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# A hybrid adaptive synchronization protocol for nondeterministic perturbed fractional-order chaotic nonlinear systems

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## Abstract

In this paper, we investigate hybrid adaptive synchronization issue for a class of perturbed fractional-order chaotic systems with nondeterministic nonlinear terms. On the basis of fractional-order extended version of Lyapunov stability criterion, a novel fuzzy adaptive synchronization control protocol coupled with backstepping-based method is constructed, ensuring that the synchronization errors converge to a sufficiently small region of the origin. In order to avert the occurrence of “explosion of complexity”, we take advantage of a fuzzy logic system to estimate the unknown systematic term approximately in every backstepping step. Finally, some numerical simulations are given to exemplify the effectiveness of the proposed approach.

**Keywords:** Adaptive synchronization; Backstepping-based method; Explosion of complexity; Fuzzy logic system; Nondeterministic perturbed fractional-order chaotic system

## 1 Introduction

Fractional calculus [1], as a greatly ancient subject, is overwhelmingly superior than integer-order calculus in various applications. This is ascribed to that fractional calculus provides not only a powerful algorithmic tool to facilitate complex numerical computing, but also a comprehensive mathematical model of enormous practical problems [2]. In view of heredity and memristive feature, fractional-order calculus can be utilized to model most of complex dynamic behaviors or specific materials (such as chaos, anomalous diffusions, viscoelastic damping structures, neural networks, and so on, see [3–8]) more precisely, beyond the integer-order calculus in general. Due to this, the topic of synchronization protocol design for fractional-order nonlinear systems has dramatically stirred plenty of excitement in many research fields. The synchronization issues can be dealt with by applying abundant control methods, including resilient control [9, 10], output-feedback control [11, 12], sliding mode control [13–17], fuzzy control [18–21], dynamic surface control [22–25], etc.

Synchronization issue of nonlinear systems is widely considered due to its valuable significance in both theoretical and practical aspects. The goal of synchronization is to design an active controller to synchronize the so-called slave dynamical system with another

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diverse one, namely the master. Various synchronization protocols have been proposed, including lag synchronization [26], projective synchronization [27], fixed time synchronization [28], and chaos synchronization [29, 30]. In essence, chaos synchronization generalizes chaos control [31], which enables the chaotic master–slave error dynamics trajectories to be asymptotically stable.

In real life, especially in control procedure, we are frequently confronted with a slew of information with ambiguity, randomness, incompatibility, incompleteness, and so forth. This led to the invention of many mathematical approaches (e.g., Zadeh's fuzzy set approach [32], backstepping approach [33–36]) to dispose of nondeterministic systematic parameters. As a kind of recursive control strategy, backstepping control has engaged attention because of its efficient performance in handling mismatched parametric uncertainties of integer-order nonlinear systems. Unfortunately, this control method has an inherent drawback, namely “explosion of complexity”, which is triggered by iteratively differentiating virtual control inputs (see [22]). Additionally, it requires complicated analysis to compute a so-called “regression matrix” (see [37]). Dawson et al. [38] pointed out the fact that the size of the regression matrix displayed too large when backstepping technique was applied to manipulate DC motors in a conventional manner. Such a complexity might be augmented significantly for a fractional-order nonlinear system. An available remedy for relaxing the limitation of backstepping control is to incorporate fuzzy inference approach [39] into backstepping proceedings. For instance, Tong et al. [40] put forward an observer-based adaptive backstepping control protocol for nondeterministic stochastic strict-feedback integer-order systems via fuzzy inference approach, and they also developed a simplified control protocol. Liu et al. [41] introduced a robust fuzzy backstepping control method for fractional-order nonlinear systems with triangle structures. Shukla et al. [42] exploited a backstepping technique to synchronize the tracking signals of fractional-order chaotic systems with constant parameters. However, their works seldom took into account fractional-order chaotic systems with functional uncertainties and external perturbations.

Motivated by this, we aim to address the backstepping-based synchronization issue of a class of fractional-order chaotic master–slave nonlinear systems. Compared with the previous works, our problem model involves nondeterministic external perturbations and more complicated parametric uncertainties, which expands the scope of applications. In order to achieve this goal, we propose a hybrid adaptive control method combined with backstepping technique and fuzzy inference approach. The contributions of our synchronization protocol are outlined to be twofold:

- (1) An appropriate fuzzy logic system is adopted as an estimation function routinely for the nondeterministic nonlinear term in each backstepping step;
- (2) A reasonable fuzzy adaptive control strategy based on backstepping method is established to attenuate all estimation errors and realize the synchronization between master and slave systems. With the aid of the proposed protocol, the occurrence of the drawback of “explosion of complexity” will be denied in every backstepping step.

The arrangement of this paper is listed as below. In Sect. 2, some fundamental notions and results involving with fractional calculus are recalled and a concrete description of model for the research issue is presented. In Sect. 3, we construct an adaptive backstepping-based controller via fuzzy inference approach, and analyze the systematic

synchronization on the basis of our proposed synchronization scheme. The validity of this synchronization scheme is demonstrated by numerical simulation in Sect. 4. Finally, we summarize the research in this paper and present an outlook for our further research in Sect. 5.

## 2 Preliminaries

### 2.1 Fractional calculus fundamental

In the full context, we denote the space of all real numbers (resp. complex numbers,  $n$ -dimensional real vectors) by  $\mathbb{R}$  (resp.  $\mathbb{C}$ ,  $\mathbb{R}^n$ ). For a vector  $x \in \mathbb{R}^n$ ,  $x^T$  denotes its transpose.

A fractional-order integral  ${}_0^C D_t^{-\beta} f(t)$  (denoted by  $D^{-\beta} f(t)$ , briefly) of order  $\beta \in (0, 1)$  is expressed by

$${}_0^C D_t^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} f(u) du,$$

where  $f(t)$  is a time-dependent function with  $t \geq 0$ ,  $\Gamma$  denotes the Gamma function, that is,

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt.$$

The Caputo derivative  ${}_0^C D_t^\beta f(t)$  (denoted by  $D^\beta f(t)$ , briefly) of order  $\beta \in (0, 1)$  is defined by

$${}_0^C D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta} f'(u) du. \quad (2.1)$$

In [1], the Laplace transform of Eq. (2.1) is represented as

$$\mathcal{L}\{D^\beta f(t); s\} = \int_0^\infty e^{-st} D^\beta f(t) dt = s^\beta F(s) - s^{\beta-1} f(0),$$

where  $\mathcal{L}$  denotes the Laplace transform operator,  $F(s) = \mathcal{L}\{f(t); s\}$ .

**Definition 2.1** ([43]) The two-parameter Mittag-Leffler function  $E_{\alpha, \beta}$ , with  $\alpha, \beta > 0$ , is given by

$$E_{\alpha, \beta}(z) = \sum_{t=0}^{\infty} \frac{z^t}{\Gamma(\alpha t + \beta)}, \quad z \in \mathbb{C}.$$

It is immediately seen that  $E_{1,1}(z) = e^z$ .

**Lemma 2.2** ([43]) Let  $s$  be a variable of the Laplace domain and  $v \in \mathbb{R}$ . Then

$$\mathcal{L}^{-1} \left\{ \frac{s^{\alpha-\beta}}{s^\alpha + v}; t \right\} = t^{\beta-1} E_{\alpha, \beta}(-vt^\alpha) \quad (\operatorname{Re}(s) > |v|^{1/\alpha}),$$

where  $\mathcal{L}^{-1}$  denotes the inverse Laplace transform operator,  $\operatorname{Re}(s)$  means the real part of  $s$ .

**Lemma 2.3** ([1]) *The inverse Laplace transform of the product of functions  $F_i : [0, +\infty) \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) is*

$$\mathcal{L}^{-1}\{F_1(s)F_2(s); t\} = f_1(t) * f_2(t),$$

where  $f_i(t) = \mathcal{L}^{-1}\{F_i(s); t\}$  ( $i = 1, 2$ ), and  $f_1(t) * f_2(t)$  denotes the convolution of  $f_i$ ,  $i = 1, 2$ , that is,

$$f_1(t) * f_2(t) = \int_0^t f_1(t-u)f_2(u) \, du.$$

**Lemma 2.4** ([1]) *Let  $\alpha \in (0, 1)$ ,  $\beta \in \mathbb{C}$ . Suppose there is a  $p \in \mathbb{R}$  such that*

$$\frac{\pi\alpha}{2} < p < \min\{\pi, \pi\alpha\}. \quad (2.2)$$

*Then, for all positive integers  $n$ ,*

$$E_{\alpha, \beta}(z) = - \sum_{t=1}^n \frac{z^{-t}}{\Gamma(-\alpha t + \beta)} + o\left(\frac{1}{|z|^{n+1}}\right),$$

where  $p \leq |\arg(z)| \leq \pi$  as  $|z|$  tends to  $\infty$ .

