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# The controllability of nonlinear fractional differential system with pure delay

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## Abstract

In this study, we are currently investigating the controllability of nonlinear fractional differential control systems with delays in the state function. The solution representations of fractional delay differential equations have been established by using the delayed Mittag-Leffler function. Firstly we obtain the result of the controllability of a linear fractional control system with delay. Then, for the controllability criteria of nonlinear fractional delay system, we establish the set of sufficient conditions of nonlinear fractional differential systems with delay in their state function by using Schauder's fixed point theorem. In the end, a numerical example is constructed to support the results.

**MSC:** 34K05; 26A33

**Keywords:** Controllability; Delayed Mittag-Leffler type matrix; State delay; Caputo fractional derivative

## 1 Introduction

The fractional calculus and its applications have become popular because of the differential integral. It is an operator which consists of both integer-order derivatives and integrals as special cases. The fractional integral is used for better understanding of the summation of a given quantity. Podlubny [1] described it as a parameter of a regression model in case of an unknown order of integration. Conversely, the fractional derivative is generally used for describing damping. Fractional derivatives have several kinds, such as Caputo, Riemann–Liouville, Grunwald–Letnikov, and Hadamard, etc. [2–6].

During the previous decades, fractional differential equations acquired tremendous prominence and significance. In order to generate the mathematical modeling of several physical phenomena, the category of fractional differential equations of different types plays a significant role, and methods are used from not only mathematics but also dynamical systems, physics, engineering, biology, and control systems [7–13]. There are delays in the development of operational systems such as industrial systems, transmission lines, chemical processes, and rolling mill systems. Several researchers have considered applications of fractional differential equations to different areas, and some fundamental results have been obtained on fractional differential equations. There are different methods to solve the differential equations such as numerical methods and spectral methods,

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etc. However, the spectral method is advantageous over the numerical method due to its high accuracy. Podlubny, Bellman, and Cooke conducted a similar analysis on analytical solutions for linear delay differential equations [1, 14].

Many processes around the world not only depend upon the present state but also based on their complete past state. These types of processes are present in the industrial systems, transmission lines, chemical processes, and the rolling mill systems. Delay, integral, and integro-differential equations are used to mathematically present these systems. Laplace and Condorcet firstly introduced delay differential equations during the eighteenth century. Graphical tools, asymptotic solutions, and numerical methods are generally used to solve delay differential equations. Many researchers have attempted to find the solution of these equations with delays in state and control functions, respectively [15–17].

Controllability is among the key concepts of mathematical control theory. Controllability plays a key role in the development of complex theories of mathematical control. It is used to manipulate the actions of an object to achieve the desired outcome. In infinite and finite-dimensional systems this plays a major role. The fixed point technique for nonlinear systems is one of the most effective approaches for the controllability problem. Based on the nonlinear functions involved, some fixed point theorems were specifically used to determine the conditions of controllability. Sakthivel et al. [18] established the controllability conditions about nonlinear neutral fractional control systems. The controllability problem in the finite-dimensional spaces for the nonlinear fractional dynamical system was discussed by [19]. Moreover, Balachandran discussed the controllability conditions for fractional nonlinear control systems with distributed delay and multiple delays in [20, 21], respectively. Controllability problem for the linear and nonlinear control systems was discussed by Balachandran and Dauer in [22]. Approximate controllability results for the nonlinear fractional differential system were discussed by Sakthivel et al. in [23, 24]. In [25] controllability results were obtained for fractional neutral stochastic dynamical systems with Poisson jumps in the finite dimension, and controllability for the nonlinear fractional differential system in a Banach space was discussed by Muslims et al. in [26], whereas controllability results for the nonlinear fractional system with control delay were discussed in [27]. Recently, controllability results for linear and nonlinear fractional delay dynamical systems with time-varying delay in control have been discussed by Rajagopal [28]. Controllability results of nonlinear neutral-type fractional-order differential delay systems with impulsive effects have been studied by Sundara et al. [29]. Recent research work on controllability results for the nonlinear fractional differential system for higher-order fractional delay dynamical systems, damped delay systems with multiple delays in control, integro-differential equations with small unknown parameters, and damping in Hilbert spaces has been carried out by researchers [30–33], respectively. Moreover, research work about the controllability results of linear and nonlinear fractional-order integrodifferential delay system by using Schauder's fixed point theorem and Arzela–Ascoli theorem has been discussed by Yi et al. in [34]. The previous studies mainly focused on controllability of nonlinear different types of fractional differential systems, but no work has reported on the controllability of nonlinear fractional differential system that discusses pure delay, which is presented in this manuscript. Motivated mainly by [34] and the above-mentioned works, by using the idea of a pure delay from research work [35], we consider the controllability conditions of nonlinear fractional systems with time delay in the state function represented by two-parameter delayed Mittag-Leffler type matrix by

