Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Numerical approximation of time-fractional Burgers-type equation

Miaomiao Yang[1*]

[*]Correspondence:
mathxinxi@163.com
[1]General Education Center,
Zhengzhou Business University,
Zhengzhou, P.R. China

Abstract

In this work, we analyze and test a local discontinuous Galerkin method for solving the Burgers-type equation. The proposed numerical method, which is high-order accurate, is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. We prove that the scheme is unconditionally stable and convergent. Some numerical tests are provided to illustrate the accuracy and capability of the scheme.

Keywords: Time-fractional Burgers-type equation; Local discontinuous Galerkin method; Stability analysis; Error estimates

1 Introduction

Burgers equation, which is used to model turbulent fluid motion, was first introduced by Harry Bateman and later studied by Burgers [1] in 1948. It plays an important role in various areas of applied mathematics, such as traffic flow, fluid mechanics, and nonlinear acoustics. Fractional calculus is a branch of mathematical analysis that studies several different possibilities of defining real number powers or complex number powers of the differentiation operator. Fractional differential equations, also known as extraordinary differential equations, are a generalization of differential equations through the application of fractional calculus. In the past decades, fractional differential equations have been applied in many fields, and, in fact, some physical processes can be modeled more accurately by using fractional derivatives [4, 7–12, 19, 21]. Recently, fractional Burgers-type equation was introduced and studied since the adding effect of the wall friction through the boundary layer can be modeled by fractional derivatives.

Some methods have been introduced to solve fractional Burgers-type equation. Vong and Lyu [13] studied a linearized finite difference scheme for a time-fractional Burgers-type equation and proved that the scheme was unconditionally convergent with second order in maximum-norm. In [3], parametric spline functions were used to solve the time-fractional Burgers equation. Wang [14] studied the Adomian decomposition method for the equation. In [5], Inc presented a variational iteration method for space- and time-fractional Burgers equations and obtained some exact and approximate solutions.

Springer

In this paper, we consider the following time-fractional Burgers-type equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u u_x - \lambda_1 u_{xx} = f(x,t), \quad (x,t) \in [a,b] \times [0,T],$$

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.1}$$

where $0 < \alpha < 1$ is the order of the fractional derivatives, f, u_0 are given smooth functions, $\lambda_1 \geq 0$ is a real parameter. The solution is considered to be either periodic or compactly supported in this paper.

The Caputo fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is defined as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1, \tag{1.2}$$

where $\Gamma(\cdot)$ is the gamma function.

The paper is organized as follows. In Sect. 2, some notations and auxiliary results are described. In Sect. 3, we present the local discontinuous Galerkin method for fractional equation (1.1) and prove that the scheme is unconditionally stable and the numerical solution is convergent. Some numerical examples, given in Sect. 4, are presented to illustrate the accuracy of the method, and concluding remarks are provided in Sect. 5.

2 Notations and auxiliary results

In this section we introduce some notations, projections, and the numerical flux.

2.1 Notations and projections

Let $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$ be a partition of $\Omega = [a,b]$, denote $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, \ldots, N$, and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \leq j \leq N$, $h = \max_{1 \leq j \leq N} h_j$.

Let us denote $u_{j+\frac{1}{2}}^+ = \lim_{t \to 0^+} u(x_{j+\frac{1}{2}} + t)$, $u_{j+\frac{1}{2}}^- = \lim_{t \to 0^+} u(x_{j+\frac{1}{2}} - t)$, and $[u]_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}^+ - u_{j+\frac{1}{2}}^+$.

Denote by V_h^k the space of piecewise discontinuous polynomials of the degree up to k:

$$V_h^k = \left\{ v : v \in P^k(I_j), x \in I_j, j = 1, 2, \ldots, N \right\}.$$

The following two projections in one dimension $[a,b]$ will be used when proving error estimate: projection $\mathcal{P}$

$$\int_{I_j} \left(\mathcal{P}\omega(x) - \omega(x) \right) v(x) = 0, \quad \forall v \in P^k(I_j), \tag{2.1}$$

and projection $\mathcal{P}^\pm$

$$\int_{I_j} \left(\mathcal{P}^+\omega(x) - \omega(x) \right) v(x) = 0, \quad \forall v \in P^{k-1}(I_j),$$

$$\mathcal{P}^+\omega\left(x_{j-\frac{1}{2}}^+\right) = \omega(x_{j-\frac{1}{2}}), \tag{2.2}$$

and

$$\int_{I_j} \big(\mathcal{P}^- \omega(x) - \omega(x)\big) v(x) = 0, \quad \forall v \in P^{k-1}(I_j),$$

$$\mathcal{P}^- \omega\big(x^-_{j+\frac{1}{2}}\big) = \omega(x_{j+\frac{1}{2}}). \tag{2.3}$$