**Lemma 2.5** ([1]) *Let  $\alpha \in (0, 2)$  and  $\beta \in \mathbb{R}$ . If Eq. (2.2) holds for some constant  $p > 0$ , then there is  $M > 0$  such that*

$$|E_{\alpha, \beta}(z)| \leq \frac{M}{1 + |z|}$$

for all  $z \in \mathbb{C}$  with  $|z| \geq 0$  and  $\beta \leq |\arg(z)| \leq \pi$ .

**Lemma 2.6** ([44]) *For arbitrary  $w(t) \in \mathbb{R}^n$  and  $t \in [0, +\infty)$ , it holds that*

$$\frac{1}{2}D^\alpha(w^T(t)w(t)) \leq w^T(t)D^\alpha w(t).$$

## 2.2 Fuzzy logic systems

Let  $\mathbf{x}(t) \in \mathbb{R}^n$  at time instant  $t \in [0, +\infty)$ , and  $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \dots, \mathcal{R}^{(m)}$  be fuzzy rules, which can be interpreted as [18–21, 39]:

$\mathcal{R}^{(k)}$ : IF  $x_1$  is  $E_1^k$  and  $x_2$  is  $E_2^k$  and ... and  $x_n$  is  $E_n^k$ ,

THEN  $\hat{f}(\mathbf{x}(t))$  is  $F^k$  ( $k = 1, 2, \dots, m$ ),

where  $x_k$  is the  $k$ th component of  $\mathbf{x}(t)$ ,  $E_j^k$  ( $j = 1, 2, \dots, n$ ) and  $F^k$  are fuzzy sets. A fuzzy logic system, with  $\mathbf{x}(t)$  and  $\hat{f}(\mathbf{x}(t))$  being the input-variable and the output-variable, respectively, is given by

$$\hat{f}(\mathbf{x}(t)) = \frac{\sum_{k=1}^m \eta_k(t) [\prod_{j=1}^n \mu_{E_j^k}(x_j(t))]}{\sum_{k=1}^m [\prod_{j=1}^n \mu_{E_j^k}(x_j(t))]}, \quad (2.3)$$

where  $\mu_{E_j^k}$  is the fuzzy membership function of  $E_j^k$  defined from  $\mathbb{R}$  to the interval  $[0, 1]$ , and  $\eta_k(t)$  (called the centroid of the fuzzy rule  $\mathcal{R}^{(k)}$ ) is a real number at which the fuzzy membership degree  $\mu_{F^k}$  for  $F^k$  is maximized, i.e.,

$$\eta_k(t) = \arg \max_{z \in \mathbb{R}} \mu_{F^k}(z).$$

In general, we set  $\mu_{F^k}(\eta_k(t)) = 1$  for simplicity.

A fuzzy basis function  $\phi_k : \mathbb{R}^n \rightarrow \mathbb{R}$  based on the fuzzy rule  $\mathcal{R}^{(k)}$  is defined by

$$\phi_k(\mathbf{x}(t)) = \frac{\prod_{j=1}^n \mu_{E_j^k}(x_j(t))}{\sum_{k=1}^m [\prod_{j=1}^n \mu_{E_j^k}(x_j(t))]}.$$

Denote  $\boldsymbol{\phi}(\mathbf{x}(t)) = (\phi_1(\mathbf{x}(t)), \phi_2(\mathbf{x}(t)), \dots, \phi_m(\mathbf{x}(t)))^T$ ,  $\boldsymbol{\eta}(t) = (\eta_1(t), \eta_2(t), \dots, \eta_m(t))^T$ . Then the output-variable of the fuzzy logic system (2.3) can be simplified as

$$\hat{f}(\mathbf{x}(t)) = \boldsymbol{\eta}^T(t) \boldsymbol{\phi}(\mathbf{x}(t)). \quad (2.4)$$

**Theorem 2.7** ([39]) *Let  $h : \Psi \rightarrow \mathbb{R}$  be a continuous function, where  $\Psi$  is compact in  $\mathbb{R}^n$ . Given a constant  $\zeta > 0$ , there is a fuzzy logic system (2.4) such that*

$$\sup_{\Psi} |h(\mathbf{x}(t)) - \hat{f}(\mathbf{x}(t))| \leq \zeta.$$

### 3 Problem description

In this paper, we concentrate on chaos synchronization issue of a type of uncertain fractional-order master system, which is represented as

$$\begin{cases} D^\alpha x_1(t) = f_1(\bar{\bar{x}}_1(t)), \\ \vdots \\ D^\alpha x_{n-1}(t) = f_{n-1}(\bar{\bar{x}}_{n-1}(t)), \\ D^\alpha x_n(t) = f_n(\bar{\bar{x}}_n(t)), \end{cases} \quad (3.1)$$

where  $\bar{\bar{x}}_i(t) = (x_1(t), x_2(t), \dots, x_i(t))^T \in \mathbb{R}^i$  ( $i = 1, 2, \dots, n$ ) describe variables of master pseudo-states at time instant  $t \in [0, +\infty)$ , and  $f_i$  denote nondeterministic smooth nonlinear functions.

The slave system coupled with system (3.1) is expressed by

$$\begin{cases} D^\alpha y_1(t) = q_1(\bar{\bar{y}}_1(t)) + y_2(t) + d_1(t), \\ D^\alpha y_2(t) = q_2(\bar{\bar{y}}_2(t)) + y_3(t) + d_2(t), \\ \vdots \\ D^\alpha y_{n-1}(t) = q_{n-1}(\bar{\bar{y}}_{n-1}(t)) + y_n(t) + d_{n-1}(t), \\ D^\alpha y_n(t) = q_n(\bar{\bar{y}}_n(t)) + u(t) + d_n(t), \end{cases} \quad (3.2)$$

where  $\bar{\bar{y}}_i(t) = (y_1(t), y_2(t), \dots, y_i(t))^T \in \mathbb{R}^i$  ( $1 \leq i \leq n$ ) stand for variables of slave pseudo-states,  $q_i$  denote nondeterministic smooth nonlinear real-valued functions,  $d_i(t) \in \mathbb{R}$  represent external perturbations, and  $u(t) \in \mathbb{R}$  is a synchronization controller which will be

specified later. The relationship between systems (3.1) and (3.2) is that slave (3.2) subjected to the dynamic behaviors of master (3.1) is asymptotically synchronized with master (3.1) with the aid of controller  $u(t)$ .

**Assumption 3.1** For each  $i$  ( $1 \leq i \leq n$ ), the external perturbation  $d_i(t)$  is bounded, that is,

$$|d_i(t)| \leq \bar{d}_i$$

for some known positive constant  $\bar{d}_i$ .

#### 4 Controller construction and stability analysis

The synchronization error  $e_i(t)$  for the master variable  $x_i(t)$  and its slave  $y_i(t)$  is defined as

$$e_i(t) = y_i(t) - x_i(t) \quad (i = 1, 2, \dots, n).$$

Subtracting (3.1) from (3.2) leads to

$$\begin{cases} D^\alpha e_1(t) = h_1(\bar{y}_1(t), \bar{x}_2(t)) + e_2(t) + d_1(t), \\ D^\alpha e_2(t) = h_2(\bar{y}_2(t), \bar{x}_3(t)) + e_3(t) + d_2(t), \\ \vdots \\ D^\alpha e_{n-1}(t) = h_{n-1}(\bar{y}_{n-1}(t), \bar{x}_n(t)) + e_n(t) + d_{n-1}(t), \\ D^\alpha e_n(t) = h_n(\bar{y}_n(t), \bar{x}_n(t)) + u(t) + d_n(t), \end{cases} \quad (4.1)$$

where

$$h_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) = q_i(\bar{y}_i(t)) - f_i(\bar{x}_i(t)) + x_{i+1}(t)$$

for  $i = 1, 2, \dots, n-1$ , and

$$h_n(\bar{y}_n(t), \bar{x}_n(t)) = q_n(\bar{y}_n(t)) - f_n(\bar{x}_n(t)).$$

Next, let us focus on the design of a fuzzy adaptive backstepping-based control protocol step by step.