using Schauder’s fixed point theorem. Li and Wang [36] considered pure delay for linear fractional differential equations and gave a representation of the solution by using two-parameter delayed Mittag-Leffler type matrix

$$\begin{cases} {}^c D_{0^+}^\delta y(x) = Ay(x-h), & y(x) \in \mathcal{R}^n, x \in J := [0, x_1], h > 0, \\ y(x) = \varphi(x), & -h \leq x \leq 0, \varphi \in C_h^1 := C^1([-h, 0], \mathcal{R}^n), \end{cases} \tag{1}$$

where  ${}^c D_{0^+}^\delta y(x)$  stands for  $\delta$  order Caputo fractional derivative of  $y(x)$  where zero is a lower limit,  $x_1$  is the integral multiple of  $h$ ,  $A \in \mathcal{R}^{n \times n}$ ,  $h > 0$  is a time delay,  $n \in \mathcal{N}$  stands for a constant matrix.  $\mathcal{E}_h^{A,\delta}$  is a new notation to express (delayed Mittag-Leffler type matrix) reported in Definition 2.3, any solution  $y \in C([-h, x_1], \mathcal{R}^n)$  of (1) can be established by Li

$$y(x) = \mathcal{E}_h^{Ax^\delta} \varphi(-h) + \int_{-h}^0 \mathcal{E}_h^{A(x-h-\lambda)^\delta} \varphi'(\lambda) d\lambda. \tag{2}$$

Motivated by the previous study, in this research work we deal with the fractional differential systems with state delay by using the virtue of explicit formula governed by

$$\begin{cases} {}^c D_{0^+}^\delta y(x) = Ay(x-h) + Bu(x), & y(x) \in J := [0, x_1], h > 0, x_1 \geq 0, \\ y(x) = \varphi(x), & -h \leq x \leq 0, \end{cases} \tag{3}$$

and the corresponding nonlinear system

$$\begin{cases} {}^c D_{0^+}^\delta y(x) = Ay(x-h) + Bu(x) + f(x, y(x-h), u(x)), & y(x) \in \mathcal{R}^n, x \in J, \\ y(x) = \varphi(x), & -h \leq x \leq 0, \end{cases} \tag{4}$$

where  $y : [-h, x_1] \rightarrow \mathcal{R}^n$  is continuous differentiable on  $[0, x_1]$  with  $x_1 > (n-1)h$ ,  $0 < \delta \leq 1$ ,  $A \in \mathcal{R}^{n \times n}$ ,  $B \in \mathcal{R}^{n \times m}$  are any matrices,  $h > 0$  shows time delay,  $y(x) \in \mathcal{R}^n$  denotes state vector,  $u(x) \in \mathcal{R}^m$  shows control vector,  $\varphi(x)$  shows the initial state function, and the nonlinear function  $f : J \times \mathcal{R}^n \times \mathcal{R}^m \rightarrow \mathcal{R}^n$  is continuous.

The assembling of this article is as follows. Section 2 includes some useful definitions, preliminary results, and lemmas about a delayed Mittag-Leffler type matrix to establish the controllability of a fractional differential system with state delay. In Sect. 3 we obtain sufficient conditions for controllability criteria of nonlinear fractional differential delay system (4) by using Schauder’s fixed point theorem. Section 4 presents one example to explain the applicability of the theoretical results.

## 2 Preliminaries and essential lemmas

This part includes some basic definitions and results used throughout this paper and some lemmas for the main results. We recall some known definitions. For more details, see [8, 37, 38].

**Definition 2.1** ([8]) A function  $f : [0, \infty) \rightarrow \mathcal{R}$  where its Caputo fractional derivative of order  $(0 < \delta < 1)$  is defined as

$$({}^c D_{0^+}^\delta y)(x) = \frac{1}{\Gamma(1-\delta)} \int_0^x \frac{y'(\theta)}{(x-\theta)^\delta} d\theta, \quad x > 0.$$

Here the gamma function is denoted by  $\Gamma(\cdot)$ .

