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Some oscillation theorems for nonlinear second-order differential equations with an advanced argument

G.E. Chatzarakis[1], O. Moaaz[2], T. Li[3,4,5] and B. Qaraad[2,6*]

*Correspondence:
belgeesmath2016@gmail.com
[2]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt
[6]Department of Mathematics, Faculty of Science, Saadah University, Saadah, Yemen
Full list of author information is available at the end of the article

Abstract

The objective in this work is to study oscillation criteria for second-order quasi-linear differential equations with an advanced argument. We establish new oscillation criteria using both the comparison technique with first-order advanced differential inequalities and the Riccati transformation. The established criteria improve, simplify and complement results that have been published recently in the literature. We illustrate the results by an example.

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1 Introduction

In this work, we study sufficient conditions for the oscillation of the solutions of second-order nonlinear differential equations with an advanced argument of the form

$$\left(r\left(u'\right)^{\alpha}\right)'(t)+p(t)f\left(u\left(g(t)\right)\right)=0, \tag{1.1}$$

where we assume that the following conditions hold:

(H_1) α and β are quotients of odd positive integers;

(H_2) $r \in C^1([t_0,\infty),(0,\infty))$, satisfies

$$\mu(t_0):=\int_{t_0}^{\infty}\frac{1}{r^{1/\alpha}(s)}\,\mathrm{d}s<\infty;$$

(H_3) $g \in C^1([t_0,\infty),\mathbb{R})$, and we suppose that, for all $t \geq t_0$, $g(t) \geq t$, $g'(t) \geq 0$ and $p \in C[t_0,\infty),[0,\infty)$ does not vanish identically.

(H_4) $f \in (\mathbb{R},\mathbb{R})$ is such that $uf(u)>0$ for $u \neq 0$ and satisfies the following condition:

$$\text{There exists a constant } \kappa>0 \text{ such that } f(u)>\kappa u^{\beta} \text{ for all } u\neq 0. \tag{1.2}$$

A solution of (1.1) is an $x \in C([t_0,\infty),[0,\infty))$ with $t_a = \min\{\tau(t_b), g(t_b)\}$, for some $t_b > t_0$, which satisfies the property $r(u')^\alpha \in C^1([t_a,\infty),[0,\infty))$ and moreover satisfies (1.1) on $[t_b,\infty)$. We consider the nontrivial solutions of (1.1) existing on some half-line $[t_b,\infty)$ and satisfying the condition

$$\sup\{|x(t)| : t_c \le t < \infty\} > 0 \quad \text{for any } t_c \ge t_b.$$

If x is neither positive nor negative eventually, then $x(t)$ is called *oscillatory*. Otherwise, it is a *non-oscillatory* solution.

Differential equations with advanced arguments are used in many applied problems where the rate of development depends on the future, as well as on the present time. In a delay equation, delays represent the retrospective memory of the past. In differential equations with an advanced argument, advances represent the prospective memory of the future, accounting for the influence on the system of potential future actions, which are available at the present time. For instance, population dynamics, economics problems, or mechanical control engineering are typical fields where such phenomena are thought to occur, see [14, 20].

The many applications of functional differential equations have been the motive behind the active research movement in recent times, see [1–13, 22, 24–33] and [34, 35, 37]. In recent decades, a great amount of work has been done on the oscillation theory of the different order differential equations with delay and advanced argument [4–13, 15–21, 23] and [24–33, 36].

This work aims at further developing the oscillation theory of second-order quasi-linear equations with advanced argument. We use an approach that combines the comparison with first-order advanced differential inequalities and the Riccati transformation. That enables us to get various conditions, ensuring the oscillation of (1.1). In this paper, we simplify and improve the results in [36, Theorem 1.7.8] and obtain a new criterion for ensuring the oscillation of the solutions of (1.1). We illustrate the improvement obtained by the results in this paper, through an example.

Lemma 1.1 ([7]) *Let $\alpha \ge 1$ be a ratio of two odd numbers. Then*

$$A^{(\alpha+1)/\alpha} - (A-B)^{(\alpha+1)/\alpha} \le \frac{1}{\alpha} B^{1/\alpha}\bigl[(1+\alpha)A - B\bigr], \quad AB \ge 0 \tag{1.3}$$

and

$$DV - CV^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{D^{\alpha+1}}{C^\alpha}, \quad C > 0.$$

Lemma 1.2 ([33]) *Let $\alpha, \beta > 0$ and assume that u is an eventually non-increasing positive solution of (1.1). Then, $u^{\beta-\alpha}(t) \ge \eta(t)$ holds, where $\eta(t)$ is defined by*

$$\eta(t) = \begin{cases} 1 & \text{for } \alpha = \beta; \\ a_1 & \text{for } \alpha > \beta; \\ a_2 \mu^{\beta-\alpha}(t) & \text{for } \alpha < \beta, \end{cases}$$

and a, a_2 are positive constants.