*Step1.* To approximate the nondeterministic continuous function  $h_1$ , we construct the following fuzzy logic system, whose output-variable is determined by an estimation function  $\hat{h}_1$  for  $h_1$ :

$$\hat{h}_1(\eta_1(t), \bar{y}_1(t)) = \eta_1^T(t) \phi_1(\bar{y}_1(t)), \quad (4.2)$$

where  $\eta_1(t)$  is an adjustable 1-dimensional parameter vector and  $\phi_1$  is a fuzzy basis function. Moreover, with respect to  $\eta_1(t)$ , Theorem 2.7 guarantees the existence of the optimal parameter  $\eta_1^*$ , which is given by

$$\eta_1^* = \arg \min_{\eta_1(t)} \left[ \sup_{\bar{y}_1(t)} |h_1(\bar{y}_1(t), \bar{x}_2(t)) - \hat{h}_1(\eta_1(t), \bar{y}_1(t))| \right].$$

Here, the employment of  $\eta_1^*$  just only facilitates the analysis of the systematic stability, but it is not mandatory in the control protocol.

Define the optimal parametric error  $\tilde{\eta}_1$  and the optimal estimation error  $\varepsilon_1$  by

$$\tilde{\eta}_1(t) = \eta_1(t) - \eta_1^*, \quad (4.3)$$

$$\varepsilon_1(\bar{y}_1(t), \bar{x}_2(t)) = h_1(\bar{y}_1(t), \bar{x}_2(t)) - \hat{h}_1(\eta_1^*, \bar{y}_1(t)), \quad (4.4)$$

respectively. It can be easily seen that the boundedness of the estimation error  $\varepsilon_1$  is guaranteed by [45, 46] and Theorem 2.7, that is,

$$|\varepsilon_1(\bar{y}_1(t), \bar{x}_2(t))| \leq \varepsilon_1^*,$$

where  $\varepsilon_1^* > 0$  is a given constant.

Let  $\varrho_1(t) = e_1(t)$ . Based on Eqs. (4.2), (4.3), and (4.4), one gets

$$\begin{aligned} D^\alpha \varrho_1(t) &= h_1(\bar{y}_1(t), \bar{x}_2(t)) + e_2(t) + [\hat{h}_1(\eta_1^*, \bar{y}_1(t)) - \hat{h}_1(\eta_1^*, \bar{y}_1(t))] \\ &\quad + (\lambda_1 - \lambda_1) + d_1(t) \\ &= [h_1(\bar{y}_1(t), \bar{x}_2(t)) - \hat{h}_1(\eta_1^*, \bar{y}_1(t))] + \hat{h}_1(\eta_1^*, \bar{y}_1(t)) \\ &\quad + (e_2(t) - \lambda_1) + \lambda_1 + d_1(t) \\ &= \varepsilon_1(\bar{y}_1(t), \bar{x}_2(t)) - \tilde{\eta}_1^T(t) \phi_1(\bar{y}_1(t)) + \eta_1^T(t) \phi_1(\bar{y}_1(t)) \\ &\quad + \varrho_2(t) + \lambda_1 + d_1(t), \end{aligned} \quad (4.5)$$

where  $\varrho_2(t) = e_2(t) - \lambda_1$ , and  $\lambda_1$  is a virtual control input to be constructed later.

Define

$$\lambda_1 = -\eta_1^T(t) \phi_1(\bar{y}_1(t)) - k_{11} \varrho_1(t) - k_{21} \text{sign}(\varrho_1(t)) - k_{31} \text{sign}(\varrho_1(t)), \quad (4.6)$$

where  $k_{11}$ ,  $k_{21}$  and  $k_{31}$  are designed parameters with  $k_{11} > 0$ ,  $k_{21} \geq \varepsilon_1^*$ , and  $k_{31} \geq \bar{d}_1$ .

Substituting (4.6) into (4.5) yields

$$\begin{aligned} D^\alpha \varrho_1(t) &= \varepsilon_1(\bar{y}_1(t), \bar{x}_2(t)) - \tilde{\eta}_1^T(t) \phi_1(\bar{y}_1(t)) + \varrho_2(t) + d_1(t) \\ &\quad - k_{11} \varrho_1(t) - (k_{21} + k_{31}) \text{sign}(\varrho_1(t)). \end{aligned} \quad (4.7)$$

Multiplying both sides of (4.7) by  $\varrho_1(t)$  yields

$$\begin{aligned} \varrho_1(t) D^\alpha \varrho_1(t) &= \varrho_1(t) \varepsilon_1(\bar{y}_1(t), \bar{x}_2(t)) - \varrho_1(t) \tilde{\eta}_1^T(t) \phi_1(\bar{y}_1(t)) + \varrho_1(t) \varrho_2(t) \\ &\quad + \varrho_1(t) d_1(t) - k_{11} \varrho_1^2(t) - k_{21} |\varrho_1(t)| - k_{31} |\varrho_1(t)| \\ &\leq |\varrho_1(t)| \varepsilon_1^* - \varrho_1(t) \tilde{\eta}_1^T(t) \phi_1(\bar{y}_1(t)) + \varrho_1(t) \varrho_2(t) + |\varrho_1(t)| \bar{d}_1 \\ &\quad - k_{11} \varrho_1^2(t) - k_{21} |\varrho_1(t)| - k_{31} |\varrho_1(t)| \\ &\leq -\varrho_1(t) \tilde{\eta}_1^T(t) \phi_1(\bar{y}_1(t)) + \varrho_1(t) \varrho_2(t) - k_{11} \varrho_1^2(t). \end{aligned} \quad (4.8)$$

Now, construct a Lyapunov function as follows:

$$V_1(t) = \frac{1}{2} \varrho_1^2(t) + \frac{1}{2\xi_1} \tilde{\eta}_1^T(t) \tilde{\eta}_1(t). \quad (4.9)$$

The associated adaptation law can be designed as

$$D^\alpha \eta_1(t) = \xi_1 \varrho_1(t) \phi_1(\bar{y}_1(t)) - \hat{\xi}_1 \eta_1(t), \quad (4.10)$$

where  $\xi_1$  and  $\hat{\xi}_1$  are positive designed parameters.

Note that  $D^\alpha \tilde{\eta}_1(t) = D^\alpha \eta_1(t)$ . Taking fractional-order derivative on both sides of (4.9) and substituting (4.8) and (4.10) into it, by Lemma 2.6, we deduce that

$$\begin{aligned} D^\alpha V_1(t) &\leq \varrho_1(t) D^\alpha \varrho_1(t) + \frac{1}{\xi_1} \tilde{\eta}_1^T(t) D^\alpha \tilde{\eta}_1(t) \\ &\leq -\varrho_1(t) \tilde{\eta}_1^T(t) \phi_1(\bar{y}_1(t)) + \varrho_1(t) \varrho_2(t) - k_{11} \varrho_1^2(t) + \frac{1}{\xi_1} \tilde{\eta}_1^T(t) D^\alpha \tilde{\eta}_1(t) \\ &= -k_{11} \varrho_1^2(t) + \varrho_1(t) \varrho_2(t) + \frac{1}{\xi_1} \tilde{\eta}_1^T(t) [D^\alpha \tilde{\eta}_1(t) - \xi_1 \varrho_1(t) \phi_1(\bar{y}_1(t))] \\ &\leq -k_{11} \varrho_1^2(t) - \frac{\hat{\xi}_1}{\xi_1} \tilde{\eta}_1^T(t) \tilde{\eta}_1(t) + \varrho_1(t) \varrho_2(t) + \frac{\hat{\xi}_1}{2\xi_1} (\eta_1^*)^T \eta_1^* \\ &\leq -a_{11} V_1(t) + a_{21} + \varrho_1(t) \varrho_2(t), \end{aligned}$$

where  $a_{11} = \min\{2k_{11}, 2\hat{\xi}_1\}$  and  $a_{21} = \frac{\hat{\xi}_1}{2\xi_1} (\eta_1^*)^T \eta_1^*$ .