**Definition 2.2** ([8]) A function  $f : [0, \infty) \rightarrow \mathcal{R}$  where its fractional integral of order  $\delta > 0$  is defined as

$$(I_{0^+}^\delta f)(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x - \theta)^{\delta-1} f(\theta) d\theta.$$

Here  $\Gamma(\cdot)$  denotes the gamma function.

**Definition 2.3** ([38]) A matrix  $\mathcal{E}_h^{A,\delta} : \mathcal{R} \rightarrow \mathcal{R}^{n \times n}$  known as a delayed Mittag-Leffler type matrix is defined as

$$\mathcal{E}_h^{Ax^\delta} = \begin{cases} \Theta, & -\infty < x < -h, \\ I, & -h \leq x \leq 0, \\ I + A \frac{(x)^\delta}{\Gamma(\delta+1)} + A^2 \frac{(x-h)^{2\delta}}{\Gamma(2\delta+1)} \\ \quad + \dots + A^k \frac{(x-(k-1)h)^{k\delta}}{\Gamma(k\delta+1)}, & (k-1)h < x \leq kh, k \in \mathcal{N}, \end{cases} \tag{5}$$

where zero and identity matrices are shown by  $\Theta$  and  $I$ , respectively.

**Definition 2.4** ([36]) A matrix  $\mathcal{E}_{h,\beta}^{A,\delta} : \mathcal{R} \rightarrow \mathcal{R}^{n \times n}$  known as delayed two-parameter Mittag-Leffler type matrix is defined as

$$\mathcal{E}_{h,\beta}^{Ax^\delta} = \begin{cases} \Theta, & -\infty < x < -h, \\ I \frac{(h+x)^{\delta-1}}{\Gamma(\beta)}, & -h \leq x \leq 0, \\ I \frac{(h+x)^{\delta-1}}{\Gamma(\beta)} + A \frac{x^{2\delta-1}}{\Gamma(\delta+\beta)} + A^2 \frac{(x-h)^{3\delta-1}}{\Gamma(2\delta+\beta)} \\ \quad + \dots + A^k \frac{(x-(k-1)h)^{(k+1)\delta-1}}{\Gamma(k\delta+\beta)}, & (k-1)h < x \leq kh, k \in \mathcal{N}. \end{cases} \tag{6}$$

**Lemma 2.5** ([36]) Let  $f : J \rightarrow \mathcal{R}^n$  be a continuous vector-valued function. A solution  $y \in C([-h, x_1], \mathcal{R}^n)$  of the following system. Consider the linear fractional delay differential equation of the form:

$$\begin{cases} {}^c D_{0^+}^\delta y(x) = Ay(x-h) + f(x), & y(x) \in \mathcal{R}^n, x \in J := [0, x_1], h > 0, \\ y(x) = \varphi(x), & -h \leq x \leq 0, \varphi \in C_h^1. \end{cases} \tag{7}$$

It can be written in the form of integral equation by using the method in [35]:

$$y(x) = \mathcal{E}_h^{Ax^\delta} \varphi(-h) + \int_{-h}^0 \mathcal{E}_h^{A(x-h-\lambda)^\delta} \varphi'(\lambda) d\lambda + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} f(\lambda) d\lambda.$$

By Lemma 2.5, a solution  $y \in C([-h, x_1], \mathcal{R}^n)$  of system (3) can be composed as a form

$$\begin{aligned} y(x) &= \mathcal{E}_h^{Ax^\delta} \varphi(-h) + \int_{-h}^0 \mathcal{E}_h^{A(x-h-\lambda)^\delta} \varphi'(\lambda) d\lambda \\ &\quad + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} Bu(\lambda) d\lambda. \end{aligned} \tag{8}$$

Similarly, a solution for nonlinear system (4) is as follows:

$$\begin{aligned}
 y(x) &= \mathcal{E}_h^{Ax^\delta} \varphi(-h) + \int_{-h}^0 \mathcal{E}_h^{A(x-h-\lambda)^\delta} \varphi'(\lambda) d\lambda \\
 &\quad + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} [Bu(\lambda) + f(\lambda, y(\lambda-h), u(\lambda))] d\lambda.
 \end{aligned}
 \tag{9}$$

### 3 Controllability results

In this part, controllability results of system (3) are investigated as in [35]. Firstly we prove one lemma, then by using this lemma the main results are constructed for nonlinear fractional differential delay system (4).