The approximation results for the projections $\mathcal{P}$ and $\mathcal{P}^\pm$ hold [2, 15, 18, 20]

$$\|\vartheta\| + h\|\vartheta\|_\infty + h^{\frac{1}{2}} \|\vartheta\|_{\tau_h} \leq Ch^{k+1}, \tag{2.4}$$

where $\vartheta = \mathcal{P}\omega - \omega$ or $\vartheta = \mathcal{P}^\pm\omega - \omega$. The positive constant C, solely depending on ω, is independent of h. Here τ_h is the union of all element interface points, and the L^2-norm on τ_h is defined by

$$\|\vartheta\|_{\tau_h} = \left[\sum_{1 \leq j \leq N} \left(\big(\vartheta^+_{j+\frac{1}{2}}\big)^2 + \big(\vartheta^-_{j+\frac{1}{2}}\big)^2 \right) \right]^{\frac{1}{2}}.$$

In the present paper the usual notation of norms in Sobolev spaces is used. Let the scalar inner product on $L^2(\Omega)$ be denoted by $(\cdot, \cdot)_\Omega$, and the associated norm by $\|\cdot\|_\Omega$.

2.2 A quantity related to the numerical flux

$\widehat{g}(w^-, w^+)$ is a given monotone numerical flux which is related to the discontinuous Galerkin discretization in space. It depends on the two values of the function g at the point $x_{j+\frac{1}{2}}$, that is, $w^\pm_{j+\frac{1}{2}} = w(x^\pm_{j+\frac{1}{2}})$, and satisfies:

  (i) locally Lipschitz continuous;
 (ii) consistent with the flux $g(w)$, that is, $\widehat{g}(w, w) = g(w)$;
(iii) a nondecreasing function of its first argument and a nonincreasing function of its second argument.

3 Fully discrete LDG scheme

Set $\Delta t = T/M$ to be the time mesh size, M is a positive integer, $t_j = j\Delta t, j = 0, 1, \ldots, M$, is a mesh point. First we estimate the time-fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ at t_n as follows [6, 17]:

$$\frac{\partial^\alpha u(x, t_n)}{\partial t^\alpha} = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} s_i \frac{u(x, t_{n-i}) - u(x, t_{n-i-1})}{\Delta t} + \gamma^n(x), \tag{3.1}$$

where $s_i = (i+1)^{1-\alpha} - i^{1-\alpha}$, $\gamma^n(x)$ is the truncation error

$$\gamma^n(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t_n - s)^\alpha} - \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n} \frac{\partial u(x, t_{i-\frac{1}{2}})}{\partial t} \int_{t_{i-1}}^{t_i} \frac{ds}{(t_n - s)^\alpha}$$

$$+ \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} s_i \left(\frac{\partial}{\partial t} u(x, t_{n-i-\frac{1}{2}}) - \frac{u(x, t_{n-i}) - u(x, t_{n-i-1})}{\Delta t} \right). \tag{3.2}$$

By some analysis, we know [16]

$$\|\gamma^n(x)\| \leq C(\Delta t)^{2-\alpha}, \tag{3.3}$$

where C is a constant depending on u, T, α.

Rewrite Eq. (1.1) as a first-order system:

$$p = u_x, \qquad \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + g(u)_x - \lambda_1 p_x = f, \tag{3.4}$$

where $g(u) = \frac{1}{2}u^2$.

Let $u_h^n, p_h^n \in V_h^k$ be the approximation of $u(\cdot, t_n), p(\cdot, t_n)$, respectively, $f^n(x) = f(x, t_n)$. Then a fully discrete local discontinuous Galerkin method is given: find $u_h^n, p_h^n \in V_h^k$ such that, for all test functions $v, w \in V_h^k$,

$$\int_\Omega u_h^n v \, dx - \chi \left(\int_\Omega g(u_h^n) v_x \, dx - \sum_{j=1}^N \left((\widehat{g}v^-)_{j+\frac{1}{2}} - (\widehat{g}v^+)_{j-\frac{1}{2}} \right) \right)$$

$$+ \chi \lambda_1 \left(\int_\Omega p_h^n v_x \, dx - \sum_{j=1}^N \left((\widehat{p_h^n}v^-)_{j+\frac{1}{2}} - (\widehat{p_h^n}v^+)_{j-\frac{1}{2}} \right) \right)$$

$$= \sum_{i=1}^{n-1} (s_{i-1} - s_i) \int_\Omega u_h^{n-i} v \, dx + s_{n-1} \int_\Omega u_h^0 v \, dx + \chi \int_\Omega f^n v \, dx, \tag{3.5}$$

$$\int_\Omega p_h^n w \, dx + \int_\Omega u_h^n w_x \, dx - \sum_{j=1}^N \left((\widehat{u_h^n}w^-)_{j+\frac{1}{2}} - (\widehat{u_h^n}w^+)_{j-\frac{1}{2}} \right) = 0,$$

where $\chi = (\Delta t)^\alpha \Gamma(2 - \alpha)$.