*Step2.* Let  $\varrho_3(t) = e_3(t) - \lambda_2$ , where  $\lambda_2$  is a virtual control input which will be defined later. Observe that

$$D^\alpha \varrho_2(t) = D^\alpha (e_2(t) - \lambda_1) = D^\alpha e_2(t) - D^\alpha \lambda_1. \quad (4.11)$$

Substituting (4.1) into (4.11) gives

$$\begin{aligned} D^\alpha \varrho_2(t) &= h_2(\bar{y}_2(t), \bar{x}_3(t)) + e_3(t) - \lambda_2 + \lambda_2 + d_2(t) - D^\alpha \lambda_1 \\ &= H_2(\bar{y}_2(t), \bar{x}_3(t)) + \varrho_3(t) + \lambda_2 + d_2(t), \end{aligned} \quad (4.12)$$

where  $H_2$  is a nondeterministic continuous function given by  $H_2(\bar{y}_2(t), \bar{x}_3(t)) = h_2(\bar{y}_2(t), \bar{x}_3(t)) - D^\alpha \lambda_1$ . In analogy to Step 1, to approximate unknown  $H_2$ , we adopt the employment of fuzzy logic system as follows:

$$\hat{H}_2(\eta_2(t), \bar{y}_2(t)) = \eta_2^T(t) \phi_2(\bar{y}_2(t)). \quad (4.13)$$

Establish the adaptation law and the virtual control input  $\lambda_2$  respectively by

$$D^\alpha \eta_2(t) = \xi_2 \varrho_2(t) \phi_2(\bar{y}_2(t)) - \hat{\xi}_2 \eta_2(t), \quad (4.14)$$

$$\lambda_2 = -\eta_2^T(t) \phi_2(\bar{y}_2(t)) - k_{12} \varrho_2(t) - (k_{22} + k_{32}) \text{sign}(\varrho_2(t)) - \varrho_1(t), \quad (4.15)$$



where  $\xi_2$  and  $\hat{\xi}_2$  are positive designed parameters;  $\eta_2(t)$  is an adjustable 2-dimensional parameter vector,  $\phi_2$  is a fuzzy basis function, while  $k_{12} > 0$ ,  $k_{22} \geq \varepsilon_2^*$ , and  $k_{32} \geq \bar{d}_2$ . Here,  $\varepsilon_2^* > 0$  is a certain constant satisfying  $|\varepsilon_2(\bar{y}_2(t), \bar{x}_3(t))| \leq \varepsilon_2^*$  with  $\varepsilon_2(\bar{y}_2(t), \bar{x}_3(t)) = H_2(\bar{y}_2(t), \bar{x}_3(t)) - \hat{H}_2(\eta_2^*, \bar{y}_2(t))$ .

Multiply both sides of (4.12) by  $\varrho_2(t)$ . Then, according to (4.13), (4.14), and (4.15),

$$\begin{aligned} \varrho_2(t) D^\alpha \varrho_2(t) &= \varrho_2(t) \varepsilon_2(\bar{y}_2(t), \bar{x}_3(t)) + \varrho_2(t) \varrho_3(t) - \varrho_2(t) \tilde{\eta}_2^T(t) \phi_2(\bar{y}_2(t)) \\ &\quad + \varrho_2(t) d_2(t) - k_{12} \varrho_2^2(t) - k_{22} |\varrho_2(t)| - k_{32} |\varrho_2(t)| - \varrho_1(t) \varrho_2(t) \\ &\leq |\varrho_2(t)| \varepsilon_2^* - \varrho_2(t) \tilde{\eta}_2^T(t) \phi_2(\bar{y}_2(t)) + \varrho_2(t) \varrho_3(t) + |\varrho_2(t)| \bar{d}_2 \\ &\quad - k_{12} \varrho_2^2(t) - k_{22} |\varrho_2(t)| - k_{32} |\varrho_2(t)| - \varrho_1(t) \varrho_2(t) \\ &\leq -\varrho_2(t) \tilde{\eta}_2^T(t) \phi_2(\bar{y}_2(t)) + \varrho_2(t) \varrho_3(t) - k_{12} \varrho_2^2(t) - \varrho_1(t) \varrho_2(t). \end{aligned} \quad (4.16)$$

Consider the Lyapunov candidate

$$V_2(t) = V_1(t) + \frac{1}{2} \varrho_2^2(t) + \frac{1}{2\xi_2} \tilde{\eta}_2^T(t) \tilde{\eta}_2(t). \quad (4.17)$$

Apply  $D^\alpha$  on both sides of (4.17). An application of Eqs. (4.14), (4.16), and Lemma 2.6 gives

$$\begin{aligned} D^\alpha V_2(t) &\leq D^\alpha V_1(t) + \varrho_2(t) D^\alpha \varrho_2(t) + \frac{1}{\xi_2} \tilde{\eta}_2^T(t) D^\alpha \tilde{\eta}_2(t) \\ &\leq D^\alpha V_1(t) - \varrho_2(t) \tilde{\eta}_2^T(t) \phi_2(\bar{y}_2(t)) + \varrho_2(t) \varrho_3(t) - k_{12} \varrho_2^2(t) \\ &\quad - \varrho_1(t) \varrho_2(t) + \frac{1}{\xi_2} \tilde{\eta}_2^T(t) D^\alpha \tilde{\eta}_2(t) \\ &\leq -a_{11} V_1(t) + a_{21} + \varrho_2(t) \varrho_3(t) - k_{12} \varrho_2^2(t) \\ &\quad + \frac{1}{\xi_2} \tilde{\eta}_2^T(t) [D^\alpha \tilde{\eta}_2(t) - \xi_2 \varrho_2(t) \phi_2(\bar{y}_2(t))] \\ &\leq -a_{11} V_1(t) + a_{21} + \varrho_2(t) \varrho_3(t) - k_{12} \varrho_2^2(t) \\ &\quad - \frac{\hat{\xi}_2}{\xi_2} \tilde{\eta}_2^T(t) \tilde{\eta}_2(t) + \frac{\hat{\xi}_2}{2\xi_2} (\eta_2^*)^T \eta_2^* \\ &\leq -a_{12} V_2(t) + a_{22} + \varrho_2(t) \varrho_3(t), \end{aligned}$$

where  $a_{12} = \min\{a_{11}, 2k_{12}, 2\hat{\xi}_2\}$  and  $a_{22} = a_{21} + \frac{\hat{\xi}_2}{2\xi_2} (\eta_2^*)^T \eta_2^*$ .

Step  $i$  ( $i = 3, 4, \dots, n-1$ ). Let  $\varrho_i(t) = e_i(t) - \lambda_{i-1}$ . Then

$$\begin{aligned} D^\alpha \varrho_i(t) &= h_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) + e_{i+1}(t) + d_i(t) - D^\alpha \lambda_{i-1} \\ &= H_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) + \varrho_{i+1}(t) + \lambda_i + d_i(t), \end{aligned} \quad (4.18)$$

where  $H_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) = h_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) - D^\alpha \lambda_{i-1}$  is an unknown function,  $\lambda_i$  is a virtual control input which is pending design. In analogy to Step 2, the functional uncertainty

can be approximated by the fuzzy logic system

$$\hat{H}_i(\eta_i(t), \bar{y}_i(t)) = \eta_i^T(t) \phi_i(\bar{y}_i(t)), \quad (4.19)$$

where  $\eta_i(t)$  is an adjustable parameter vector and  $\phi_i$  is a fuzzy basis function.

Put

$$\lambda_i = -\eta_i^T(t) \phi_i(\bar{y}_i(t)) - k_{1i} \varrho_i(t) - (k_{2i} + k_{3i}) \text{sign}(\varrho_i(t)) - \varrho_{i-1}(t), \quad (4.20)$$

and, in addition, select the adaptation law as

$$D^\alpha \eta_i(t) = \xi_i \varrho_i(t) \phi_i(\bar{y}_i(t)) - \hat{\xi}_i \eta_i(t), \quad (4.21)$$

where  $\xi_i$  and  $\hat{\xi}_i$  are positive designed parameters,  $k_{1i} > 0$ ,  $k_{2i} \geq \varepsilon_i^*$  with  $\varepsilon_i^*$  being a known positive constant such that  $|\varepsilon_i(\bar{y}_i(t), \bar{x}_{i+1}(t))| \leq \varepsilon_i^*$  for  $\varepsilon_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) = H_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) - \hat{H}_i(\eta_i^*, \bar{y}_i(t))$ ,  $k_{3i} \geq \bar{d}_i$ .

