes(A^2_k)^{-1})\cdots((A^1_{t-1})^{-1}\otimes(A^2_{t-1})^{-1})$ for $k<t$, which shows $\Phi^1(k,t)\otimes\Phi^2(k,t)$ is the evolution operator of the system $x_{k+1}=(A^1_k\otimes A^2_k)x_k$. Here I_1 and I_2 are identity matrices.

The remaining part is similar to Proposition 4 in [13] just by using Definition 2.1 here. Taking the invariant linear space $V^1_l\otimes V^2_j$ of the operator $\Phi^1\otimes\Phi^2$, when $k>t$, we have that

$$\begin{aligned}\left\|\left(\Phi^1(k,t)\otimes\Phi^2(k,t)\right)(u\times v)\right\| &= \left\|\Phi^1(k,t)u\right\|\cdot\left\|\Phi^2(k,t)v\right\|\\ &\le K_1\left(\alpha^1_l\right)^{(k-t)}\|u\|\cdot K_2\left(\alpha^2_j\right)^{(k-t)}\|v\|\\ &= K_1K_2\left(\alpha^1_l\alpha^2_j\right)^{(k-t)}\|u\otimes v\|\end{aligned}$$

for all possible l and j, where $u\in V^1_l$ and $v\in V^2_j$, $a^{(i)}_j<\alpha^i_j<a^{(i)}_j+\mu$, $i=1,2$ and $\mu\ll 1$. The case is similar for $k<t$. So making $\mu\to 0$, this completes the proof. □

Then we show the properties of two matrix operators $N(\cdot)$ and $T(\cdot)$, which are key to the formal cancellation of the nonlinearities. Denote by $H^l_n(\mathbb{R}^n)$ the vector space of homogeneous polynomials of degree l in n variables with values in $\mathbb{R}^n$. A basis $\{u_1,\dots,u_n\}$ in $\mathbb{R}^n$ and the basis $\{x^\tau\}|_{|\tau|=l}$ of $H^l_n(\mathbb{R}^1)$ give a basis $\{x^\tau u_i\}$ in $H^l_n(\mathbb{R}^n)$. By the following equivalence:

$$H^l_n\left(\mathbb{R}^n\right)\ni f=\sum_{i=1}^n\sum_{|\tau|=l}f_{\tau,i}x^\tau u_i\to(f_{\tau,i})\in\mathbb{R}^\Delta,$$

it admits $H^l_n(\mathbb{R}^n)\cong\mathbb{R}^\Delta$, where $\Delta=\dim H^l_n(\mathbb{R}^n)$. Moreover, we can make the identity $H^l_n(\mathbb{R}^n)=H^l_n(\mathbb{R}^1)\otimes\mathbb{R}^n$. Thus denote a $\widetilde{d}\times\widetilde{d}$ matrix by

$$N(A)_l=\left(N^{(l)}_{\tau\zeta}(A)\right),\qquad (Ax)^\tau=\sum_{\zeta\in\mathbb{Z}^n_+,|\zeta|=l}N^l_{\zeta\tau}(A)x^\zeta.$$

So the entries of $N(A)_l$ depend nonlinearly on the ones of A. The following results are summarised from [13] and [12] or [9].

Lemma 2.3 *Let A and B be $n\times n$ matrices and set $l\ge 2$, then the following statements hold*:

(i) $\|N(A)_l\|\le C\|A\|^l$, $N(I_2)_l=I_1$, $N(AB)_l=N(B)_lN(A)_l$, *where the constant C is independent of A, I_2 and I_1 are $n\times n$ and $\widetilde{d}\times\widetilde{d}$ unit matrices, respectively.*

(ii) *If A is invertible, then* $N(A^{-1})_l=(N(A))^{-1}_l$.

(iii) *If $A_k=\operatorname{diag}\{A^1_k,\dots,A^d_k\}$ is bounded with $A^i_k:\mathbb{Z}\to\mathbb{R}^{n_i\times n_i}$ for $i=1,\dots,d$, then there exists a permutation matrix P independent of k in $\mathbb{R}^{\widetilde{d}\times\widetilde{d}}$, which makes $N(A)_l$ similar to a block diagonal matrix*

$$\operatorname{diag}\left\{N(A)_{\tau\in\mathbb{N}^d_l},\tau\in\mathbb{Z}^d_+,|\tau|=l\right\}$$

with $N(A)_\tau : \mathbb{Z} \to \mathbb{R}^{q_\tau \times q_\tau}$, $q_\tau = \prod_{i=1}^{d} \frac{(\tau_i+n_i-1)!}{\tau_i!(n_i-1)!}$. *Moreover, we have*

$$\|N(A_k)_\tau\| \le C\prod_{i=1}^{d}\|A_k^i\|^{\tau_i},$$

where the constant C is dependent of A_k.

The above is from Proposition 5 (p. 2965) of [13], which stems from propositions in [2]. Then comes the operator $T(\cdot)$. Precisely, the linear operator $T(A)_l$ on $H_n^l(\mathbb{R}^1)$ is given by

$$T(A)_l : h(x) \mapsto \sum_{|\tau|=l} h_\tau (Ax)^\tau, \quad h(x) = \sum_{\tau\in\mathbb{Z}_+^d, |\tau|=l} h_\tau x^\tau \in H_n^l(\mathbb{R}^1).$$

So we can regard $N(A)_l$ as the matrix representation of the operator $T(A)_l$. Denote by $\Phi_{T(A_k^{-1})_l}(k,t)$ the evolution operator of the system $x_{k+1} = T(A_k^{-1})_l x_k$. According to the definitions of matrices $N(\cdot)_l$ and $T(\cdot)_l$, obviously, we have the following.

Lemma 2.4 *It admits* $\Phi_{T(A_k^{-1})_l}(k,t) = N(\Phi(k,t))_l^{-1}$, *where* $\Phi(k,t)$ *is the evolution operator of system* (1).