**Definition 3.1** Systems (3) and (4) are said to be controllable on  $J$  if, for every initial function  $\varphi(x)$  and  $y_1 \in \mathcal{R}^n$ , there exists a continuous control function  $u$  such that the solutions of (3) and (4) satisfy  $y(x_1) = y_1$ .

The solution of linear system (3) that corresponds to  $u$  can be written as follows:

$$\begin{aligned}
 y_s(x) &= \mathcal{E}_h^{Ax^\delta} \varphi(-h) + \int_{-h}^0 \mathcal{E}_h^{A(x-h-\lambda)^\delta} \varphi'(\lambda) d\lambda \\
 &\quad + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} Bu(\lambda) d\lambda, \quad x \in J,
 \end{aligned}
 \tag{10}$$

where

$$\chi(x; \varphi) = \mathcal{E}_h^{Ax^\delta} \varphi(-h) + \int_{-h}^0 \mathcal{E}_h^{A(x-h-\lambda)^\delta} \varphi'(\lambda) d\lambda.$$

Similarly, the solution of system (4) that corresponds to  $u$  takes the form

$$\begin{aligned}
 y(x) &= \chi(x; \varphi) + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} Bu(\lambda) d\lambda \\
 &\quad + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} f(\lambda, y(\lambda-h), u(\lambda)) d\lambda,
 \end{aligned}$$

for  $x \in J$  and  $y(x) = \varphi(x)$  for  $x \in [-h, 0]$  the Gramian matrix is defined as

$$W_{h,\delta}[0, x_1] = \int_0^{x_1} \mathcal{E}_{h,\delta}^{A(x_1-h-\lambda)^\delta} BB^T \mathcal{E}_{h,\delta}^{A^T(x_1-h-\lambda)^\delta} d\lambda,
 \tag{11}$$

where the symbol  $T$  indicates that the matrix is transposed. And the control function is taken as

$$u(x) = B^T \mathcal{E}_{h,\delta}^{A^T(x_1-h-\lambda)^\delta} W_{h,\delta}^{-1}[0, x_1] \chi(x_1; \varphi).
 \tag{12}$$

**Lemma 3.2** Linear system (3) is controllable on  $J$  if and only if  $W_{h,\delta}[0, x_1]$  is nonsingular.

*Proof* The proof is similar as given in [35], so it is omitted. □

Now we get our key outcomes on the controllability of nonlinear fractional control delay system (4), for which we take  $f : J \times \mathcal{R}^n \times \mathcal{R}^m \rightarrow \mathcal{R}^n$  is continuous,  $h > 0$  is time delay. Let us propose the following notation, labeling  $Q$  as a Banach space of continuous  $\mathcal{R}^n \times \mathcal{R}^m$ -valued functions identified with the norm on the interval  $J$ :

$$\|(y, u)\| = \|y\| + \|u\|,$$

where

$$\|y\| = \sup\{|y(x)| : x \in J\}$$

and

$$\|u\| = \sup\{|u(x)| : x \in J\},$$

that is,  $Q = C_n(J) \times C_m(J)$ , where  $C_n(J)$  is the continuous  $\mathcal{R}^n$ -valued function's Banach space defined on the interval  $J$  with the sup norm for each  $(z, v) \in Q$

$$p = (y, u) : [-h, x_1] \times [0, x_1] \in \mathcal{R}^n \times \mathcal{R}^m,$$

and let

$$|p| = |y| + |u|,$$

where  $\|y\| = \sup\{|y(x)| \text{ for } x \in [-h, x_1]\}$  and  $\|u\| = \sup\{|u(x)| \text{ for } x \in [0, x_1]\}$ .

**Theorem 3.3** *Let the continuous function  $f$  satisfy the condition*

$$\lim_{|p| \rightarrow \infty} \frac{|f(x, p)|}{|p|} = 0$$

*uniformly in  $x \in J$ , and suppose that linear system (3) is controllable, then nonlinear system (4) is also controllable on  $J$ .*

