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Attractivity for Hilfer fractional stochastic evolution equations

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Abstract

This article is devoted to the study of the attractivity of solutions to a class of stochastic evolution equations involving Hilfer fractional derivative. By employing the semigroup theory, fractional calculus and the fixed point technique, we establish new alternative criteria to ensure the existence of globally attractive solutions for the Cauchy problem when the associated semigroup is compact.

Keywords: Stochastic evolution equations; Hilfer fractional derivative; Attractivity

1 Introduction

In the past three decades, fractional differential equations received much attention. The growing interest in the subject is due to its extensive applications in diverse fields such as physics, fluid mechanics, viscoelasticity, heat conduction in materials with memory, chemistry and engineering. Much of the work is devoted to the existence and uniqueness of solutions for fractional differential equations; see, for example, Kilbas et al. [10], Miller and Ross [13], Podlubny [14], Zhou [22] and [1, 5, 19, 21, 23, 24] and the references cited therein. Since Hilfer [9] proposed the generalized Riemann–Liouville fractional derivative (Hilfer fractional derivative), there has been shown some interest in studying evolution equations involving Hilfer fractional derivatives (see [2, 4, 7, 8, 18, 20]).

Recently, the stability of fractional-order systems has been discussed by several authors. Abbas et al. [2] investigated the Ulam stability for Hilfer fractional differential inclusions via the weakly Picard operators theory. Chen et al. [6] established the global attractivity for nonlinear fractional differential equations. Losada, et al. [11] studied the attractivity of solutions for a class of multi-term fractional functional differential equations. Rajivganthi, Rihan et al. [15–17] studied the stability of a fractional-order prey–predator system. Moreover, stochastic perturbation is unavoidable in nature and sometimes useful in research, for example, the existence of stochastic perturbation in the mathematical model has been found to be quite effective in preventing the explosion of the population. Hence, it is important and necessary to consider stochastic perturbation into the investigation of fractional differential equations (see [3, 12]. However, it seems that there is less literature related to the stability of Hilfer fractional stochastic evolution equations. Therefore, the attractivity of solutions of Hilfer fractional stochastic evolution equations might be a fascinating and useful problem.

In this work, we study the attractivity of solutions for the following Hilfer fractional stochastic evolution equations:

$$\begin{cases} D_{0^+}^{\mu,\beta}x(t) = Ax(t) + f(t,x(t)) + \sigma(t,x(t))\frac{d\omega(t)}{dt}, & t > 0, \\ I_{0^+}^{(1-\mu)(1-\beta)}x(0) = x_0, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\mu,\beta}$ denotes the Hilfer fractional derivative of order μ and type β which will be given in next section, $\frac{1}{2} < \mu < 1$, $0 \le \beta < 1$, A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $(t \ge 0)$ on a separable Hilbert space X with inner product $\langle\cdot\rangle$, and norm $\|\cdot\|$. Let $\{\omega(t)\}_{t\ge0}$ denote a K-valued Wiener process with a finite trace nuclear covariance operator $Q \ge 0$ defined on the filtered complete probability space $(\Omega, \mathcal{F}, P)$. The functions f, σ are given functions satisfying some appropriate assumptions. x_0 is an element of the Hilbert space $L_2^0(\Omega, X)$ which will be specified later.

The objective of this paper is to discuss the attractivity of solutions for Cauchy problem (1.1). In fact, we establish sufficient conditions for the global attractivity of mild solutions for system (1.1) in cases that semigroup associated with A is compact. The obtained results essentially reveal certain characteristics of solutions for Hilfer fractional evolution equations, in contrast to integer-order evolution equations.

The rest of this paper is organized as follows. Section 2 contains some basic notations and essential preliminary results. In Sect. 3, we obtain alternative sufficient conditions for the attractivity of fractional stochastic evolution equations. Some conclusions are presented in Sect. 4.

2 Preliminaries

In this section, we provide some basic definitions, notations, lemmas, properties of semigroup theory and fractional calculus which are used throughout this paper.

Definition 2.1 ([10]) The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\,ds, \quad t > 0, \alpha > 0,$$

provided the right side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([10]) The left-sided Riemann–Liouville fractional-order derivative of order α with the low limit 0 for a function $f : [0,+\infty) \to \mathbb{R}$ is defined as

$${}^LD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}\,ds, \quad t > 0, n-1 < \alpha \le n.$$

Definition 2.3 ([10]) The left-sided Caputo derivative of order $\alpha \in (n-1,n)$, $n \in \mathbb{Z}^+$ for a function $f : [a,+\infty) \to \mathbb{R}$ is defined by

$${}^cD_{a^+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\,ds, \quad t > a, n-1 < \alpha < n.$$

Definition 2.4 ([9]; Hilfer fractional derivative) The left-sided Hilfer fractional derivative of order $0 < \mu < 1$ and $0 \leq \beta \leq 1$ of the function $f(t)$ is defined as

$$D_{a^+}^{\mu,\beta} f(t) = \bigl(I_{a^+}^{\beta(1-\mu)} D\bigl(I_{a^+}^{(1-\mu)(1-\beta)} f\bigr)\bigr)(t), \quad \text{where } D := \frac{d}{dt}.$$

Lemma 2.1 ([9])

(i) *When* $\beta = 0$, $0 < \mu < 1$ *and* $a = 0$, *the Hilfer fractional derivative reduces to the classical Riemann–Liouville fractional derivative*:

$$D_{0^+}^{\mu,0} f(t) = \frac{d}{dt} I_{0^+}^{1-\mu} f(t) =^L D_{0^+}^{\mu} f(t).$$

(ii) *For* $\beta = 1$, $0 < \mu < 1$ *and* $a = 0$, *the Hilfer fractional derivative becomes the classical Caputo fractional derivative*:

$$D_{0^+}^{\mu,1} f(t) = I_{0^+}^{1-\mu} \frac{d}{dt} f(t) =^c D_{0^+}^{\mu} f(t).$$

Lemma 2.2 ([10]) *For* $\sigma \in (0,1]$ *and* $0 < a \leq b$, *we have* $|a^\sigma - b^\sigma| \leq (b-a)^\sigma$.

For convenience, let $\nu = \beta + \mu - \beta\mu$, by a simple calculation, we show that $0 < \nu < 1$.

Lemma 2.3 *The Cauchy problem* (1.1) *is equivalent to the integral equation*

$$\begin{aligned} x(t) &= \frac{x_0}{\Gamma(\beta(1-\mu)+\mu)} t^{\nu-1} \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \bigl[Ax(s) + f\bigl(s, x(s)\bigr)\bigr]\, ds \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \sigma\bigl(s, x(s)\bigr)\, d\omega(s), \quad t > 0. \end{aligned} \tag{2.1}$$

Proof We omit it and refer the reader to [22].

To define a mild solution of system (1.1), we use the Wright function $M_\mu(\theta)$ defined by

$$M_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-\mu n)}, \quad 0 < \mu < 1, \theta \in \mathbb{C},$$

and $\int_0^\infty \theta^\tau M_\mu(\theta)\, d\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\mu\tau)}$, for $\theta \geq 0$.

Denote by $L_2(\Omega, X)$ the collection of all strongly-measurable, square-integrable, and X-valued random variables, which is a Banach space equipped with the norm $\|x(\cdot)\|_{L_2(\Omega,X)} = (E\|x(\cdot,\omega)\|^2)^{\frac{1}{2}}$, where the expectation E is defined by $E(x) = \int_\Omega x(\omega)\, dP$. An important subspace of $L_2(\Omega, X)$ is $L_2^0(\Omega, X) = \{x \in L_2(\Omega, X), x \text{ is } \mathcal{F}_0\text{-measurable}\}$. Let $C((0,+\infty), L_2(\Omega, X))$ be the Banach space of all continuous maps from $(0,+\infty)$ into $L_2(\Omega, X)$ with $\|x\|_{C((0,+\infty),L_2(\Omega,X))} = (\sup_{t\in(0,\infty)} E|x(t)|^2)^{\frac{1}{2}} < \infty$, for each $x \in C((0,+\infty), L_2(\Omega, X))$. Let $C_0((0,\infty), L_2(\Omega, X)) = \{x \in C((0,\infty), L_2(\Omega, X)) : \lim_{t\to\infty} E|x(t)|^2 = 0\}$ be endowed with the norm $\|x\|_0 = (\sup_{t\in(0,\infty)} E|x(t)|^2)^{\frac{1}{2}} < \infty$. Obviously, $C_0((0,\infty), L_2(\Omega, X))$ is a Banach space. □

Lemma 2.4 *By a mild solution of system* (1.1) *we mean the $\mathcal{F}_t$-adapted stochastic progress $x:(0,+\infty)\to L_2(\Omega,H)$ that satisfies*

$$x(t)=S_{\mu,\beta}(t)x_0+\int_0^t T_\mu(t-s)f\bigl(s,x(s)\bigr)\,ds+\int_0^t T_\mu(t-s)\sigma\bigl(s,x(s)\bigr)\,d\omega(s),\quad t>0, \tag{2.2}$$

where $T_\mu(t)=t^{\mu-1}P_\mu(t)$, $P_\mu(t)=\int_0^\infty \mu\theta M_\mu(\theta)S(t^\mu\theta)\,d\theta$, and $S_{\mu,\beta}(t)=I_{0^+}^{\beta(1-\mu)}T_\mu(t)$.