Multiply both sides of (4.18) by  $\varrho_i(t)$ . Using Eqs. (4.19), (4.20), and (4.21), this generates

$$\begin{aligned} \varrho_i(t) D^\alpha \varrho_i(t) &= \varrho_i(t) \varepsilon_i(\bar{y}_i(t), \bar{x}_{i+1}(t)) - \varrho_i(t) \tilde{\eta}_i^T(t) \phi_i(\bar{y}_i(t)) + \varrho_i(t) \varrho_{i+1}(t) \\ &\quad + \varrho_i(t) d_i(t) - k_{1i} \varrho_i^2(t) - k_{2i} |\varrho_i(t)| - k_{3i} |\varrho_i(t)| - \varrho_{i-1}(t) \varrho_i(t) \\ &\leq |\varrho_i(t)| \varepsilon_i^* - \varrho_i(t) \tilde{\eta}_i^T(t) \phi_i(\bar{y}_i(t)) + \varrho_i(t) \varrho_{i+1}(t) + |\varrho_i(t)| \bar{d}_i \\ &\quad - k_{1i} \varrho_i^2(t) - k_{2i} |\varrho_i(t)| - k_{3i} |\varrho_i(t)| - \varrho_{i-1}(t) \varrho_i(t) \\ &\leq -\varrho_i(t) \tilde{\eta}_i^T(t) \phi_i(\bar{y}_i(t)) + \varrho_i(t) \varrho_{i+1}(t) - k_{1i} \varrho_i^2(t) - \varrho_{i-1}(t) \varrho_i(t). \end{aligned} \quad (4.22)$$

Select the Lyapunov candidate of the form

$$V_i(t) = V_{i-1}(t) + \frac{1}{2} \varrho_i^2(t) + \frac{1}{2\xi_i} \tilde{\eta}_i^T(t) \tilde{\eta}_i(t). \quad (4.23)$$

Apply the derivative operator  $D^\alpha$  on both sides of (4.23) and substitute Eqs. (4.21) and (4.22) into it. By Lemma 2.6, one obtains

$$\begin{aligned} D^\alpha V_i(t) &\leq D^\alpha V_{i-1}(t) + \varrho_i(t) D^\alpha \varrho_i(t) + \frac{1}{\xi_i} \tilde{\eta}_i^T(t) D^\alpha \tilde{\eta}_i(t) \\ &\leq D^\alpha V_{i-1}(t) - \varrho_i(t) \tilde{\eta}_i^T(t) \phi_i(\bar{y}_i(t)) + \varrho_i(t) \varrho_{i+1}(t) \\ &\quad - k_{1i} \varrho_i^2(t) - \varrho_{i-1}(t) \varrho_i(t) + \frac{1}{\xi_i} \tilde{\eta}_i^T(t) D^\alpha \tilde{\eta}_i(t) \\ &\leq -a_{1,i-1} V_{i-1}(t) + a_{2,i-1} + \varrho_i(t) \varrho_{i+1}(t) \\ &\quad - k_{1i} \varrho_i^2(t) + \frac{1}{\xi_i} \tilde{\eta}_i^T(t) [D^\alpha \tilde{\eta}_i(t) - \xi_i \varrho_i(t) \phi_i(\bar{y}_i(t))] \\ &\leq -a_{1,i-1} V_{i-1}(t) + a_{2,i-1} + \varrho_i(t) \varrho_{i+1}(t) - k_{1i} \varrho_i^2(t) \\ &\quad - \frac{\hat{\xi}_i}{\xi_i} \tilde{\eta}_i^T(t) \tilde{\eta}_i(t) + \frac{\hat{\xi}_i}{2\xi_i} (\eta_i^*)^T \eta_i^* \end{aligned}$$

$$\leq -a_{1i}V_i(t) + a_{2i} + \varrho_i(t)\varrho_{i+1}(t),$$

where  $a_{1i} = \min\{a_{1,i-1}, 2k_{1i}, 2\hat{\xi}_i\}$  and  $a_{2i} = a_{2,i-1} + \frac{\hat{\xi}_i}{2\xi_i}(\eta_i^*)^T \eta_i^*$ .

Step  $n$ . From  $\varrho_n(t) = e_n(t) - \lambda_{n-1}$ , we derive

$$\begin{aligned} D^\alpha \varrho_n(t) &= h_n(\bar{y}_n(t), \bar{x}_n(t)) + u(t) + d_n(t) - D^\alpha \lambda_{n-1} \\ &= H_n(\bar{y}_n(t), \bar{x}_n(t)) + u(t) + d_n(t), \end{aligned} \quad (4.24)$$

where  $H_n(\bar{y}_n(t), \bar{x}_n(t)) = h_n(\bar{y}_n(t), \bar{x}_n(t)) - D^\alpha \lambda_{n-1}$  is an uncertain function. Similar to Step  $n-1$ , one may utilize an approximator and an adaptation law as

$$\hat{H}_n(\eta_n(t), \bar{y}_n(t)) = \eta_n^T(t) \phi_n(\bar{y}_n(t)), \quad (4.25)$$

$$D^\alpha \eta_n(t) = \xi_n \varrho_n(t) \phi_n(\bar{y}_n(t)) - \hat{\xi}_n \eta_n(t), \quad (4.26)$$

respectively, where  $\xi_n$  and  $\hat{\xi}_n$  are positive designed parameters,  $\varepsilon_n(\bar{y}_n(t), \bar{x}_n(t)) = H_n(\bar{y}_n(t), \bar{x}_n(t)) - \hat{H}_n(\eta_n^*, \bar{y}_n(t))$  satisfies  $|\varepsilon_n(\bar{y}_n(t), \bar{x}_n(t))| \leq \varepsilon_n^*$  with  $\varepsilon_n^* > 0$  being a known constant.

To accomplish the remaining protocol design procedure, the active controller may be constructed as in the following expression:

$$u(t) = -\eta_n^T(t) \phi_n(\bar{y}_n(t)) - k_{1n} \varrho_n(t) - k_{2n} \text{sign}(\varrho_n(t)) - k_{3n} \text{sign}(\varrho_n(t)) - \varrho_{n-1}(t), \quad (4.27)$$

where  $k_{1n} > 0, k_{2n} \geq \varepsilon_n^*$  and  $k_{3n} \geq \bar{d}_n$ .

Multiply both sides of (4.24) with  $\varrho_n(t)$ . By (4.25), (4.26), and (4.27), we get

$$\begin{aligned} \varrho_n(t) D^\alpha \varrho_n(t) &= \varrho_n(t) \varepsilon_n(\bar{y}_n(t), \bar{x}_n(t)) + \varrho_n(t) d_n(t) - \varrho_n(t) \tilde{\eta}_n^T(t) \phi_n(\bar{y}_n(t)) \\ &\quad - \varrho_{n-1}(t) \varrho_n(t) - k_{1n} \varrho_n^2(t) - k_{2n} |\varrho_n(t)| - k_{3n} |\varrho_n(t)| \\ &\leq |\varrho_n(t)| \varepsilon_n^* + |\varrho_n(t)| \bar{d}_n - \varrho_n(t) \tilde{\eta}_n^T(t) \phi_n(\bar{y}_n(t)) \\ &\quad - \varrho_{n-1}(t) \varrho_n(t) - k_{1n} \varrho_n^2(t) - k_{2n} |\varrho_n(t)| - k_{3n} |\varrho_n(t)| \\ &\leq -\varrho_n(t) \tilde{\eta}_n^T(t) \phi_n(\bar{y}_n(t)) - \varrho_{n-1}(t) \varrho_n(t) - k_{1n} \varrho_n^2(t). \end{aligned} \quad (4.28)$$

Consider the following Lyapunov candidate:

$$V_n(t) = V_{n-1}(t) + \frac{1}{2} \varrho_n^2(t) + \frac{1}{2\xi_n} \tilde{\eta}_n^T(t) \tilde{\eta}_n(t). \quad (4.29)$$