Proof Straightforwardly, by Lemma 2.3(i) we compute

$$\begin{aligned} N\big(\Phi(k+1,t)\big)_l^{-1} \cdot \big(N\big(\Phi(k,t)\big)_l^{-1}\big)^{-1} &= N\big(\Phi(k+1,t)^{-1}\big)_l \cdot N\big(\Phi(k,t)\big)_l \\ &= N\big(\Phi(k,t)\Phi(k+1,t)^{-1}\big)_l = N\big(A_k^{-1}\big)_l, \end{aligned}$$

which shows that $N(\Phi(k,t))_l$ is the matrix solution of the system $x_{k+1} = T(A_k^{-1})_l x_k$. So it completes the proof. □

Compared with the continuous case in [13, 14], ours is simple as the one in [2]. Therefore, by Lemmas 2.2, 2.3 and 2.4, the following statements hold.

Lemma 2.5 *Let* $\Phi(k,t)$ *be the evolution operator of system* (1).

(i) *We have that* $\Phi_{T(A_k^{-1})_l}(k,t) \otimes \Phi(k,t) = N(\Phi(t,k))_l \otimes \Phi(k,t)$.
(ii) *If* $A_k = \operatorname{diag}\{A_k^1,\dots,A_k^d\}$, *then there exists a permutation matrix* $P \in \mathbb{R}^{\widetilde{d}\times\widetilde{d}}$ *independent of k such that* $N(\Phi(t,k))_l$ *is similar to a block diagonal matrix*

$$\{\Lambda_\tau\}_{\tau\in\mathbb{N}_l^d} = \operatorname{diag}\big\{N\big(\Phi(t,k)\big)\big\}_{\tau\in\mathbb{N}_l^d}$$

and

$$\big\|N\big(\Phi(t,k)\big)_\tau\big\| \le C(n,l)\prod_{i=1}^{d}\big\|\Phi^i(t,k)\big\|^{\tau_i},$$

where $\tau \in \mathbb{Z}_+^d$, $|\tau| = l$, $i = 1,\dots,d$, $C(n,l)$ *depends on* n, l.

Proof The result (i) is from Lemma 2.4, while (ii) is a direct application of Lemma 2.3. This completes the proof. □

The last one is the classical Gronwall type inequality for the discrete case.

Lemma 2.6 (Gronwall inequality [1]) *Let, for all $k \in \mathbb{N}$ and $k \geq a$, the following inequality be satisfied*:

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{k-1} f(l)u(l).$$

Then, for all $k \in \mathbb{N}$ and $k \geq a$, we have

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{k-1} p(l)f(l) \prod_{\tau=l+1}^{k-1} \bigl(1 + q(\tau)f(\tau)\bigr).$$

3 Proofs of the main results

The proof consists of two parts. Firstly, we get the normal form of (1) under formal conjugacy. Then we continue to deal with its analytic normal form.

At the beginning we make the estimation of the exponential dichotomy spectrum for the linear equation deduced from the cancellation of non-resonant terms. Define a linear operator $L^l_{A_k}$ on $H^l_n(\mathbb{R}^n)$ as follows:

$$L^l_{A_k} h(x) = A_k h\bigl(A_k^{-1} x\bigr) \tag{4}$$

for $h(x) \in H^l_n(\mathbb{R}^n)$. Since $H^l_n(\mathbb{R}^n) = H^l_n(\mathbb{R}^1) \otimes \mathbb{R}^n$, the matrix representation of $L^l_{A_k}$ is $(I_1 \otimes A_k) \cdot (T(A_k^{-1})_l \otimes I_2)$, where I_1 and I_2 are the identity matrices in $H^l_n(\mathbb{R}^1)$ and $\mathbb{R}^n$. See also Proposition 8.2.6 (p. 415) in [2].

Proposition 3.1 *The non-autonomous system $x_{k+1} = L^l_{A_k} x_k$ has the exponential dichotomy spectrum estimation*

$$\bigcup_{\tau \in \mathbb{N}^d_l} \bigcup_{j=1,\dots,d} \left[a_j \prod_{i=1}^{d} b_i^{-\tau_i}, b_j \prod_{i=1}^{d} a_i^{-\tau_i} \right]. \tag{5}$$

Proof Notice that the matrix representation of the linear operator $L^l_{A_k}$ given by (4) is

$$\begin{aligned} L^l_{A_k} &= (I_1 \otimes A_k) \cdot \bigl(T\bigl(A_k^{-1}\bigr)_l \otimes I_2\bigr) \\ &= \bigl(I_1 \cdot T\bigl(A_k^{-1}\bigr)_l\bigr) \otimes (A_k \cdot I_2) = T\bigl(A_k^{-1}\bigr)_l \otimes A_k, \end{aligned}$$

where I_1, I_2 are identity matrices in $H^l_n(\mathbb{R}^1)$ and $\mathbb{R}^n$. By Lemmas 2.2, 2.4 and 2.5, we obtain that

$$\Phi_{L^l_{A_k}}(k,t) = \Phi_{T(A_k^{-1})_l}(k,t) \otimes \Phi(k,t) = N\bigl(\Phi(k,t)\bigr)_l^{-1} \otimes \Phi(k,t),$$

which leads to

$$\begin{aligned}
\Phi_{L^l_{A_k}}(k,t) &= N\big(\Phi(k,t)\big)_l^{-1} \otimes \Phi(k,t) \\
&= N\big(S_k\Psi(k,t)S_t^{-1}\big)_l^{-1} \otimes \big(S_k\Psi(k,t)S_t^{-1}\big) \\
&= N\big(S_t\Psi^{-1}(k,t)S_k^{-1}\big)_l \otimes \big(S_k\Psi(k,t)S_t^{-1}\big) \\
&= \big(N\big(S_k^{-1}\big)_l \cdot N\big(\Psi^{-1}(k,t)\big)_l \cdot N(S_t)_l\big) \otimes \big(S_k \cdot \Psi(k,t) \cdot S_t^{-1}\big) \\
&= \big(N\big(S_k^{-1}\big)_l \otimes S_k\big) \cdot \big(N\big(\Psi^{-1}(k,t)\big)_l \otimes \Psi(k,t)\big) \cdot \big(N(S_t)_l \otimes S_t^{-1}\big),
\end{aligned}$$

where evolution operators $\Phi(k,t)$ and $\Psi(k,t)$ are kinematically similar to an invertible matrix S, $\Psi(k,t)$ is a block diagonal matrix with blocks $\Psi^i(k,t)$ for $i = 1,\dots,d$. Moreover, each block $\Psi^i(k,t)$ satisfies

$$\begin{aligned}
&\big\|\Psi^i(k,t)\big\| \le K\beta_i^{k-t}, \quad k \ge t; \\
&\big\|\Psi^i(k,t)\big\| \le K\alpha_i^{k-t}, \quad k \le t.
\end{aligned}$$