Lemma 2.5 ([10]) *If $a>0$ and $b>0$, then*

$$\int_0^t (t-s)^{a-1}s^{b-1}\,ds=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}t^{a+b-1}. \tag{2.3}$$

To formulate some essential propositions, we introduce the assumption:

(H_0) $S(t)$ is continuous in the uniform operator topology for $t\geq 0$, and $\{S(t)\}_{t\geq 0}$ is uniformly bounded, i.e., there exists $M>1$ such that $\sup_{t\in[0,+\infty)}|S(t)|<M$.

Remark 2.1 ([7]) Under the assumption (H_0), $P_\mu(t)$ is continuous in the uniform operator topology for $t>0$.

Remark 2.2 ([22]) Under the assumption (H_0), for any fixed $t>0$, $\{T_\mu(t)\}_{t>0}$ and $\{S_{\mu,\beta}(t)\}_{t>0}$ are linear operators, and, for any $x\in X$,

$$\bigl\|T_\mu(t)x\bigr\|\leq\frac{Mt^{\mu-1}}{\Gamma(\mu)}\|x\|,\qquad \bigl\|S_{\mu,\beta}(t)x\bigr\|\leq\frac{Mt^{\nu-1}}{\Gamma(\beta(1-\mu)+\mu)}\|x\|,$$

where $\nu=\beta+\mu-\beta\mu$.

Remark 2.3 ([7]) Under the assumption (H_0), $\{T_\mu(t)\}_{t>0}$ and $\{S_{\mu,\beta}(t)\}_{t>0}$ are strongly continuous, that is, for any $x\in X$, and $0<t'<t''\leq b$, we have

$$\bigl\|T_\mu(t')x-T_\mu(t'')x\bigr\|\to 0\quad\text{and}\quad\bigl\|S_{\mu,\beta}(t')x-S_{\mu,\beta}(t'')x\bigr\|\to 0,\quad\text{as } t''\to t'.$$

We also need the following generalization of the Ascoli–Arzela theorem.

Lemma 2.6 *The set $H\subset C_0((0,\infty),X)$ is relatively compact if and only if the following conditions hold*:

(i) *for any $b>0$, the function in H is equicontinuous on $[0,b]$*;
(ii) *for any $t\in[0,\infty)$, $H(t)=\{x(t):x\in H\}$ is relatively compact in X*;
(iii) $\lim_{t\to\infty}|x(t)|=0$ *uniformly for $x\in H$.*

Theorem 2.1 ([5]) *Let S be a nonempty, closed, convex and bounded subset of the Banach space X and let $A:X\to X$ and $B:S\to X$ be two operators such that*

(a) *A is a contraction with constant $L<1$,*
(b) *B is continuous, BS resides in a compact subset of X,*
(c) $[x=Ax+By, y\in S]\Rightarrow x\in S$.

Then the operator equation $Ax+Bx=x$ has a solution in S.

3 Main results

In this section, we establish the attractivity of solutions for system (1.1).

Since $T_\mu(t) = t^{\mu-1}P_\mu(t)$, (2.2) takes the form

$$x(t) = S_{\mu,\beta}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)f\bigl(s,x(s)\bigr)\,ds + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)\sigma\bigl(s,x(s)\bigr)\,d\omega(s), \quad t>0. \tag{3.1}$$

In order to establish the attractivity of solutions for system (1.1), we need the following assumptions:

(H_1) $S(t)$ is a compact operator for each $t>0$.

(H_2) $f(t,x(t))$ and $\sigma(t,x(t))$ are Lebesgue measurable with respect to t on $(0,\infty)$, and $f(t,x(t))$, $\sigma(t,x(t))$ are continuous with respect to x on $C((0,\infty),L_2(\Omega,X))$.

(H_3) There exists a constant $\mu_1 \in (\frac{1}{2},\mu)$ such that

$$\int_0^h E\bigl|f\bigl(t,x(t)\bigr)\bigr|^{\frac{2}{2\mu_1-1}}\,dt<\infty, \qquad \int_0^h E\bigl|\sigma\bigl(t,x(t)\bigr)\bigr|^{\frac{2}{2\mu_1-1}}\,dt<\infty, \quad \text{for all } h<\infty.$$

(H_4) $E|f(t,x)|^2 \le Lt^{-\eta}$, $E|\sigma(t,x)|^2 \le Lt^{-\eta}$ for $t\in(0,\infty)$, $L\ge 0$ and $x\in C((0,\infty), L_2(\Omega,X))$, where $2\mu-1<\eta<\min\{1,2\mu+1-2\nu\}$.

By a simple calculation, we can infer that $c\in(-1,0)$ when $c=\frac{\mu-1}{1-\mu_1}$.

For any $x\in C((0,\infty),L_2(\Omega,X))$, we define an operator F as

$$\begin{aligned}(Fx)(t) &= (F_1x)(t)+(F_2x)(t)\\ &= S_{\mu,\beta}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)f\bigl(s,x(s)\bigr)\,ds\\ &\quad + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)\sigma\bigl(s,x(s)\bigr)\,d\omega(s), \quad t>0,\end{aligned} \tag{3.2}$$

where

$$(F_1x)(t) = S_{\mu,\beta}(t)x_0, \quad t>0, \tag{3.3}$$

and

$$(F_2x)(t) = \int_0^t (t-s)^{\mu-1}P_\mu(t-s)f\bigl(s,x(s)\bigr)\,ds + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)\sigma\bigl(s,x(s)\bigr)\,d\omega(s), \quad t>0. \tag{3.4}$$

Observe that x is a mild solution of (1.1) if and only there exists a fixed point x^* such that the operator equation $x^* = Fx^* = F_1x^* + F_2x^*$ holds for $t>0$.

Lemma 3.1 *Assume that* (H_0) *and* (H_2)–(H_4) *hold. Then the operator* F_2 *is continuous and for any* $b>0$, F_2S_1 *is equicontinuous on* $[0,b)$, *where* $D_1 = \{y(t) \mid y(t)\in C((0,\infty),L_2(\Omega,X)), E|y(t)|^2 \le t^{-\delta}$ *for* $t\ge T_1\}$, $\delta=\frac{1}{2}(1+\eta-2\mu)$ *and* T_1 *satisfies the in-*

equality

$$3\frac{M^2|x_0|^2}{\Gamma^2(\beta(1-\mu)+\mu)}T_1^{\nu-1}+3\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)T_1^{-\frac{1}{2}(1+\eta-2\mu)}\leq 1. \quad (3.5)$$

Proof Obviously, D_1 is a nonempty bounded closed and convex subset of $C_0((0,\infty), L_2(\Omega,X))$. The proof will be completed in three steps.

Step 1. F_2 maps D_1 into itself for $t\geq T_1$.

For any $t>0$, from (H_4) and Lemma 2.5, we obtain

$$\begin{aligned}
E\|F_2x\|^2 &\leq 2E\left\|\int_0^t(t-s)^{\mu-1}P_\mu(t-s)f\bigl(s,x(s)\bigr)\,ds\right\|^2\\
&\quad+2E\left\|\int_0^t(t-s)^{\mu-1}P_\mu(t-s)\sigma\bigl(s,x(s)\bigr)\,d\omega(s)\right\|^2\\
&\leq 2\left(\frac{M}{\Gamma(\mu)}\right)^2\int_0^t(t-s)^{2\mu-2}E\bigl\|f\bigl(s,x(s)\bigr)\bigr\|^2\,ds\\
&\quad+2TrQ\left(\frac{M}{\Gamma(\mu)}\right)^2\int_0^t(t-s)^{2\mu-2}E\bigl\|\sigma\bigl(s,x(s)\bigr)\bigr\|^2\,ds\\
&\leq 2\left(\frac{ML}{\Gamma(\mu)}\right)^2(1+TrQ)\int_0^t(t-s)^{2\mu-2}s^{-\eta}\,ds\\
&\leq 2\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{-(1+\eta-2\mu)}.
\end{aligned}$$

Note that the above inequity is restricted to the integrability of $s^{-\eta}$, which is indeed true for $\eta<1$.

For $t>T_1$, taking into account the fact that $\eta>2\mu-1$ and inequality (3.5), we infer that

$$\begin{aligned}
&2\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{-\frac{1}{2}(1+\eta-2\mu)}\\
&\quad\leq 2\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)T_1^{-\frac{1}{2}(1+\eta-2\mu)}\leq 1.
\end{aligned}$$

Then, for $t\geq T_1$, we have

$$\begin{aligned}
E\bigl|F_2y(t)\bigr|^2 &\leq \left[2\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{-\frac{1}{2}(1+\eta-2\mu)}\right]t^{-\frac{1}{2}(1+\eta-2\mu)}\\
&\leq t^{-\frac{1}{2}(1+\eta-2\mu)}.
\end{aligned}$$

Thus $F_2D_1\subset D_1$ for $t\geq T_1$.

Step 2. F_2 is continuous.

For any $y_m(t),y(t)\in D_1$, $m=1,2,\ldots$ with $\lim_{m\to\infty}E|y_m(t)-y(t)|^2=0$, we get $\lim_{m\to\infty}E|y_m(t)|^2=E|y(t)|^2$ and $\lim_{m\to\infty}E|f(t,y_m(t))|^2=E|f(t,y(t))|^2$ for $t\geq T_1$. By direct computation, we infer that $(t-s)^{2\mu-2}\in L^{\frac{1}{2-2\mu_1}}$ for $t>0$ and $\mu_1\in(\frac{1}{2},\mu)$.

