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Estimations of fractional integral operators for convex functions and related results

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Abstract

This research investigates the bounds of fractional integral operators containing an extended generalized Mittag-Leffler function as a kernel via several kinds of convexity. In particular, the established bounds are studied for convex functions and further connected with known results. Furthermore, these results applied to the parabolic function and consequently recurrence relations for Mittag-Leffler functions are obtained. Moreover, some fractional differential equations containing Mittag-Leffler functions are constructed and their solutions are provided by Laplace transform technique.

Keywords: Convex function; Mittag-Leffler function; Generalized fractional integral operators; Fractional differential equations

1 Introduction

Fractional calculus is the generalization of classical calculus. Fractional integral/derivative operators play a key role in the development of fractional calculus. They have been used to formulate various physical and dynamic problems in fractional models. The complex behavior of physical systems can be represented in terms of fractional models. For applications of fractional calculus operators in sciences and engineering we refer to reader to [5–7, 19, 26, 51]. Physical properties of viscoelastic material can be interpreted by a model of fractional derivatives [4]. Furthermore, fractional calculus is applied to physics [21], bioengineering [29] optics [9, 18, 25], fluid flow [12], energy systems [11, 28] and biology [22–24].

On the other hand fractional integral/derivative operators have been used to construct and formulate new results in the theory of inequalities. Many of the well-known inequalities and related results are generalized and extended via fractional integral/derivative operators; see [2, 14–17, 30–32, 44, 52] and the references therein. At the same time convexity plays a vital role in enhancing the theory of inequalities, and facilitates optimization theory, mathematical analysis, mathematical statistics, graph theory with many other subjects. Fractional integral inequalities being suitable constraints provide existence and uniqueness of solutions for several mathematical problems in the form of fractional models.

The goal of this paper is the study of fractional integral operators containing Mittag-Leffler functions in their kernels. These operators are comprised in a single definition

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(Eq. (2.4) and Remark 1). We will analyze them for a generalized notion of convexity called $(h - m)$ -convexity. The method of proving the results of this paper can be utilized to get the results for other kinds of fractional and conformable integrals/derivatives already exist in the literature which authors will may consider for their future work; for instance for convenience Caputo–Fabrizio derivatives [3, 10] can be used.

2 Preliminary results

In this section we give definitions and notions which will be useful to establish the results of this paper.

Definition 1 Let I be an interval in \mathbb{R} . A function $f : I \rightarrow \mathbb{R}$ is said to be convex if, for all $a, b \in I$ and $0 \leq t \leq 1$, the following inequality holds:

$$f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b).$$

Convex functions are further generalized in different ways. One of the generalizations of convex functions is called $(h - m)$ -convexity that contains several kinds of convexity for example h -convexity, m -convexity, s -convexity defined on the right half of real line including zero (see [35, 45]).

Definition 2 Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \rightarrow \mathbb{R}$ be a non-negative function. A function $f : [0, b] \rightarrow \mathbb{R}$ is called $(h - m)$ -convex function, if f is non-negative and for all $x, y \in [0, b], m \in [0, 1]$ and $\alpha \in (0, 1)$, one has

$$f(\alpha x + m(1 - \alpha)y) \leq h(\alpha)f(x) + mh(1 - \alpha)f(y).$$

In the solution of integral and differential equations, the exponential function arises while in the solutions of fractional integral and differential equations, Mittag-Leffler function appears naturally. The Mittag-Leffler function is defined as follows [33]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}; \quad z \in \mathbb{C}, \alpha \in \mathbb{C}; \Re(\alpha) > 0. \tag{2.1}$$

The Mittag-Leffler functions are used in many areas of science and engineering, especially in the theory of fractional differential equations, in solutions of generalized fractional kinetic equations (see [40]). The Mittag-Leffler function was generalized by many mathematicians: for example Wiman [46], Agarwal [1], Prabhakar [36], Shukla and Prajapati [41], Salim [38], Salim and Faraj [39], Rahman et al. [37]. For a detailed study of this function see [20, 27, 36, 37, 39, 41–43].

Recently in [2], Andrić et al. defined the extended generalized Mittag-Leffler function $E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\cdot; p)$ as follows.

Definition 3 Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\alpha), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(t; p)$ is defined by

$$E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{2.2}$$

where β_p is the generalized beta function defined by

$$\beta_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt$$

and $(c)_{nk}$ is the Pochhammer symbol, $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

Lemma 1 ([2]) *If $m \in \mathbb{N}$, $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k < \delta + \Re(\mu)$, then*

$$\left(\frac{d}{dt}\right)^m [t^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega t^\mu; p)] = t^{\alpha-m-1} E_{\mu, \alpha-m, l}^{\gamma, \delta, k, c}(\omega t^\mu; p); \quad \Re(\alpha) > m. \tag{2.3}$$

The left and right sided fractional integral operators corresponding to the Mittag-Leffler function (2.2) are defined as follows.

Definition 4 ([2]) *Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators $\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} f$ and $\epsilon_{\mu, \alpha, l, \omega, b^-}^{\gamma, \delta, k, c} f$ are defined by*

$$(\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} f)(x; p) = \int_a^x (x-t)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) f(t) dt, \tag{2.4}$$

$$(\epsilon_{\mu, \alpha, l, \omega, b^-}^{\gamma, \delta, k, c} f)(x; p) = \int_x^b (t-x)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t-x)^\mu; p) f(t) dt. \tag{2.5}$$

The following remark provides the connection of integral operators with already known fractional integral operators.

Remark 1 The operator in (2.4) contains various fractional operators:

- (i) Setting $p = 0$, it reduces to the fractional integral operator defined by Salim–Faraj in [39].
- (ii) Setting $l = \delta = 1$, it reduces to the fractional integral operator defined by Rahman et al. in [37].
- (iii) Setting $p = 0$ and $l = \delta = 1$, it reduces to the fractional integral operator defined by Srivastava and Tomovski in [42].
- (iv) Setting $p = 0$ and $l = \delta = k = 1$, it reduces to the fractional integral operator defined by Prabhakar in [36].
- (v) Setting $p = \omega = 0$, it reduces to the Riemann–Liouville fractional integral.

