Tian and Fan *Advances in Difference Equations* (2020) 2020:142
https://doi.org/10.1186/s13662-020-02596-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Nonlinear integral inequality with power and its application in delay integro-differential equations

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Abstract

New nonlinear integral inequalities (NII) are presented in this paper. Based on mathematical analysis technique, several estimation results are obtained, which not only complement the aforementioned results, but also generalize the inequalities to the more general nonlinearities. As an application, they can be employed to estimate the bound on the solutions of power integro-differential equations (IDE).

MSC: 26D10; 26D15

Keywords: Nonlinear; Integral inequality; Power; Delay IDE

1 Introduction

As everyone knows, there exists a class of mathematical models described by differential equations, such as Malthus population model. However, a lot of differential equations do not possess the exact solution. Under this case, integral inequalities are significant for investigating the boundedness, stability, asymptotic behavior of solutions to differential equations. Gronwall [1] put forward the well-known Gronwall inequality to estimate the solution of linear differential equation. Bihari inequality [2] extended [1] to nonlinear one, and many authors have been devoted to studying NII in recent years [3–25]. For example, based on the generalized Gronwall inequality, Tian et al. [3] investigated the asymptotic behavior of switched delay systems that represent a class of systems in practical engineering and have wide application in automated highways, power systems, and so on. Pachpatte [4] considered a linear integral inequality (1.1).

Theorem 1.1 ([4]) *Let $c_0 \geq 0$ and $u, b, c, d \in C(R^+, R^+)$, $R^+ = [0, +\infty)$. If*

$$u(t) \leq c_0 + \int_0^t \big(b(s)u(s) + d(s)\big)\,ds + \int_0^t b(s)\left(\int_0^s c(\xi)u(\xi)\,d\xi\right)ds, \tag{1.1}$$

then

$$u(t) \le c_0 + \int_0^t \biggl[d(s) + b(s) \biggl(c_0 \exp\biggl(\int_0^s \bigl(b(\sigma) + c(\sigma)\bigr)\,d\sigma \biggr) + \int_0^s d(\sigma) \exp\biggl(\int_\sigma^s \bigl(b(\tau) + c(\tau)\bigr)\,d\tau \biggr)\,d\sigma \biggr) \biggr]\,ds, \quad t \in R^+.$$

After that, Abdeldaim and El-Deeb [12] generalized (1.1) and investigated the delay integral inequality (1.2).

Theorem 1.2 ([12, Theorem 2.1]) *Assume that $c_0 \ge 0$, $u, b, c, d \in C(R^+, R^+)$, and $\alpha \in C^1(R^+, R^+)$ are nondecreasing functions with $\alpha(t) \le t$, $\alpha(0) = 0$. If*

$$u(t) \le c_0 + \int_0^{\alpha(t)} \bigl(b(s)u(s) + d(s)\bigr)\,ds + \int_0^{\alpha(t)} b(s) \biggl(\int_0^s c(\xi)u(\xi)\,d\tau \biggr)\,ds, \tag{1.2}$$

then

$$u(t) \le c_0 + \int_0^t \biggl(\alpha'(s)d\bigl(\alpha(s)\bigr) + \alpha'(s)b\bigl(\alpha(s)\bigr) \exp\biggl(\int_0^{\alpha(s)} \bigl(b(\xi) + c(\xi)\bigr)\,d\xi \biggr) \times \biggl(c_0 + \int_0^{\alpha(s)} d(\sigma) \exp\biggl(- \int_0^\sigma \bigl(b(\xi) + c(\xi)\bigr)\,d\xi \biggr) d\sigma \biggr) \biggr)\,ds, \quad \forall t \in R^+.$$

Very recently, Li and Wang [21] studied the power integral inequality (1.3).