The flux $\widehat{g}((u_h^n)^-, (u_h^n)^+)$ is a monotone flux as described in (2.2). In order to simplify the computation, we can choose the flux

$$\widehat{g}^{\mathrm{LF}}(w^-, w^+) = \frac{1}{2} \left(g(w^-) + g(w^+) - \lambda_0(w^+ - w^-) \right), \quad \lambda_0 = \max_w |g'(w)|,$$

$$\widehat{u_h^n} = (u_h^n)^-, \qquad \widehat{p_h^n} = (p_h^n)^+. \tag{3.6}$$

Remark that the crucial part for flux (3.6) is taking $\widehat{u_h^n}$ and $\widehat{p_h^n}$ from opposite sides.

Because the problem is nonlinear, an iterative method should be used when computing: find $u_h^{n,m}, p_h^{n,m} \in V_h^k$ such that, for all test functions $v, w \in V_h^k$,

$$\int_\Omega u_h^{n,m} v \, dx + \chi \lambda_1 \left(\int_\Omega p_h^{n,m} v_x \, dx - \sum_{j=1}^N \left((\widehat{p_h^{n,m}}v^-)_{j+\frac{1}{2}} - (\widehat{p_h^{n,m}}v^+)_{j-\frac{1}{2}} \right) \right)$$

$$= \sum_{i=1}^{n-1} (s_{i-1} - s_i) \int_\Omega u_h^{n-i} v \, dx + s_{n-1} \int_\Omega u_h^0 v \, dx + \chi \int_\Omega f^n v \, dx$$

$$+ \chi \left(\int_\Omega g(u_h^{n,m-1}) v_x \, dx - \sum_{j=1}^N \left((g \widehat{(u_h^{n,m-1})} v^-)_{j+\frac{1}{2}} - (g \widehat{(u_h^{n,m-1})} v^+)_{j-\frac{1}{2}} \right) \right), \tag{3.7}$$

$$\int_\Omega p_h^{n,m} w \, dx + \int_\Omega u_h^{n,m} w_x \, dx - \sum_{j=1}^N \left((\widehat{u_h^{n,m}}w^-)_{j+\frac{1}{2}} - (\widehat{u_h^{n,m}}w^+)_{j-\frac{1}{2}} \right) = 0,$$

where m is the iterative step.

Without loss of generality, we consider the case $f = 0$ in numerical analysis. For the stability of scheme (3.5), we have the following result.

Theorem 3.1 *Suppose that u_h^n is the solution of numerical scheme* (3.5), *if the numerical flux* (3.6) *is used, it will hold that*

$$\left\| u_h^n \right\|_\Omega^2 + \chi \lambda_1 \left\| p_h^n \right\|_\Omega^2 \leq \left\| u_h^0 \right\|_\Omega^2, \quad n = 0, 1, \dots, M. \tag{3.8}$$

Proof We prove Theorem 3.1 by mathematical induction. First, for notational convenience, we introduce the following notations:

$$
\begin{aligned}
\widetilde{G}(u_h^n) &= -\left(\int_\Omega g(u_h^n)(u_h^n)_x \, dx - \sum_{j=1}^N \left(\left(\widehat{g}(u_h^n)^- \right)_{j+\frac{1}{2}} - \left(\widehat{g}(u_h^n)^+ \right)_{j-\frac{1}{2}} \right) \right), \\
\Lambda(p_h^n, q_h^n, u_h^n) &= (p_h^n)^- (u_h^n)^- - (p_h^n)^+ (u_h^n)^- - (u_h^n)^- (p_h^n)^-, \\
\Upsilon(p_h^n, q_h^n, u_h^n) &= (p_h^n)^- (u_h^n)^- - (p_h^n)^+ (u_h^n)^+ - (p_h^n)^+ (u_h^n)^- + (p_h^n)^+ (u_h^n)^+ \\
&\quad - (u_h^n)^- (p_h^n)^- + (u_h^n)^- (p_h^n)^+.
\end{aligned}
\tag{3.9}
$$

We consider the case $n = 1$ in (3.5) and take the test functions $v = u_h^1$, $w = \chi \lambda_1 p_h^1$, then we have

$$
\begin{aligned}
&\left\| u_h^1 \right\|_\Omega^2 + \chi \lambda_1 \left\| p_h^1 \right\|_\Omega^2 + \chi \widetilde{G}(u_h^1) + \sum_{j=1}^N \chi \lambda_1 \left(\Lambda(p_h^1, q_h^1, u_h^1)_{j+\frac{1}{2}} \right. \\
&\quad \left. - \Lambda(p_h^1, q_h^1, u_h^1)_{j-\frac{1}{2}} + \Upsilon(p_h^1, q_h^1, u_h^1)_{j-\frac{1}{2}} \right) \\
&\leq \frac{1}{2} \left\| u_h^1 \right\|_\Omega^2 + \frac{1}{2} \left\| u_h^0 \right\|_\Omega^2.
\end{aligned}
\tag{3.10}
$$