Here, for each spectrum interval $[a_i,b_i]$, we have that $a_i - \mu_1 \le \alpha_i < a_i$ and $b_i < \beta_i \le b_i + \mu_1$ for $i = 1,\dots,d$, where the positive parameter μ_1 can be chosen arbitrarily small. By the way, we specially note that $\Psi(k,t)$ is just an evolution operator of system (2).

For arbitrary $a \in \mathbb{Z}$, we note that

$$\Psi(k,a)\Psi(a,t) = \Psi(k,t), \qquad \Psi(k,t)\Psi(t,k) = I.$$

Therefore, it yields that

$$\begin{aligned}
N\big(\Psi^{-1}(k,a)\big)_l \cdot N\big(\Psi^{-1}(t,a)\big)_l^{-1} &= N\big(\Psi^{-1}(k,a)\big)_l \cdot N\big(\Psi(t,a)\big)_l \\
&= N\big(\Psi(t,a)\Psi^{-1}(k,a)\big)_l \\
&= N\big(\Psi(t,a)\Psi(a,k)\big)_l \\
&= N\big(\Psi(t,k)\big)_l = N\big(\Psi(k,t)\big)_l^{-1}.
\end{aligned}$$

By Lemmas 2.3 and 2.5, we have that $N(\Psi(k,t))_l^{-1} = N(\Psi(t,k))_l$ and $N(\Psi(t,k))_l$ is similar to the block diagonal one

$$\{\Lambda_\tau\}_{\tau\in\mathbb{N}_l^d} = \operatorname{diag}\big\{N\big(\Psi(t,k)\big)\big\}_{\tau\in\mathbb{N}_l^d}, \quad |\tau| = l,$$

where $\|N(\Psi(t,k))_\tau\| \le C(n,l)\prod_{i=1}^d \|\Psi^i(t,k)\|^{\tau_i}$ and $C(n,l)$ depends on n and l . Furthermore, these lead to

$$\begin{aligned}
\big\|N\big(\Psi(t,k)\big)_\tau\big\| &\le C(n,l)\prod_{i=1}^d \big\|\Psi^i(t,k)\big\|^{\tau_i} \\
&\le C(n,l)\prod_{i=1}^d K\alpha_i^{(t-k)\tau_i} = K'\prod_{i=1}^d \alpha_i^{-\tau_i(k-t)}
\end{aligned}$$

for $k \geq t$ and

$$\begin{aligned}\left\|N\big(\Psi(t,k)\big)_\tau\right\| &\leq C(n,l)\prod_{i=1}^{d}\left\|\Psi^i(t,k)\right\|^{\tau_i}\\ &\leq C(n,l)\prod_{i=1}^{d}K\beta_i^{(t-k)\tau_i} = K'\prod_{i=1}^{d}\beta_i^{-\tau_i(k-t)}\end{aligned}$$

for $k \leq t$, where $K' = C(n,l)K^d$. Hence, let $\mu_1 \to 0$, and the spectrum estimation of $N(\Psi(k,t))_l^{-1}$ is

$$\bigcup_{\tau\in\mathbb{N}_l^d}\left[\prod_{i=1}^{d}b_i^{-\tau_i},\prod_{i=1}^{d}a_i^{-\tau_i}\right].$$

Note again that $\Phi_{L^l_{A_k}}(k,t)$ is kinematically similar to $N(\Psi^{-1}(k,t))_l \otimes \Psi(k,t)$ with an invertible matrix $N(S_t)_l \otimes S_t^{-1}$. Since the kinematic similarity affects nothing about the exponential dichotomy spectrum of the system, the spectrum estimation of $N(\Phi(k,t))_l^{-1}$ is

$$\bigcup_{\tau\in\mathbb{N}_l^d}\left[\prod_{i=1}^{d}b_i^{-\tau_i},\prod_{i=1}^{d}a_i^{-\tau_i}\right].$$

At the end, by Lemma 2.2, we can obtain that the spectrum estimation of linear operator $L^l_{A_k}$ is

$$\bigcup_{\tau\in\mathbb{N}_l^d}\bigcup_{j=1,\dots,d}\left[a_j\prod_{i=1}^{d}b_i^{-\tau_i},b_j\prod_{i=1}^{d}a_i^{-\tau_i}\right].$$

This completes the proof. □

Now we do formal cancellations. Applying the scheme of Poincaré–Dulac type formal reductions, we can assume that a near identity formal change of variables $x_k = y_k + h_k(y_k)$ transforms system (3) into

$$y_{k+1} = R_k(y_k) = A_k y_k + \bar{g}_k(y_k), \tag{6}$$

which leads to

$$h_{k+1}\big(A_k y_k + \bar{g}_k(y_k)\big) = A_k h_k(y_k) + f_k\big(y_k + h_k(y_k)\big) - \bar{g}_k(y_k),$$

or equivalently

$$h_{k+1}\big(y_k + \bar{g}_k\big(A_k^{-1}y_k\big)\big) = A_k h_k\big(A_k^{-1}y_k\big) + f_k\big(A_k^{-1}y_k + h_k\big(A_k^{-1}y_k\big)\big) - \bar{g}_k\big(A_k^{-1}y_k\big). \tag{7}$$

Expanding $h_k, \bar{g}_k, f_k$ as formal Taylor series

$$h_k \sim \sum_{|\nu|\geq 2}^{\infty} h_{k,\nu}y_k^\nu, \qquad f_k \sim \sum_{|\nu|\geq 2}^{\infty} f_{k,\nu}y_k^\nu, \qquad \bar{g}_k \sim \sum_{|\nu|\geq 2}^{\infty} \bar{g}_{k,\nu}y_k^\nu,$$

where $\nu = (\nu_1, \ldots, \nu_n)$, $l = |\nu| \geq 2$, $h_{k,\nu}$, $f_{k,\nu}$ and $\bar{g}_{k,\nu}$ are bounded vector-valued functions from $\mathbb{Z}$ to $\mathbb{R}^n$. Inserting these expressions into equality (7), we shall solve equation (7) for $h_{k,\nu}, k \in \mathbb{Z}$ inductively by comparing terms of degree l for $l = 2, 3, \ldots$.