Theorem 1.3 ([21, Theorem 2.1]) *Suppose that $m, n, p \in (0, 1]$ are nonnegative constants, $u, a, b, c \in C(R^+, R^+)$, $\alpha \in C^1(R^+, R^+)$, $\alpha(t)$ is nondecreasing with $\alpha(t) \le t$, $\alpha(0) = 0$. If*

$$u(t) \le a(t) + \int_0^{\alpha(t)} b(s) \biggl(u^m(s) + \int_0^s c(\xi)u^n(\xi)\,d\xi \biggr)^p\,ds, \tag{1.3}$$

then

$$u(t) \le a(t) + A(t) \exp\biggl(\int_0^{\alpha(t)} pmb(s)\,ds + \int_0^{\alpha(t)} pb(s) \biggl(\int_0^s nc(\xi)\,d\xi \biggr)\,ds \biggr), \quad t \in R^+,$$

where

$$A(t) = \int_0^{\alpha(t)} b(s)\bigl[(1-p) + p\bigl(ma(s) + (1-m)\bigr)\bigr]\,ds + \int_0^{\alpha(t)} pb(s) \int_0^s c(\xi)\bigl[na(\xi) + 1 - n\bigr]\,d\xi\,ds.$$

Note that inequalities (1.2) and (1.3) have been proved in the cases $p = 1$ and $p \in (0, 1]$, respectively, how about $p > 1$? The aforementioned results are not covered, and it would also be interesting to generalize the inequalities considered in [4, 12, 21] to the more general nonlinearities, which is the motivation why we further study the above inequalities and their general cases.

We study some new power NII and establish several estimation results under the condition of $p > 1$, which not only complement the ones established in [4, 12, 21] but also generalize inequalities (1.1)–(1.3) to the more general nonlinearities. The obtained results can be employed to study the boundedness of the delay IDE. As an application, two illustrative examples are also presented.

2 Main results

Throughout the paper, $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$, $C(D, E)$ and $C^1(D, E)$ defined on D with range in E are continuous functions and continuously differentiable function sets, respectively. The three lemmas are essential to proving the main results.

Lemma 2.1 ([21]) *Let $a \geq 0$ and $m \geq n > 0$. Then*

$$a^{n/m} \leq \frac{n}{m} a + \frac{m-n}{m}.$$

Lemma 2.2 ([25]) *Assume that $u, v \geq 0$ and $p \geq 0$. Then*

$$(u+v)^p \leq K_p\bigl(u^p + v^p\bigr),$$

where $K_p = 1$, $0 \leq p \leq 1$, and $K_p = 2^{p-1}$, $p > 1$.

Lemma 2.3 *Suppose that $p > 0$ is a constant and $\alpha(t)$ is a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$, $\alpha \in C^1(R^+, R^+)$, $u, a, b, c, d \in C(R^+, R^+)$, and*

$$u(t) \leq a(t) + \int_0^{\alpha(t)} b(s)\bigl(c(s)u(s) + d(s)\bigr)^p \, ds. \tag{2.1}$$

Then

$$u(t) \leq \begin{cases} a(t) + g(t) + \exp(\int_0^{\alpha(t)} h(s)\,ds) \int_0^{\alpha(t)} g(s)h(s) \exp(-\int_0^s h(\xi)\,d\xi)\,ds, & 0 < p \leq 1, \\ a(t) + (k^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1} b(s)c^p(s)\,ds)^{\frac{1}{1-p}}, & p > 1, \end{cases} \tag{2.2}$$

with

$$k^{1-p}(t) > (p-1)\int_0^{\alpha(t)} 2^{p-1} b(s)c^p(s)\,ds,$$

where

$$\begin{aligned} h(t) &= pb(t)c(t), \\ g(t) &= \int_0^{\alpha(t)} \bigl[pb(s)\bigl(a(s)c(s) + d(s)\bigr) + (1-p)b(s)\bigr]\,ds, \\ k(t) &= \int_0^{\alpha(t)} 2^{p-1} b(s)\bigl(a(s)c(s) + d(s)\bigr)^p \, ds. \end{aligned} \tag{2.3}$$