Next we deal with the term $\widetilde{G}(u_h^1)$. We define

$$F(u) = \int^u g(u) \, du,$$

then

$$\int_\Omega g(u_h^1)(u_h^1)_x \, dx = \sum_{j=1}^N \left(F\left((u_h^1)_{j+\frac{1}{2}}^- \right) - F\left((u_h^1)_{j-\frac{1}{2}}^+ \right) \right).$$

Based on the monotonicity of flux $\widehat{g}$ and the mean value theorem, we have

$$\widetilde{G}(u_h^1) = \left(F'(\eta) - \widehat{g} \right) [u_h^1] \geq 0. \tag{3.11}$$

By some manual calculation, we have

$$\Upsilon(p_h^1, q_h^1, u_h^1)_{j-\frac{1}{2}} = 0, \tag{3.12}$$

that is,

$$\left\| u_h^1 \right\|_\Omega^2 + \chi \lambda_1 \left\| p_h^1 \right\|_\Omega^2 \leq \left\| u_h^0 \right\|_\Omega^2 \tag{3.13}$$

and

$$\left\| u_h^1 \right\|_\Omega \le \left\| u_h^0 \right\|_\Omega. \tag{3.14}$$

Suppose that

$$\left\| u_h^k \right\|_\Omega \le \left\| u_h^0 \right\|_\Omega, \quad k = 1, 2, \ldots, P. \tag{3.15}$$

Considering the case $n = P + 1$ in scheme (3.5) and taking the test functions $v = u_h^{P+1}$, $\omega = \chi \lambda_1 p_h^{P+1}$, we have

$$\int_\Omega \left(u_h^{P+1} \right)^2 dx + \chi \widetilde{G}\left(u_h^{P+1} \right) + \sum_{j=1}^N \chi \lambda_1 \left(\Lambda \left(p_h^{P+1}, q_h^{P+1}, u_h^{P+1} \right)_{j+\frac{1}{2}} \right.$$
$$- \Lambda \left(p_h^{P+1}, q_h^{P+1}, u_h^{P+1} \right)_{j-\frac{1}{2}} + \Upsilon \left(p_h^{P+1}, q_h^{P+1}, u_h^{P+1} \right)_{j-\frac{1}{2}} \right)$$
$$\le \left(\sum_{i=1}^P (s_{i-1} - s_i) + s_P \right) \left\| u_h^0 \right\|_\Omega \left\| u_h^{P+1} \right\|_\Omega. \tag{3.16}$$

Analogous to the proofs of inequalities (3.11) and (3.12), we know

$$\widetilde{G}\left(u_h^{P+1} \right) \ge 0, \qquad \Upsilon_{j-\frac{1}{2}} \left(p_h^{P+1}, q_h^{P+1}, u_h^{P+1} \right) = 0.$$

Then the inequality follows

$$\left\| u_h^{P+1} \right\|_\Omega^2 + \chi \lambda_1 \left\| p_h^{P+1} \right\|_\Omega^2 \le \left\| u_h^0 \right\|_\Omega^2. \tag{3.17}$$

This finishes the proof of the stability result. □

In order to simplify the analysis, the linear case $g(u) = u$ is studied.

Theorem 3.2 *Suppose that $u(x, t_n)$ is the exact solution of problem* (1.1), *which is sufficiently smooth with bounded derivatives, and u_h^n is the numerical solution of the fully discrete LDG scheme* (3.5), *it holds that*

$$\left\| u(x, t_n) - u_h^n \right\| \le C \left(h^{k+1} + (\Delta t)^{2-\alpha} \right), \tag{3.18}$$

where C is a constant depending on u, T, α.

Proof Consider the separation of numerical error in the form

$$e_u^n = u(x, t_n) - u_h^n = \xi_u^n - \eta_u^n, \qquad \xi_u^n = \mathcal{P}^- e_u^n, \qquad \eta_u^n = \mathcal{P}^- u(x, t_n) - u(x, t_n),$$
$$e_p^n = p(x, t_n) - p_h^n = \xi_p^n - \eta_p^n, \qquad \xi_p^n = \mathcal{P} e_p^n, \qquad \eta_p^n = \mathcal{P} p(x, t_n) - p(x, t_n). \tag{3.19}$$

Here η_u^n and η_p^n have been estimated by inequality (2.4). In what follows we are going to estimate ξ_u^n and ξ_p^n.