Comparing the monomials of degree l in equality (7), we can get that

$$h_{k+1,l} = L^l_{A_k} h_{k,l} + T_{k,l} - \hat{G}_{k,l}, \tag{8}$$

where $L^l_{A_k}$ is the linear operator defined by (4), $\hat{G}_{k,l}$ is the coefficient of $\bar{g}_k(A_k^{-1}y_k)$ of degree l and $T_{k,l}$ is the coefficient of degree l of the expression

$$f_k\left(A_k^{-1}y_k + \sum_{|\nu|=2}^{l-1} h_{k,\nu}\left(A_k^{-1}y_k\right)^\nu\right) - \sum_{|\nu|=2}^{l-1} h_{k+1,\nu}\left(y_k + \sum_{|\nu|=2}^{l-1} \bar{g}_k\left(A_k^{-1}y_k\right)^\nu\right),$$

which has been known already by the induction assumption.

First we consider the non-resonant case. If the spectrum $\Sigma(A)$ is lth non-resonant, i.e. $1 \notin [a_{j'} \prod_{i=1}^d b_i^{-\tau_i}, b_{j'} \prod_{i=1}^d a_i^{-\tau_i}]$ for all $|\tau| = l$, the linear operator $L^l_{A_k}$ has the exponential dichotomy spectrum (5) by Proposition 3.1. So we can choose $\hat{G}_{k,l} = 0$ because equation (8) has a unique solution $h_{k,l}$ for arbitrary $\hat{G}_{k,l}$. Especially, if $\Sigma(A)$ is non-resonant for $\forall |l| \geq 2$, system (3) is formally conjugated to its linear part.

We now consider the resonant case. Assume that A_k is in block diagonal form corresponding to $\Sigma(A)$. By Lemmas 2.2, 2.4 and Proposition 3.1, $(T_{k,\nu} - \hat{G}_{k,\nu})e_j$ corresponds to the diagonal block of $L^l_{A_k}$ with exponential dichotomy spectrum estimation $[a_{j'} \prod_{i=1}^d b_i^{-\tau_i}, b_{j'} \prod_{i=1}^d a_i^{-\tau_i}]$, where $e_{j'} = \Gamma(e_j)$, $\tau = \Gamma(\nu)$. Moreover, by Proposition 3.1, the linear operator $L^l_{A_k}$ is similar to the following matrix:

$$\begin{pmatrix} M_k^+ & 0 & 0 \\ 0 & M_k^- & 0 \\ 0 & 0 & M_k^c \end{pmatrix},$$

where M_k^+ is made of the blocks corresponding to exponential dichotomy spectrum estimation satisfying $a_{j'} \prod_{i=1}^d b_i^{-\tau_i} > 1$; M_k^- has ones fulfilling $b_{j'} \prod_{i=1}^d a_i^{-\tau_i} < 1$ and others are in M_k^c.

In order to solve system (8), let $h_{k,l} = (h_k^+, h_k^-, h_k^c)$, $T_{k,l} = (T_k^+, T_k^-, T_k^c)$, $\hat{G}_{k,l} = (\hat{G}_k^+, \hat{G}_k^-, \hat{G}_k^c)$, we divide the evolution operator of linear system

$$h_{k+1,l} = L^l_{A_k} h_{k,l}$$

into $\Phi_{L^l_{A_k}}(k,t) = (\Phi^+_{L^l_{A_k}}(k,t), \Phi^-_{L^l_{A_k}}(k,t), \Phi^c_{L^l_{A_k}}(k,t))$. So system (8) can be decomposed into three subsystems as follows:

$$h^+_{k+1,l} = L^l_{A_k} h^+_{k,l} + T^+_{k,l} - \hat{G}^+_{k,l}, \tag{9}$$

$$h^-_{k+1,l} = L^l_{A_k} h^-_{k,l} + T^-_{k,l} - \hat{G}^-_{k,l}, \tag{10}$$

$$h^c_{k+1,l} = L^l_{A_k} h^c_{k,l} + T^c_{k,l} - \hat{G}^c_{k,l}. \tag{11}$$

For subsystems (9) and (10), the conditions

$$a_{j'}\prod_{i=1}^{d} b_i^{-\tau_i} > 1 \quad \text{and} \quad b_{j'}\prod_{i=1}^{d} a_i^{-\tau_i} < 1$$

imply the non-resonance by $1 \notin [a_{j'}\prod_{i=1}^{d} b_i^{-\tau_i}, b_{j'}\prod_{i=1}^{d} a_i^{-\tau_i}]$ for $l = \sum_{i=1}^{d} \tau_i \geq 2$, then by setting $\hat{G}^+_{k,l} = \hat{G}^-_{k,l} = 0$ these subsystems have the unique solutions

$$h^+_{k,l} = -\sum_{t=k}^{\infty} \Phi^+_{L^l_{A_k}}(k,t+1)T^+_{t,l}, \quad a_{j'} > \prod_{i=1}^{d} b_i^{\tau_i},$$
$$h^-_{k,l} = \sum_{t=-\infty}^{k-1} \Phi^-_{L^l_{A_k}}(k,t+1)T^-_{t,l}, \quad b_{j'} < \prod_{i=1}^{d} a_i^{\tau_i},$$

respectively. For subsystem (11) we can make $T^c_{k,l} = \hat{G}^c_{k,l}$. It especially has the trivial solution $h^c_{k,l} = 0$.