Proof Define

$$v(t) = \int_0^{\alpha(t)} b(s)\bigl(c(s)u(s) + d(s)\bigr)^p \, ds.$$

Then $v(t)$ is a nondecreasing function, and

$$u(t) \le a(t) + v(t). \tag{2.4}$$

Therefore,

$$v(t) \le \int_0^{\alpha(t)} b(s)\bigl(c(s)v(s) + a(s)c(s) + d(s)\bigr)^p \, ds. \tag{2.5}$$

Next we will prove the following two cases $0 < p \le 1$ and $p > 1$, respectively.

Case 1: $0 < p \le 1$.

By Lemma 2.1, we have

$$\bigl(c(s)v(s) + a(s)c(s) + d(s)\bigr)^p \le p\bigl(c(s)v(s) + a(s)c(s) + d(s)\bigr) + 1 - p.$$

This combined with (2.5) yields

$$\begin{aligned} v(t) &\le \int_0^{\alpha(t)} b(s)\bigl(c(s)v(s) + a(s)c(s) + d(s)\bigr)^p \, ds \\ &\le \int_0^{\alpha(t)} \bigl[pb(s)\bigl(c(s)v(s) + a(s)c(s) + d(s)\bigr) + (1-p)b(s)\bigr] \, ds \\ &= g(t) + \int_0^{\alpha(t)} h(s)v(s)\, ds, \end{aligned}$$

where $h(t)$ and $g(t)$ are defined by (2.3). Define $J(t) = \int_0^{\alpha(t)} h(s)v(s)\, ds$, then $J(0) = 0$, $J(t)$ is nondecreasing, $v(t) \le g(t) + J(t)$, and

$$\begin{aligned} J'(t) &= h\bigl(\alpha(t)\bigr)\alpha'(t)v\bigl(\alpha(t)\bigr) \\ &\le h\bigl(\alpha(t)\bigr)\alpha'(t)\bigl(g\bigl(\alpha(t)\bigr) + J\bigl(\alpha(t)\bigr)\bigr) \\ &\le h\bigl(\alpha(t)\bigr)g\bigl(\alpha(t)\bigr)\alpha'(t) + h\bigl(\alpha(t)\bigr)\alpha'(t)J(t), \end{aligned}$$

i.e.,

$$J'(t) - h\bigl(\alpha(t)\bigr)\alpha'(t)J(t) \le h\bigl(\alpha(t)\bigr)g\bigl(\alpha(t)\bigr)\alpha'(t). \tag{2.6}$$

Multiplying (2.6) by $\exp(-\int_0^{\alpha(t)} h(s)\, ds)$ produces

$$\begin{aligned} &\exp\biggl(-\int_0^{\alpha(t)} h(s)\, ds\biggr)\bigl[J'(t) - h\bigl(\alpha(t)\bigr)\alpha'(t)J(t)\bigr] \\ &\quad \le \exp\biggl(-\int_0^{\alpha(t)} h(s)\, ds\biggr)h\bigl(\alpha(t)\bigr)\alpha'(t)g\bigl(\alpha(t)\bigr). \end{aligned}$$

Integrating the above inequality from 0 to t, we have

$$J(t) \le \exp\biggl(\int_0^{\alpha(t)} h(s)\, ds\biggr) \int_0^{\alpha(t)} g(s)h(s)\exp\biggl(-\int_0^{s} h(\xi)\, d\xi\biggr)\, ds.$$

Since $v(t) \leq g(t) + J(t)$, we have

$$v(t) \leq g(t) + \exp\left(\int_0^{\alpha(t)} h(s)\,ds\right) \int_0^{\alpha(t)} g(s)h(s) \exp\left(-\int_0^s h(\xi)\,d\xi\right) ds.$$

This together with (2.4) produces

$$u(t) \leq a(t) + g(t) + \exp\left(\int_0^{\alpha(t)} h(s)\,ds\right) \int_0^{\alpha(t)} g(s)h(s) \exp\left(-\int_0^s h(\xi)\,d\xi\right) ds.$$

Case 2: $p > 1$.