With flux (3.6), we can get the error equation

$$
\int_\Omega e_u^n v \, dx - \chi \left(\int_\Omega e_u^n v_x \, dx - \sum_{j=1}^N \left(\left((e_u^n)^- v^- \right)_{j+\frac{1}{2}} - \left((e_u^n)^- v^+ \right)_{j-\frac{1}{2}} \right) \right)
$$

$$
+ \chi \lambda_1 \left(\int_\Omega e_p^n v_x \, dx - \sum_{j=1}^N \left(\left((e_p^n)^+ v^- \right)_{j+\frac{1}{2}} - \left((e_p^n)^+ v^+ \right)_{j-\frac{1}{2}} \right) \right)
$$

$$
- \sum_{i=1}^{n-1} (s_{i-1} - s_i) \int_\Omega e_u^{n-i} v \, dx - s_{n-1} \int_\Omega e_u^0 v \, dx
$$

$$
+ \chi \int_\Omega \gamma^n(x) v \, dx + \int_\Omega e_p^n w \, dx + \int_\Omega e_u^n w_x \, dx
$$

$$
- \sum_{j=1}^N \left(\left((e_u^n)^- w^- \right)_{j+\frac{1}{2}} - \left((e_u^n)^- w^+ \right)_{j-\frac{1}{2}} \right) = 0. \tag{3.20}
$$

Using (3.19), error equation (3.20) can be written as follows:

$$
\int_\Omega \xi_u^n v \, dx - \chi \left(\int_\Omega \xi_u^n v_x \, dx - \sum_{j=1}^N \left(\left((\xi_u^n)^- v^- \right)_{j+\frac{1}{2}} - \left((\xi_u^n)^- v^+ \right)_{j-\frac{1}{2}} \right) \right)
$$

$$
+ \chi \lambda_1 \left(\int_\Omega \xi_p^n v_x \, dx - \sum_{j=1}^N \left(\left((\xi_p^n)^+ v^- \right)_{j+\frac{1}{2}} - \left((\xi_p^n)^+ v^+ \right)_{j-\frac{1}{2}} \right) \right)
$$

$$
+ \int_\Omega \xi_p^n w \, dx + \int_\Omega \xi_u^n w_x \, dx - \sum_{j=1}^N \left(\left((\xi_u^n)^- w^- \right)_{j+\frac{1}{2}} - \left((\xi_u^n)^- w^+ \right)_{j-\frac{1}{2}} \right)
$$

$$
= \sum_{i=1}^{n-1} (s_{i-1} - s_i) \int_\Omega \xi_u^{n-i} v \, dx + s_{n-1} \int_\Omega \xi_u^0 v \, dx - \chi \int_\Omega \gamma^n(x) v \, dx
$$

$$
+ \int_\Omega (\eta_u^n) v \, dx - \chi \left(\int_\Omega (\eta_u^n) v_x \, dx - \sum_{j=1}^N \left(\left((\eta_u^n)^- v^- \right)_{j+\frac{1}{2}} - \left((\eta_u^n)^- v^+ \right)_{j-\frac{1}{2}} \right) \right)
$$

$$
+ \chi \lambda_1 \left(\int_\Omega (\eta_p^n) v_x \, dx - \sum_{j=1}^N \left(\left((\eta_p^n)^+ v^- \right)_{j+\frac{1}{2}} - \left((\eta_p^n)^+ v^+ \right)_{j-\frac{1}{2}} \right) \right)
$$

$$
- \sum_{i=1}^{n-1} (s_{i-1} - s_i) \int_\Omega (\eta_u^{n-i}) v \, dx - s_{n-1} \int_\Omega (\eta_u^0) v \, dx + \int_\Omega (\eta_p^n) w \, dx + \int_\Omega (\eta_u^n) w_x \, dx
$$

$$
- \sum_{j=1}^N \left(\left((\eta_u^n)^- w^- \right)_{j+\frac{1}{2}} - \left((\eta_u^n)^- w^+ \right)_{j-\frac{1}{2}} \right). \tag{3.21}
$$