Applying Hölder's inequality, we have

$$\begin{aligned}
&E\big|F_2y_m(t)-F_2y(t)\big|^2\\
&\quad\leq 2\int_0^t(t-s)^{2\mu-2}\big|P_\mu(t-s)\big|^2E\big|f\big(s,y_m(s)\big)-f\big(s,y(s)\big)\big|^2\,ds\\
&\qquad+2TrQ\int_0^t(t-s)^{2\mu-2}\big|P_\mu(t-s)\big|^2E\big|\sigma\big(s,y_m(s)\big)-\sigma\big(s,y(s)\big)\big|^2\,ds\\
&\quad\leq 2\bigg(\frac{M}{\Gamma(\mu)}\bigg)^2\int_0^t(t-s)^{2\mu-2}\big(E\big|f\big(s,y_m(s)\big)-f\big(s,y(s)\big)\big|^2\big)\,ds\\
&\qquad+2TrQ\bigg(\frac{M}{\Gamma(\mu)}\bigg)^2\int_0^t(t-s)^{2\mu-2}\big(E\big|\sigma\big(s,y_m(s)\big)-\sigma\big(s,y(s)\big)\big|^2\big)\,ds\\
&\quad\leq 2\bigg(\frac{M}{\Gamma(\mu)}\bigg)^2\frac{t^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}\bigg(\int_0^t\big|f\big(s,y_m(s)\big)-f\big(s,y(s)\big)\big|^{\frac{2}{2\mu_1-1}}\,ds\bigg)^{2\mu_1-1}\\
&\qquad+2TrQ\bigg(\frac{M}{\Gamma(\mu)}\bigg)^2\frac{t^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}\bigg(\int_0^t\big|\sigma\big(s,y_m(s)\big)-\sigma\big(s,y(s)\big)\big|^{\frac{2}{2\mu_1-1}}\,ds\bigg)^{2\mu_1-1}\\
&\quad\leq 2\bigg(\frac{M}{\Gamma(\mu)}\bigg)^2\frac{t^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}T^{2\mu_1-1}\big\|f\big(\cdot,y_m(\cdot)\big)-f\big(\cdot,y(\cdot)\big)\big\|_{L^{\frac{1}{2\mu_1-1}}}\\
&\qquad+2TrQ\bigg(\frac{M}{\Gamma(\mu)}\bigg)^2\frac{t^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}T^{2\mu_1-1}\big\|\sigma\big(\cdot,y_m(\cdot)\big)-\sigma\big(\cdot,y(\cdot)\big)\big\|_{L^{\frac{1}{2\mu_1-1}}}\to 0\\
&\qquad\text{as } m\to\infty,
\end{aligned}$$

which implies that F_2 is continuous.

Step 3. F_2D_1 is equicontinuous.

Let $\varepsilon>0$ be given. Since $\lim_{t\to\infty}t^{-(1+\eta-2\mu)}=0$, there exists a $T'>T_1$ such that for $t>T'$

$$\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)(1+TrQ)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}t^{-(1+\eta-2\mu)}<\frac{\varepsilon}{8}.$$

Let $T_1\leq t'<t''\leq T'$, then

$$\begin{aligned}
&E\big|F_2y(t'')-F_2y(t')\big|^2\\
&\quad=2E\bigg|\int_0^{t''}\big(t''-s\big)^{\mu-1}P_\mu\big(t''-s\big)f\big(s,y(s)\big)\,ds-\int_0^{t'}\big(t'-s\big)^{\mu-1}P_\mu\big(t'-s\big)f\big(s,y(s)\big)\,ds\bigg|^2\\
&\qquad+2E\bigg|\int_0^{t''}\big(t''-s\big)^{\mu-1}P_\mu\big(t''-s\big)\sigma\big(s,y(s)\big)\,d\omega(s)\\
&\qquad-\int_0^{t'}\big(t'-s\big)^{\mu-1}P_\mu\big(t'-s\big)\sigma\big(s,y(s)\big)\,d\omega(s)\bigg|^2\\
&\quad\leq 4\int_0^{t'}\big(t'-s\big)^{2\mu-2}\big|P_\mu\big(t'-s\big)\big|^2E\big|f\big(s,y(s)\big)\big|^2\,ds\\
&\qquad+4\int_0^{t''}\big(t''-s\big)^{2\mu-2}\big|P_\mu\big(t''-s\big)\big|^2E\big|f\big(s,y(s)\big)\big|^2\,ds\\
&\qquad+4TrQ\int_0^{t'}\big(t'-s\big)^{2\mu-2}\big|P_\mu\big(t'-s\big)\big|^2E\big|\sigma\big(s,y(s)\big)\big|^2\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ 4TrQ \int_0^{t''} (t''-s)^{2\mu-2} \left| P_\mu(t''-s) \right|^2 E\left| \sigma(s,y(s)) \right|^2 ds \\
&\le 4\frac{M^2L^2}{\Gamma^2(\mu)}(1+TrQ)\left[\int_0^{t''} (t''-s)^{2\mu-2} s^{-\eta}\, ds + \int_0^{t'} (t'-s)^{2\mu-2} s^{-\eta}\, ds \right] \\
&\le 4\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)\left(t''^{-(1+\eta-2\mu)} + t'^{-(1+\eta-2\mu)}\right) \le \varepsilon.
\end{aligned}
$$

Moreover, for $0 < t' < t'' \le T_1$, by using Hölder's inequality and Remark 2.3, we have

$$
\begin{aligned}
&E\left| F_2y(t'') - F_2y(t') \right|^2 \\
&\le 2E\left| \int_0^{t''} (t''-s)^{\mu-1} P_\mu(t''-s) f(s,y(s))\, ds - \int_0^{t'} (t'-s)^{\mu-1} P_\mu(t'-s) f(s,y(s))\, ds \right|^2 \\
&\quad + 2E\left| \int_0^{t''} (t''-s)^{\mu-1} P_\mu(t''-s)\sigma(s,y(s))\, d\omega(s) \right. \\
&\quad \left. - \int_0^{t'} (t'-s)^{\mu-1} P_\mu(t'-s)\sigma(s,y(s))\, d\omega(s) \right|^2 \\
&\le 6E\left| \int_{t'}^{t''} (t''-s)^{\mu-1} P_\mu(t''-s) f(s,y(s))\, ds \right|^2 \\
&\quad + 6E\left| \int_{t'}^{t''} (t''-s)^{\gamma-1} P_\mu(t''-s)\sigma(s,y(s))\, d\omega(s) \right|^2 \\
&\quad + 6E\left| \int_0^{t'} \left[(t''-s)^{\mu-1} - (t'-s)^{\mu-1} \right] P_\mu(t''-s) f(s,y(s))\, ds \right|^2 \\
&\quad + 6E\left| \int_0^{t'} \left[(t''-s)^{\mu-1} - (t'-s)^{\mu-1} \right] P_\mu(t''-s)\sigma(s,y(s))\, d\omega(s) \right|^2 \\
&\quad + 6E\left| \int_0^{t'} (t'-s)^{\mu-1} \left[P_\mu(t''-s) - P_\mu(t'-s) \right] f(s,y(s))\, ds \right|^2 \\
&\quad + 6E\left| \int_0^{t'} (t'-s)^{\mu-1} \left[P_\mu(t''-s) - P_\mu(t'-s) \right] \sigma(s,y(s))\, d\omega(s) \right|^2 \\
&\le 6\int_{t'}^{t''} (t''-s)^{2\mu-2} \left| P_\mu(t''-s) \right|^2 E\left| f(s,y(s)) \right|^2 ds \\
&\quad + 6TrQ\int_{t'}^{t''} (t''-s)^{2\mu-2} \left| P_\mu(t''-s) \right|^2 E\left| \sigma(s,y(s)) \right|^2 ds \\
&\quad + 6\int_0^{t'} \left| t''-s)^{\mu-1} - (t'-s)^{\mu-1} \right|^2 \left| P_\mu(t''-s) \right|^2 E\left| f(s,y(s)) \right|^2 ds \\
&\quad + 6TrQ\int_0^{t'} \left| t''-s)^{\mu-1} - (t'-s)^{\mu-1} \right|^2 \left| P_\mu(t''-s) \right|^2 E\left| \sigma(s,y(s)) \right|^2 ds \\
&\quad + 6\int_0^{t'} (t''-s)^{2\mu-2} \left| P_\mu(t'-s) - P_\mu(t''-s) \right|^2 E\left| f(s,y(s)) \right|^2 ds \\
&\quad + 6TrQ\int_0^{t'} (t''-s)^{2\mu-2} \left| P_\mu(t'-s) - P_\mu(t''-s) \right|^2 E\left| \sigma(s,y(s)) \right|^2 ds \\
&= \sum_{i=1}^{6} 6I_i,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_{t'}^{t''} (t''-s)^{2\mu-2}\left|P_\mu(t''-s)\right|^2 E\left|f(s,y(s))\right|^2 ds \\
&\le \left(\frac{M}{\Gamma(\mu)}\right)^2 \left(\int_{t'}^{t''} (t''-s)^{\frac{2\mu-2}{2-2\mu_1}}\right)^{2-2\mu_1} \left(\int_{t'}^{t''} \left|f(s,y(s))\right|^{\frac{1}{2\mu_1-1}} ds\right)^{2\mu_1-1}, \\
I_2 &= TrQ\int_{t'}^{t''} (t''-s)^{2\mu-2}\left|P_\mu(t''-s)\right|^2 E\left|\sigma(s,y(s))\right|^2 ds \\
&\le \left(\frac{M}{\Gamma(\mu)}\right)^2 TrQ\left(\int_{t'}^{t''} (t''-s)^{\frac{2\mu-2}{2-2\mu_1}}\right)^{2-2\mu_1} \left(\int_{t'}^{t''} \left|\sigma(s,y(s))\right|^{\frac{1}{2\mu_1-1}} ds\right)^{2\mu_1-1}, \\
I_3 &= \int_0^{t'} \left|(t''-s)^{\mu-1} - (t'-s)^{\mu-1}\right|^2 \left|P_\mu(t''-s)\right|^2 E\left|f(s,y(s))\right|^2 ds \\
&\le \left(\frac{M}{\Gamma(\mu)}\right)^2 \frac{1}{(1+c)(2-2\mu_1)}\left[t'^{(1+c)(2-2\mu_1)} - t''^{(1+c)(2-2\mu_1)} + (t''-t')^{(1+c)(2-2\mu_1)}\right] \\
&\quad \times \left(\int_0^{t'} \left|f(s,y(s))\right|^{\frac{1}{2\mu_1-1}} ds\right)^{2\mu_1-1}, \\
I_4 &= TrQ\int_0^{t'} \left|t''-s)^{\mu-1} - (t'-s)^{\mu-1}\right|^2 \left|P_\mu(t''-s)\right|^2 E\left|\sigma(s,y(s))\right|^2 ds \\
&\le \left(\frac{M}{\Gamma(\mu)}\right)^2 TrQ\frac{1}{(1+c)(2-2\mu_1)}\left[t'^{(1+c)(2-2\mu_1)} - t''^{(1+c)(2-2\mu_1)} + (t''-t')^{(1+c)(2-2\mu_1)}\right] \\
&\quad \times \left(\int_0^{t'} E\left|\sigma(s,y(s))\right|^{\frac{1}{2\mu_1-1}} ds\right)^{2\mu_1-1}, \\
I_5 &= \int_0^{t'} (t''-s)^{2\mu-2}\left|P_\mu(t'-s) - P_\mu(t''-s)\right|^2 E\left|f(s,y(s))\right|^2 ds \\
&\le \left|P_\mu(t''-s) - P_\mu(t'-s)\right|^2 \int_0^{t'} (t''-s)^{2\mu-2} E\left\|f(s,y(s))\right\|^2 ds, \\
I_6 &= TrQ\int_0^{t'} (t''-s)^{2\mu-2}\left|P_\mu(t'-s) - P_\mu(t''-s)\right|^2 E\left|\sigma(s,y(s))\right|^2 ds \\
&\le TrQ\left|P_\mu(t''-s) - P_\mu(t'-s)\right|^2 \int_0^{t'} (t''-s)^{2\mu-2} E\left\|\sigma(s,y(s))\right\|^2 ds.
\end{aligned}
$$