Fractional integral/derivative operators containing Mittag-Leffler functions are studied extensively with the prospect of their utilization in different fields; see [8, 27, 36–38, 41, 42].

The following formulas are frequently used [16] in this paper:

$$(\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} 1)(x; p) = (x-a)^\alpha E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p) := C_{\alpha, a^+}(x; p), \tag{2.6}$$

$$(\epsilon_{\mu, \beta, l, \omega, b^-}^{\gamma, \delta, k, c} 1)(x; p) = (b-x)^\beta E_{\mu, \beta+1, l}^{\gamma, \delta, k, c}(\omega(b-x)^\mu; p) := C_{\beta, b^-}(x; p). \tag{2.7}$$

The following lemmas are useful in establishing Hadamard type estimations.

Lemma 2 ([14]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq f(x), \quad x \in [a, b]. \tag{2.8}$$

Lemma 3 ([15]) *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a $(h - m)$ -convex function. If $0 \leq a < b$ and $f(x) = f\left(\frac{a+b-x}{m}\right)$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq (m + 1)h\left(\frac{1}{2}\right)f(x), \quad x \in [a, b]. \tag{2.9}$$

The rest of the paper is organized as follows: In Sect. 2 the bounds of sum of the left and right sided generalized fractional integral operators (2.4) and (2.5) via $(h - m)$ -convex functions are established. These bounds hold for several kinds of convexity as well as for several fractional integral operators. Further in Sect. 3 they are computed for the parabolic function $y = x^2$, and as a result recurrence relations for Mittag-Leffler functions are obtained. Section 4 consists of generalized fractional differential equations and their solutions are computed in terms of the Mittag-Leffler function.

3 Bounds of generalized fractional integral operators

Theorem 1 *Let $f : [a, b] \rightarrow \mathbb{R}$, $0 \leq a < b$, be a real valued function. If f is positive and $(h - m)$ -convex, then, for $\alpha, \beta \geq 1$, we have*

$$\begin{aligned} & (\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} f)(x; p) + (\epsilon_{\mu, \beta, l, \omega, b^-}^{\gamma, \delta, k, c} f)(x; p) \\ & \leq \left((x - a)f(a)C_{\alpha-1, a^+}(x; p) + (b - x)f(b)C_{\beta-1, b^-}(x; p) \right. \\ & \quad \left. + mf\left(\frac{x}{m}\right)\left((x - a)C_{\alpha-1, a^+}(x; p) + (b - x)C_{\beta-1, b^-}(x; p)\right) \right) \int_0^1 h(z) dz. \end{aligned} \tag{3.1}$$

Proof Let $x \in [a, b]$. Then first we observe the function f on the interval $[a, x]$; for $t \in [a, x]$ and $\alpha \geq 1$, one has the following inequality:

$$(x - t)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - t)^\mu; p) \leq (x - a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p). \tag{3.2}$$

As f is $(h - m)$ -convex, so for $t \in [a, x]$, we have

$$f(t) \leq h\left(\frac{x - t}{x - a}\right)f(a) + mh\left(\frac{t - a}{x - a}\right)f\left(\frac{x}{m}\right). \tag{3.3}$$

Multiplying (3.2) and (3.3), then integrating over $[a, x]$, we get

$$\begin{aligned} & \int_a^x (x - t)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - t)^\mu; p) f(t) dt \\ & \leq f(a)(x - a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) \int_a^x h\left(\frac{x - t}{x - a}\right) dt \\ & \quad + mf\left(\frac{x}{m}\right)(x - a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) \int_a^x h\left(\frac{t - a}{x - a}\right) dt. \end{aligned} \tag{3.4}$$

By using (2.4) on left hand side and (2.6) on right hand side, we have

$$(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x;p) \leq (x-a)C_{\alpha-1,a^+}(x;p) \left(f(a) + mf\left(\frac{x}{m}\right) \right) \int_0^1 h(z) dz. \tag{3.5}$$

Now on the other hand we address the function f on the interval $[x, b]$; for $t \in [x, b]$ and $\beta \geq 1$, one has the following inequality:

$$(t-x)^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(t-x)^\mu;p) \leq (b-x)^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(b-x)^\mu;p). \tag{3.6}$$

Again from $(h-m)$ -convexity of f for $t \in [x, b]$, we have

$$f(t) \leq h\left(\frac{t-x}{b-x}\right)f(b) + mh\left(\frac{b-t}{b-x}\right)f\left(\frac{x}{m}\right). \tag{3.7}$$

Similarly multiplying (3.6) and (3.7), then integrating over $[x, b]$, we get

$$(\epsilon_{\mu,\beta,l,\omega,b^-}^{\gamma,\delta,k,c} f)(x;p) \leq (b-x)C_{\beta-1,b^-}(x;p) \left(f(b) + mf\left(\frac{x}{m}\right) \right) \int_0^1 h(z) dz. \tag{3.8}$$

Adding (3.5) and (3.8), inequality (3.1) is obtained. □

If $m = 1$ and $h(z) = z$ in (3.1), then the following result holds for a convex function.

Corollary 1 *Let $f : [a, b] \rightarrow \mathbb{R}$, $a < b$, be a real valued function. If f is positive and convex, then, for $\alpha, \beta \geq 1$, we have*

$$\begin{aligned} & (\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x;p) + (\epsilon_{\mu,\beta,l,\omega,b^-}^{\gamma,\delta,k,c} f)(x;p) \\ & \leq \frac{(x-a)f(a)C_{\alpha-1,a^+}(x;p) + (b-x)f(b)C_{\beta-1,b^-}(x;p)}{2} \\ & \quad + f(x) \left[\frac{(x-a)C_{\alpha-1,a^+}(x;p) + (b-x)C_{\beta-1,b^-}(x;p)}{2} \right]. \end{aligned} \tag{3.9}$$

Remark 2 If $\omega = p = 0$ in (3.9), then [14, Theorem 1] is obtained.