Applying Lemma 2.2 to (2.5), we get

$$\begin{aligned} v(t) &\leq \int_0^{\alpha(t)} 2^{p-1} b(s)\bigl(c^p(s)v^p(s) + \bigl(a(s)c(s) + d(s)\bigr)^p\bigr)\,ds \\ &= k(t) + \int_0^{\alpha(t)} 2^{p-1} b(s)c^p(s)v^p(s)\,ds, \end{aligned}$$

where $k(t)$ is defined by (2.3). Since $k(t)$ is a nondecreasing function, for fixed T,

$$v(t) \leq k(T) + \int_0^{\alpha(t)} 2^{p-1} b(s)c^p(s)v^p(s)\,ds, \quad t \in [0, T].$$

Define

$$w(t) = k(T) + \int_0^{\alpha(t)} 2^{p-1} b(s)c^p(s)v^p(s)\,ds.$$

Then $w(0) = k(T)$, w is a nondecreasing function, and

$$v(t) \leq w(t), \qquad v\bigl(\alpha(t)\bigr) \leq w\bigl(\alpha(t)\bigr) \leq w(t). \tag{2.7}$$

Differentiating w and using (2.7), we get

$$\begin{aligned} w'(t) &= 2^{p-1}\alpha'(t)b\bigl(\alpha(t)\bigr)c^p\bigl(\alpha(t)\bigr)v^p\bigl(\alpha(t)\bigr) \\ &\leq 2^{p-1}\alpha'(t)b\bigl(\alpha(t)\bigr)c^p\bigl(\alpha(t)\bigr)w^p(t) \end{aligned}$$

and

$$\frac{w'(t)}{w^p(t)} \leq 2^{p-1}\alpha'(t)b\bigl(\alpha(t)\bigr)c^p\bigl(\alpha(t)\bigr).$$

The above inequality multiplied by $1 - p$ gives

$$(1-p)\frac{w'(t)}{w^p(t)} \geq (1-p)2^{p-1}\alpha'(t)b\bigl(\alpha(t)\bigr)c^p\bigl(\alpha(t)\bigr). \tag{2.8}$$

By simple calculation of (2.8),

$$w(t) \leq \left(k^{1-p}(T) + (1-p)\int_0^{\alpha(t)} 2^{p-1} b(s)c^p(s)\,ds\right)^{\frac{1}{1-p}}, \quad t \in [0, T].$$

Letting $t = T$ in the above inequality, we have

$$w(T) \leq \left(k^{1-p}(T) + (1-p)\int_0^{\alpha(T)} 2^{p-1}b(s)c^p(s)\,ds\right)^{\frac{1}{1-p}}, \quad t \in [0,T].$$

Because T is arbitrary,

$$w(t) \leq \left(k^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1}b(s)c^p(s)\,ds\right)^{\frac{1}{1-p}}.$$

This together with (2.4), (2.7) implies

$$u(t) \leq a(t) + \left(k^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1}b(s)c^p(s)\,ds\right)^{\frac{1}{1-p}}.$$

Based on Cases 1 and 2, we can draw a conclusion that $u(t)$ satisfies (2.2). □

Theorem 2.1 *Assume that m, n, p are nonnegative constants satisfying $0 < m, n \leq 1$, $p > 1$, $\alpha(t)$ is nondecreasing with $\alpha \in C^1(R^+, R^+)$, $\alpha(t) \leq t$, $\alpha(0) = 0$, $u, a, b, c \in C(R^+, R^+)$, and*

$$u(t) \leq a(t) + \int_0^{\alpha(t)} b(s)\left(u^m(s) + \int_0^s c(\xi)u^n(\xi)\,d\xi\right)^p ds. \tag{2.9}$$