By a standard calculation, we have

$$
I_1 \le \frac{M^2}{\Gamma^2(\mu)} \frac{(t''-t')^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}} \left(\int_{t'}^{t''} E\left|f(s,y(s))\right|^{\frac{1}{2\mu_1-1}} ds\right)^{2\mu_1-1} \to 0, \quad \text{as } t''-t' \to 0.
$$

In a similar manner, one can show that $I_2 \to 0$ as $t''-t' \to 0$. Moveover, from Lemma 2.2 and the fact that $1+c \in (0,1)$, it follows that $I_3, I_4 \to 0$ as $t''-t' \to 0$. As for I_5, I_6, by the strong continuity of $S(t)$, $(t>0)$, we can infer that $I_5, I_6 \to 0$ as $t''-t' \to 0$.

On the other hand, if $T_1 \le t' < T < t''$ and $t''-t' \to 0$, then $t'' \to T$ and $t' \to T$. Thus one can easily get

$$
\begin{aligned}
E\left|F_2y(t'') - F_2y(t')\right|^2 &\le 2E\left|F_2y(t'') - F_2y(T)\right|^2 + 2E\left|F_2y(T) - F_2y(t')\right|^2 \\
&\to 0, \quad \text{as } t''-t' \to 0.
\end{aligned}
$$

□

Lemma 3.2 *Suppose the assumptions* (H_0), (H_2) *and* (H_4) *are satisfied, then a solution of* (1.1) *is in* D_1 *for* $t \geq T_1$.

Proof It is clear that $x(t)$ is a fixed point of $L_2(\Omega, X)$ if and only if it is a solution of system (1.1). To prove this assertion, we only need to show that, for each fixed $y \in D_1$ and for $\forall x \in C((0,\infty), L_2(\Omega, X))$, $x = F_1x + F_2y \Rightarrow x \in D_1$ holds. In fact, if $x = F_1x + F_2y$, according to (H_4), we can obtain

$$\begin{aligned}
E\left|x(t)\right|^2 &\leq E\left|F_1x(t) + F_2y(t)\right|^2 \\
&\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 \\
&\quad + 3\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t-s)\right|^2 E\left|f\left(s,y(s)\right)\right|^2\,ds \\
&\quad + 3TrQ\int_0^t (t-s)^{\mu-1}\left|P_\mu(t-s)\right|^2 E\left|\sigma\left(s,y(s)\right)\right|^2\,ds \\
&\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 \\
&\quad + 3\left(\frac{ML}{\Gamma(\mu)}\right)^2(1+TrQ)\int_0^t (t-s)^{2\mu-2}s^{-\eta}\,ds \\
&= 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 \\
&\quad + 3(1+TrQ)\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}t^{-(1+\eta-2\mu)}.
\end{aligned}$$

For $t \geq T_1$, from inequality (3.5) and $0 < 2\mu - 1 < \eta$, it follows that

$$\begin{aligned}
&3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{\nu-1}E|x_0|^2 + 3(1+TrQ)\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}t^{-\frac{1}{2}(1+\eta-2\mu)} \\
&\quad\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 T_1^{\nu-1}E|x_0|^2 \\
&\qquad + 3(1+TrQ)\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}T_1^{-\frac{1}{2}(1+\eta-2\mu)} \\
&\quad\leq 1.
\end{aligned}$$

In addition, since $\eta < \min\{1, 2\mu + 1 - 2\nu\}$ and $\delta = \frac{1}{2}(1+\eta-2\mu)$, we can deduce $\nu - 1 - \delta - (2\nu - 2) > 0$, then it follows that

$$\begin{aligned}
E\left|x(t)\right|^2 &\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 \\
&\quad + 3\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{-(1+\eta-2\mu)} \\
&\leq 3\left[\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{\nu-1}E|x_0|^2 \right.\\
&\quad \left. + \frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{-\frac{1}{2}(1+\eta-2\mu)}\right]t^{-\delta}
\end{aligned}$$

$$\leq t^{-\delta},$$

which implies that $x(t) \in D_1$ for $t \geq T_1$. □

Lemma 3.3 *Assume that* (H_1)–(H_3) *hold. Then the set* $V_2(t) = \{(F_2x)(t) : x \in D_1\}$ *is relatively compact in* $L_2(\Omega, X)$ *for any* $t \in (0,\infty)$.

Proof Define $F_2^{\lambda,\delta}x$ as

$$\begin{aligned}
&\bigl(F_2^{\lambda,\delta}x\bigr)(t)\\
&\quad= S\bigl(\lambda^\mu\theta\bigr)\int_0^{t-\lambda}\int_\delta^\infty \mu\theta M_\gamma(\theta)S\bigl((t-s)^\gamma\theta-\lambda^\gamma\theta\bigr)(t-s)^{\mu-1}f\bigl(s,x(s)\bigr)\,d\theta\,ds\\
&\qquad+S\bigl(\lambda^\mu\theta\bigr)\int_0^{t-\lambda}\int_\delta^\infty \mu\theta M_\mu(\theta)S\bigl((t-s)^\mu\theta-\lambda^\mu\theta\bigr)(t-s)^{\mu-1}\sigma\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s).
\end{aligned}$$

From the compactness of $S(\lambda^\mu\theta)$, $(\lambda^\mu\theta > 0)$, it is easy to see that, for $\forall\lambda \in (0,t)$ and $\forall\delta > 0$, the set $V^{\lambda,\delta}(t) = \{(F_2^{\lambda,\delta}x)(t), x \in D_1\}$ is relatively compact in $L_2(\Omega, X)$. Then we obtain

$$\begin{aligned}
&E\bigl|F_2x(t)-F_2^{\lambda,\delta}x(t)\bigr|^2\\
&\quad\leq E\biggl|\int_0^t\int_0^\delta \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds\\
&\qquad+\int_0^t\int_\delta^\infty \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds\\
&\qquad-\int_0^{t-\lambda}\int_\delta^\infty \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds\\
&\qquad+\int_0^t\int_0^\delta \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\sigma\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s)\\
&\qquad+\int_0^t\int_\delta^\infty \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\sigma\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s)\\
&\qquad-\int_0^{t-\lambda}\int_\delta^\infty \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\sigma\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s)\biggr|^2\\
&\quad\leq 4E\biggl|\int_0^t\int_0^\delta \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds\biggr|^2\\
&\qquad+4E\biggl|\int_{t-\lambda}^t\int_\delta^\infty \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds\biggr|^2\\
&\qquad+4TrQE\biggl|\int_0^t\int_0^\delta \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\sigma\bigl(s,x(s)\bigr)\,d\theta\,ds\biggr|^2\\
&\qquad+4TrQE\biggl|\int_{t-\lambda}^t\int_\delta^\infty \mu\theta(t-s)^{\mu-1}M_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\sigma\bigl(s,x(s)\bigr)\,d\theta\,ds\biggr|^2\\
&\quad\leq 4\mu^2M^2\frac{t^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}\biggl(\int_0^t E\bigl|f\bigl(s,y(s)\bigr)\bigr|^{\frac{1}{2\mu_1-1}}\,ds\biggr)^{2\mu_1-1}\biggl|\int_0^\delta\theta M_\mu(\theta)\,d\theta\biggr|^2\\
&\qquad+4\mu^2M^2TrQ\frac{t^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}\biggl(\int_0^t E\bigl|\sigma\bigl(s,y(s)\bigr)\bigr|^{\frac{1}{2\mu_1-1}}\,ds\biggr)^{2\mu_1-1}\biggl|\int_0^\delta\theta M_\mu(\theta)\,d\theta\biggr|^2
\end{aligned}$$