2 Auxiliary lemmas

The proofs of our main results are essentially based on the following lemmas.

Lemma 2.1 *Assume that* (1.1) *has an eventually positive solution* u. *If*

$$\int_{t_0}^{\infty} p(s)\,\mathrm{d}s = \infty, \tag{2.1}$$

then

($\mathbf{P}_1$) *u is decreasing and $\left(r\left(u'\right)^{\alpha}\right)'$ is non-increasing, eventually.*

Proof Assume that there exists a $t_1 \geq t_0$ such that equation (1.1) has a positive solution u on $[t_1, \infty]$. Hence, from (H_4), we obtain

$$\left(r\left(u'\right)^{\alpha}\right)'(t) \leq -\kappa p(t) u^{\beta}\left(g(t)\right). \tag{2.2}$$

Thus, we get that u' is of fixed sign, eventually. Now, we will prove that $u' < 0$. To the contrary, suppose there exists a $t_2 \geq t_1$ such that $u' > 0$ for $t \geq t_2$. Define a positive function w by

$$w(t) = \frac{r(t)(u'(t))^{\alpha}}{u^{\beta}(t)}. \tag{2.3}$$

Differentiating (2.3), we get

$$w'(t) = \frac{(r(t)(u'(t))^{\alpha})'}{u^{\beta}(t)} - \beta \frac{r(t)(u'(t))^{\alpha} u'(t)}{u^{\beta+1}(t)}.$$

From (2.2) and $g(t) \geq t$, it follows that

$$\begin{aligned} w'(t) &\leq -\kappa p(t) \left(\frac{u(g(t))}{u(t)}\right)^{\beta} - \beta \frac{r(t)(u'(t))^{\alpha}}{u^{\beta}(t)} \frac{u'(t)}{u(t)} \\ &\leq -\kappa p(t) - \beta w(t) \frac{u'(t)}{u(t)}. \end{aligned}$$

Thus

$$w'(t) \leq -\kappa p(t). \tag{2.4}$$

Integrating (2.4) from t_2 to t, we get

$$w(t) \leq w(t_2) - \kappa \int_{t_2}^{t} p(s)\,\mathrm{d}s,$$

which yields a contradiction to w being positive. Hence, $u'(t) < 0$, therefore, the proof is complete. □

Lemma 2.2 *If equation* (1.1) *has an eventually decreasing positive solution* u, *then*

$(\mathbf{P}_2)$ $\quad \dfrac{u}{\mu}$ *is non-decreasing, eventually.*

Proof Assume that there exists a $t_1 \geq t_0$ such that equation (1.1) has a positive solution u on $[t_1, \infty]$ and $u' < 0$. Hence, from (H_4), we get that (2.2) holds and therefore

$$r^{1/\alpha}(s)u'(s) \leq r^{1/\alpha}(t)u'(t),$$

for all $s \geq t$. Integrating this inequality from t to v yields

$$u(v) - u(t) \leq r^{1/\alpha}(t)u'(t) \int_t^v r^{-1/\alpha}(s)\,\mathrm{d}s, \quad v \geq t.$$

Letting $v \to \infty$ in the above inequality, we see that

$$r^{-1/\alpha}(t)u(t) + u'(t)\mu(t) \geq 0, \tag{2.5}$$

and consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{\mu}\right) = \frac{u'}{\mu} + \frac{r^{-1/\alpha}u}{\mu^2} \geq 0.$$

Thus, the proof is complete. □

Lemma 2.3 *Assume that equation* (1.1) *has an eventually positive solution* u *and*

$$\int_{t_0}^{\infty} \left(\frac{1}{r(v)} \int_{t_0}^{v} p(s)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}v = \infty. \tag{2.6}$$

Then u *satisfies* $(\mathbf{P}_1)$, *and*

$(\mathbf{P}_3)$ $\quad \lim\limits_{t\to\infty} u(t) = 0.$

Proof Assume that there exists a $t_1 \geq t_0$ such that $u(t) > 0$ for all $t \geq t_1$. From (2.6) and (H_2), we conclude that condition (2.1) holds. From Lemma 2.1, it follows that u satisfies $(\mathbf{P}_1)$.