Taking the test functions $v = \xi_u^n$, $w = \chi \lambda_1 \xi_p^n$ in (3.21), using properties (2.1) and (2.3), then the following equality holds:

$$
\int_\Omega \left(\xi_u^n \right)^2 dx + \chi \lambda_1 \int_\Omega \left(\xi_p^n \right)^2 dx + \sum_{j=1}^N \frac{\chi \lambda_1}{2} \left[\xi_u^n \right]_{j-\frac{1}{2}}^2
$$

$$
= \sum_{i=1}^{n-1} (s_{i-1} - s_i) \int_\Omega \xi_u^{n-i} \xi_u^n \, dx + s_{n-1} \int_\Omega \xi_u^0 \xi_u^n \, dx - \chi \int_\Omega \gamma^n(x) \xi_u^n \, dx
$$

$$+ \int_\Omega (\eta_u^n)\xi_u^n\, dx - \chi\left(\int_\Omega (\eta_u^n)(\xi_u^n)_x\, dx - \sum_{j=1}^{N}\left(((\eta_u^n)^-(\xi_u^n)^-)_{j+\frac{1}{2}} - ((\eta_u^n)^-(\xi_u^n)^+)_{j-\frac{1}{2}}\right)\right)$$

$$+ \chi\lambda_1\left(\int_\Omega (\eta_p^n)(\xi_u^n)_x\, dx - \sum_{j=1}^{N}\left(((\eta_p^n)^+(\xi_u^n)^-)_{j+\frac{1}{2}} - ((\eta_p^n)^+(\xi_u^n)^+)_{j-\frac{1}{2}}\right)\right)$$

$$- \sum_{i=1}^{n-1}(s_{i-1}-s_i)\int_\Omega (\eta_u^{n-i})\xi_u^n\, dx - s_{n-1}\int_\Omega (\eta_u^0)\xi_u^n\, dx + \int_\Omega (\eta_p^n)(\chi\lambda_1\xi_p^n)\, dx$$

$$+ \int_\Omega (\eta_u^n)(\chi\lambda_1\xi_p^n)_x\, dx - \sum_{j=1}^{N}\left(((\eta_u^n)^-(\chi\lambda_1\xi_p^n)^-)_{j+\frac{1}{2}} - ((\eta_u^n)^-(\chi\lambda_1\xi_p^n)^+)_{j-\frac{1}{2}}\right)$$

$$= \sum_{i=1}^{n-1}(s_{i-1}-s_i)\int_\Omega \xi_u^{n-i}\xi_u^n\, dx + s_{n-1}\int_\Omega \xi_u^0\xi_u^n\, dx - \chi\int_\Omega \gamma^n(x)\xi_u^n\, dx$$

$$+ \int_\Omega (\eta_u^n)\xi_u^n\, dx + \chi\lambda_1\sum_{j=1}^{N}\left((\eta_p^n)^+[\xi_u^n]\right)_{j-\frac{1}{2}}$$

$$- \sum_{i=1}^{n-1}(s_{i-1}-s_i)\int_\Omega (\eta_u^{n-i})\xi_u^n\, dx - s_{n-1}\int_\Omega (\eta_u^0)\xi_u^n\, dx. \tag{3.22}$$

Error estimates (3.18) will be proved by mathematical induction. First, we consider the case $n=1$, that is,

$$\int_\Omega (\xi_u^1)^2\, dx + \chi\lambda_1\int_\Omega (\xi_p^1)^2\, dx + \sum_{j=1}^{N}\frac{\chi}{2}[\xi_u^1]_{j-\frac{1}{2}}^2$$

$$= \int_\Omega \xi_u^0\xi_u^1\, dx - \chi\int_\Omega \gamma^1(x)\xi_u^1\, dx + \int_\Omega (\eta_u^1)\xi_u^1\, dx$$

$$+ \chi\lambda_1\sum_{j=1}^{N}\left((\eta_p^1)^+[\xi_u^1]\right)_{j-\frac{1}{2}} - \int_\Omega (\eta_u^0)\xi_u^1\, dx. \tag{3.23}$$

We know

$$\left\|\xi_u^0\right\| \le Ch^{k+1}, \qquad ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2,$$

and based on (2.4), we have

$$\int_\Omega (\xi_u^1)^2\, dx + \chi\lambda_1\int_\Omega (\xi_p^1)^2\, dx + \sum_{j=1}^{N}\frac{\chi}{2}[\xi_u^1]_{j-\frac{1}{2}}^2$$

$$\le \left(\left\|\xi_u^0\right\| + \chi\left\|\gamma^1(x)\right\| + \left\|(\eta_u^1)\right\| + \left\|(\eta_u^0)\right\|\right)\left\|\xi_u^1\right\|$$

$$+ \frac{\chi\lambda_1}{4}\sum_{j=1}^{N}((\eta_p^1)^+)_{j-\frac{1}{2}}^2 + \varepsilon\chi\lambda_1\sum_{j=1}^{N}([\xi_u^1])_{j-\frac{1}{2}}$$

$$\le C\left(h^{k+1} + (\Delta t)^{2-\alpha}\right)^2 + \varepsilon\left\|\xi_u^1\right\|^2 + \varepsilon\chi\lambda_2\sum_{j=1}^{N}([\xi_u^1])_{j-\frac{1}{2}}. \tag{3.24}$$