$$
\begin{aligned}
&+4\mu^2M^2\frac{\lambda^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}\left(\int_{t-\lambda}^{t}E\big|f\big(s,y(s)\big)\big|^{\frac{1}{2\mu_1-1}}\,ds\right)^{2\mu_1-1}\left|\frac{1}{\Gamma^2(1+\mu)}\right.\\
&+4\mu^2M^2TrQ\frac{\lambda^{(1+c)(2-2\mu_1)}}{(1+c)^{2-2\mu_1}}\left(\int_{t-\lambda}^{t}E\big|\sigma\big(s,y(s)\big)\big|^{\frac{1}{2\mu_1-1}}\,ds\right)^{2\mu_1-1}\left|\frac{1}{\Gamma^2(1+\mu)}\right.\\
\to{}& 0,\quad \text{as } \lambda\to 0,\delta\to 0.
\end{aligned}
$$

Therefore, there do exist relatively compact sets arbitrarily close to the set $\{V(t),t>0\}$. Thus we deduce that the set $\{V(t),t>0\}$ is also relatively compact in $L_2(\Omega,X)$.

Hence, $\{F_2x,x\in D_1\}$ is a relatively compact set in $L_2(\Omega,X)$ by the Arzola–Ascoli theorem. As a consequence of Lemmas 3.1–3.3 and Theorem 2.1, there exists a $y\in D_1$ such that $y=F_1y+F_2y$, that is, H has a fixed point in D_1 which is a solution of system (1.1) for $t\geq T_1$. □

Now, we are well prepared to present our first attractivity result for system (1.1).

Theorem 3.1 *Suppose that assumptions* (H_1)–(H_4) *hold. Then the zero solution of system* (1.1) *is globally attractive.*

Proof From Lemmas 3.1–3.3 and the properties of D_1, for $t\geq T_1$, we know that the solution of (1.1) does exist which is still in D_1. Moreover, all functions in D_1 tend to 0 as $t\to\infty$. Therefore the solution of system (1.1) tends to zero as $t\to\infty$. The proof is completed. □

To give our second attractivity result, we require the following hypothesis:

(H_5) $E|f(t,x)|^2\leq Lt^{-\eta_1}E|x(t)|^2$, $E|\sigma(t,x)|^2\leq Lt^{-\eta_1}E|x(t)|^2$, for $\forall t\in(0,\infty)$ and each $x\in C((0,\infty),L_2(\Omega,X))$, where $L\geq 0$, $2\mu-1<\eta_1<\mu$.

Theorem 3.2 *Suppose that assumptions* (H_1)–(H_3) *and* (H_5) *hold, then the zero solution of* (1.1) *is globally attractive.*

Proof Set $D_2=\{y(t)\mid y(t)\in C((0,\infty),L_2(\Omega,X)),E|y(t)|^2\leq t^{-\delta_1}\text{ for }t\geq T_2\}$, where $\delta_1=1-\mu$. We choose constant T_2 large enough such that the following inequality holds:

$$
3\frac{M^2E|x_0|^2}{\Gamma^2(\beta(1-\mu)+\mu)}T_2^{\nu-1}+3\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)T_2^{-(1+\eta_1-2\mu)}\leq 1. \tag{3.6}
$$

One can easily see that operator F_1 is contraction. In addition, for each fixed $y\in D_2$ and for $\forall x\in L_2(\Omega,X)$, $x=F_1x+F_2y\Rightarrow x\in D_2$ holds. If $x=F_1x+F_2y$, then the application of (H_5) yields

$$
\begin{aligned}
E\big|x(t)\big|^2&\leq E\big|F_1x(t)+F_2y(t)\big|^2\\
&\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2t^{2\nu-2}E|x_0|^2+3\int_0^t(t-s)^{2\mu-2}\big|P_\mu(s)\big|^2E\big|f\big(s,y(s)\big)\big|^2\,ds\\
&\quad+3TrQ\int_0^t(t-s)^{2\mu-2}\big|P_\mu(s)\big|^2E\big|\sigma\big(s,y(s)\big)\big|^2\,ds\\
&\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2t^{2\nu-2}E|x_0|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 3\left(\frac{ML}{\Gamma(\mu)}\right)^2 (1 + TrQ) \int_0^t (t-s)^{2\mu-2} s^{-\eta_1} E\bigl|y(s)\bigr|^2\, ds \\
&\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2} E|x_0|^2 \\
&\quad + 3\left(\frac{ML}{\Gamma(\mu)}\right)^2 (1 + TrQ) \int_0^t (t-s)^{2\mu-2} s^{-\eta_1-\delta_1}\, ds \\
&\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2} E|x_0|^2 \\
&\quad + 3\frac{M^2L^2\Gamma(1-\eta_1-\delta_1)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta_1-\delta_1)} (1+TrQ) t^{-(1+\eta_1+\delta_1-2\mu)}.
\end{aligned}
\tag{3.7}
$$

Since $\eta_1 < \mu$ and $\delta_1 = 1 - \mu$, we have $-\eta_1 - \delta_1 > -1$, therefore $s^{-\eta_1-\delta_1}$ in (3.7) is integrable.

Furthermore, for $t > T_2$, we have

$$
\begin{aligned}
&3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{\nu-1} + 3\frac{M^2L^2\Gamma(1-\eta_1-\delta_1)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta_1-\delta_1)} (1+TrQ) t^{-\frac{1}{2}(1+\eta_1+\delta_1-2\mu)} \\
&\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 T_2^{\nu-1} \\
&\quad + 3\frac{M^2L^2\Gamma(1-\eta_1-\delta_1)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta_1-\delta_1)} (1+TrQ) T_2^{-\frac{1}{2}(1+\eta_1+\delta_1-2\mu)} \\
&\leq 1.
\end{aligned}
\tag{3.8}
$$

From inequality (3.6) and $\nu - 1 < \delta_1$, we have

$$
\begin{aligned}
E\bigl|x(t)\bigr|^2 &\leq 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2} E|x_0|^2 \\
&\quad + \frac{M^2L^2\Gamma(1-\eta_1-\delta_1)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta_1-\delta_1)} (1+TrQ) t^{-(1+\eta_1+\delta_1-2\mu)} \\
&\leq \biggl[3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{\nu-1} E|x_0|^2 \\
&\quad + \frac{M^2L^2\Gamma(1-\eta_1-\delta_1)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta_1-\delta_1)} (1+TrQ) t^{-(1+\eta_1-2\mu)}\biggr] t^{-\delta_1} \\
&\leq t^{-\delta_1},
\end{aligned}
\tag{3.9}
$$

which implies that $x(t) \in D_2$ for $t \geq T_2$. Moreover, taking (3.6) and (3.9) into account, we can also get $E|F_2y(t)|^2 \leq t^{-\delta_1}$, which implies that $F_2D_2 \subset D_2$ for $t \geq T_2$.

By applying a similar argument to the one used in Lemma 3.1, we can deduce that the operator F_2 is continuous and F_2D_2 resides in a compact subset of $L_2(\Omega, X)$ for $t \geq T_2$. Using Theorem 2.1 and Krasnoselskii's theorem, there exists some $y \in D_2$ such that $y = F_1y + F_2y$, that is, F has a fixed point in D_2 which is indeed a solution of (1.1). Moreover, it is obvious that all functions in D_2 tend to 0 as $t \to \infty$, therefore, the solution of (1.1) tends to zero as $t \to \infty$ which implies the zero solution of (1.1) is globally attractive. This completes the proof. □

Corollary 3.1 *Suppose that assumptions* (H_1)–(H_3) *hold and that*

(H$_6$) $E|f(t,x(t))-f(t,y(t))|^2 \leq Lt^{-\eta_2}E|x(t)-y(t)|^2, E|\sigma(t,x(t))-\sigma(t,y(t))|^2 \leq Lt^{-\eta_2}E|x(t)-y(t)|^2$ *for* $t \in (0,\infty)$ *and* $x(t),y(t) \in C((0,\infty),L_2(\Omega,X))$, $L \geq 0$ *and* $2\mu-1<\eta_2<\mu$, $f(t,0)\equiv 0$, $\sigma(t,0)\equiv 0$.

Then the zero solution of system (1.1) *is globally attractive.*

Proof From condition (H$_6$), we have

$$E\left|f\left(t,x(t)\right)\right|^2 = E\left|f\left(t,x(t)\right)-f(t,0)\right|^2 \leq Lt^{-\eta_1}E\left|x(t)-0\right|^2 = Lt^{-\eta_1}E\left|x(t)\right|^2, \tag{3.10}$$

$$E\left|\sigma\left(t,x(t)\right)\right|^2 = E\left|\sigma\left(t,x(t)\right)-\sigma(t,0)\right|^2 \leq Lt^{-\eta_1}E\left|x(t)-0\right|^2 = Lt^{-\eta_1}E\left|x(t)\right|^2, \tag{3.11}$$

which imply that condition (H$_5$) holds. According to Theorem 3.2, the solution of (1.1) is global attractive. □

Lemma 3.4 *Suppose that assumptions* (H$_0$), (H$_2$) *and* (H$_4$) *hold. Then F maps* $\bar{D}$ *into* $\bar{D}$ *and F is continuous in* $\bar{D}$.