Summarising the above arguments, if A_k is block diagonal corresponding to $\Sigma(A)$, then system (1) is analytically conjugated to

$$x_{k+1} = \mathrm{Jet}^{l-1}_{x_k=0} F_k(x_k) + g_k(x_k) + O\bigl(\|x_k\|^{l+1}\bigr), \tag{12}$$

where g_k is a polynomial consisting of resonant monomials only with respect to the variable x_k of degree l. Here $\mathrm{Jet}^l_{\cdot=0} F(\cdot)$ denotes the part of the Taylor expansion of the function F of order no more than l with respect to the variable $\cdot$ at $\cdot = 0$.

At this moment, we can deal with analytical cancellations, when $\Sigma(A)$ is in the Poincaré domain. Now we take the following system into account:

$$x_{k+1} = A_k x_k + g_k(x_k) + O\bigl(\|x_k\|^{l+1}\bigr),$$

which has the same smoothness as system (1). Notice that the fact $a_1 > 1$ or $b_d < 1$ means that the remainder $O(\|\cdot\|^{l+1})$ contains non-resonant terms only for the case $l + 1 > \ln b_d / \ln a_1$ or $l + 1 > \ln a_1 / \ln b_d$, respectively.

After doing formal normal form reductions, we do the cancellation of the remainder analytically. First comes the homotopy method. In order to apply the homotopy method from [8], technically we consider this s-parametric system instead

$$x_{k+1} = \mathcal{F}_k(x_k, s) = A_k x_k + g_k(x_k) + s e_k(x_k), \tag{13}$$

where the smoothness is the same as system (1), $s \in [0,1]$, A_k and g_k are the ones as mentioned above and $e_k(\cdot) = O(\|\cdot\|^{l+1})$ is the remainder.

Lemma 3.1 *If there exists a series mappings r_k analytic for (x_k, s) in $O_{\rho'} \times [0,1]$ and fulfilling $\|r_k(x_k,s)\| = o(\|x\|^2)$ as $x_k \to 0$, such that r_k satisfies the following equation*:

$$D_{x_k}\mathcal{F}_k(x_k,s)r_k(x_k,s) - r_{k+1}\bigl(\mathcal{F}_k(x_k,s),s\bigr) = -e_k(x_k), \tag{14}$$

where $D_\cdot$ is the Jacobian matrix with respect to $\cdot$, then systems $x_{k+1} = \mathcal{F}_k(x_k, 0)$ and $x_{k+1} = \mathcal{F}_k(y_k, 1)$ are locally conjugated analytically. Here $\mathcal{F}_k$ is given by (13).

Proof Let $V_k = (\mathcal{F}_k(x_k,s),s)$, $U_k = (r_k(x_k,s),1)$ for $k \in \mathbb{Z}$. We have that

$$D_{(x_k,s)}V_kU_k = \begin{pmatrix} \frac{\partial \mathcal{F}_k(x_k,s)}{\partial x_k} r_k(x_k,s) + \frac{\partial \mathcal{F}_k(x_k,s)}{\partial s} \\ 1 \end{pmatrix}.$$

By using (14) and $\frac{\partial \mathcal{F}_k(x_k,s)}{\partial s} = e_k(x_k)$, the following holds:

$$D_{(x_k,s)}V_kU_k = U_{k+1} \circ V_k. \tag{15}$$

Denote by $\varphi^\tau_{U_k}$ the local flow generated by U_k, which is analytic. From (15) we have

$$V_k \circ \varphi^\tau_{U_k} = \varphi^\tau_{U_{k+1}} \circ V_k.$$

Set $\varphi^1_{U_k}(x_k,0) = (h_k(x_k),1)$, it yields that

$$V_k \circ \varphi^1_{U_k}(x_k,0) = \big(\mathcal{F}_k(x_k,s),s\big) \circ \big(h_k(x_k),1\big) = \big(\mathcal{F}_k\big(h_k(x_k),1\big),1\big)$$

and

$$\varphi^1_{U_{k+1}} \circ V_k(x_k,0) = \big(h_{k+1}(x_k),1\big) \circ \big(\mathcal{F}_k(x_k,0),0\big) = \big(h_{k+1}\big(\mathcal{F}_k(x_k,0)\big),1\big),$$

which is equivalent to

$$\mathcal{F}_k\big(h_k(x_k),1\big) = h_{k+1}\big(\mathcal{F}_k(x_k,0)\big),$$

where h_k is analytic locally. This completes the proof. □

So we set $\phi^m_k(\cdot,s) = \mathcal{F}_{k+m-1}(\mathcal{F}_{k+m-2}(\cdots\mathcal{F}_{k+1}(\mathcal{F}_k(\cdot,s),s)\cdots),s)$ analytically from $\mathbb{R}^n$ to $\mathbb{R}^n$, and denote $\phi^m_k(\cdot,s)$ and $r_k(\cdot,s)$ by $\phi^m_{k,s}(\cdot)$ and $r_{k,s}(\cdot)$ for convenience, respectively. To simplify notations, we use $f_*h = (Df \cdot h) \circ f^{-1}$.Therefore, system (13) is in the new form

$$x_{k+1} = \mathcal{F}_k(x_k,s) = \phi_k(x_k,s) = \phi_{k,s}(x_k),$$

and equation (14) is

$$(\phi_{k,s})_* r_{k,s}(\cdot) = r_{k+1,s}(\cdot) - \widetilde{e}_k(\cdot), \tag{16}$$

where $\widetilde{e}_k(\cdot) = e_k \circ \phi^{-1}_{k,s}(\cdot)$. Then we write down a formal solution of equation (14) from [8].

Lemma 3.2 *The function*

$$r_{k,s}(\cdot) = -\sum_{m=1}^{\infty} \big(\phi^{-m}_{k,s}\big)_* \widetilde{e}_{k+m-1}(\cdot)$$

is a formal solution of (16).