Theorem 2 *Let $f : [a, b] \rightarrow \mathbb{R}$, $0 \leq a < b$, be a real valued function. If f is differentiable and $|f'|$ is $(h-m)$ -convex, then, for $\alpha, \beta \geq 1$, we have*

$$\begin{aligned} & \left| (\epsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x;p) + (\epsilon_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} f)(x;p) \right. \\ & \quad \left. - (f(a)C_{\alpha-1,a^+}(x;p) + f(b)C_{\beta-1,b^-}(x;p)) \right| \\ & \leq \left((x-a)|f'(a)|C_{\alpha-1,a^+}(x;p) + (b-x)|f'(b)|C_{\beta-1,b^-}(x;p) \right. \\ & \quad \left. + m \left| f'\left(\frac{x}{m}\right) \right| \left((x-a)C_{\alpha-1,a^+}(x;p) + (b-x)C_{\beta-1,b^-}(x;p) \right) \right) \int_0^1 h(z) dz. \end{aligned} \tag{3.10}$$

Proof Let $x \in [a, b]$ and $t \in [a, x]$. Then using $(h-m)$ -convexity of $|f'|$, we have

$$|f'(t)| \leq h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{3.11}$$

From (3.11), one has

$$f'(t) \leq h\left(\frac{x-t}{x-a}\right) |f'(a)| + mh\left(\frac{t-a}{x-a}\right) \left|f'\left(\frac{x}{m}\right)\right|. \tag{3.12}$$

Multiplying (3.2) and (3.12), then integrating over $[a, x]$, we get

$$\begin{aligned} & \int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-t)^\mu; p) f'(t) dt \\ & \leq (x-a)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu; p) \\ & \quad \times \left[|f'(a)| \int_a^x h\left(\frac{x-t}{x-a}\right) dt + m \left|f'\left(\frac{x}{m}\right)\right| \int_a^x h\left(\frac{t-a}{x-a}\right) dt \right]. \end{aligned} \tag{3.13}$$

The left hand side is calculated thus: Put $x-t = z$, that is, $t = x-z$, also using the derivative property (2.3) of the Mittag-Leffler function, we have

$$\begin{aligned} & \int_0^{x-a} z^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega z^\mu; p) f'(x-z) dz \\ & = -(x-a)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu; p) f(a) + \int_0^{x-a} z^{\alpha-2} E_{\mu,\alpha-1,l}^{\gamma,\delta,k,c}(\omega z^\mu; p) f(x-z) dz, \end{aligned}$$

now by putting $x-z = t$, in second term of the right hand side of the above equation and by using (2.4) and (2.6), we get

$$\int_0^{x-a} z^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega z^\mu; p) f'(x-z) dz = -f(a) C_{\alpha-1,a^+}(x; p) + (\epsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x; p).$$

Therefore, (3.13) takes the form

$$\begin{aligned} & (\epsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x; p) - f(a) C_{\alpha-1,a^+}(x; p) \\ & \leq (x-a) C_{\alpha-1,a^+}(x; p) \left(|f'(a)| + m \left|f'\left(\frac{x}{m}\right)\right| \right) \int_0^1 h(z) dz. \end{aligned} \tag{3.14}$$

Also from (3.11), one has

$$f'(t) \geq -\left(h\left(\frac{x-t}{x-a}\right) |f'(a)| + mh\left(\frac{t-a}{x-a}\right) \left|f'\left(\frac{x}{m}\right)\right| \right). \tag{3.15}$$

Following the same procedure as one did for (3.12), we also have

$$\begin{aligned} & f(a) C_{\alpha-1,a^+}(x; p) - (\epsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x; p) \\ & \leq (x-a) C_{\alpha-1,a^+}(x; p) \left(|f'(a)| + m \left|f'\left(\frac{x}{m}\right)\right| \right) \int_0^1 h(z) dz. \end{aligned} \tag{3.16}$$

From (3.14) and (3.16), we get

$$\begin{aligned} & |(\epsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x; p) - f(a) C_{\alpha-1,a^+}(x; p)| \\ & \leq (x-a) C_{\alpha-1,a^+}(x; p) \left(|f'(a)| + m \left|f'\left(\frac{x}{m}\right)\right| \right) \int_0^1 h(z) dz. \end{aligned} \tag{3.17}$$

Now let $t \in [x, b]$. Then using $(h - m)$ -convexity of $|f'|$, we have

$$|f'(t)| \leq h \left(\frac{t-x}{b-x} \right) |f'(b)| + mh \left(\frac{b-t}{b-x} \right) \left| f' \left(\frac{x}{m} \right) \right|. \tag{3.18}$$

Along the same lines as for (3.2), (3.12) and (3.15), one can get from (3.6) and (3.18) the following inequality:

$$\begin{aligned} & \left| (\epsilon_{\mu, \beta-1, l, \omega, b^-}^{\gamma, \delta, k, c} f)(x; p) - f(b) C_{\beta-1, b^-}(x; p) \right| \\ & \leq (b-x) C_{\beta-1, b^-}(x; p) \left(|f'(b)| + m \left| f' \left(\frac{x}{m} \right) \right| \right) \int_0^1 h(z) dz. \end{aligned} \tag{3.19}$$

From (3.17) and (3.19) via the triangular inequality, inequality (3.10) is obtained. □

If $m = 1$ and $h(z) = z$ in (3.10), then the following result holds for a convex function.