Proof Since $0<2\mu-1<\eta$ and $0<\nu<1$, we can choose a constant $\xi>0$ sufficiently small such that $\xi<1-\nu$ and $\xi-\eta+2\mu-1<0$. Let $T>0$ be sufficiently large such that

$$\begin{aligned} &3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 T^{\xi+2\nu-2}E|x_0|^2 \\ &\quad + 3\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)T^{\xi-(1+\eta-2\mu)} < 1. \end{aligned} \tag{3.12}$$

Define a set $\bar{D} = \{x(t): x \in C((0,\infty),L_2(\Omega,X)), E|x(t)|^2 \leq t^{-\xi} \text{ for } t \geq T\}$, we note that $\bar{D}$ is a nonempty closed bounded and convex subset of $C_0((0,\infty),L_2(\Omega,X))$.

We now show that F maps $\bar{D}$ into $\bar{D}$. For $x \in \bar{D}$, from (H$_2$), (H$_4$) and (3.12), we can deduce that

$$\begin{aligned} &E\left|(Fx)(t)\right|^2 \\ &\quad \leq 3E\left|S_{\mu,\beta}(t)x_0\right|^2 + 3E\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t_2-s)\right|^2 E\left|f\left(s,x(s)\right)\right|^2\,ds \\ &\qquad + 3TrQE\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t_2-s)\right|^2 E\left|\sigma\left(s,x(s)\right)\right|^2\,d\omega(s) \\ &\quad = \left(3t^\xi\left|S_{\mu,\beta}(t)x_0\right|^2 + 3t^\xi\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t_2-s)\right|^2 E\left|f\left(s,x(s)\right)\right|^2\,ds\right. \\ &\qquad \left. + 3t^\xi\,TrQ\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t_2-s)\right|^2 E\left|\sigma\left(s,x(s)\right)\right|^2\,d\omega(s)\right)t^{-\xi} \\ &\quad \leq \left(\frac{3M^2}{\Gamma^2(\beta(1-\mu)+\mu)}t^{\xi+2\nu-2}|x_0|\right. \\ &\qquad \left. + 3\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{\xi-(1+\eta-2\mu)}\right)t^{-\xi} \\ &\quad \leq t^{-\xi}, \quad t>T, \end{aligned}$$

which implies that $F\bar{D} \subset \bar{D}$. We next shall prove that F is continuous in $\bar{D}$ by considering F on two intervals.

Firstly, for $t > T_1 > T$, for each $x_m \in \bar{D}$, $m = 1, 2, \ldots$, with $\lim_{m\to\infty} E|x^{(m)}(t)|^2 = E|x(t)|^2$, we have

$$\begin{aligned}
&E\left|Fx_m(t) - Fx(t)\right|^2 \\
&\quad\leq 2\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t-s)\right|^2 E\left|\left(f\left(s, x^{(m)}(s)\right) - f\left(s, x(s)\right)\right|^2 ds \\
&\qquad + 2TrQ\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t-s)\right|^2 E\left|\left(\sigma\left(s, x^{(m)}(s)\right) - \sigma\left(s, x(s)\right)\right|^2 ds \\
&\quad\leq 2\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t-s)\right|^2 \left(E\left|\left(f\left(s, x^{(m)}(s)\right)\right|^2 + E\left|f\left(s, x(s)\right)\right|^2\right) ds \\
&\qquad + 2TrQ\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t-s)\right|^2 \left(E\left|\left(\sigma\left(s, x^{(m)}(s)\right)\right|^2 + E\left|\sigma\left(s, x(s)\right)\right|^2\right) ds \\
&\quad\leq 4\left(\frac{ML}{\Gamma(\mu)}\right)^2 (1 + TrQ)\int_0^t (t-s)^{2\mu-2}s^{-\eta}\, ds \\
&\quad\leq 4\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1 + TrQ)T_1^{-(1+\eta-2\mu)} < \varepsilon.
\end{aligned}$$

Next, for $0 < t \leq T_1$, we have

$$\begin{aligned}
&E\left|Fx_m(t) - Fx(t)\right|^2 \\
&\quad\leq 2\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t-s)\right|^2 E\left|\left(f\left(s, x^{(m)}(s)\right) - f\left(s, x(s)\right)\right|^2 ds\, ds \\
&\qquad + 2TrQ\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t-s)\right|^2 E\left|\left(\sigma\left(s, x^{(m)}(s)\right) - \sigma\left(s, x(s)\right)\right|^2 ds \\
&\quad\leq 2\frac{M^2}{\Gamma^2(\mu)}\int_0^t (t-s)^{2\mu-2} E\left|\left(f\left(s, x^{(m)}(s)\right) - f\left(s, x(s)\right)\right|^2 ds \\
&\qquad + 2TrQ\frac{M^2}{\Gamma^2(\mu)}\int_0^t (t-s)^{2\mu-2} E\left|\left(\sigma\left(s, x^{(m)}(s)\right) - \sigma\left(s, x(s)\right)\right|^2 ds.
\end{aligned}$$

Since $\lim_{m\to\infty} E|f(t, x_m(t)) - f(t, x(t))|^2 = 0$, $\lim_{m\to\infty} E|\sigma(t, x_m(t)) - \sigma(t, x(t))|^2 = 0$ as $m \to \infty$, it follows from the Lebesgue dominated convergence theorem that $E|(Fx_m)(t) - (Fx)(t)|^2 \to 0$ as $m \to \infty$. The above arguments lead to the fact that the operator F is continuous. □

Lemma 3.5 *Suppose that assumptions* (H_0), (H_2) *and* (H_4) *hold. Then* $\{Fx : x \in \bar{D}\}$ *is equicontinuous and* $\lim_{t\to\infty} E|(Fx)(t)|^2 = 0$ *uniformly for* $x \in \bar{D}$.

Proof Since $\nu < 1$ and $1 + \eta - 2\mu > 0$, there exists a $T_1 > T$ such that, for $t > T_1$,

$$\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 < \frac{\varepsilon}{6}$$

and

$$\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}t^{-(1+\eta-2\mu)} < \frac{\varepsilon}{6},$$
$$\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}TrQt^{-(1+\eta-2\mu)} < \frac{\varepsilon}{6}.$$

For any $x \in \bar{D}$ and $t_1, t_2 > T_1$, we can deduce that

$$\begin{aligned}
&E\bigl|(Fx)(t_2)-(Fx)(t_1)\bigr|^2 \\
&\quad\le 6E\bigl|S_{\mu,\beta}(t_2)x_0\bigr|^2 + 6E\bigl|S_{\mu,\beta}(t_1)x_0\bigr|^2 + 6\int_0^{t_2}(t_2-s)^{2\gamma-2}\bigl|P_\mu(t_2-s)\bigr|^2E\bigl|f\bigl(s,x(s)\bigr)\bigr|^2\,ds \\
&\qquad + 6\int_0^{t_1}(t_1-s)^{2\mu-2}\bigl|P_\mu(t_1-s)\bigr|^2E\bigl|f\bigl(s,x(s)\bigr)\bigr|^2\,ds \\
&\qquad + 6TrQ\int_0^{t_2}(t_2-s)^{2\mu-2}\bigl|P_\mu(t_2-s)\bigr|^2E\bigl|\sigma\bigl(s,x(s)\bigr)\bigr|^2\,ds \\
&\qquad + 6TrQ\int_0^{t_1}(t_1-s)^{2\mu-2}\bigl|P_\mu(t_1-s)\bigr|^2E\bigl|\sigma\bigl(s,x(s)\bigr)\bigr|^2\,ds \\
&\quad\le 6\frac{M^2}{\Gamma^2(\beta(1-\mu)+\mu)}t_2^{2\nu-2}E|x_0|^2 + 6\frac{M^2}{\Gamma^2(\beta(1-\mu)+\mu)}t_1^{2\nu-2}E|x_0|^2 \\
&\qquad + 6\frac{M^2L^2}{\Gamma^2(\mu)}(1+TrQ)\int_0^{t_2}(t_2-s)^{2\mu-2}s^{-\eta}\,ds \\
&\qquad + 6\frac{M^2L^2}{\Gamma^2(\mu)}(1+TrQ)\int_0^{t_1}(t_1-s)^{2\mu-2}s^{-\eta}\,ds \\
&\quad\le 6\frac{M^2}{\Gamma^2(\beta(1-\mu)+\mu)}t_2^{2\nu-2}E|x_0|^2 + 6\frac{M^2}{\Gamma^2(\beta(1-\mu)+\mu)}t_1^{2\nu-2}E|x_0|^2 \\
&\qquad + 6\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)\bigl[t_2^{-(1+\eta-2\mu)}+t_1^{-(1+\eta-2\mu)}\bigr].
\end{aligned}$$