Choosing small ε, we know

$$\left\| \xi_u^1 \right\|_\Omega \le C\big(h^{k+1} + (\Delta t)^{2-\alpha}\big).$$

Then suppose that

$$\left\| \xi_u^m \right\|_\Omega \le C\big(h^{k+1} + (\Delta t)^{2-\alpha}\big), \quad m = 1, 2, \ldots, K.$$

Let $n = K + 1$ in (3.22), we have

$$
\begin{aligned}
\int_\Omega \big(\xi_u^{K+1}\big)^2 \, dx &+ \chi\lambda_1 \int_\Omega \big(\xi_p^{K+1}\big)^2 \, dx + \sum_{j=1}^N \frac{\chi}{2} \big[\xi_u^{K+1}\big]_{j-\frac{1}{2}}^2 \\
&= \sum_{i=1}^K (s_{i-1} - s_i) \int_\Omega \xi_u^{K+1-i} \xi_u^{K+1} \, dx + s_K \int_\Omega \xi_u^0 \xi_u^{K+1} \, dx \\
&\quad - \chi \int_\Omega \gamma^{K+1}(x) \xi_u^{K+1} \, dx + \int_\Omega \big(\eta_u^{K+1}\big) \xi_u^{K+1} \, dx + \chi\lambda_1 \sum_{j=1}^N \big(\big(\eta_p^{K+1}\big)^+ \big[\xi_u^{K+1}\big]\big)_{j-\frac{1}{2}} \\
&\quad - \sum_{i=1}^K (s_{i-1} - s_i) \int_\Omega \big(\eta_u^{K+1-i}\big) \xi_u^{K+1} \, dx - s_K \int_\Omega \big(\eta_u^0\big) \xi_u^{K+1} \, dx \\
&\le C\big(h^{k+1} + (\Delta t)^{2-\alpha}\big)^2 + \varepsilon \big\| \xi_u^{K+1} \big\|^2 + \varepsilon\chi\lambda_1 \sum_{j=1}^N \big(\big[\xi_u^{K+1}\big]\big)_{j-\frac{1}{2}}.
\end{aligned}
\tag{3.25}
$$

We can obtain

$$\big\| \xi_u^{K+1} \big\| \le C\big(h^{k+1} + (\Delta t)^{2-\alpha}\big).$$

By using the triangle inequality and the interpolating property (2.4), we have Theorem 3.2.   □

4 Numerical examples

In this section, we present some numerical experiments for the proposed local discontinuous Galerkin method to illustrate its capability.

Example 4.1 Consider problem (1.1) when $\lambda_1 = 1$ in $\Omega = [0, 2]$, the forcing term

$$f(x, t) = \left(\frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} + \pi^2\big(t^4 + 1\big) \right) \sin(\pi x) + \pi \sin(\pi x)\cos(\pi x)\big(t^4 + 1\big)^2, \tag{4.1}$$

and the exact solution is $u(x, t) = (t^4 + 1)\sin(\pi x)$. The L^2 and L^∞ errors and the numerical orders of accuracy at time $T = 1$ for several α are contained in Tables 1–2. We can see that the convergence rates are well displayed and justify our theoretical results.

5 Conclusion

In this paper we have presented a fully discrete local discontinuous Galerkin method for solving a class of fractional Burgers-type equations. Numerical experiments show that

Table 1 Numerical accuracy of scheme (3.5) with $M = 1000$, $\alpha = 0.3$

	N	L^2-error	Order	L^∞-error	Order
P^0	5	4.743511647465447e−01	–	5.230485703724679e−01	–
	10	2.453322326385340e−01	0.95	2.745599439848157e−01	0.92
	15	1.653406665596814e−01	0.97	1.860855920209763e−01	0.95
	20	1.242678198594019e−01	0.99	1.415701279740494e−01	0.95
P^1	5	7.256517536855258e−02	–	9.635412347824566e−02	–
	10	1.873884491779974e−02	1.95	2.552822346426505e−02	1.91
	15	8.432294834666999e−03	1.97	1.166039812718010e−02	1.93
	20	4.766101927280031e−03	1.98	6.630816791277043e−03	1.96
P^2	5	1.097532348354286e−02	–	2.164416564453414e−02	–
	10	1.477316800540677e−03	2.89	2.900434572823438e−03	2.90
	15	4.535585735151620e−04	2.91	8.863974464363477e−04	2.92
	20	1.956222641594544e−04	2.92	3.793487019026590e−04	2.95