Corollary 2 *Let $f : [a, b] \rightarrow \mathbb{R}$, $a < b$, be a real valued function. If f is differentiable and $|f'|$ is convex, then, for $\alpha, \beta \geq 1$, we have*

$$\begin{aligned} & \left| (\epsilon_{\mu, \alpha-1, l, \omega, a^+}^{\gamma, \delta, k, c} f)(x; p) + (\epsilon_{\mu, \beta-1, l, \omega, b^-}^{\gamma, \delta, k, c} f)(x; p) \right. \\ & \quad \left. - (f(a) C_{\alpha-1, a^+}(x; p) + f(b) C_{\beta-1, b^-}(x; p)) \right| \\ & \leq \frac{(x-a) |f'(a)| C_{\alpha-1, a^+}(x; p) + (b-x) |f'(b)| C_{\beta-1, b^-}(x; p)}{2} \\ & \quad + |f'(x)| \left(\frac{(x-a) C_{\alpha-1, a^+}(x; p) + (b-x) C_{\beta-1, b^-}(x; p)}{2} \right). \end{aligned} \tag{3.20}$$

Remark 3

- (i) If $\omega = p = 0$ and replace α by $\alpha + 1$ in (3.20), then [14, Theorem 2] is obtained.
- (ii) If $\omega = p = 0$, $\alpha = \beta = 1$ and f' passes through $x = \frac{a+b}{2}$, then from (3.20) [13, Theorem 2.2] is obtained.

Theorem 3 *Let $f : [a, b] \rightarrow \mathbb{R}$, $0 \leq a < b$, be a real valued function. If f is positive, $(h - m)$ -convex and $f(x) = f(\frac{a+b-x}{m})$, then, for $\alpha, \beta > 0$, we have*

$$\begin{aligned} & \frac{f(\frac{a+b}{2})}{(m+1)h(\frac{1}{2})} [C_{\beta+1, b^-}(a; p) + C_{\alpha+1, a^+}(b; p)] \\ & \leq (\epsilon_{\mu, \beta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f)(a; p) + (\epsilon_{\mu, \alpha+1, l, \omega, a^+}^{\gamma, \delta, k, c} f)(b; p) \\ & \leq (b-a)^2 [C_{\beta-1, b^-}(a; p) + C_{\alpha-1, a^+}(b; p)] \left(f(a) + mf \left(\frac{b}{m} \right) \right) \int_0^1 h(z) dz. \end{aligned} \tag{3.21}$$

Proof For $x \in [a, b]$, we have

$$(x-a)^\beta E_{\mu, \beta, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p) \leq (b-a)^\beta E_{\mu, \beta, l}^{\gamma, \delta, k, c}(\omega(b-a)^\mu; p), \quad \beta > 0. \tag{3.22}$$

As f is $(h - m)$ -convex, for $x \in [a, b]$ we have

$$f(x) \leq mh \left(\frac{x-a}{b-a} \right) f \left(\frac{b}{m} \right) + h \left(\frac{b-x}{b-a} \right) f(a). \tag{3.23}$$

Multiplying (3.22) and (3.23), then integrating over $[a, b]$, we get

$$\begin{aligned} & \int_a^b (x-a)^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu; p) f(x) dx \\ & \leq m(b-a)^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(b-a)^\mu; p) f\left(\frac{b}{m}\right) \int_a^b h\left(\frac{x-a}{b-a}\right) dx \\ & \quad + (b-a)^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(b-a)^\mu; p) f(a) \int_a^b h\left(\frac{b-x}{b-a}\right) dx. \end{aligned}$$

From this we have

$$\left(\epsilon_{\mu,\beta+1,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(a; p) \leq (b-a)^2 C_{\beta-1,b^-}(a; p) \left(f(a) + mf\left(\frac{b}{m}\right)\right) \int_0^1 h(z) dz. \tag{3.24}$$

On the other hand for $x \in [a, b]$, we have

$$(b-x)^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-x)^\mu; p) \leq (b-a)^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-a)^\mu; p), \quad \alpha > 0. \tag{3.25}$$

Similarly multiplying (3.23) and (3.25), then integrating over $[a, b]$, we get

$$\left(\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(b; p) \leq (b-a)^2 C_{\alpha-1,a^+}(b; p) \left(f(a) + mf\left(\frac{b}{m}\right)\right) \int_0^1 h(z) dz. \tag{3.26}$$

Adding (3.24) and (3.26), we get

$$\begin{aligned} & \left(\epsilon_{\mu,\beta+1,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(a; p) + \left(\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(b; p) \\ & \leq (b-a)^2 (C_{\beta-1,b^-}(a; p) + C_{\alpha-1,a^+}(b; p)) \left(f(a) + mf\left(\frac{b}{m}\right)\right) \int_0^1 h(z) dz. \end{aligned} \tag{3.27}$$

Multiplying (2.9) with $(x-a)^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu; p)$, then integrating over $[a, b]$, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \int_a^b (x-a)^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu; p) dx \\ & \leq (m+1)h\left(\frac{1}{2}\right) \int_a^b (x-a)^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu; p) f(x) dx, \end{aligned}$$

by using (2.5) and (2.7), we get

$$\frac{f\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)} C_{\beta+1,b^-}(a; p) \leq \left(\epsilon_{\mu,\beta+1,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(a; p). \tag{3.28}$$

Similarly multiplying (2.9) with $(b-x)^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-x)^\mu; p)$, then integrating over $[a, b]$ and using (2.4) and (2.6), one can get

$$\frac{f\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)} C_{\alpha+1,a^+}(b; p) \leq \left(\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(b; p). \tag{3.29}$$

Adding (3.28) and (3.29), we get

$$\begin{aligned} & \frac{f\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)} [C_{\beta+1,b^-}(a;p) + C_{\alpha+1,a^+}(b;p)] \\ & \leq (\epsilon_{\mu,\beta+1,l,\omega,b^-}^{\gamma,\delta,k,c} f)(a;p) + (\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f)(b;p). \end{aligned} \tag{3.30}$$

From inequalities (3.27) and (3.30), inequality (3.21) is obtained. □

If $m = 1$ and $h(z) = z$ in (3.21), then the following result holds for a convex function.