Compute fractional-order derivative of both sides of (4.29), then substitute (4.26) and (4.28) into it. By Lemma 2.6, this yields

$$\begin{aligned} D^\alpha V_n(t) &\leq D^\alpha V_{n-1}(t) + \varrho_n(t) D^\alpha \varrho_n(t) + \frac{1}{\xi_n} \tilde{\eta}_n^T(t) D^\alpha \tilde{\eta}_n(t) \\ &\leq D^\alpha V_{n-1}(t) - \varrho_{n-1}(t) \varrho_n(t) - k_{1n} \varrho_n^2(t) \\ &\quad - \varrho_n(t) \tilde{\eta}_n^T(t) \phi_n(\bar{y}_n(t)) + \frac{1}{\xi_n} \tilde{\eta}_n^T(t) D^\alpha \tilde{\eta}_n(t) \\ &\leq -a_{1,n-1} V_{n-1}(t) + a_{2,n-1} - k_{1n} \varrho_n^2(t) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\xi_n} \tilde{\eta}_n^T(t) [D^\alpha \tilde{\eta}_n(t) - \xi_n Q_n(t) \phi_n(\bar{y}_n(t))] \\
& \leq -a_{1,n-1} V_{n-1}(t) + a_{2,n-1} - k_{1n} Q_n^2(t) \\
& \leq -\frac{\hat{\xi}_n}{\xi_n} \tilde{\eta}_n^T(t) \tilde{\eta}_n(t) + \frac{\hat{\xi}_n}{2\xi_n} (\eta_n^*)^T \eta_n^* \\
& \leq -a_{1n} V_n(t) + a_{2n},
\end{aligned} \tag{4.30}$$

where  $a_{1n} = \min\{a_{1,n-1}, 2k_{1n}, 2\hat{\xi}_n\}$  and  $a_{2n} = a_{2,n-1} + \frac{\hat{\xi}_n}{2\xi_n} (\eta_n^*)^T \eta_n^*$ .

**Theorem 4.1** *Let Assumption 3.1 be fulfilled. If the controller  $u(t)$  is chosen as in (4.27), the virtual control inputs satisfy Eqs. (4.6), (4.15), and (4.20), and, moreover, the adaptation laws are determined by (4.10), (4.14), (4.21), and (4.26), then every synchronization error  $e_i(t)$  ( $i = 1, 2, \dots, n$ ) for systems (3.1) and (3.2) converges to a sufficiently small region of the origin as  $t \rightarrow +\infty$ .*

*Proof* By virtue of Eq. (4.30), we derive

$$D^\alpha V_n(t) + k(t) = -a_{1n} V_n(t) + a_{2n}, \tag{4.31}$$

for some function  $k(t) \geq 0$ . Applying the Laplace transform  $\mathcal{L}$  to (4.31), we obtain

$$W_n(s) = \frac{s^{\alpha-1}}{s^\alpha + a_{1n}} V_n(0) + \frac{s^{-1} a_{2n}}{s^\alpha + a_{1n}} - \frac{K(s)}{s^\alpha + a_{1n}}, \tag{4.32}$$

where  $W_n(s) = \mathcal{L}\{V_n(t); s\}$  and  $K(s) = \mathcal{L}\{k(t); s\}$ .

Take the inverse Laplace transform  $\mathcal{L}^{-1}$  of Eq. (4.32). Based on Lemmas 2.2 and 2.3, this yields

$$V_n(t) = V_n(0) E_{\alpha,1}(-a_{1n} t^\alpha) + a_{2n} t^\alpha E_{\alpha,\alpha+1}(-a_{1n} t^\alpha) - k(t) * t^{\alpha-1} E_{\alpha,\alpha}(-a_{1n} t^\alpha).$$

Trivially,  $k(t) * t^{\alpha-1} E_{\alpha,\alpha}(-a_{1n} t^\alpha) \geq 0$ , since  $k(t)$  and  $E_{\alpha,\alpha}(-a_{1n} t^\alpha)$  are nonnegative. Thereby,

$$|V_n(t)| \leq |V_n(0)| E_{\alpha,1}(-a_{1n} t^\alpha) + a_{2n} t^\alpha E_{\alpha,\alpha+1}(-a_{1n} t^\alpha).$$

Noting that  $\arg(-a_{1n} t^\alpha) = -\pi$  and  $|-a_{1n} t^\alpha| \geq 0$  for all  $t \geq 0$  with  $0 < \alpha < 2$ , by Lemma 2.5, we conclude that there is an  $M > 0$  with

$$|E_{\alpha,1}(-a_{1n} t^\alpha)| \leq \frac{M}{1 + a_{1n} t^\alpha}.$$

Hence,

$$\lim_{t \rightarrow \infty} |V_n(0)| E_{\alpha,1}(-a_{1n} t^\alpha) = 0.$$

Therefore, for each  $\epsilon > 0$ , there is a constant  $T_1 > 0$  satisfying

$$|V_n(0)| E_{\alpha,1}(-a_{1n} t^\alpha) \leq \frac{\epsilon}{3}$$

for any  $t \geq T_1$ .

On the other hand, it follows from Lemma 2.4 that

$$E_{\alpha, \alpha+1}(-a_{1n}t^\alpha) = \frac{1}{\Gamma(1)a_{1n}t^\alpha} + o\left(\frac{1}{|a_{1n}t^\alpha|^2}\right).$$

As a consequence, for each  $\epsilon > 0$ , there is a constant  $T_2 > 0$  satisfying

$$a_{2n}t^\alpha E_{\alpha, \alpha+1}(-a_{1n}t^\alpha) \leq \frac{a_{2n}}{a_{1n}} + \frac{\epsilon}{3}$$

for all  $t \geq T_2$ .

Select  $a_{1n}$  and  $a_{2n}$  such that  $\frac{a_{2n}}{a_{1n}} \leq \frac{\epsilon}{3}$ . Based on the preceding argument, we have

$$|V_n(t)| \leq \epsilon, \quad \forall t \geq \max\{T_1, T_2\}. \quad (4.33)$$

From (4.33), it can be inferred that all state variables and estimation errors are bounded in the closed-loop system according to (4.29). Hence, by the arbitrariness of  $\epsilon$ , every synchronization error  $e_i(t)$  tends towards a sufficiently small region of the origin ultimately.  $\square$

**Remark 4.2** In order to realize the systematic synchronization, we should adjust  $a_{2n}/a_{1n}$  to be as small as possible by means of parameterizing the fuzzy logic system properly. For instance, we can enlarge  $\hat{\xi}_i$  and reduce  $\hat{\xi}_i$  simultaneously.

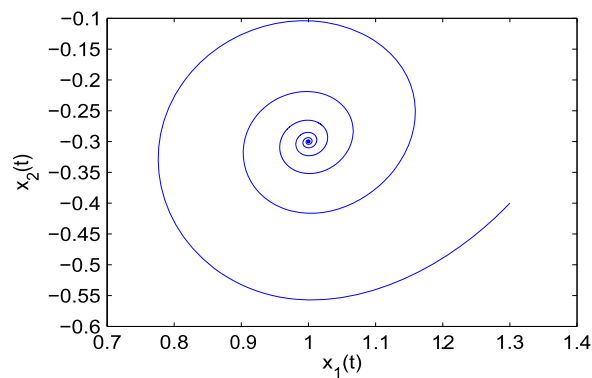
**Remark 4.3** Note that the proposed method is also valid if system (3.2) is described as

$$\begin{cases} D^\alpha y_1(t) = q_1(\bar{y}_1(t)) + b_1 y_2(t) + d_1(t), \\ D^\alpha y_2(t) = q_2(\bar{y}_2(t)) + b_2 y_3(t) + d_2(t), \\ \vdots \\ D^\alpha y_{n-1}(t) = q_{n-1}(\bar{y}_{n-1}(t)) + b_n y_n(t) + d_{n-1}(t), \\ D^\alpha y_n(t) = q_n(\bar{y}_n(t)) + u(t) + d_n(t), \end{cases}$$

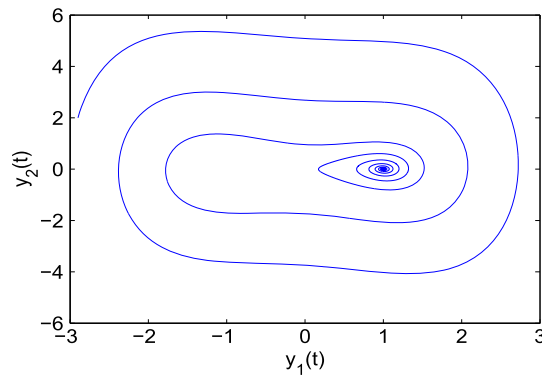
where  $b_1, b_2, \dots, b_n$  are known constants.