Proof We do the computation straightforwardly and get

$$\begin{aligned}
(\phi_{k,s})_* r_{k,s}(\cdot) &= -\sum_{m=1}^{\infty} (\phi_{k,s})_* \big(\phi_{k,s}^{-m}\big)_* \widetilde{e}_{k+m-1}(\cdot) \\
&= -\sum_{m=1}^{\infty} \big(\phi_{k+1,s}^{-m+1}\big)_* \widetilde{e}_{k+m-1}(\cdot) \\
&= -\big(\phi_{k+1,s}^{0}\big)_* \widetilde{e}_{k}(\cdot) - \sum_{m=2}^{\infty} \big(\phi_{k+1,s}^{-m+1}\big)_* \widetilde{e}_{k+m-1}(\cdot) \\
&= -\widetilde{e}_{k}(\cdot) - \sum_{m=1}^{\infty} \big(\phi_{k+1,s}^{-m}\big)_* \widetilde{e}_{k+m}(\cdot) \\
&= r_{k+1,s}(\cdot) - \widetilde{e}_{k}(\cdot).
\end{aligned}$$

That completes the proof. □

Without loss of generality, we assume that $b_p < 1$. Let $U_\delta = \{x \in \mathbb{C}^n : \|x\| \leq \delta\}$, $\Delta = \{u \in \mathbb{C} \mid |u| \leq 2\}$ and $l = \ln a_1 / \ln b_p$. Otherwise, if $a_1 > 1$, the following proof procedure is similar only by using another solution

$$r_{k,s}(\cdot) = \sum_{m=0}^{\infty} \big(\phi_{k,s}^{m}\big)_* \widetilde{e}_{k-m-1}(\cdot)$$

of equation (16). Note again here $\Sigma(A) = \bigcup_{i=1}^{d} [a_i, b_i]$. The following estimations are used to control the norm of the formal solution.

Lemma 3.3 *In system* (13) *there exists* $\delta_0 > 0$ *such that*

(i) $\|e_k(x)\| \leq C\|x\|^{l+1}$ *for* $(x,k) \in U_{\delta_0} \times \mathbb{Z}$,

(ii) $\|\phi_{k,s}^{m}(x)\| \leq \widetilde{C}(b_d + \mu_1)^{md} e^{m\mu_2}$ *for* $(x,k) \in U_{\delta_0} \times \mathbb{Z}$, $s \in \Delta$, $m \geq 0$,

(iii) $\|D\phi_{k,s}^{-m}(x)\| \leq \widetilde{C}(a_1 - \mu_1)^{-md} e^{m\mu_3}$ *for* $(x,k) \in U_{\delta_0} \times \mathbb{Z}$, $s \in \Delta$, $m \geq 0$, *where* $\mu_i(\delta_0)$ *are positive constants and* $\mu_i \ll 1$, $i = 1, 2, 3$.

Proof There exists $\delta_0 > 0$ such that $\mathcal{F}_{k,s}(x)$ is analytic in the region $(x,s) \in U_{\delta_0} \times \Delta$ uniformly for $k \in \mathbb{Z}$. Set $M = \sup_{U_{\delta_0} \times \Delta} \|\mathcal{F}\| < \infty$. By Cauchy's integral representation

$$\partial_x^\omega f_k(x) = \frac{\partial^{|\omega|} f_k(x_1 \cdots x_n)}{\partial x_1^{\omega_1} \cdots \partial x_n^{\omega_n}} = \frac{\omega!}{(2\pi i)^n} \int_\gamma \frac{f_k(u)\, du}{(u-x)^{\omega+e}},$$

we have that

$$g_k(x) = \sum_{|\omega|=2}^{l} \frac{\partial_x^\omega f_k(0)}{\omega!} x^\omega,$$

where $|\omega| = \sum_{i=1}^{n} \omega_i$, $e = (1, \ldots, 1) \in \mathbb{Z}_+^n$, $\gamma = \{u : |u_i| = \delta - \varepsilon, i = 1, 2, \ldots, n\}$, $0 < \varepsilon \ll 1$ and $g_k(x) = f_k(x) - e_k(x)$. Let $L_{k,s}(x) = g_k(x) + s e_k(x)$, $s \in [0,1] \subset \Delta$. Thus, it admits

$$\sup_{U_{\delta_0} \times \Delta} \big\| \partial_x L_{k,s}(x) \big\| = \rho \leq \frac{CM}{\delta^2} \delta_0$$

and

$$\left\| e_k(x) \right\| \le n^{l+1}(l+1)!C_0 M \frac{1}{\delta^{l+1}} \|x\|^{l+1} = C\|x\|^{l+1},$$

where $\|x\| \le \delta_0 < \delta$, $0 < \delta_0 \le \frac{2}{3}\delta$ and C is a constant depending only on n. So conclusion (i) is proved.

Then set $\Gamma_k(t,w)$ to be the evolution operator of the linear part of system (14). Using linear variation, it admits

$$x_{k+m} = \phi^m_{k,s}(x_k) = \Gamma_k(m,0)x_k + \sum_{i=0}^{m-1} \Gamma_k(m,i+1)L_{k+i,s}(x_{k+i}), \tag{17}$$

where $L_{k,s}(\cdot) = g_k(\cdot) + se_k(\cdot)$. As we have shown, from [11] there exists kinematic similarity $S : \mathbb{Z} \to \mathbb{R}^{n\times n}$ such that $\Gamma_k(t,w)$ is kinematically similar to a block diagonal matrix with invertible block matrices $\psi^i_k(t,w)$ for $i = 1,\dots,d$. Moreover, it admits

$$\left\| \Gamma_k(t,w) \right\| \le \begin{cases} \widetilde{C}\beta_d^{d(t-w)}, & t \ge \omega, \\ \widetilde{C}\alpha_1^{d(t-w)}, & t \le \omega, \end{cases}$$

where $b_d < \beta_d \le b_d + \mu_1$, $a_1 - \mu_1 \le \alpha_1 < a_1$, μ_1 is a small constant and $\widetilde{C} > 1$.