Corollary 3 *Let $f : [a, b] \rightarrow \mathbb{R}$, $a < b$, be a real valued function. If f is positive, convex and symmetric about $\frac{a+b}{2}$, then, for $\alpha, \beta > 0$, we have*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) [C_{\beta+1,b^-}(a;p) + C_{\alpha+1,a^+}(b;p)] \\ & \leq (\epsilon_{\mu,\beta+1,l,\omega,b^-}^{\gamma,\delta,k,c} f)(a;p) + (\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f)(b;p) \\ & \leq (b-a)^2 [C_{\beta-1,b^-}(a;p) + C_{\alpha-1,a^+}(b;p)] \left[\frac{f(a) + f(b)}{2} \right]. \end{aligned} \tag{3.31}$$

Remark 4

- (i) If $\omega = p = 0$ in (3.31), then [14, Theorem 3] is obtained.
- (ii) If $\alpha = \beta \rightarrow 0$ and $\omega = p = 0$, then from the above inequality, we get the Hadamard inequality.

4 Inequalities for the extended generalized Mittag-Leffler functions

In this section, results of previous section are applied for the function $f(x) = x^2$. The function f is convex and $|f'(x)| = 2|x|$ is also convex. By virtue of this function we succeeded to establish recurrence relations among Mittag-Leffler functions which may be useful in the solutions of fractional boundary value problems and fractional differential equations. Ullah et al. computed generalized fractional integral operators for the function $f(x) = x^2$, in [44], as follows:

$$\begin{aligned} (\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f)(x;p) &= (x-a)^\alpha [a^2 E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu;p) \\ & \quad + 2a(x-a) E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu;p) \\ & \quad + 2(x-a)^2 E_{\mu,\alpha+3,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu;p)], \end{aligned} \tag{4.1}$$

$$\begin{aligned} (\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f)(x;p) &= (b-x)^\alpha [b^2 E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega(b-x)^\mu;p) \\ & \quad - 2b(b-x) E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega(b-x)^\mu;p) \\ & \quad + 2(b-x)^2 E_{\mu,\alpha+3,l}^{\gamma,\delta,k,c}(\omega(b-x)^\mu;p)]. \end{aligned} \tag{4.2}$$

Below, the results of Sect. 2 are applied to obtain recurrence inequalities for Mittag-Leffler functions.

Theorem 4 Mittag-Leffler functions satisfy the following recurrence relation:

$$\begin{aligned} & \frac{(a^2 + b^2)}{(b - a)^2} E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) + E_{\mu, \alpha+3, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) \\ & \leq \frac{((2m + 1)(a^2 + b^2) + 2ab)}{2m(b - a)^2} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) \int_0^1 h(z) dz \\ & \quad + E_{\mu, \alpha+2, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p), \end{aligned} \tag{4.3}$$

where $\omega' = \frac{\omega}{2^\mu}$.

Proof By using (4.1), (4.2) and $f(x) = x^2$ in (3.1) of Theorem 1, we have

$$\begin{aligned} & (x - a)^\alpha [a^2 E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) + 2a(x - a) E_{\mu, \alpha+2, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) \\ & \quad + 2(x - a)^2 E_{\mu, \alpha+3, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p)] + (b - x)^\beta [b^2 E_{\mu, \beta+1, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p) \\ & \quad - 2b(b - x) E_{\mu, \beta+2, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p) + 2(b - x)^2 E_{\mu, \beta+3, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p)] \\ & \leq \left(\left(a^2 + \frac{x^2}{m} \right) (x - a)^\alpha E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) \right. \\ & \quad \left. + \left(b^2 + \frac{x^2}{m} \right) (b - x)^\beta E_{\mu, \beta, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p) \right) \int_0^1 h(z) dz. \end{aligned} \tag{4.4}$$

Now by putting $x = \frac{a+b}{2}$ and $\alpha = \beta$ in (4.4), then after simplification, inequality (4.3) is obtained. □

Corollary 4 If $m = 1$ and $h(z) = z$ in (4.3), then we have

$$\begin{aligned} & \frac{(a^2 + b^2)}{(b - a)^2} E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) + E_{\mu, \alpha+3, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) \\ & \leq \frac{(3a^2 + 3b^2 + 2ab)}{4(b - a)^2} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) + E_{\mu, \alpha+2, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p). \end{aligned} \tag{4.5}$$

Theorem 5 Mittag-Leffler functions satisfy the following recurrence relation:

$$\begin{aligned} & |E_{\mu, \alpha+2, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) - E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p)| \\ & \leq \frac{1}{m(b - a)} (ma + mb + (a + b)) E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega'(b - a)^\mu; p) \int_0^1 h(z) dz, \end{aligned} \tag{4.6}$$

where $\omega' = \frac{\omega}{2^\mu}$.

Proof By using (4.1), (4.2) and $|f'(x)| = 2|x|$ in (3.10) of Theorem 2, we have

$$\begin{aligned} & |(x - a)^\alpha [a^2(x - a)^{-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) + 2a E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) \\ & \quad + 2(x - a) E_{\mu, \alpha+2, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p)] + (b - x)^\beta [b^2(b - x)^{-1} E_{\mu, \beta, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p) \\ & \quad - 2b E_{\mu, \beta+1, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p) + 2(b - x) E_{\mu, \beta+2, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p)] \\ & \quad - (a^2(x - a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x - a)^\mu; p) + b^2(b - x)^{\beta-1} E_{\mu, \beta, l}^{\gamma, \delta, k, c}(\omega(b - x)^\mu; p))| \end{aligned}$$

$$\begin{aligned} &\leq \left(2 \left(|a| + \frac{|x|}{m} \right) (x - a)^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x - a)^\mu; p) \right. \\ &\quad \left. + 2 \left(|b| + \frac{|x|}{m} \right) (b - x)^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(b - x)^\mu; p) \right) \int_0^1 h(z) dz. \end{aligned} \tag{4.7}$$

Now by putting $x = \frac{a+b}{2}$ and $\alpha = \beta$ in (4.7), then after simplification, inequality (4.6) is obtained. \square