Then

$$u(t) \leq a(t) + \left(\tilde{k}^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1}b(s)\left(m + n\int_0^s c(\xi)\,d\xi\right)^p ds\right)^{\frac{1}{1-p}}$$

with

$$\tilde{k}^{1-p}(t) > (p-1)\int_0^{\alpha(t)} 2^{p-1}b(s)\left(m + n\int_0^s c(\xi)\,d\xi\right)^p ds,$$

where

$$\tilde{k}(t) = \int_0^{\alpha(t)} 2^{p-1}b(s)\left(ma(s) + 1 - m + \int_0^s c(\xi)\big(na(\xi) + 1 - n\big)\,d\xi\right)^p ds. \tag{2.10}$$

Proof Construct

$$y(t) = \int_0^{\alpha(t)} b(s)\left(u^m(s) + \int_0^s c(\xi)u^n(\xi)\,d\xi\right)^p ds.$$

Then $y(0) = 0$, y is a nondecreasing function, and

$$u(t) \leq a(t) + y(t). \tag{2.11}$$

By Lemma 2.1,

$$\begin{aligned} \big(a(t) + y(t)\big)^m &\leq m\big(a(t) + y(t)\big) + 1 - m, \\ \big(a(t) + y(t)\big)^n &\leq n\big(a(t) + y(t)\big) + 1 - n. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12), we have

$$\begin{aligned} y(t) &\leq \int_0^{\alpha(t)} b(s)\left(\left(a(s)+y(s)\right)^m + \int_0^s c(\xi)\left(a(\xi)+y(\xi)\right)^n d\xi\right)^p ds \\ &\leq \int_0^{\alpha(t)} b(s)\left(m\left(a(s)+y(s)\right)+1-m+\int_0^s c(\xi)\left(n\left(a(\xi)+y(\xi)\right)+1-n\right) d\xi\right)^p ds \\ &= \int_0^{\alpha(t)} b(s)\biggl[\left(m+n\int_0^s c(\xi)\,d\xi\right)y(s) \\ &\quad + ma(s)+1-m+\int_0^s c(\xi)\left(na(\xi)+1-n\right) d\xi\biggr]^p ds. \end{aligned}$$

Using Lemma 2.3,

$$y(t) \leq \left(\tilde{k}^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1} b(s)\left(m+n\int_0^s c(\xi)\,d\xi\right)^p ds\right)^{\frac{1}{1-p}}, \quad t \geq 0,$$

where $\tilde{k}(t)$ is defined as in (2.10). This associated with (2.11) yields

$$u(t) \leq a(t) + \left(\tilde{k}^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1} b(s)\left(m+n\int_0^s c(\xi)\,d\xi\right)^p ds\right)^{\frac{1}{1-p}}. \qquad \square$$

Remark 2.1 When $0 < p \leq 1$, inequality (2.9) has been studied in [12, Theorem 2.1] and [21, Theorem 2.1]. However, the above results cannot be applied to the case $p > 1$. In Theorem 2.1, we further investigate (2.9) under the condition of $p > 1$. To some extent, our result extends the results in [12, Theorem 2.1] and [21, Theorem 2.1].

Theorem 2.2 *Suppose that p, q, m, n are nonnegative constants with $q \geq m > 0$, $q \geq n > 0$, $p > 0$, $\alpha(t)$ is nondecreasing with $\alpha \in C^1(R^+, R^+)$, $\alpha(t) \leq t$, $\alpha(0) = 0$, $u, a, b, c \in C(R^+, R^+)$, and*

$$u^q(t) \leq a(t) + \int_0^{\alpha(t)} b(s)\left(u^m(s) + \int_0^s c(\xi) u^n(\xi)\,d\xi\right)^p ds. \tag{2.13}$$