Next, since u is a positive decreasing function, we get that $\lim_{t\to\infty} u(t) = c \geq 0$. Suppose that $c > 0$. Then, there exists $t_2 \geq t_1$ such that $u(g(t)) \leq c$, and so

$$-\big(r\big(u'\big)^{\alpha}\big)'(t) \geq \kappa p(t)u^{\beta}\big(g(t)\big) \geq \kappa c^{\beta} p(t),$$

for $t \geq t_2$. We integrate this inequality twice from t_2 to t. Then, after the first integration, we get

$$r(t_2)\big(u'(t_2)\big)^{\alpha} - r(t)\big(u'(t)\big)^{\alpha} \geq \kappa c^{\beta} \int_{t_2}^{t} p(s)\,\mathrm{d}s.$$

Therefore,

$$u'(t) \leq \kappa c^{\beta} \left(\frac{1}{r(t)} \int_{t_2}^{t} p(s)\,\mathrm{d}s \right)^{\frac{1}{\alpha}}. \tag{2.7}$$

After the second integration, we obtain

$$u(t) - u(t_2) \leq -\kappa c^{\beta} \int_{t_2}^{t} \left(\frac{1}{r(\nu)} \int_{t_2}^{\nu} p(s)\,\mathrm{d}s \right)^{\frac{1}{\alpha}} \mathrm{d}\nu.$$

This implies that $\lim_{t\to\infty} u(t) = -\infty$, which contradicts $c > 0$. The proof of the lemma is complete. □

Lemma 2.4 *Assume that equation* (1.1) *has an eventually positive solution u and* (2.6) *holds. Then, there exist positive constants δ_1 and δ_2 and $t_\delta \geq t_1$ such that*

$$\begin{aligned} \delta_1 \mu(t) &\leq u(t) \\ &\leq \delta_2 \exp\left(-\kappa^{\frac{1}{\alpha}} \int_{t_0}^{t} \left(\frac{1}{r(s)\eta(s)} \right)^{\frac{1}{\alpha}} \left(\frac{\mu(g(s))}{\mu(s)} \right)^{\frac{\beta}{\alpha}} \left(\int_{t_0}^{s} p(\zeta)\,\mathrm{d}\xi \right)^{\frac{1}{\alpha}} \right), \end{aligned} \tag{2.8}$$

for $t \geq t_\delta$.

Proof As in the proof of Lemma 2.3, we get that $(\mathbf{P}_1)$, $(\mathbf{P}_2)$ and $(\mathbf{P}_3)$ hold. From $(\mathbf{P}_2)$, there exist $t_2 \geq t_1$ and $\delta_1 > 0$ such that $u(t)/\mu(t) \geq \delta_1$ for all $t \geq t_2$. Next, by integrating (1.1) from t_2 to t, we get

$$\begin{aligned} -r(t)\big(u'(t)\big)^{\alpha} &\geq -r(t_2)\big(u'(t_2)\big)^{\alpha} + \kappa \int_{t_2}^{t} p(s) u^{\beta}\big(g(s)\big)\,\mathrm{d}s \\ &\geq -r(t_2)\big(u'(t_2)\big)^{\alpha} + \kappa u^{\beta}\big(g(t)\big) \int_{t_2}^{t} p(s)\,\mathrm{d}s. \end{aligned}$$

Therefore,

$$-r(t)\big(u'(t)\big) - \kappa u^{\beta}\big(g(t)\big) \int_{t_0}^{t} p(s)\,\mathrm{d}s \geq -r(t_2)\big(u'(t_2)\big)^{\alpha} - \kappa u^{\beta}\big(g(t)\big) \int_{t_0}^{t_2} p(s)\,\mathrm{d}s.$$

From $(\mathbf{P}_3)$, there exists a $t_3 \geq t_2$ such that the left-hand side of this inequality is positive for $t \geq t_3$, and thus

$$-r(t)\big(u'(t)\big)^{\alpha} \geq \kappa u^{\beta}\big(g(t)\big) \int_{t_0}^{t} p(s)\,\mathrm{d}s,$$

for $t \geq t_3$. By Lemma 2.2, the last inequality gives

$$\begin{aligned} -r(t)\big(u'(t)\big)^{\alpha} &\geq \kappa \frac{u^{\beta}(g(t))}{\mu^{\beta}(g(t))} \big(\mu^{\beta}\big(g(t)\big)\big) \int_{t_0}^{t} p(s)\,\mathrm{d}s \\ &\geq \kappa \frac{u^{\beta}(t)}{\mu^{\beta}(t)} \mu^{\beta}\big(g(t)\big) \int_{t_0}^{t} p(s)\,\mathrm{d}s. \end{aligned}$$

Hence,

$$\frac{(u'(t))^{\alpha}}{u^{\beta}(t)} \leq \frac{-\kappa}{r(t)}\left(\frac{\mu(g(t))}{\mu(t)}\right)^{\beta}\int_{t_0}^{t}p(s)\,\mathrm{d}s.$$