Corollary 5 *If $m = 1$ and $h(z) = z$ in (4.6), then we have*

$$\begin{aligned} &|E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega'(b - a)^\mu; p) - E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega'(b - a)^\mu; p)| \\ &\leq \frac{1}{b - a} (a + b) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega'(b - a)^\mu; p). \end{aligned} \tag{4.8}$$

Theorem 6 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$\begin{aligned} &E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) - \left(1 + \frac{1}{m} \right) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) \int_0^1 h(z) dz \\ &\leq \frac{2(b - a)^2}{(a^2 + b^2)} (E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) - 2E_{\mu,\alpha+3,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p)). \end{aligned} \tag{4.9}$$

Proof In (4.4), putting $x = a$ and $x = b$, then adding for $\alpha = \beta$, inequality (4.9) is obtained. \square

Corollary 6 *If $m = 1$ and $h(z) = z$ in (4.9), then we have*

$$\begin{aligned} &E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) - E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) \\ &\leq \frac{2(b - a)^2}{(a^2 + b^2)} (E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) - 2E_{\mu,\alpha+3,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p)). \end{aligned} \tag{4.10}$$

Theorem 7 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$\begin{aligned} &\left| E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) - \frac{1}{2} E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) \right| \\ &\leq \frac{(1 + \frac{1}{m})}{2(b - a)} (a + b) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) \int_0^1 h(z) dz. \end{aligned} \tag{4.11}$$

Proof In (4.7), putting $x = a$ and $x = b$, then adding for $\alpha = \beta$, inequality (4.11) is obtained. \square

Corollary 7 *If $m = 1$ and $h(z) = z$ in (4.11), then we have*

$$\begin{aligned} &\left| E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) - \frac{1}{2} E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p) \right| \\ &\leq \frac{1}{2(b - a)} (a + b) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b - a)^\mu; p). \end{aligned} \tag{4.12}$$

Theorem 8 Mittag-Leffler functions satisfy the following recurrence relation:

$$\begin{aligned}
 & E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega;p) - \left(1 + \frac{1}{m}\right) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega;p) \int_0^1 h(z) dz \\
 & \leq 2(E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega;p) - 2E_{\mu,\alpha+3,l}^{\gamma,\delta,k,c}(\omega;p)).
 \end{aligned}
 \tag{4.13}$$

Proof In (4.9), putting $a = 0$ and $b = 1$, then inequality (4.13) is obtained. □

Corollary 8 If $m = 1$ and $h(z) = z$ in (4.13), then we have

$$\begin{aligned}
 & E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega;p) - E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega;p) \\
 & \leq 2(E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega;p) - 2E_{\mu,\alpha+3,l}^{\gamma,\delta,k,c}(\omega;p)).
 \end{aligned}
 \tag{4.14}$$

Theorem 9 Mittag-Leffler functions satisfy the following recurrence relation:

$$\begin{aligned}
 & |2E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega;p) - E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega;p)| \\
 & \leq \left(1 + \frac{1}{m}\right) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega;p) \int_0^1 h(z) dz.
 \end{aligned}
 \tag{4.15}$$

Proof In (4.11), putting $a = 0$ and $b = 1$, then inequality (4.15) is obtained. □

Corollary 9 If $m = 1$ and $h(z) = z$ in (4.15), then we have

$$|2E_{\mu,\alpha+2,l}^{\gamma,\delta,k,c}(\omega;p) - E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega;p)| \leq E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega;p).
 \tag{4.16}$$

By applying Theorem 3 similar relations can be established; we leave these for the reader.

5 Fractional differential equations involving extended generalized Mittag-Leffler function

In this section, generalized fractional differential equations are solved. The Riemann–Liouville fractional derivative operator $D_{a^+}^\nu$ is defined as follows:

$$(D_{a^+}^\nu f)(x) = \left(\frac{d}{dx}\right)^n (I_{a^+}^{n-\nu} f)(x), \quad \nu \in \mathbb{C}; \Re(\nu) > 0 \ (n = [\Re(\nu)] + 1),
 \tag{5.1}$$

where $(I_{a^+}^\nu f)(x)$ is the Riemann–Liouville fractional integral operator defined as follows:

$$(I_{a^+}^\nu f)(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt, \quad x > a.
 \tag{5.2}$$

For $a = 0$ the operator $(D_{a^+}^\nu f)(x)$ is represented by $(D_{0^+}^\nu f)(x)$ and $(I_{a^+}^\nu f)(x)$ is represented by $(I_{0^+}^\nu f)(x)$.

The Laplace transform of a function $f(x)$ is defined as follows:

$$L[f(x); s] = \int_0^\infty e^{-sx} f(x) dx = F(s).
 \tag{5.3}$$

In [34], the Laplace transform of fractional derivative $(D_{0^+}^\nu f)(x)$ is found to be

$$L[D_{0^+}^\nu f; s] = s^\nu F(s) - \sum_{k=1}^n s^{k-1} D_{0^+}^{\nu-k} f(0^+) \quad (n-1 < \nu < n) \Re(s) > 0. \tag{5.4}$$

For more information related to differential equations see Refs. [47–50].