Then

$$u(t) \leq \begin{cases} [a(t) + \hat{g}(t) + \exp(\int_0^{\alpha(t)} \hat{h}(s)\,ds)\int_0^{\alpha(t)} \hat{g}(s)\hat{h}(s)\exp(-\int_0^s \hat{h}(\xi)\,d\xi)\,ds]^{1/q}, & 0 < p \leq 1, \\ [a(t) + (\hat{k}^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1} b(s)(\frac{m}{q} + \frac{n}{q}\int_0^s c(\xi)\,d\xi)^p\,ds)^{\frac{1}{1-p}}]^{1/q}, & p > 1, \end{cases} \tag{2.14}$$

with

$$\hat{k}^{1-p}(t) > (p-1)\int_0^{\alpha(t)} 2^{p-1} b(s)\left(\frac{m}{q} + \frac{n}{q}\int_0^s c(\xi)\,d\xi\right)^p ds,$$

where

$$\begin{aligned} \hat{h}(t) &= pb(t)\left(\frac{m}{q} + \frac{n}{q}\int_0^t c(\xi)\,d\xi\right), \\ \hat{g}(t) &= \int_0^{\alpha(t)} \left[pb(s)\left(\frac{m}{q}a(s) + \frac{q-m}{q}\right.\right. \\ &\quad \left.\left. + \int_0^s c(\xi)\left(\frac{n}{q}a(\xi) + \frac{q-n}{q}\right)d\xi\right) + (1-p)b(s)\right]ds, \\ \hat{k}(t) &= \int_0^{\alpha(t)} 2^{p-1}b(s)\left(\frac{m}{q}a(s) + \frac{q-m}{q} + \int_0^s c(\xi)\left(\frac{n}{q}a(\xi) + \frac{q-n}{q}\right)d\xi\right)^p ds. \end{aligned} \tag{2.15}$$

Proof Construct

$$z(t) = \int_0^{\alpha(t)} b(s)\left(u^m(s) + \int_0^s c(\xi)u^n(\xi)\,d\xi\right)^p ds. \tag{2.16}$$

Then $z(0) = 0$, z is a nondecreasing function, and

$$u(t) \le \big(a(t) + z(t)\big)^{1/q}. \tag{2.17}$$

By Lemma 2.1,

$$\begin{aligned} u^m(t) &\le \big(a(t) + z(t)\big)^{m/q} \le \frac{m}{q}\big(a(t) + z(t)\big) + \frac{q-m}{q}, \\ u^n(t) &\le \big(a(t) + z(t)\big)^{n/q} \le \frac{n}{q}\big(a(t) + z(t)\big) + \frac{q-n}{q}. \end{aligned} \tag{2.18}$$

From (2.16)–(2.18),

$$\begin{aligned} z(t) &\le \int_0^{\alpha(t)} b(s)\left(\big(a(s) + z(s)\big)^{m/q} + \int_0^s c(\xi)\big(a(\xi) + z(\xi)\big)^{n/q}d\xi\right)^p ds \\ &\le \int_0^{\alpha(t)} b(s)\left(\frac{m}{q}\big(a(s) + z(s)\big) + \frac{q-m}{q} + \int_0^s c(\xi)\left(\frac{n}{q}\big(a(\xi) + z(\xi)\big) + \frac{q-n}{q}\right)d\xi\right)^p ds \\ &= \int_0^{\alpha(t)} b(s)\left[\left(\frac{m}{q} + \frac{n}{q}\int_0^s c(\xi)\,d\xi\right)z(s)\right. \\ &\quad \left. + \frac{m}{q}a(s) + \frac{q-m}{q} + \int_0^s c(\xi)\left(\frac{n}{q}a(\xi) + \frac{q-n}{q}\right)d\xi\right]^p ds. \end{aligned} \tag{2.19}$$