**Remark 4.4** It should be mentioned that Theorem 4.1 can be extended to analyze the stability of many other fractional-order nonlinear systems. Based on fractional Lyapunov stability criterion, it is not difficult to show that if there exist positive constants  $\tau_1, \tau_2$  with  $D^\alpha V(t) \leq -\tau_1 V(t) + \tau_2$ , where  $V(t) = \frac{1}{2}e^T(t)e(t)$  is a quadratic Lyapunov function, then  $e(t) \in \mathbb{R}^n$  is globally bounded and  $e(t) \leq \frac{\tau_2}{\tau_1}$  holds whenever the time instant  $t$  is sufficiently large.

**Remark 4.5** Under the proposed fuzzy adaptive backstepping control protocol, it can be apparently seen that the superfluous terms which might appear by recurring fractional derivations on virtual control inputs are fully averted, which is also suitable for many other fractional-order nonlinear systems. For the details, the readers may refer to Appendix B of [35].



**Figure 1** The master system in Example 1



**Figure 2** The slave system in Example 1

## 5 Numerical simulations

### 5.1 Example 1

Suppose that a master system is formed by

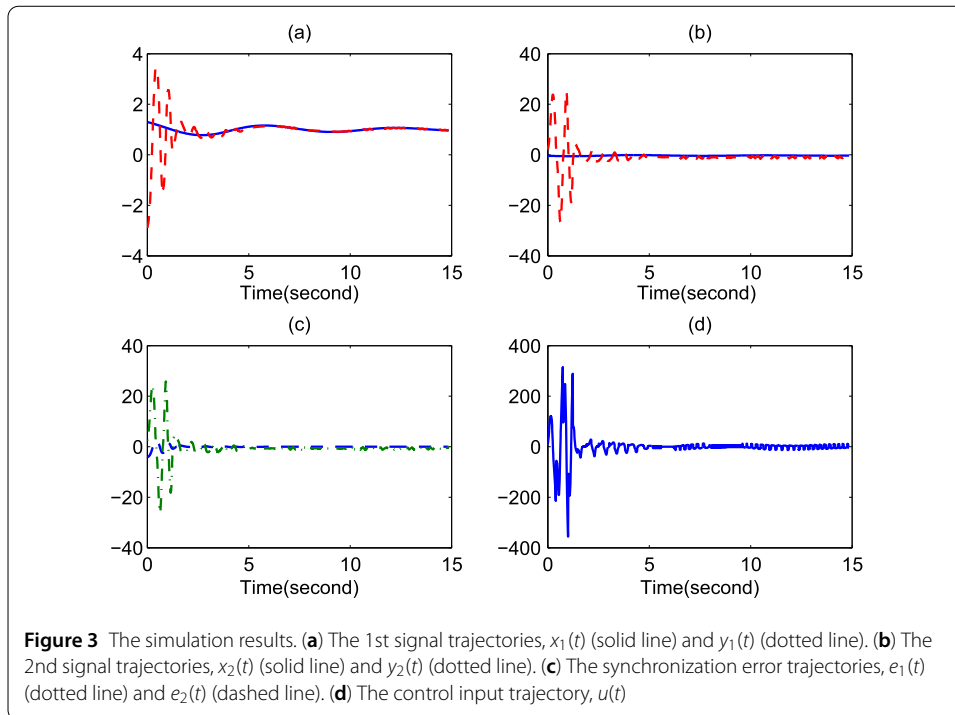
$$\begin{cases} D^\alpha x_1(t) = -0.5x_1^2(t) + x_2(t) + 0.8x_1(t), \\ D^\alpha x_2(t) = \frac{x_1(t) - x_1^3(t)}{1 + x_1^4(t)}. \end{cases} \quad (5.1)$$

The relevant slave system is formulated by

$$\begin{cases} D^\alpha y_1(t) = y_2(t) + d_1(t), \\ D^\alpha y_2(t) = y_1(t) - y_1^3(t) - 0.15y_2(t) + d_2(t) + u(t). \end{cases} \quad (5.2)$$

Let  $\alpha = 0.98$ ,  $d_i(t) = u(t) \equiv 0$ . Figures 1 and 2 exhibit the uncontrolled phenomena of system (5.1) with the initial value  $(x_1(0), x_2(0)) = (0.3, -0.4)$  and system (5.2) with the initial value  $(y_1(0), y_2(0)) = (-2.9, 2)$ , respectively.

Take two fuzzy logic systems into consideration in this simulation. Assume that  $y_1(t)$  is the input of the first fuzzy logic system with the Gaussian membership functions, which are expressed by  $\exp(-(x - c_i)^2/(2\sigma_i^2))$  for  $i = 1, 2, 3, 4$ . Suppose they are uniformly distributed on  $[-3, 3]$ , and consider the initial value  $\eta_1(0) = (1, 1, 1, 1)^T$ . The secondary one



treats  $y_1(t)$  and  $y_2(t)$  as its inputs. In terms of  $y_2(t)$ , there are five Gaussian functions treated as fuzzy membership functions uniformly distributed on  $[-4, 4]$  with the initial value  $\eta_2(0) = (1, \dots, 1)^T \in \mathbb{R}^{20}$ .

Let  $k_{11} = k_{12} = 1.2$ ,  $k_{13} = 1$ ,  $k_{21} = k_{22} = 1.6$ ,  $k_{23} = 2$ ,  $k_{31} = k_{32} = 2$ ,  $k_{33} = 5$ ,  $\xi_1 = \xi_2 = \xi_3 = 100$ , and  $\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi}_3 = 0.1$ , which are the parameters of the synchronization controller. Choose  $d_1(t) = 1$  and  $d_2(t) = 0.1 + \cos(t)$  as the external disturbances. The simulation results are revealed in Fig. 3.

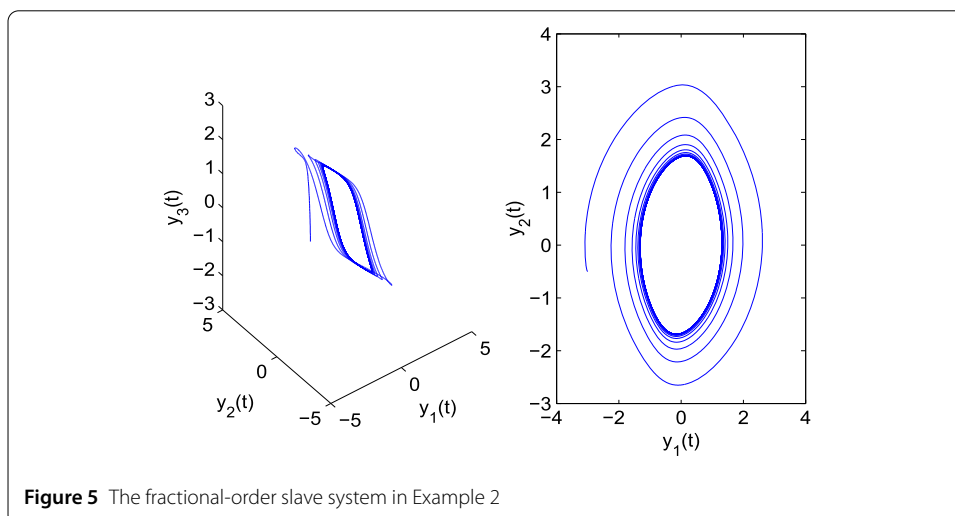
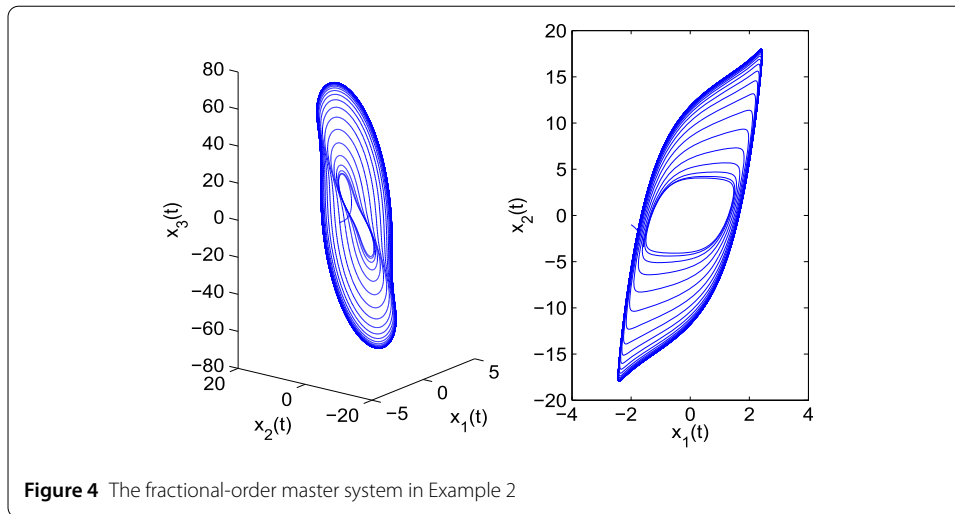
In view of the above-mentioned results, we observe that the synchronization errors reduce very rapidly and thereafter converge to a sufficiently small region as time elapses, which shows the outstanding performance of the fuzzy logic system in practical applications. Such results meet our expectation.