From Lemma 1.2, we obtain

$$\left(\frac{u'(t)}{u(t)}\right)^{\alpha}\eta(t) \leq \frac{-\kappa}{r(t)}\left(\frac{\mu(g(t))}{\mu(t)}\right)^{\beta}\int_{t_0}^{t}p(s)\,\mathrm{d}s.$$

Then

$$\frac{u'(t)}{u(t)} \leq -\left(\frac{\kappa}{\eta(t)r(t)}\right)^{\frac{1}{\alpha}}\left(\frac{\mu(g(t))}{\mu(t)}\right)^{\frac{\beta}{\alpha}}\left(\int_{t_0}^{t}p(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}. \tag{2.9}$$

Integrating (2.9) from t_3 to t, we have

$$\begin{aligned}u(t) &\leq u(t_3)\exp\left(-\int_{t_3}^{t}\left(\frac{\mu(g(\nu))}{\mu(\nu)}\right)^{\frac{\beta}{\alpha}}\left(\frac{\kappa}{\eta(\nu)r(\nu)}\int_{t_0}^{\nu}p(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}\nu\right)\\ &\leq \delta_2\exp\left(-\int_{t_0}^{t}\left(\frac{\mu(g(\nu))}{\mu(\nu)}\right)^{\frac{\beta}{\alpha}}\left(\frac{\kappa}{\eta(\nu)r(\nu)}\int_{t_0}^{\nu}p(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}\nu\right),\end{aligned}$$

where

$$\delta_2 = u(t_3)\exp\left(-\int_{t_0}^{t_3}\left(\frac{\mu(g(\nu))}{\mu(\nu)}\right)^{\frac{\beta}{\alpha}}\left(\frac{\kappa}{\eta(\nu)r(\nu)}\int_{t_0}^{\nu}p(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}\nu\right) > 0.$$

The proof is complete. □

Lemma 2.5 *Assume that (2.1) holds and (1.1) has a positive solution u on $[t_1,\infty)$. Let there exist constants γ and δ such that $\gamma+\delta \in [0,1)$,*

$$p(t)\mu^{\alpha}\big(g(t)\big)\mu(t)\eta\big(g(t)\big)r^{\frac{1}{\alpha}}(t) \geq \frac{\gamma}{\kappa} \tag{2.10}$$

and

$$\eta^{-1/\alpha}(t)\mu^{1-\beta/\alpha}(t)\mu^{\beta/\alpha}\big(g(t)\big)\left(\int_{t_0}^{t}p(s)\,\mathrm{d}s\right)^{1/\alpha} \geq \frac{\delta}{\kappa^{1/\alpha}}. \tag{2.11}$$

Then, there exists a $t_2 \geq t_1$ such that

$$\frac{u}{\mu^{1-\gamma}} \text{ and } \frac{u}{\mu^{\delta}} \text{ are non-decreasing and non-increasing, respectively,}$$

for $t \geq t_2$.

Proof From (2.1) and Lemma 2.1, we get that $(\mathbf{P}_1)$ holds, and hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(-r(t)\big(u'(t)^{\alpha}\big)\mu^{\gamma}(t)\big) &= \big(-r(t)\big(u'(t)\big)^{\alpha}\big)'\mu^{\gamma}(t)\\ &\quad+\gamma\big(r(t)\big(u'(t)\big)^{\alpha}\big)\mu^{\gamma-1}(t)r^{-1/\alpha}(t).\end{aligned}$$

Using (2.10), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(-r(t)\left(u'(t)\right)^{\alpha}\mu^{\gamma}(t)\right)\\
&\quad\geq -\kappa p(t)u^{\alpha}\left(g(t)\right)\eta\left(g(t)\right)\mu^{\gamma}(t)+\gamma r\left(u'(t)\right)^{\alpha}\frac{\mu^{\gamma-1}(t)}{r^{1/\alpha}(t)}\\
&\quad\geq -\kappa p(t)r(t)\left(u'(t)\right)^{\alpha}\eta\left(g(t)\right)\mu^{\alpha}\left(g(t)\right)\mu^{\gamma}(t)+\gamma r\left(u'(t)\right)^{\alpha}\frac{\mu^{\gamma-1}(t)}{r^{1/\alpha}(t)}\\
&\quad= -r(t)\left(u'(t)\right)^{\alpha}\mu^{\gamma}(t)\left(\kappa p(t)\mu^{\alpha}\left(g(t)\right)\eta\left(g(t)\right)-\frac{\gamma}{\mu(t)r^{1/\alpha}(t)}\right)\geq 0.
\end{aligned}$$