From (17) we obtain

$$\left\| \phi^m_{k,s}(x_k) \right\| \le \left\| \Gamma_k(m,0) \right\| + \sum_{i=0}^{m-1} \left\| \Gamma_k(m,i+1) \right\| \rho \left\| \phi^i_{k,s}(x_k) \right\|,$$

which by Lemma 2.6 implies

$$\left\| \phi^m_{k,s}(x_k) \right\| \le \widetilde{C}(b_d + \mu_1)^{md}(1 + \widetilde{C}\rho)^m \le \widetilde{C}(b_p + \mu_1)^{mp} e^{m\mu_2}$$

for $\mu_2 = \widetilde{C}\rho$. Of course, μ_2 is a positive constant that can be chosen sufficiently small, if so is δ_0. This verifies result (ii).

At last, set $M_{k,s}(x_{k+1}) = -A_k^{-1}L_{k,s}(x_k)$ and $\rho' = \sup_{U_{\delta_0}\times\Delta} \|\partial_x M_{k,s}(x)\|$. We can obtain that

$$\phi^{-m}_{k,s}(x_{k+m}) = \Gamma_k(0,m)x_{k+m} + \sum_{i=0}^{m-1} \Gamma_k(0,m-i-1)M_{k+m-1-i,s}\bigl(\phi^{-i}_{k,s}(x_{k+m})\bigr).$$

Taking derivatives with respect to variable x_{k+m} on both sides, we have that

$$\left\| D\phi^{-m}_{k,s}(x_{k+m}) \right\| \le \left\| \Gamma_k(0,m) \right\| + \sum_{i=0}^{m-1} \left\| \Gamma_k(0,m-i-1) \right\| \rho' \left\| D\phi^{-i}_{k,s}(x_{k+m}) \right\|.$$

Utilising Lemma 2.6 again and by similar arguments as before it yields that

$$\left\| D\phi^{-m}_{k,s}(x_{k+m}) \right\| \le \widetilde{C}(a_1 - \mu)^{-mp} e^{m\mu_3},$$

where μ_3 is a positive constant and $\mu_3 \ll 1$. So result (iii) is confirmed. □

Summarising all the above arguments, we can provide the proof of our main results.

Proof of Theorem 1.2 Doing the change $x_k = y_k + h_k(y_k)$ to system (3), we get system (6). If $h_k(\cdot)$ is the homogeneous polynomial of degree l with respect to the variable $\cdot$, by comparing the lth order terms on both sides of equality (7), we can obtain equation (8). From Proposition 3.1 we can solve it for the proper $T_{k,l}$ and $\hat{G}_{k,l}$, which finally leads to system (12). Since this transformation has nothing to do with monomials of degree less than l, we can do such transformations in the order by l and have the new system without non-resonant terms formally, which proves (i).

Now $\Sigma(A)$ is in the Poincaré domain. We do the above changes till it reaches $l = \max\{[\ln a_1/\ln b_d], [\ln b_d/\ln a_1]\}$, where $[\cdot]$ is the classical integer part function. Hence, it leads to system (13) after s-parameterising, where the remainder $e_k(\cdot)$ only contains non-resonant terms. Without loss of generality, we assume that $b_d < 1$. Then we begin to prove that the formal solution $r_{k,s}$ of (14) is locally analytic. By Lemma 3.3 it yields the norm estimation

$$\|r_{k,s}(x)\| \leq \sum_{m=1}^{\infty} C\widetilde{C}^{l+2}(a_1-\mu_1)^{-md}e^{m\mu_3}(b_d+\mu_1)^{md(l+1)}e^{m\mu_2(l+1)},$$

where $l = [\ln a_1/\ln b_d]$. By choosing the positive number μ_1 small enough, we can make $(b_d+\mu_1)^l > a_1 > a_1 - \mu_1$, which implies that

$$\begin{aligned}\|r_{k,s}(x)\| &\leq \overline{C}\sum_{m=1}^{\infty}\left[\left(\frac{(b_d+\mu_1)^l}{a_1-\mu}\right)^d (b_d+\mu_1)^d e^{\mu_3}e^{\mu_2(l+1)}\right]^m \\ &\leq \overline{C}\sum_{m=1}^{\infty} e^{(\mu_4 d+\mu_3+\mu_2(l+1)+d\ln(b_d+\mu_1))m}\end{aligned}$$

for $\overline{C} = C\widetilde{C}^{l+2}$ and $\mu_i \ll 1$ for $i = 1,\ldots,4$. Therefore, $r_{k,s}(\cdot)$ is analytic in $O_{\rho'} \times [0,1] \subset U_{\delta_0} \times \Delta$. So by Lemma 3.1 we get result (ii). □

At last, we illustrate two examples for the applications.

First comes a singular perturbed system

$$\varepsilon\Delta_k x = A_k x_k + f_k(x_k), \tag{18}$$

where $\Sigma(A) = \bigcup_{i=1}^d [a_i, b_i]$ with $0 < a_1 \leq b_1 < \cdots < a_d \leq b_d$, $\Delta_k x = x_{k+1} - x_k$ is the classical first order difference operator, ε is a positive small parameter and $f_k(x) = O(\|x\|^2)$ as $x \to 0$ is analytic with respect to the variable $x \in U_{\delta_0}$ uniformly for $k \in \mathbb{Z}$.

Corollary 3.4 *There exists* $\varepsilon_0 > 0$ *such that system* (18) *can be analytically linearised for* $0 < \varepsilon < \varepsilon_0$.

Proof Rewrite system (18) into

$$x_{k+1} = \varepsilon^{-1}(A_k + \varepsilon I)x_k + \varepsilon^{-1}f_k(x_k).$$

By the roughness of the spectrum or using the method in the proof of Lemma 6(b) in [8], we obtain that the spectrum $\Sigma(B(\varepsilon)) = \bigcup_{i=1}^{d}[a'_i, b'_i]$, where $B_k(\varepsilon) = A_k + \varepsilon I$. Moreover, there exists $\varepsilon_1 > 0$ such that $|a_i - a'_i| \leq \rho$ and $|b_i - b'_i| \leq \rho$ for $\varepsilon < \varepsilon_1$, where $\rho_0 = \min_{i=1,\dots,d-1}\{|a_1|, |a_{i+1} - b_i|\}$ and $\rho = \rho_0/4$. So $\Sigma(\varepsilon^{-1}B(\varepsilon)) = \bigcup_{i=1}^{d}[\varepsilon^{-1}a'_i, \varepsilon^{-1}b'_i]$. When $\varepsilon^{-1}a'_1 > 1$, system (18) is in the Poincaré domain. Additionally, there are no resonant terms, provided that $\varepsilon^{-2}a_1'^2 > \varepsilon^{-1}b'_d$. Therefore, choosing $\varepsilon < \varepsilon_0 = \min\{\varepsilon_1, \frac{(a_1-\rho)^2}{b_d+\rho}\}$ and applying Theorem 1.2, system (18) can be analytically linearised, which completes the proof. □