Furthermore, for $0 < t_1 < t_2 \le T_1$, we have

$$\begin{aligned}
&E\bigl|(Fx)(t_2)-(Fx)(t_1)\bigr|^2 \\
&\quad\le E\biggl|S_{\mu,\beta}(t_2)x_0 - S_{\mu,\beta}(t_1)x_0 \\
&\qquad + \int_0^{t_2}(t_2-s)^{\mu-1}P_\mu(t_2-s)f\bigl(s,x(s)\bigr)\,ds - \int_0^{t_1}(t_1-s)^{\gamma-1}P_\mu(t_1-s)f\bigl(s,x(s)\bigr)\,ds \\
&\qquad + \int_0^{t_2}(t_2-s)^{\mu-1}P_\mu(t_2-s)\sigma\bigl(s,x(s)\bigr)\,d\omega(s) \\
&\qquad - \int_0^{t_1}(t_1-s)^{\mu-1}P_\mu(t_1-s)f\bigl(s,x(s)\bigr)\,d\omega(s)\biggr|^2 \\
&\quad\le 7\bigl|S_{\mu,\beta}(t_2)-S_{\mu,\beta}(t_1)\bigr|^2E|x_0|^2 \\
&\qquad + 7\int_0^{t_1}\bigl[(t_2-s)^{2\mu-2}-(t_1-s)^{2\mu-2}\bigr]\bigl|P_\mu(t_2-s)\bigr|^2E\bigl|f\bigl(s,x(s)\bigr)\bigr|^2\,ds \\
&\qquad + 7TrQ\int_0^{t_1}\bigl[(t_2-s)^{2\mu-2}-(t_1-s)^{2\mu-2}\bigr]^2\bigl|P_\mu(t_2-s)\bigr|^2E\bigl|\sigma\bigl(s,x(s)\bigr)\bigr|^2\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ 7\int_0^{t_1}(t_1-s)^{2\mu-2}\left|P_\mu(t_2-s)-P_\mu(t_2-s)\right|^2 E\left|f\left(s,x(s)\right)\right|^2\,ds\\
&+ 7TrQ\int_0^{t_1}(t_1-s)^{2\mu-2}\left|P_\mu(t_2-s)-P_\mu(t_2-s)\right|^2 E\left|\sigma\left(s,x(s)\right)\right|^2\,ds\\
&+ 7\int_{t_1}^{t_2}\left[(t_2-s)^{2\mu-2}-(t_1-s)^{2\mu-2}\right]\left|P_\mu(t_2-s)\right|^2 E\left|f\left(s,x(s)\right)\right|^2\,ds\\
&+ 7TrQ\int_{t_1}^{t_2}\left[(t_2-s)^{2\mu-2}-(t_1-s)^{2\mu-2}\right]\left|P_\mu(t_2-s)\right|^2 E\left|\sigma\left(s,x(s)\right)\right|^2\,ds.
\end{aligned}
$$

By using the arguments employed in Lemma 3.1, the Lebesgue dominated convergence theorem and Remark 2.3, we get $E|(Fx)(t_2)-(Fx)(t_1)|^2 \to 0$ as $t_2-t_1\to 0$. Therefore, we conclude that the family of functions $\{Fx : x\in\bar{D}\}$ is equicontinuous.

We now verify that $\lim_{t\to\infty} E|(Fx)(t)|^2 = 0$ uniformly for $x\in\bar{D}$. Indeed, by a standard calculation, we have

$$
\begin{aligned}
&E\left|(Fx)(t)\right|^2\\
&\le 3E\left|S_{\mu,\beta}(t)x_0\right| + 3E\left|\int_0^t (t-s)^{\mu-1}P_\mu(t-s)f\left(s,x(s)\right)ds\right|^2\\
&\quad + 3E\left|\int_0^t (t-s)^{\mu-1}P_\mu(t-s)\sigma\left(s,x(s)\right)d\omega(s)\right|^2\\
&\le 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 + 3\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t_2-s)\right|^2 E\left|f\left(s,x(s)\right)\right|^2 ds\\
&\quad + 3TrQ\int_0^t (t-s)^{2\mu-2}\left|P_\mu(t_2-s)\right|^2 E\left|\sigma\left(s,x(s)\right)\right|^2 ds\\
&\le 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 + 3\left(\frac{ML}{\Gamma(\mu)}\right)^2(1+TrQ)\int_0^t (t-s)^{2\mu-2}s^{-\eta}\,ds\\
&\le 3\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{2\nu-2}E|x_0|^2 + 3\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{-(1+\eta-2\mu)}\\
&\le 3\left[\left(\frac{M}{\Gamma(\beta(1-\mu)+\mu)}\right)^2 t^{\nu-1}E|x_0|^2\right.\\
&\quad \left.+\frac{M^2L^2\Gamma(1-\eta)\Gamma(2\mu-1)}{\Gamma^2(\mu)\Gamma(2\mu-\eta)}(1+TrQ)t^{-\frac{1}{2}(1+\eta-2\mu)}\right]t^{-\frac{1}{2}(1+\eta-2\gamma)}\\
&\le t^{-\frac{1}{2}(1+\eta-2\mu)} \to 0,\quad \text{as } t\to\infty.
\end{aligned}
$$

Thus, we deduce that $\lim_{t\to\infty} E|(Fx)(t)|^2 = 0$ uniformly for $x\in\bar{D}$. □

We now proceed to our third attractivity result for system (1.1).

Theorem 3.3 *Suppose that assumptions* (H_1)–(H_4) *hold. Then the Cauchy problem* (1.1) *admits at least one attractive solution.*

Proof In order to verify that x is a mild solution of (1.1) in $\bar{D}$, we need to verify that the operator $Fx = x$ has a fixed point in $\bar{D}$. By Lemmas 3.4–3.5, it follows that $F:\bar{D}\to\bar{D}$ is continuous and F maps $\bar{D}$ into itself. Moreover, $\{Fx : x\in\bar{D}\}$ is equicontinuous and $\lim_{t\to\infty} E|Fx(t)|^2 = 0$ is uniformly valid for $x\in\bar{D}$. We next show that F is relatively compact. From the definition of $S_{\mu,\beta}(t)$, we have

$$\begin{aligned} S_{\mu,\beta}(t)x_0 &= I_{0^+}^{\beta(1-\mu)} T_\mu(t)x_0 \\ &= \frac{1}{\Gamma(\beta(1-\mu))}\int_0^t (t-s)^{\beta(1-\mu)-1} T_\mu(s)x_0\,ds \\ &= \frac{1}{\Gamma(\beta(1-\mu))}\int_0^t (t-s)^{\beta(1-\mu)-1} s^{\mu-1} P_\mu(s)x_0\,ds \\ &= \frac{1}{\Gamma(\beta(1-\mu))}\int_0^t (t-s)^{\beta(1-\mu)-1} s^{\mu-1}\int_0^\infty \mu\theta M_\mu(\theta)x_0\,d\theta\,ds. \end{aligned}$$

For $\forall\lambda \in (0,t)$ and $\forall\delta > 0$, we define

$$\begin{aligned} \bigl(F^{\lambda,\delta}x\bigr)(t) &= \bigl(F_1^{\lambda,\delta}x\bigr)(t) + \bigl(F_2^{\lambda,\delta}x\bigr)(t) \\ &= \frac{1}{\Gamma(\beta(1-\mu))}\int_0^{t-\lambda} (t-s)^{\beta(1-\mu)-1} s^{\mu-1} S\bigl(s^\mu\theta\bigr)\int_\delta^\infty \mu\theta M_\mu(\theta)x_0\,d\theta\,ds \\ &\quad + S\bigl(\lambda^\mu\theta\bigr)\int_0^{t-\lambda}\int_\delta^\infty \mu\theta M_\mu(\theta) S\bigl((t-s)^\mu\theta - \lambda^\mu\theta\bigr)(t-s)^{\mu-1} f\bigl(s,x(s)\bigr)\,d\theta\,ds \\ &\quad + S\bigl(\lambda^\mu\theta\bigr)\int_0^{t-\lambda}\int_\delta^\infty \mu\theta M_\mu(\theta) S\bigl((t-s)^\mu\theta - \lambda^\mu\theta\bigr)(t-s)^{\mu-1} \\ &\quad \times \sigma\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s). \end{aligned}$$

Since $\{S(t)\}_{t>0}$ is compact, it is easy to see that $\bar{V}_1^{\lambda,\delta}(t) = \{(F_1^{\lambda,\delta}x)(t), x \in \bar{D}\}$ is relatively compact for $t > 0$. Next, we verify that $\bar{V}_1(t) = \{(F_1x)(t), x \in \bar{D}\}$ is relatively compact for $t > 0$. Observe that

$$\begin{aligned} &E\bigl|(F_1(t) - \bigl(F_1^{\lambda,\delta}x\bigr)(t)\bigr|^2 \\ &\quad \le E\biggl|\frac{1}{\Gamma(\beta(1-\mu))}\int_0^t (t-s)^{\beta(1-\mu)-1} s^{\mu-1} S\bigl(s^\mu\theta\bigr)\int_0^\delta \mu\theta M_\mu(\theta)x_0\,d\theta\,ds \\ &\qquad + \frac{1}{\Gamma(\beta(1-\mu))}\int_0^t (t-s)^{\beta(1-\mu)-1} s^{\mu-1} S\bigl(s^\mu\theta\bigr)\int_\delta^\infty \mu\theta M_\mu(\theta)x_0\,d\theta\,ds \\ &\qquad - \frac{1}{\Gamma(\beta(1-\mu))}\int_0^{t-\lambda} (t-s)^{\beta(1-\mu)-1} s^{\mu-1} S\bigl(s^\mu\theta\bigr)\int_\delta^\infty \mu\theta M_\mu(\theta)x_0\,d\theta\,ds\biggr|^2 \\ &\quad \le 2\frac{M}{\Gamma(\beta(1-\mu)+\mu)}E\biggl|\int_0^t (t-s)^{\beta(1-\mu)-1} s^{\mu-1}\int_0^\delta \mu\theta M_\mu(\theta)x_0\,d\theta\,ds\biggr|^2 \\ &\qquad + 2\frac{M}{\Gamma(\beta(1-\mu)+\mu)}E\biggl|\int_{t-\lambda}^t (t-s)^{\beta(1-\mu)-1} s^{\mu-1}\int_\delta^\infty \mu\theta M_\mu(\theta)x_0\,d\theta\,ds\biggr|^2 \\ &\quad \le 2\biggl(\frac{M\mu}{\Gamma(\beta(1-\mu)+\mu)}\biggr)^2 E|x_0|^2 B^2\bigl(\beta(1-\mu),\mu\bigr)\biggl|\int_0^\delta \theta M_\mu(\theta)\,d\theta\biggr|^2 \\ &\qquad + 2\biggl(\frac{M\mu}{\Gamma(\beta(1-\mu)+\mu)}\frac{1}{\Gamma(1+\mu)}\biggr)^2 E|x_0|^2\biggl|\int_{t-\lambda}^t (t-s)^{\beta(1-\mu)-1} s^{\mu-1}\,ds\biggr|^2 \to 0, \\ &\qquad \text{as } \lambda,\delta \to 0^+, \end{aligned}$$