Hence, $-r(u')^{\alpha}\mu^{\gamma}$ is non-decreasing, and thus there exists a $t_2\geq t_1$ such that

$$\begin{aligned}
u(t)&\geq -r^{1/\alpha}(t)u'(t)\mu^{\gamma}(t)\int_t^{\infty}\frac{\mu^{-\gamma}(s)}{r^{1/\alpha}(s)}\,\mathrm{d}s\\
&= -\frac{1}{1-\gamma}r^{1/\alpha}(t)\mu(t)u'(t).
\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu^{1-\gamma}(t)}u(t)\right)=\frac{1}{(1-\gamma)r^{\frac{1}{\alpha}}(t)\mu^{2-\gamma}(t)}\left(u+\frac{1}{1-\gamma}r^{1/\alpha}(t)\mu(t)u'(t)\right)\geq 0.$$

Proceeding as in the proof of Lemma 2.4, we obtain that (2.9) holds, and so

$$\frac{u(t)}{\mu^{\delta+1}(t)r^{1/\alpha}(t)}\leq -\frac{u'(t)}{\mu^{\delta+1}(t)}\left(\frac{\eta(t)}{\kappa}\right)^{1/\alpha}\left(\frac{\mu(t)}{\mu(g(t))}\right)^{\beta/\alpha}\left(\int_{t_0}^{t}p(s)\,\mathrm{d}s\right)^{-1/\alpha}. \tag{2.12}$$

Therefore, we arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u(t)}{\mu^{\delta}(t)}\right)&=\frac{u'(t)}{\mu^{\delta}(t)}+\frac{\delta u(t)}{\mu^{\delta+1}(t)r^{1/\alpha}(t)}\\
&\leq \frac{u'(t)}{\mu^{\delta}(t)}\left(1-\frac{\delta}{\mu(t)}\left(\frac{\eta(t)}{\kappa}\right)^{1/\alpha}\left(\frac{\mu(t)}{\mu(g(t))}\right)^{\beta/\alpha}\left(\int_{t_0}^{t}p(s)\,\mathrm{d}s\right)^{-1/\alpha}\right)\\
&\leq 0.
\end{aligned}$$

The proof is complete. □

3 Main results

In this section, we shall establish some oscillation criteria for (1.1). Let us define

$$\widehat{\mu}(t):=\left(\mu(t)+\frac{\kappa}{\alpha}\int_t^{\infty}\mu(s)\mu^{\alpha}\left(g(s)\right)p(s)\,\mathrm{d}s\right).$$

We are now ready to state and prove the main theorems.

Theorem 3.1 *If*

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\int_{t_0}^{s}p(s)\mu^{\beta}\big(g(s)\big)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}t=\infty, \tag{3.1}$$

then every solution of (1.1) *is oscillatory.*

Proof Suppose, against the theorem's statement, that equation (1.1) has a non-oscillatory solution u on $[t_0,\infty)$. Without loss of generality, we may assume that $u(t)>0$, $u(g(t))>0$ for $t\geq t_1\geq t_0$. Now, a necessary result to satisfy Condition (3.1) is that $\int_{t_0}^{\infty}p(s)\mu^{\beta}(g(s))\,\mathrm{d}s$ is unbounded. Thus, from (H_2) and $\mu'(t)<0$, it is easy to note that (2.1) is valid. So, by Lemmas 2.1 and 2.2, we get that $(\mathbf{P}_1)$ and $(\mathbf{P}_2)$ hold. Therefore, there exist $a>0$ and $t_2\geq t_1$ such that $u(t)\geq a\mu(t)$ for $t\geq t_1$, and then

$$\begin{aligned}-\big(r\big(u'\big)^{\alpha}\big)'(t)&\geq\kappa p(t)u^{\beta}\big(g(t)\big)\\&\geq\kappa a^{\beta}p(t)\mu^{\beta}\big(g(t)\big).\end{aligned} \tag{3.2}$$

Integrating (3.2) from t_2 to t, we get

$$-r(t)\big(u'(t)\big)^{\alpha}\geq r(t_2)\big(u'(t_2)\big)^{\alpha}+\kappa a^{\beta}\int_{t_2}^{t}p(s)\mu^{\beta}\big(g(s)\big)\,\mathrm{d}s,$$

i.e.,

$$-u'(t)\geq\big(\kappa a^{\beta}\big)^{\frac{1}{\alpha}}\left(\frac{1}{r(t)}\int_{t_2}^{t}p(s)\mu^{\beta}\big(g(s)\big)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}.$$

Integrating this inequality from t_2 to t, letting $t\to\infty$, and using (3.1), we get a contradiction to u being positive. The proof is complete. □