*Proof* Let  $y \in \mathcal{R}^n$  and  $Q$  be the Banach space of all functions where  $y$  is continuous and  $u$  is an admissible control function, with norm defined by

$$\|(y, u)\| = \|y\| + \|u\|.$$

Define the operator

$$T : Q \rightarrow Q$$

by

$$T(y, u) = (z, \mu),$$

where

$$\begin{aligned} \mu(x) = & B^T \mathcal{E}_{h,\delta}^{A^T(x_1-h-x)^\delta} W_{h,\delta}^{-1}[0, x_1] \left[ y_1 - \chi(x_1; \varphi) \right. \\ & \left. - \int_0^{x_1} \mathcal{E}_{h,\delta}^{A(x_1-h-x)^\delta} f(x, y(x-h), u(x)) dx \right] \end{aligned} \tag{13}$$

and

$$\begin{aligned} z(x) = & \chi(x; \varphi) + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} B \mu(\lambda) d\lambda \\ & + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} f(\lambda, y(\lambda-h), u(\lambda)) d\lambda \end{aligned} \tag{14}$$

for  $x \in J$ ,  $z(x) = \varphi(x)$ ,  $x \in [-h, 0]$ . Let

$$\begin{aligned} a_1 = & \sup_{0 \leq \lambda \leq x \leq x_1} \left\| \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} B \right\|, & a_2 = & \left\| W_{h,\delta}^{-1}[0, x_1] \right\|, \\ a_3 = & \sup \left\| \chi(x; \varphi) \right\| + \|y_1\|, & a_4 = & \sup_{x, \lambda \in J} \left\| \mathcal{E}_{h,\delta}^{A(x-h-\lambda)^\delta} \right\|, \\ a = & \max\{a_1 x_1, 1\}, & c_1 = & 4a a_1 a_2 a_4 x_1, \\ c_2 = & 4a_4 x_1, & c = & \max\{c_1, c_2\}, \\ d_1 = & 4a a_1 a_2 a_3, & d_2 = & 4a_3, & d = & \max\{d_1, d_2\}, \\ \sup |f| = & \sup |f(\lambda, y(\lambda-h), u(\lambda))|, & & \text{where } \lambda \in J. \end{aligned}$$

Then

$$\begin{aligned} |\mu(x)| \leq & a_1 a_2 \left[ a_3 + a_4 x_1 \sup_{\lambda \in J} |f(\lambda, y(\lambda-h), u(\lambda))| \right] \\ \leq & \left[ \frac{d_1}{4a} + \frac{c_1}{4a} \sup_{\lambda \in J} |f(\lambda, y(\lambda-h), u(\lambda))| \right] \\ \leq & \frac{1}{4} \left[ d + c \sup_{\lambda \in J} |f(\lambda, y(\lambda-h), u(\lambda))| \right] \end{aligned}$$

for  $x \in J$  and

$$\begin{aligned} |z(x)| \leq & a_3 + a_1 \|\mu\| x_1 + a_4 x_1 \sup_{\lambda \in J} |f(\lambda, y(\lambda-h), u(\lambda))| \\ \leq & \frac{d_2}{4} + a \|\mu\| + \frac{c_2}{4} \sup_{\lambda \in J} |f(\lambda, y(\lambda-h), u(\lambda))| \\ \leq & \frac{d}{4} + \frac{r}{4} + \frac{c}{4} \sup_{\lambda \in J} |f(\lambda, y(\lambda-h), u(\lambda))|. \end{aligned}$$

The function  $f$  satisfies the following conditions by the proposition in [39],  $f$  satisfies the following conditions: for each pair of positive constants  $c$  and  $d$ , there exists a positive constant  $r$  such that if  $|p| \leq r$ , then

$$c|f(x, p)| + d \leq r \quad \text{for all } x \in J. \tag{15}$$

For given  $c$  and  $d$ , if  $r$  is a constant, then the inequality in (15) is satisfied. Then any  $r_1$  such that  $r < r_1$  will also satisfy the inequality. Now, take  $c$  and  $d$  as given above, and let  $r$  be chosen so that the implication in (15) is satisfied. Therefore if

$$\|y\| \leq r/2$$

and

$$\|u\| \leq r/2,$$

then

$$|y(\lambda - h)| + |u(\lambda)| \leq r \quad \text{for all } \lambda \in J.$$

It follows that

$$d + c \sup |f(\lambda, y(\lambda - h), u(\lambda))| \leq r \quad \text{for } \lambda \in J.$$