Applying Lemma 2.3 to (2.19), we can obtain

$$z(t) \le \begin{cases} \hat{g}(t) + \exp(\int_0^{\alpha(t)} \hat{h}(s)\,ds)\int_0^{\alpha(t)} \hat{g}(s)\hat{h}(s)\exp(-\int_0^s \hat{h}(\xi)\,d\xi)\,ds, & 0 < p \le 1, \\ (\hat{k}^{1-p}(t) + (1-p)\int_0^{\alpha(t)} 2^{p-1}b(s)(\frac{m}{q} + \frac{n}{q}\int_0^s c(\xi)\,d\xi)^p\,ds)^{\frac{1}{1-p}}, & p > 1, \end{cases}$$

where $\hat{h}(t)$, $\hat{g}(t)$, and $\hat{k}(t)$ are defined by (2.15). This associated with (2.17) yields (2.14). □

Remark 2.2 Inequality (2.13) generalizes the ones in [4, 12, 21] to the more general nonlinear case.

3 Examples

Now, we study the boundedness of the integral equation and IDE with delay.

Example 3.1 Consider the Volterra type integral equation with delay

$$x(t) = a(t) + \int_0^{\alpha(t)} b(s)\left(x(s) + \int_0^s c(\xi)x(\xi)\,d\xi\right)^3 ds, \quad t \in R^+, \tag{3.1}$$

where $a, b, c \in C(R, R)$, $\alpha \in C^1(R^+, R^+)$, $\alpha(t) \le t$, $\alpha(0) = 0$, $\alpha(t)$ is a nondecreasing function. Then (3.1) satisfies

$$|x(t)| \le |a(t)| + \int_0^{\alpha(t)} |b(s)|\left(|x(s)| + \int_0^s |c(\xi)||x(\xi)|\,d\xi\right)^3 ds. \tag{3.2}$$

Let $u(t) = |x(t)|$ and rewrite (3.2):

$$u(t) \le |a(t)| + \int_0^{\alpha(t)} |b(s)|\left(u(s) + \int_0^s |c(\xi)|u(\xi)\,d\xi\right)^3 ds.$$

By Theorem 2.1,

$$u(t) \le |a(t)| + \left(\tilde{k}^{-2}(t) - 8\int_0^{\alpha(t)} |b(s)|\left(1 + \int_0^s |c(\xi)|\,d\xi\right)^3 ds\right)^{-\frac{1}{2}}$$

with

$$\tilde{k}(t) < \frac{\sqrt{2}}{4}\left(\int_0^{\alpha(t)} |b(s)|\left(1 + \int_0^s |c(\xi)|\,d\xi\right)^3 ds\right)^{-\frac{1}{2}},$$

where

$$\tilde{k}(t) = 4\int_0^{\alpha(t)} |b(s)|\left(|a(s)| + \int_0^s |c(\xi)||a(\xi)|\,d\xi\right)^3 ds,$$

which illustrates that the solution of (3.1) is bounded.

Example 3.2 Consider the delay IDE

$$\left(x^q(t)\right)' = F\left(t, x(\tau(t)), \int_0^t G(\xi, x(\tau(\xi)))\,d\xi\right), \tag{3.3}$$

and $x(t) = \varphi(t)$, $t \in [d, 0]$ with $-\infty < d = \inf\{\tau(t), t \in I\} \le 0$, $\tau(t) \le t$, where $x(t)$ and $x(\tau(t))$ are the state and state delay, respectively. $F \in C(R^+ \times R \times R, R)$ and $G \in C(R^+ \times R, R)$ satisfy

$$\left|F(t, U, V)\right| \le b(t)\left(|U|^m + |V|\right)^p,$$
$$\left|G(t, W)\right| \le c(t)|W|^n, \quad t \in R^+,$$

where $b, c \in C(R^+, R^+)$, $q \ge m > 0$, $q \ge n > 0$, $p > 0$. Integrating (3.3) produces

$$x^q(t) = x^q(0) + \int_0^t F\left(s, x(\tau(s)), \int_0^s G(\xi, x(\tau(\xi)))\,d\xi\right) ds.$$