Table 2 Numerical accuracy of scheme (3.5) with $M = 1000$, $\alpha = 0.7$

	N	L^2-error	Order	L^∞-error	Order
P^0	5	4.348211568435824e−01	–	5.042185301727834e−01	–
	10	2.424520324538541e−01	0.95	2.624532134506741e−01	0.92
	15	1.352400215461305e−01	0.97	1.589644242186741e−01	0.95
	20	1.121874628534372e−01	0.99	1.289403452797642e−01	0.95
P^1	5	7.684216542820121e−02	–	9.154712784824911e−02	–
	10	1.687534622310582e−02	1.95	2.164722782426461e−02	1.91
	15	8.534804436315641e−03	1.97	1.247106712856485e−02	1.93
	20	4.521403201237941e−03	1.98	6.503791571074942e−03	1.96
P^2	5	1.164727942789012e−02	–	2.237484026942101e−02	–
	10	1.348554214954641e−03	2.89	2.899243648502348e−03	2.90
	15	4.054195123465723e−04	2.91	8.057494485268102e−04	2.92
	20	1.264756425642313e−04	2.92	3.752416915626942e−04	2.95

our scheme is very effective. The advantages of the method are its flexibility in terms of mesh and shape functions, and it can achieve a high order of convergence. In future we would study a local discontinuous Galerkin method and an alternating direction implicit/iterative scheme to solve the two- or higher-dimensional problems.

Publisher's Note

References
 1. Burgers, J.: A Mathematical Model Illustrating the Theory of Turbulence. Academic Press, New York (1948)

2. Ciarlet, P.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
3. El-Danaf, T., Hadhoud, A.: Parametric spline functions for the solution of the one time fractional Burgers' equation. Appl. Math. Model. **36**, 4557–4564 (2012)
4. Gu, X., Huang, T., Ji, C., Carpentieri, B., Alikhanov, A.: Fast iterative method with a second-order implicit difference scheme for time-space fractional convection-diffusion equation. J. Sci. Comput. **72**, 957–985 (2017)
5. Inc, M.: The approximate and exact solutions of the space- and time-fractional Burgers equation with initial conditions by variational iteration method. J. Math. Anal. Appl. **345**, 476–484 (2008)
6. Jiang, Y., Ma, J.: High-order finite element methods for time-fractional partial differential equations. J. Comput. Appl. Math. **235**, 3285–3290 (2011)
7. Li, M., Gu, X., Huang, C., Fei, M., Zhang, G.: A fast linearized conservative finite element method for the strongly coupled nonlinear fractional Schrödinger equations. J. Comput. Phys. **358**, 256–282 (2018)
8. Meerschaert, M.M., Nane, E., Vellaisamy, P.: Distributed-order fractional diffusions on bounded domains. J. Math. Anal. Appl. **379**, 216–228 (2011)
9. Morgado, M.L., Rebelo, M.: Numerical approximation of distributed order reaction-diffusion equations. J. Comput. Appl. Math. **275**, 216–227 (2015)
10. Naber, M.: Distributed order fractional sub-diffusion. Fractals **12**, 23–32 (2004)
11. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
12. Ren, Z., Wei, L., He, Y., Wang, S.: Numerical analysis of an implicit fully discrete local discontinuous Galerkin method for the fractional Zakharov–Kuznetsov equation. Math. Model. Anal. **17**, 558–570 (2012)
13. Vong, S., Lyu, P.: Unconditional convergence in maximum-norm of a second-order linearized scheme for a time-fractional Burgers-type equation. J. Sci. Comput. **76**, 1252–1273 (2018)
14. Wang, Q.: Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method. Appl. Math. Comput. **182**, 1048–1055 (2006)
15. Wei, L.L.: Analysis of a new finite difference/local discontinuous Galerkin method for the fractional diffusion-wave equation. Appl. Math. Comput. **304**, 180–189 (2017)
16. Wei, L.L.: Stability and convergence of a fully discrete local discontinuous Galerkin method for multi-term time fractional diffusion equations. Numer. Algorithms **76**, 695–707 (2017)
17. Wei, L.L.: Analysis of a new finite difference/local discontinuous Galerkin method for the fractional Cattaneo equation. Numer. Algorithms **77**, 675–690 (2018)
18. Wei, L.L., He, Y.N.: Analysis of a fully discrete local discontinuous Galerkin method for time-fractional fourth-order problems. Appl. Math. Model. **38**, 1511–1522 (2014)
19. Wei, L.L., He, Y.N., Tang, B.: Analysis of the fractional Kawahara equation using an implicit fully discrete local discontinuous Galerkin method. Numer. Methods Partial Differ. Equ. **29**, 1441–1458 (2013)
20. Zhang, Q., Shu, C.-W.: Error estimate for the third order explicit Runge–Kutta discontinuous Galerkin method for a linear hyperbolic equation with discontinuous initial solution. Numer. Math. **126**, 703–740 (2014)
21. Zhang, X., Tang, B., He, Y.: Homotopy analysis method for higher-order fractional integro-differential equations. Comput. Math. Appl. **62**, 3194–3203 (2011)