## 5.2 Example 2

Consider the master system as defined in [47]:

$$\begin{cases} D^\alpha x_1(t) = x_2(t) + \frac{10}{7}(x_1(t) - x_1^3(t)), \\ D^\alpha x_2(t) = x_3(t) + 10x_1(t) - x_2(t), \\ D^\alpha x_3(t) = -\frac{100}{7}x_2(t). \end{cases} \quad (5.3)$$

Let the initial value be  $(x_1(0), x_2(0), x_3(0)) = (-2, -1, 1)$ . According to [47], when  $\alpha = 0.98$ , chaos emerges in system (5.3), which is shown in Fig. 4.



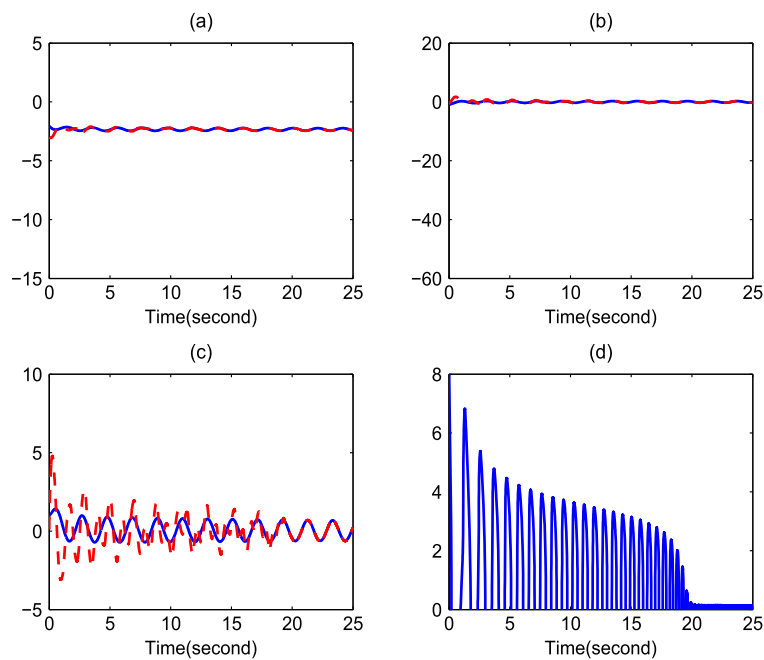
The slave system defined in [48] is

$$\begin{cases} D^\alpha y_1(t) = y_2(t) + d_1(t), \\ D^\alpha y_2(t) = y_3(t) + d_2(t), \\ D^\alpha y_3(t) = -\beta_1 y_1(t) - \beta_2 y_2(t) - \beta_3 y_3(t) + \beta_4 y_3^3(t) + d_3(t) + u(t). \end{cases} \quad (5.4)$$

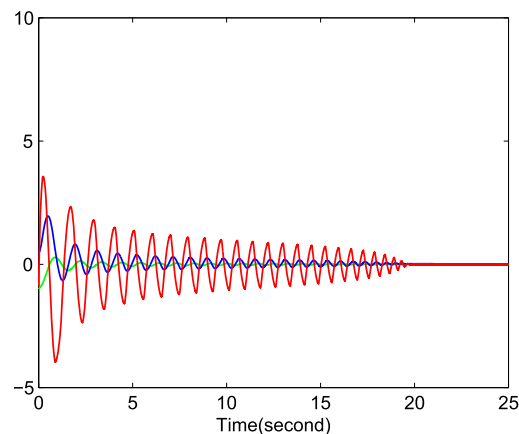
Without loss of generality, assume  $\beta_1 = \frac{100}{9}$ ,  $\beta_2 = 1.5$ ,  $\beta_3 = 1$ , and  $\beta_4 = -1$ . When  $\alpha = 0.98$ ,  $d_i(t) = u(t) \equiv 0$ , and the initial value is  $(y_1(0), y_2(0), y_3(0)) = (-3, -0.5, 0.1)$ , system (5.4) is in chaos, which is displayed in Fig. 5.

The simulation involves three fuzzy systems. The first is based on the four fuzzy membership functions as defined in Example 1, viewing  $y_1(t)$  as its input. Let the initial value be  $\eta_1(0) = (1, 1, 1, 1)^T$ . The second takes  $y_1(t)$  and  $y_2(t)$  as its inputs. For every input, the membership functions are defined similarly to that of the first fuzzy system with the initial value  $\eta_2(0) = (1, \dots, 1)^T \in \mathbb{R}^{16}$ . The last system regards  $y_1(t)$ ,  $y_2(t)$ , and  $y_3(t)$  as its inputs. With respect to  $y_3(t)$ , one defines five Gaussian membership functions uniformly distributed on  $[-4, 4]$ . The initial value is chosen as  $\eta_3(0) = (1, \dots, 1)^T \in \mathbb{R}^{80}$ .





**Figure 6** The state stabilization. **(a)** The 1st tracking trajectory,  $x_1(t)$  (solid line) and  $y_1(t)$  (dotted line). **(b)** The 2nd tracking trajectory,  $x_2(t)$  (solid line) and  $y_2(t)$  (dotted line). **(c)** The 3rd tracking trajectory,  $x_3(t)$  (solid line) and  $y_3(t)$  (dotted line). **(d)** The control input,  $u(t)$



**Figure 7** The convergency trajectory:  $e_1(t)$  (green line),  $e_2(t)$  (blue line), and  $e_3(t)$  (red line)

Choose  $k_{11} = k_{12} = 1.2$ ,  $k_{13} = 1$ ,  $k_{21} = k_{22} = 0.5$ ,  $k_{23} = 0.8$ ,  $k_{31} = k_{32} = k_{33} = 2$ ,  $\xi_1 = \xi_2 = \xi_3 = 10$ , and  $\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi}_3 = 0.05$  as the parameters of the controller. Set the external disturbances  $d_1(t) = \sin(10t)$ ,  $d_2(t) = 0.1 + \cos(t)$ , and  $d_3(t) = 0.1 + \sin(5t)$ , respectively. Figures 6 and 7 depict the state stabilization and the state synchronization under simulation, respectively. This indicates that the simulation results coincide with those of the preceding theoretical analysis.

## 6 Conclusion

This work provides a framework to study stabilization control of perturbed fractional-order chaotic systems with nondeterministic terms based on extended Lyapunov stability criterion. It is demonstrated by numerical simulations that the proposed adaptive fuzzy backstepping-based control strategy not only overcomes the inherent drawback of “explosion of complexity”, but also reflects the robust attribute for fractional-order chaotic systems consisting of parameter uncertainties and external perturbations. In the future, it is worth considering the synchronization issue of fractional-order systems with more sophisticated structures (for example, we can assume that the linear term coefficient is nondeterministic).

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## Availability of data and materials

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

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