which, together with Lemma 3.3 implies that $\bar{V}(t) = \{(Fx)(t), x \in \bar{D}\}$ is relatively compact for $t > 0$. In view of the foregoing arguments, it follows by Schauder's fixed point theorem that system (1.1) has a mild solution $x \in \bar{D}$ and that $x(t)$ tends to zero as $t \to \infty$. This completes the proof. □

To end this section, we now present the last result of our paper.

Theorem 3.4 *Suppose that assumptions* (H_1)–(H_3) *and the following assumptions are satisfied*:

(H_7) *There exists a strictly decreasing function* $\mathcal{A} : (0, \infty) \to [0, \infty)$ *such that, for all* $\psi \in C((0,\infty), L_2(\Omega, X))$, *and* $t > 0$,

$$E\bigg|S_{\mu,\beta}(t)x_0 + \int_0^t (t-s)^{\mu-1} P_\mu(t-s) f\big(s, x(s)\big)\,ds + \int_0^t (t-s)^{\mu-1} P_\mu(t-s)\sigma\big(s, x(s)\big)\,d\omega(s)\bigg|^2 \le \mathcal{A}(t),$$

and $\lim_{t\to+\infty} \mathcal{A}(t) = 0$.

Then the system (1.1) *has at least one attractive solution in* $C((0,\infty), L_2(\Omega, X))$.

Proof Define a set $D_3 := \{u \in C((0,\infty), L_2(\Omega, X)) : E|u(t)|^2 \le \mathcal{A}(t) \text{ for all } t \ge 0\}$. We note that D_3 is nonempty and convex. We next verify that D_3 is bounded. Indeed, from (H_7), we have

$$\sup_{t>0} E\big|u(t)\big|^2 \le \mathcal{A}(t), \quad t > 0 \text{ for every } u \in C\big((0,\infty), L_2(\Omega, X)\big).$$

For $t > 0$, we introduce

$$(\Psi u)(t) = S_{\mu,\beta}(t)x_0 + \int_0^t (t-s)^{\mu-1} P_\mu(t-s) f\big(s, u(s)\big)\,ds + \int_0^t (t-s)^{\mu-1} P_\mu(t-s)\sigma\big(s, u(s)\big)\,d\omega(s).$$

Clearly $\Psi(D_3) \subset D_3$ by (H_7). We next show continuity of Ψ. Let $\{u_n\}_{n\in N}$ be a sequence in D_3 such that $\lim_{n\to\infty} u_n = u$ is uniformly valid on compact subsets of $(0,\infty)$. By (H_7), we deduce that $\mathcal{A}$ tend to 0 at infinity and $\mathcal{A}$ is strictly decreasing. Thus, for an arbitrary given $\epsilon > 0$, there exists $T > 0$ such that $\mathcal{A}(t) < \epsilon$ for all $t > T$. Moreover, by (H_2), there exists $\delta_1 > 0$ such that $\|u_n - u\|_{C((0,\infty),L_2(\Omega,X))} < \delta_1$, which yields

$$\begin{aligned} &\big\|f\big(t, u_n(t)\big) - f\big(t, u(t)\big)\big\|_{C((0,\infty),L_2(\Omega,X))} < \epsilon, \\ &\big\|\sigma\big(t, u(t)\big) - \sigma\big(t, v(t)\big)\big\|_{C((0,\infty),L_2(\Omega,X))} < \epsilon. \end{aligned} \tag{3.13}$$

Then there exist $N \in \mathbb{N}$ and $\delta' > 0$ such that $E|u_n(t) - u(t)|^2 < \delta'$ is uniformly valid on compact subsets of $(0,\infty)$ for $\forall n > N$ and $\forall t > 0$. Since $\|u\|_{C((0,\infty),L_2(\Omega,X))} := (\sup_{t>0} E|u(t)|^2)^{\frac{1}{2}}$, we have $\|u_n - u\|_{C((0,\infty),L_2(\Omega,X))} := (\sup_{t>0} E|u_n(t) - u(t)|^2)^{\frac{1}{2}}$ for $\forall t > 0$.

Now we prove continuity of Ψ as follows.

For $0 < t < T$, by Hölder's inequality, (H_3) and (3.13), we obtain

$$\begin{aligned}
&E\big|\Psi(u_n) - \Psi(u)\big|^2 \\
&\leq 2\int_0^t (t-s)^{2\mu-2}\big|P_\mu(t-s)\big|^2 E\big|\big(f\big(s,u_n(s)\big) - f\big(s,u(s)\big)\big|^2\,ds \\
&\quad + 2\int_0^t (t-s)^{2\mu-2}\big|P_\mu(t-s)\big|^2 E\big|\big(\sigma\big(s,u_n(s)\big) - \sigma\big(s,u(s)\big)\big|^2\,ds \\
&\leq 2\left(\frac{M}{\Gamma(\mu)}\right)^2 \int_0^t (t-s)^{2\mu-2} E\big|\big(f\big(s,u_n(s)\big) - f\big(s,u(s)\big)\big|^2\,ds \\
&\quad + 2TrQ\left(\frac{M}{\Gamma(\mu)}\right)^2 \int_0^t (t-s)^{2\mu-2} E\big|\big(\sigma\big(s,u_n(s)\big) - \sigma\big(s,u(s)\big)\big|^2\,ds \\
&\leq 2\left(\frac{M}{\Gamma(\mu)}\right)^2 \int_0^t (t-s)^{2\mu-2}\,ds \int_0^t E\big|\big(f\big(s,u_n(s)\big) - f\big(s,u(s)\big)\big|^2\,ds \\
&\quad + 2TrQ\left(\frac{M}{\Gamma(\mu)}\right)^2 \int_0^t (t-s)^{2\mu-2}\,ds \int_0^t E\big|\big(\sigma\big(s,u_n(s)\big) - \sigma\big(s,u(s)\big)\big|^2\,ds \\
&\leq 2\left(\frac{M}{\Gamma(\mu)}\right)^2 (1+TrQ)\left(\int_0^t (t-s)^{\frac{2\mu-2}{2-2\mu_1}}\right)^{2-2\mu_1} \left(\int_0^t |\varepsilon|^{\frac{1}{2\mu_1-1}}\,ds\right)^{2\mu_1-1}.
\end{aligned}$$

Next, for $0 < T < t$, using (H_7) and the S-invariance of Ψ, we have the estimate

$$E\big|\Psi(u_n)(t) - \Psi(u)(t)\big|^2 \leq 2E\big|\Psi(u_n)(t)\big|^2 + 2E\big|\Psi(u)(t)\big|^2 \leq 2\mathcal{A}(t) < 2\varepsilon.$$

This leads to the fact that Ψ is continuous.

As argued in Lemma 3.1, we conclude that $\Psi(D_3)$ is equicontinuous on each compact interval $[0,T]$ for $\forall T > 0$, which establishes the claim.

Now, we claim that $\Psi(D_3)$ is uniformly bounded. Indeed, by the definition of the set D_3 and the fact that $\mathcal{A}$ is strictly decreasing, we can infer $\sup_{u\in\Psi(D_3)}\sup_{t>T} E|u(t)|^2 \leq \mathcal{A}(T-0)$. By ($\mathrm{H}_7$), we obtain $\lim_{T\to\infty}\sup_{u\in\Psi(D_3)}\sup_{t>T} E|u(t)|^2 = 0$. On the other hand, by similar arguments to Theorem 3.3, we can show that $V(t) = \{(\Psi u)(t), u \in D_3\}$ is relatively compact for $t > 0$. Consequently, Lemma 2.6 enables us to claim that the family $\Psi(D_3)$ is relatively compact in $C((0,\infty), L_2(\Omega, X))$. Thus, from Schauder's fixed point theorem, it shows that the operator Ψ has a fixed point in D_3. Hence the system (1.1) has at least one attractive solution $u_0 \in D_3$. This completes the proof. □

4 Conclusion

In this paper, we have revealed that a certain class of Hilfer fractional stochastic evolution equations with fast decaying nonlinear term have global attractive solutions, whereas the integer-order evolution equations do not have such attractivity. Our future work will be focused on addressing the attractivity for Hilfer fractional stochastic evolution equations with fractional Brownian motion and Poisson jumps.

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Authors' contributions
MY was involved in conceptualizing and analysis; AA contributed to the design, analysis and the acquisition of funding; BA proposed the methodology and was involved in the analysis; and YZ was involved in conceptualizing and analysis of the work. All the authors read and approved the final version for submission.

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