Letting $u(t) = |x(t)|$, then

$$\begin{aligned} u^q(t) &\le u^q(0) + \int_0^t b(s)\Big(u^m\big(\tau(s)\big) + \int_0^s c(\xi)u^n\big(\tau(\xi)\big)\,d\xi\Big)^p\,ds \\ &\le \big|\varphi(t)\big|^q + \int_0^{\tau(t)} \frac{b(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\Big(u^m(s) + \int_0^s \frac{c(\tau^{-1}(\xi))}{\tau'(\tau^{-1}(\xi))}u^n(\xi)d\xi\Big)^p\,ds. \end{aligned} \tag{3.4}$$

Employing Theorem 2.2 to (3.4) produces the following: when $0 < p \le 1$,

$$u(t) \le \Big[\big|\varphi(t)\big|^q + \hat{g}(t) + \exp\Big(\int_0^{\alpha(t)} \hat{h}(s)\,ds\Big)\int_0^{\alpha(t)} \hat{g}(s)\hat{h}(s)\exp\Big(-\int_0^s \hat{h}(\xi)\,d\xi\Big)\,ds\Big]^{1/q},$$

where

$$\begin{aligned} \hat{h}(t) &= p\frac{b(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}\Big(\frac{m}{q} + \frac{n}{q}\int_0^t \frac{c(\tau^{-1}(\xi))}{\tau'(\tau^{-1}(\xi))}\,d\xi\Big), \\ \hat{g}(t) &= \int_0^{\tau(t)}\Big[p\frac{b(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\Big(\frac{m}{q}\big|\varphi(s)\big|^q + \frac{q-m}{q} \\ &\quad + \int_0^s \frac{c(\tau^{-1}(\xi))}{\tau'(\tau^{-1}(\xi))}\Big(\frac{n}{q}\big|\varphi(\xi)\big|^q + \frac{q-n}{q}\Big)\,d\xi\Big) + (1-p)\frac{b(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\Big]\,ds, \end{aligned}$$

when $p > 1$,

$$\begin{aligned} u(t) \le \Big\{\big|\varphi(t)\big|^q + \Big[\hat{k}^{1-p}(t) \\ + (1-p)\int_0^{\tau(t)} 2^{p-1}\frac{b(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\Big(\frac{m}{q} + \frac{n}{q}\int_0^s \frac{c(\tau^{-1}(\xi))}{\tau'(\tau^{-1}(\xi))}\,d\xi\Big)^p\,ds\Big]^{\frac{1}{1-p}}\Big\}^{1/q} \end{aligned}$$

with

$$\hat{k}^{1-p}(t) > (p-1)\int_0^{\tau(t)} 2^{p-1}\frac{b(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\Big(\frac{m}{q} + \frac{n}{q}\int_0^s \frac{c(\tau^{-1}(\xi))}{\tau'(\tau^{-1}(\xi))}\,d\xi\Big)^p\,ds,$$

where

$$\begin{aligned} \hat{k}(t) &= \int_0^{\tau(t)} 2^{p-1}\frac{b(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\Big(\frac{m}{q}\big|\varphi(s)\big|^p + \frac{q-m}{q} \\ &\quad + \int_0^s \frac{c(\tau^{-1}(\xi))}{\tau'(\tau^{-1}(\xi))}\Big(\frac{n}{q}\big|\varphi(\xi)\big|^p + \frac{q-n}{q}\Big)\,d\xi\Big)^p\,ds. \end{aligned}$$

Acknowledgements
The authors sincerely thank the editor and anonymous reviewers for their careful reviews and useful suggestions on improving the presentation of the paper.

Funding
This work was supported by the National Natural Science Foundation of China under grant nos. 61807015, 61703180, the Natural Science Foundation of Shandong Province (ZR2017LF012), A Project of Shandong Province Higher Educational Science and Technology Program (J17KA157), and the Doctoral Scientific Research Foundation of University of Jinan (1008398).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally to this work. They both read and approved the final version of the manuscript.

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Received: 1 November 2019 Accepted: 20 March 2020 Published online: 31 March 2020

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