Theorem 3.2 *If*

$$\limsup_{t\to\infty}\Phi(t,t_1)>1, \tag{3.3}$$

for any $t_1\in[t_0,\infty)$, *where*

$$\Phi(t,s):=\kappa\eta\big(g(t)\big)\mu^{\alpha}\big(g(t)\big)\int_{s}^{t}p(v)\,\mathrm{d}v,$$

then (1.1) *is oscillatory. Moreover, if* (2.6) *holds and*

$$\limsup_{t\to\infty}\Phi(t,t_0)>1, \tag{3.4}$$

then (1.1) *is oscillatory.*

Proof Suppose, against the theorem's statement, that equation (1.1) has a non-oscillatory solution u on $[t_0,\infty)$. Without loss of generality, we may assume that $u(t)>0$, $u(g(t))>0$

for $t \geq t_1 \geq t_0$. We can see that (3.3) and (H_2) imply (2.1). Thus, Lemma 2.1 is valid for $t > t_1$. Integrating (1.1) from t_1 to t, we obtain

$$\begin{aligned}
-r(t)\big(u'(t)\big)^{\alpha} &\geq -r(t_1)\big(u'(t_1)\big)^{\alpha} + \kappa \int_{t_1}^{t} p(s)u^{\beta}\big(g(s)\big)\,\mathrm{d}s \\
&\geq -r(t_1)\big(u'(t_1)\big)^{\alpha} + \kappa u^{\beta}\big(g(t)\big)\int_{t_1}^{t} p(s)\,\mathrm{d}s \\
&\geq -r(t_1)\big(u'(t_1)\big)^{\alpha} + \kappa u^{\beta-\alpha}\big(g(t)\big)u^{\alpha}\big(g(t)\big)\int_{t_1}^{t} p(s)\,\mathrm{d}s \\
&\geq -r(t_1)\big(u'(t_1)\big)^{\alpha} + \kappa \eta\big(g(t)\big)u^{\alpha}\big(g(t)\big)\int_{t_1}^{t} p(s)\,\mathrm{d}s.
\end{aligned} \tag{3.5}$$

The last inequality, together with (2.5), implies that

$$-r(t)\big(u'(t)\big)^{\alpha} \geq -\kappa\eta\big(g(t)\big)r\big(g(t)\big)\mu^{\alpha}\big(g(t)\big)\big(u'\big(g(t)\big)\big)^{\alpha}\int_{t_1}^{t} p(s)\,\mathrm{d}s. \tag{3.6}$$

Since $r(t)(u'(t))^{\alpha}$ is non-increasing and $g(t) \geq t$, we get

$$\Phi(t,t_1) = \kappa\eta\big(g(t)\big)\mu^{\alpha}\big(g(t)\big)\int_{t_1}^{t} p(s)\,\mathrm{d}s \leq \frac{r(t)(u'(t))^{\alpha}}{r(g(t))(u'(g(t)))^{\alpha}} \leq 1, \tag{3.7}$$

which contradicts (3.3).

On the other hand, let (2.6) hold. From the definition of η, we note that $\eta(t)$ is bounded. Thus, from Lemma 2.3, we get that $\lim_{t\to\infty} u(t) = 0$, and hence there exists a $t_2 \in [t_1,\infty)$ large enough, such that

$$\kappa\eta\big(g(t)\big)u^{\alpha}\big(g(t)\big)\int_{t_0}^{t_1} p(s)\,\mathrm{d}s < -r(t_1)\big(u'(t_1)\big)^{\alpha}$$

for all $t \geq t_2$. Therefore, from (3.5), we obtain

$$\begin{aligned}
-r(t)\big(u'(t)\big)^{\alpha} &\geq -r(t_1)\big(u'(t_1)\big)^{\alpha} + \kappa\eta\big(g(t)\big)u^{\alpha}\big(g(t)\big)\int_{t_0}^{t} p(s)\,\mathrm{d}s \\
&\quad - \kappa\eta\big(g(t)\big)u^{\alpha}\big(g(t)\big)\int_{t_0}^{t_1} p(s)\,\mathrm{d}s \\
&\geq \kappa\eta\big(g(t)\big)u^{\alpha}\big(g(t)\big)\int_{t_0}^{t} p(s)\,\mathrm{d}s.
\end{aligned}$$

As in (3.6) and (3.7), we get a contradiction to (3.4). The proof of the theorem is complete. □