Now we take the classical logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

into account. By restricting the graphic of the map to $[0, 1] \times [0, 1]$, the parameter r can be taken for all $r > 0$. It was shown in [13] that the system is chaotic for $r > r_* = 3.570$, i.e. there exists an invariant set, which is topologically semi-conjugated to 2-shifted symbolic dynamical system Λ_2. Here, by the corollary, we know that the system can be locally linearised for the sufficiently large r in the neighbourhood of each orbit of Λ_2, although globally the system is complicated.

Then we take a global exponentially Lyapunov stable/unstable system into account

$$x_{k+1} = \frac{A_k x_k}{1 + g^{\alpha}(x_k)}, \tag{19}$$

where $\Sigma(A) = \bigcup_{i=1}^{d}[a_i, b_i]$ with $0 < a_1 \leq b_1 < \cdots < a_d \leq b_d$, $g(x)$ is a positive semi-definite quadratic form and $\alpha \geq 1$ is the power.

Corollary 3.5 *Assume that $b_d < 1$ ($a_1 > 1$) of $\Sigma(A)$ in system (19). When $b_d^{2\alpha+1}/a_1 < 1$ ($a_1^{2\alpha+1}/b_d > 1$), system (19) can be linearised by a coordinates substitution, which is analytic in $U_R \times \mathbb{Z}$ for any $R > 0$.*

Proof Without loss of generality, we can assume $b_d < 1$. To apply the homotopy method, we consider

$$x_{k+1} = A_k x_k + \hat{g}(x_k, s)$$

instead, where $\hat{g}$ is given by

$$\hat{g}(x, s) = \frac{1 + sg^{\alpha}(x)}{1 + g^{\alpha}(x)}$$

for $s \in [0, 1]$. Since $0 \leq g(x) \leq c_1\|x\|^2$ for any x, we obtain that

$$\frac{1}{1 + c_1^{\alpha}\|x\|^{2\alpha}} \leq \hat{g}(x, s) \leq 1.$$

Then we check the conditions of Lemma 3.3 one by one. First, note that $\|A_k\| = \|\Phi(k-1, k)\| \leq c_2 e^{\beta}$ for $0 > \beta > \ln b_d$. Then it yields

$$\left|\frac{g^{\alpha}(x)A_k x}{1 + g^{\alpha}(x)}\right| \leq c_3\|x\|^{2\alpha+1},$$

where $c_3 = c_1^\alpha c_2 e^\beta$. Second, easily we can get that $\|\phi_{k,s}^m(x)\| \leq c_2 e^{\beta m}\|x\|$ for $m \geq 0$. As usual, $\phi_{k,s}^m(x)$ is the trajectory of the original system fulfilling $\phi_{k,s}^0(x) = x$. At last, by the first order variation, we get that

$$\begin{aligned} D\phi_{k,s}^{m+1}(x) &= \left(\frac{A_m}{1+g^\alpha(\phi_{k,s}^m(x))} - \frac{(1-s)\alpha g^{\alpha-1}(\phi_{k,s}^m(x))A_m\phi_{k,s}^m(x)\triangledown g(\phi_{k,s}^m(x))}{(1+g^\alpha(\phi_{k,s}^m(x)))^2}\right)D\phi_{k,s}^m(x) \\ &= \frac{A_m}{1+g^\alpha(\phi_{k,s}^m(x))}\left(I - \frac{(1-s)\alpha g^{\alpha-1}(\phi_{k,s}^m(x))\phi_{k,s}^m(x)\triangledown g(\phi_{k,s}^m(x))}{1+g^\alpha(\phi_{k,s}^m(x))}\right)D\phi_{k,s}^m(x), \end{aligned}$$

where the gradients $\triangledown g = (\partial_{x_1}g, \ldots, \partial_{x_n}g)$. Note that

$$\begin{aligned} \prod_{i=k}^m\left(1+g^\alpha\left(\phi_{k,s}^m(x)\right)\right) &\leq \exp\left(\ln\left(\prod_{i=0}^m\left(1+c_4\|x\|^{2\alpha}e^{2\alpha\beta m}\right)\right)\right) \\ &\leq \exp\left(\sum_{i=0}^m c_4\|x\|^{2\alpha}e^{2\alpha\beta m}\right) \leq c_5\|x\|^{2\alpha}, \end{aligned}$$

where $c_4 = c_1^\alpha c_2^{2\alpha}$ and $c_5 = c_4/(1-e^{2\alpha\beta})$. Together with the above estimations, we obtain that

$$\left\|\frac{(1-s)\alpha g^{\alpha-1}(\phi_{k,s}^m(x))\phi_{k,s}^m(x)\triangledown g(\phi_{k,s}^m(x))}{1+g^\alpha(\phi_{k,s}^m(x))}\right\| \leq c_6\|x\|^{2\alpha}e^{2\alpha\beta m},$$

where $c_6 = 2\alpha c_1^{\alpha-1}c_2^{2\alpha}$. So it leads to

$$\left\|\phi_{k,s}^{-m}(x)\right\| \leq c_7\|x\|^{2\alpha}e^{\gamma m}e^{c_8(x)},$$

where $c_8(x) = \sum_{m\geq 0} c_6\|x\|^{2\alpha}e^{2\alpha\beta m} \leq c_6\|x\|^{2\alpha}/(1-e^{2\alpha\beta})$ and $\gamma > -\ln a_1 > 0$. Finally, by conditions of the corollary, we can choose β and γ properly such that $\gamma + (2\alpha+1)\beta < 0$. So the coordinates substitution is convergent and analytic, which completes the proof. □

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Authors' contributions
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