Theorem 10 *Let $a \in \mathbb{R}_+$; $\mu, \alpha, l, \gamma, \nu, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then, for $x > a$, we have*

$$D_{a^+}^\nu [(t-a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t-a)^\mu; p)](x) = (x-a)^{\alpha-\nu-1} E_{\mu, \alpha-\nu, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p), \tag{5.5}$$

$$I_{a^+}^\nu [(t-a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t-a)^\mu; p)](x) = (x-a)^{\alpha+\nu-1} E_{\mu, \alpha+\nu, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p). \tag{5.6}$$

Proof By using the definition of $E_{\mu, \alpha, l}^{\gamma, \delta, k, c}$ defined in (2.2), we have

$$\begin{aligned} & D_{a^+}^\nu [(t-a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t-a)^\mu; p)](x) \\ &= D_{a^+}^\nu \left[(t-a)^{\alpha-1} \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{(\omega(t-a)^\mu)^n}{(l)_{n\delta}} \right](x) \\ &= \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{\omega^n}{(l)_{n\delta}} D_{a^+}^\nu [(t-a)^{\mu n + \alpha - 1}](x). \end{aligned} \tag{5.7}$$

By using the formula $D_{a^+}^\nu [(t-a)^\eta](x) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\nu+1)}(x-a)^{\eta-\nu}$, we have

$$\begin{aligned} & D_{a^+}^\nu [(t-a)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t-a)^\mu; p)](x) \\ &= \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha - \nu)} \frac{(\omega(x-a)^\mu)^n}{(l)_{n\delta}} (x-a)^{\alpha-\nu-1} \\ &= (x-a)^{\alpha-\nu-1} E_{\mu, \alpha-\nu, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p). \end{aligned} \tag{5.8}$$

Proof of (5.6) is similar to the proof of (5.5) just using the definition of fractional integral operator $I_{a^+}^\nu$ therein. □

Theorem 11 *Let $\mu, \alpha, l, \gamma, \nu, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the differential equation*

$$(D_{0^+}^\nu y)(x) = \lambda_1 (\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} 1)(x; p) + f(x) \tag{5.9}$$

with the initial condition $(I_{0^+}^{1-\nu})(0^+) = C$, has its solution in the space $L(0, \infty)$

$$\begin{aligned} y(x) &= C \frac{x^{\nu-1}}{\Gamma(\nu)} + \lambda_1 \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha + \nu + 1)} \frac{\omega^n}{(l)_{n\delta}} x^{\mu n + \alpha + \nu} \\ &\quad + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, \end{aligned} \tag{5.10}$$

where C is an arbitrary constant.

Proof Using the generalized fractional integral operator $(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} 1)(x;p)$ given in (2.6) with $a = 0$ in (5.9), we have

$$(D_{0^+}^{\nu} y)(x) = \lambda_1 x^{\alpha} E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^{\mu}; p) + f(x). \tag{5.11}$$

Applying the Laplace transform on both sides of (5.11), we have

$$L[(D_{0^+}^{\nu} y)(x); s] = \lambda_1 L[x^{\alpha} E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^{\mu}; p); s] + L[f(x); s]. \tag{5.12}$$

First we calculate the Laplace transform of Mittag-Leffler function as follows:

$$\begin{aligned} &L[x^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega x^{\mu}; p); s] \\ &= \int_0^{\infty} x^{\alpha} e^{-sx} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega x^{\mu}; p) dx \\ &= \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{\omega^n}{(l)_{n\delta}} \int_0^{\infty} x^{\alpha+\mu n-1} e^{-sx} dx \\ &= \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{\omega^n}{(l)_{n\delta}} \Gamma(\mu n + \alpha) L\left[\frac{x^{\alpha+\mu n-1}}{\Gamma(\mu n + \alpha)}; s\right]. \end{aligned} \tag{5.13}$$

Since $L[\frac{t^{n-1}}{\Gamma(n)}] = \frac{1}{s^n}$ ($n > 0$), using it in the above, we have

$$L[x^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega x^{\mu}; p); s] = \frac{1}{s^{\alpha}} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \left(\frac{\omega}{s^{\mu}}\right)^n. \tag{5.14}$$

By using (5.3), (5.4) (for $n = 1$) and (5.14) in (5.12), we have

$$s^{\nu} y(s) = C + \lambda_1 s^{-(\alpha+1)} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \left(\frac{\omega}{s^{\mu}}\right)^n + F(s),$$

which implies that

$$y(s) = C s^{-\nu} + \lambda_1 s^{-(\alpha+\nu+1)} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \left(\frac{\omega}{s^{\mu}}\right)^n + F(s) s^{-\nu}. \tag{5.15}$$

Now taking the inverse Laplace transformation on both sides of (5.15), we have

$$\begin{aligned} y(x) &= C \frac{x^{\nu-1}}{\Gamma(\nu)} + \lambda_1 \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \omega^n L^{-1}[s^{-(\mu n + \alpha + \nu + 1)}] \\ &\quad + L^{-1}[F(s) s^{-\nu}]. \end{aligned}$$

After simplification one can get (5.10). □

Theorem 12 Let $\mu, \alpha, l, \gamma, \nu, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the differential equation

$$(D_{0^+}^{\nu} y)(x) = \lambda_1 (\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c})(x; p) + \lambda_2 x^{\alpha} E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^{\mu}; p), \tag{5.16}$$

with the initial condition $(I_{0^+}^{1-\nu})(0+) = C$, has a solution in the space $L(0, \infty)$,

$$y(x) = C \frac{x^{\nu-1}}{\Gamma(\nu)} + (\lambda_1 + \lambda_2) \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha + \nu + 1)} \frac{\omega^n}{(l)_{n\delta}} x^{\mu n + \alpha + \nu}, \tag{5.17}$$

where C is an arbitrary constant.