Theorem 3.3 *Assume that*

$$\liminf_{t\to\infty}\int_{t}^{g(t)} p(s)\mu^{\alpha}\big(g(s)\big)\,\mathrm{d}s > \frac{1}{\kappa\mathrm{e}} \quad \textit{if } \alpha = \beta, \tag{3.8}$$

or

$$\lim_{t\to\infty}\int_{t_0}^{t} p(s)\mu^{\alpha}\big(g(s)\big)\,\mathrm{d}s=\infty \quad \text{if } \alpha<\beta. \tag{3.9}$$

Then (1.1) *is oscillatory.*

Proof Suppose, against the theorem's statement, that equation (1.1) has a non-oscillatory solution u on $[t_0,\infty)$. Without loss of generality, we may assume that $u(t)>0$, $u(g(t))>0$ for $t\ge t_1\ge t_0$. We note that the following condition is necessary for (3.8) to be valid:

$$\int_{t_0}^{\infty} p(s)\mu^{\alpha}\big(g(s)\big)\,\mathrm{d}s=\infty. \tag{3.10}$$

Moreover, (3.10) with (H_2) ensure (2.1). From Lemma 2.1 and 2.2, we have that $(\mathbf{P}_1)$ and (2.5) hold. It follows from (1.1) and (2.5) that

$$-\big(r(t)\big(u'(t)\big)^{\alpha}\big)'+\kappa p(t)r^{\beta/\alpha}\big(g(t)\big)\mu^{\beta}\big(g(t)\big)\big(u'\big(g(t)\big)\big)^{\beta}\ge 0.$$

This implies that $\varphi:=-r(u')^{\alpha}$ is a positive solution of the first-order advanced differential inequality

$$\varphi'(t)-\kappa p(t)\mu^{\beta}\big(g(t)\big)\varphi^{\beta/\alpha}\big(g(t)\big)\ge 0. \tag{3.11}$$

In view of [24, Theorem 2.4.1] and [23, Theorem 1], conditions (3.8) and (3.9) imply that the advanced inequality (3.11) has no positive solutions when $\alpha=\beta$ and $\alpha<\beta$, respectively. This contradiction completes the proof. □

Theorem 3.4 *Assume that*

$$\liminf_{t\to\infty}\int_{t}^{g(t)} p(s)\widehat{\mu}^{\alpha}\big(g(s)\big)\,\mathrm{d}s>\frac{1}{\kappa\,\mathrm{e}} \quad \text{if } \alpha=\beta, \tag{3.12}$$

or

$$\lim_{t\to\infty}\int_{t_0}^{t} p(s)\widehat{\mu}^{\alpha}\big(g(s)\big)\,\mathrm{d}s=\infty \quad \text{if } \alpha<\beta, \tag{3.13}$$

where

$$\widehat{\mu}(t):=\mu(t)+\frac{\kappa}{\alpha}\int_{t}^{\infty}\mu(s)\mu^{\alpha}\big(g(s)\big)p(s)\eta\big(g(s)\big)\,\mathrm{d}s.$$

Then (1.1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 3.3, we obtain that (3.10), together with (H_2), implies (2.1). Then, from Lemma 2.1 and 2.2, we get that $(\mathbf{P}_1)$ and (2.5) hold. Now, let $\varphi:=r(u')^{\alpha}$ and

$$w:=u+\mu\varphi^{1/\alpha}>0. \tag{3.14}$$

Then,

$$\begin{aligned} w'(t) &= u'(t) - r^{-1/\alpha}(t)\varphi^{1/\alpha}(t) + \frac{1}{\alpha}\mu(t)\varphi^{(1/\alpha)-1}(t)\varphi'(t) \\ &= \frac{1}{\alpha}\mu(t)\varphi^{(1/\alpha)-1}(t)\varphi'(t) \end{aligned}$$

which, together with (1.1), implies that

$$w'(t) \leq -\frac{\kappa}{\alpha}\mu(t)p(t)\varphi^{(1/\alpha)-1}(t)u^{\beta}\big(g(t)\big) < 0. \tag{3.15}$$

Integrating (3.15) from t to ∞, we get

$$\begin{aligned} w(t) &\geq \frac{\kappa}{\alpha}\int_t^{\infty} \mu(s)p(s)\varphi^{(1/\alpha)-1}(s)u^{\beta}\big(g(s)\big)\,\mathrm{d}s \\ &\geq \frac{\kappa}{\alpha}\int_t^{\infty} \mu(s)p(s)\varphi^{(1/\alpha)-1}(s)\eta\big(g(s)\big)u^{\alpha}\big(g(s)\big)\,\mathrm{d}s. \end{aligned}$$