Therefore,  $|\mu(x)| \leq r/4a$  for all  $x \in J$ , and hence  $\|\mu\| \leq r/4a$ . It follows that  $|z(x)| \leq r/4 + r/4$  for all  $x \in J$ , and hence that  $\|z\| \leq r/2$ , thus we have proved that if

$$Q(r) = \{(y, u) \in Q / \|y\| \leq r/2, \|u\| \leq r/2\},$$

then  $p$  maps  $Q(r)$  onto itself. Our objective is to prove that  $T$  has a fixed point since  $f$  is continuous, it follows that  $T$  is continuous. Let  $Q'$  be a bounded subset of  $Q$ . Consider a sequence  $\{z_i, \mu_i\}$  contained in  $T(Q')$ , where we let

$$\{z_i, \mu_i\} = T\{y_i, u_i\}$$

for some  $(y_i, u_i) \in Q'$  for  $i = 1, 2, 3, \dots$ . Since  $f$  is continuous, then

$$|f(\lambda, y(\lambda - h), u(\lambda))|$$

is uniformly bounded for all  $\lambda \in J$  and all  $i = 1, 2, 3, \dots$ , it follows that  $\{z_i, \mu_i\}$  is a bounded sequence in  $Q$ . Hence  $\{\mu_i(x)\}$  is an equicontinuous and uniformly bounded sequence on  $[0, x_1]$ . Since  $\{z_i(x)\}$  is a uniformly bounded and equicontinuous sequence on  $[-h, x_1]$ , a further application of Ascoli's theorem yields a further subsequence of  $\{z_i, \mu_i\}$ , which converges in  $Q$  to some  $(y_0, u_0)$ . It follows that  $T(Q')$  is sequentially compact, hence the closure is sequentially compact. Thus  $T$  is completely continuous. Since  $Q(r)$  is closed, bounded, and convex, Schauder's fixed point theorem implies that  $T$  has a fixed point  $(y, u) \in Q(r)$ . It follows that

$$y(x) = \chi(x; \varphi) + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)\delta} Bu(\lambda) d\lambda + \int_0^x \mathcal{E}_{h,\delta}^{A(x-h-\lambda)\delta} f(\lambda, y(\lambda - h), u(\lambda)) d\lambda$$



for  $x \in J, y(x) = \varphi(x)$  for  $x \in [-h, 0]$ . Hence  $y(x)$  is a solution of the system and

$$\begin{aligned} y(x_1) &= \chi(x; \varphi) + \int_0^{x_1} \mathcal{E}_{h,\delta}^{A(x_1-h-\lambda)^\delta} BB^T \mathcal{E}_{h,\delta}^{A^T(x_1-h-\lambda)^\delta} d\lambda * W_{h,\delta}^{-1}[0, x_1][y_1 - \chi(x_1; \varphi)] \\ &\quad - \int_0^{x_1} \mathcal{E}_{h,\delta}^{A(x_1-h-\lambda)^\delta} f(\lambda, y(\lambda-h), u(\lambda)) d\lambda \\ &\quad + \int_0^{x_1} \mathcal{E}_{h,\delta}^{A(x_1-h-\lambda)^\delta} f(\lambda, y(\lambda-h), u(\lambda)) d\lambda \\ &= y_1. \end{aligned}$$

Hence system (4) is controllable. □

#### 4 Example

In this section, we apply the conditions that we acquired in the previous section for controllability of a nonlinear fractional differential system with delay.

*Example 1* Let  $\delta = 0.7, h = 0.3, p = 3, x_1 = 1.2$ , and  $n = 3$ , consider the following nonlinear fractional delay controlled differential system's controllability:

$$\begin{cases} {}^c D_{0^+}^{0.7} y(x) = Ay(x-0.3) + Bu(x) + f(x, y(x-h), u(x)), & y(x) \in \mathcal{R}^2, y \in J := [0, 1.2], \\ y(x) = \varphi(x), & -0.3 \leq x \leq 0, \end{cases} \quad (16)$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix},$$

and

$$f(x, p) = \begin{pmatrix} \frac{1}{1+\gamma_2^2(x-h)+u^2(x)} \\ \frac{1}{1+\gamma_1^2(x-h)+u^2(x)} \end{pmatrix}.$$