Proof Using the generalized fractional integral operator $(\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} 1)(x; p)$ given in (2.6) with $a = 0$ in (5.16), we have

$$\begin{aligned} (D_{0^+}^{\nu} y)(x) &= \lambda_1 x^{\alpha} E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega x^{\mu}; p) + \lambda_2 x^{\alpha} E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega x^{\mu}; p) \\ &= (\lambda_1 + \lambda_2) x^{\alpha} E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega x^{\mu}; p). \end{aligned} \tag{5.18}$$

Applying the Laplace transform on both sides of (5.18), we have

$$L[(D_{0^+}^{\nu} y)(x); s] = (\lambda_1 + \lambda_2) L[x^{\alpha} E_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega x^{\mu}; p); s]. \tag{5.19}$$

By using (5.3), ((5.4) for $n = 1$) and (5.14) in (5.19), we have

$$s^{\nu} y(s) = C + (\lambda_1 + \lambda_2) s^{-(\alpha+1)} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \left(\frac{\omega}{s^{\mu}}\right)^n,$$

which implies that

$$y(s) = C s^{-\nu} + (\lambda_1 + \lambda_2) s^{-(\alpha+\nu+1)} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \left(\frac{\omega}{s^{\mu}}\right)^n. \tag{5.20}$$

Now taking the inverse Laplace transformation on both sides of (5.20), we have

$$y(x) = C \frac{x^{\nu-1}}{\Gamma(\nu)} + (\lambda_1 + \lambda_2) \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \omega^n L^{-1}[s^{-(\mu n + \alpha + \nu + 1)}].$$

After simplification one can get (5.17). □

Theorem 13 Let $\mu, \alpha, l, \gamma, \nu, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the differential equation

$$(D_{0^+}^{\nu} y)(x) = \lambda (\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c})(x; p) + \sum_{j=1}^n [\lambda_j x^{\alpha_j} E_{\mu_j, \alpha_j+1, l_j}^{\gamma_j, \delta_j, k_j, c_j}(w_j x^{\mu_j}; p)], \tag{5.21}$$

with the initial condition $(I_{0^+}^{1-\nu})(0+) = C$, has a solution in the space $L(0, \infty)$,

$$\begin{aligned} y(x) &= C \frac{x^{\nu-1}}{\Gamma(\nu)} + \lambda \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha + \nu + 1)} \frac{\omega^n}{(l)_{n\delta}} x^{\mu n + \alpha + \nu} \\ &\quad + \sum_{j=1}^n \left[\lambda_j \sum_{n=0}^{\infty} \frac{\beta_p(\gamma_j + nk_j, c_j - \gamma_j)}{\beta(\gamma_j, c_j - \gamma_j)} \frac{(c_j)_{nk_j}}{\Gamma(\mu_j n + \alpha_j + \nu + 1)} \frac{\omega_j^n}{(l_j)_{n\delta_j}} x^{\mu_j n + \alpha_j + \nu} \right], \end{aligned} \tag{5.22}$$

where C is an arbitrary constant.

Proof Using the generalized fractional integral operator $(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} 1)(x;p)$ given in (2.6) with $a = 0$ in (5.21), we have

$$(D_{0^+}^v y)(x) = \lambda x^\alpha E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^\mu; p) + \sum_{j=1}^n [\lambda_j x^{\alpha_j} E_{\mu_j,\alpha_j+1,l_j}^{\gamma_j,\delta_j,k_j,c_j}(w_j x^{\mu_j}; p)]. \tag{5.23}$$

Applying Laplace transform on both sides of (5.18), we have

$$L[(D_{0^+}^v y)(x); s] = \lambda L[x^\alpha E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^\mu; p); s] + L\left[\sum_{j=1}^n [\lambda_j x^{\alpha_j} E_{\mu_j,\alpha_j+1,l_j}^{\gamma_j,\delta_j,k_j,c_j}(w_j x^{\mu_j}; p)]; s\right]. \tag{5.24}$$

By using (5.3), ((5.4) for $n = 1$) and (5.14) in (5.24), we have

$$s^v y(s) = C + \lambda s^{-(\alpha+1)} \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \left(\frac{\omega}{s^\mu}\right)^n + \sum_{j=1}^n \left[\lambda_j s^{-(\alpha_j+1)} \sum_{n=0}^\infty \frac{\beta_p(\gamma_j + nk_j, c_j - \gamma_j)}{\beta(\gamma_j, c_j - \gamma_j)} \frac{(c_j)_{nk_j}}{(l_j)_{n\delta_j}} \left(\frac{\omega_j}{s^{\mu_j}}\right)^n \right],$$

which implies that

$$y(s) = Cs^{-v} + \lambda s^{-(\alpha+v+1)} \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \left(\frac{\omega}{s^\mu}\right)^n + s^{-v} \sum_{j=1}^n \left[\lambda_j s^{-(\alpha_j+1)} \sum_{n=0}^\infty \frac{\beta_p(\gamma_j + nk_j, c_j - \gamma_j)}{\beta(\gamma_j, c_j - \gamma_j)} \frac{(c_j)_{nk_j}}{(l_j)_{n\delta_j}} \left(\frac{\omega_j}{s^{\mu_j}}\right)^n \right]. \tag{5.25}$$

Now taking the inverse Laplace transformation on both sides of (5.25), we have

$$y(x) = C \frac{x^{v-1}}{\Gamma(v)} + \lambda \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \omega^n L^{-1}[s^{-(\mu n + \alpha + v + 1)}] + \sum_{j=1}^n \left[\lambda_j \sum_{n=0}^\infty \frac{\beta_p(\gamma_j + nk_j, c_j - \gamma_j)}{\beta(\gamma_j, c_j - \gamma_j)} \frac{(c_j)_{nk_j}}{(l_j)_{n\delta_j}} \omega_j^n L^{-1}[s^{-(\mu_j n + \alpha_j + v + 1)}] \right].$$

After simplification one can get (5.22). □

6 Concluding remarks

This research computes the bounds of fractional integral operators containing an extended generalized Mittag-Leffler function in their kernel. These results provide compact formulas for bounds of several kinds of fractional integral operators via several kinds of convexity. By setting specific values to parameters involved in the Mittag-Leffler function some interesting results can be obtained. For example estimations of the fractional integral operators: by Salim and Faraj defined in [39] setting $p = 0$, by Rahman et al. defined in [37] by setting $l = \delta = 1$, by Shukla and Prajapati defined in [41] by setting $p = 0$ and $l = \delta = 1$ (see also [42]), by Prabhakar defined in [36] by setting $p = 0$ and $l = \delta = k = 1$. Also

inequalities for recurrence relations of Mittag-Leffler functions are obtained via a particular convex function x^2 . At the end some generalized fractional differential equations are solved.

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