Here

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

The following explicit form is used for the fractional delayed Gram matrix of (16) via (6) through elementary calculation:

$$\begin{aligned} W_{0.3,0.7}[0, 1.2] &= \int_0^{1.2} \mathcal{E}_{0.3,0.7}^{A(1.2-0.3-\lambda)^{(0.7)}} BB^T \mathcal{E}_{0.3,0.7}^{A^T(1.2-0.3-\lambda)^{(0.7)}} d\lambda, \\ W_{0.3,0.7}[0, 1.2] &= \int_0^{1.2} \mathcal{E}_{0.3,0.7}^{A(1.2-0.3-\lambda)^{(0.7)}} B^2 \mathcal{E}_{0.3,0.7}^{A^T(1.2-0.3-\lambda)^{(0.7)}} d\lambda, \\ W_{0.3,0.7}[0, 1.2] &= W_1 + W_2 + W_3 + W_4, \\ W_1 &= \int_0^{0.3} \left( I \frac{(1.2-\lambda)^{-0.3}}{\Gamma(0.7)} + A \frac{(0.9-\lambda)^{0.4}}{\Gamma(1.4)} + A^2 \frac{(0.6-\lambda)^{1.1}}{\Gamma(2.1)} + A^3 \frac{(0.3-\lambda)^{1.8}}{\Gamma(2.8)} \right) \end{aligned}$$

$$\begin{aligned}
 & \times B^2 \left( I \frac{(1.2 - \lambda)^{-0.3}}{\Gamma(0.7)} + A^T \frac{(0.9 - \lambda)^{0.4}}{\Gamma(1.4)} \right. \\
 & \left. + (A^T)^2 \frac{(0.6 - \lambda)^{1.1}}{\Gamma(2.1)} + (A^T)^3 \frac{(0.3 - \lambda)^{1.8}}{\Gamma(2.8)} \right) d\lambda, \tag{17} \\
 W_2 &= \int_{0.3}^{0.6} \left( I \frac{(1.2 - \lambda)^{-0.3}}{\Gamma(0.7)} + A \frac{(0.9 - \lambda)^{0.4}}{\Gamma(1.4)} + A^2 \frac{(0.6 - \lambda)^{1.1}}{\Gamma(2.1)} \right) \\
 & \times B^2 \left( I \frac{(1.2 - \lambda)^{-0.3}}{\Gamma(0.7)} + A^T \frac{(0.9 - \lambda)^{0.4}}{\Gamma(1.4)} + (A^T)^2 \frac{(0.6 - \lambda)^{1.1}}{\Gamma(2.1)} \right) d\lambda, \\
 W_3 &= \int_{0.6}^{0.9} \left( I \frac{(1.2 - \lambda)^{-0.3}}{\Gamma(0.7)} + A \frac{(0.9 - \lambda)^{0.4}}{\Gamma(1.4)} \right) \\
 & \times B^2 \left( I \frac{(1.2 - \lambda)^{-0.3}}{\Gamma(0.7)} + A^T \frac{(0.9 - \lambda)^{0.4}}{\Gamma(1.4)} \right) d\lambda, \\
 W_4 &= \int_{0.9}^{1.2} \left( I \frac{(1.2 - \lambda)^{-0.3}}{\Gamma(0.7)} \right) B^2 \left( I \frac{(1.2 - \lambda)^{-0.3}}{\Gamma(0.7)} \right) d\lambda.
 \end{aligned}$$

By calculation, we get

$$\begin{aligned}
 W_1 &= \begin{pmatrix} 0.9042 & 0.4455 \\ 0.4455 & 0.5205 \end{pmatrix}, \\
 W_2 &= \begin{pmatrix} 0.6964 & 0.2225 \\ 0.2225 & 0.5579 \end{pmatrix}, \\
 W_3 &= \begin{pmatrix} 0.2678 & 0.0811 \\ 0.0811 & 0.2386 \end{pmatrix}, \\
 W_4 &= \begin{pmatrix} 0.3299 & 0 \\ 0 & 0.3299 \end{pmatrix}, \\
 W_{0.3,0.7}[0, 1.2] &= \begin{pmatrix} 2.1983 & 0.7491 \\ 0.7491 & 1.6469 \end{pmatrix}.
 \end{aligned}$$

The Gramian matrix  $W_{0.3,0.7}[0, 1.2]$  is nonsingular according to the results of the equation above. In addition, the nonlinear function  $f(x, p)$  satisfies the above results. In summary, fractional system (16) based on Theorem 3.3 is controllable on  $J$ .

### 5 Conclusion

This paper deals with the controllability of linear and nonlinear fractional differential delay systems. It should be noted that the solution representation has been established by using delayed Mittag-Leffler type matrix function. Controllability of a linear system is derived. Consequently, sufficient conditions for the nonlinear system are established by using Schauder’s fixed point theorem. To show the effectiveness of the theory, an example is provided.

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**Authors' contributions**

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