Using (2.5) in last inequality, we have

$$\begin{aligned} w(t) &\geq -\frac{\kappa}{\alpha}\int_t^{\infty} \mu(s)\mu^{\alpha}\big(g(s)\big)p(s)\varphi^{(1/\alpha)-1}(s)\eta\big(g(s)\big)\varphi\big(g(s)\big)\,\mathrm{d}s \\ &\geq -\frac{\kappa}{\alpha}\int_t^{\infty} \mu(s)\mu^{\alpha}\big(g(s)\big)p(s)\eta\big(g(s)\big)\varphi^{1/\alpha}(s)\,\mathrm{d}s \\ &\geq -\frac{\kappa}{\alpha}\varphi^{1/\alpha}(t)\int_t^{\infty} \mu(s)\mu^{\alpha}\big(g(s)\big)p(s)\eta\big(g(s)\big)\,\mathrm{d}s. \end{aligned}$$

From (3.14), we arrive at

$$\begin{aligned} u(t) &\geq -\left(\mu(t) + \frac{\kappa}{\alpha}\int_t^{\infty} \mu(s)\mu^{\alpha}\big(g(s)\big)p(s)\eta\big(g(s)\big)\,\mathrm{d}s\right)\varphi^{1/\alpha}(t) \\ &= -\widehat{\mu}(t)\varphi^{1/\alpha}(t). \end{aligned} \tag{3.16}$$

Using (3.16) and (1.1) yields

$$\widehat{\varphi}'(t) - \kappa p(t)\widehat{\mu}^{\beta}\big(g(t)\big)\widehat{\varphi}^{\beta/\alpha}\big(g(t)\big) \geq 0,$$

where $\widehat{\varphi} := -\varphi$. The rest of proof is similar to that of Theorem 3.3, and therefore we omit it. □

Theorem 3.5 *Assume that* (2.1) *holds. If there exists a* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\left\{\frac{\mu^{\alpha}(t)}{\rho(t)}\int_T^t\left(\kappa\rho(s)p(s)\left(\frac{\mu(g(s))}{\mu(s)}\right)^{\beta}\eta(s) - \frac{r(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right)\mathrm{d}s\right\} > 1,$$

for any $T \in [t_0,\infty)$, *then* (1.1) *is oscillatory.*

Proof Suppose to the contrary of the theorem's statement that equation (1.1) has a nonoscillatory solution u on $[t_0,\infty)$. Without loss of generality, we can assume that $u(t) > 0$,

$u(g(t)) > 0$ for $t \geq t_1 \geq t_0$. By Lemma 2.1 and 2.2, we have $(\mathbf{P}_1)$ and $(\mathbf{P}_2)$ hold for $t > t_1$. Equation (1.1), together with $(\mathbf{P}_2)$, leads to

$$\begin{aligned}\frac{(r(t)(u'(t))^\alpha)'}{u^\alpha(t)} &\leq -\kappa p(t)\frac{u^\beta(g(t))}{u^\alpha(t)}\\ &\leq -\kappa p(t)\frac{u^\beta(g(t))}{u^\beta(t)}u^{\beta-\alpha}(t)\\ &\leq -\kappa p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^\beta\eta(t).\end{aligned}\tag{3.17}$$

Now, let us make the positive generalized Riccati substitution:

$$\omega(t)=\rho(t)\left[\frac{r(t)(u'(t))^\alpha}{u^\alpha(t)}+\frac{1}{\mu^\alpha(t)}\right],\quad \text{for all } t\geq t_2.\tag{3.18}$$

Differentiating (3.18), we get

$$\begin{aligned}\omega'(t) &= \frac{\rho'(t)}{\rho(t)}\omega(t)+\rho(t)\frac{(r(t)(u'(t))^\alpha)'}{u^\alpha(t)}-\alpha\rho(t)r(t)\left(\frac{u'(t)}{u(t)}\right)^{\alpha+1}\\ &\quad+\frac{\alpha\rho}{r^{\frac{1}{\alpha}}(t)\mu^{\alpha+1}(t)}\\ &= \frac{\rho'(t)}{\rho(t)}\omega(t)+\rho(t)\frac{(r(t)(u'(t))^\alpha)'}{u^\alpha(t)}+\frac{\alpha\rho}{r^{\frac{1}{\alpha}}(t)\mu^{\alpha+1}(t)}\\ &\quad-\alpha\rho(t)r(t)\left(\frac{\omega(t)}{\rho(t)r(t)}-\frac{1}{r(t)\mu^\alpha(t)}\right)^{\frac{\alpha+1}{\alpha}}.\end{aligned}\tag{3.19}$$

From (3.19) and (3.17), we have

$$\begin{aligned}\omega'(t) &\leq \frac{\rho'(t)}{\rho(t)}\omega(t)-\rho(t)\kappa p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^\beta\eta(t)+\frac{\alpha\rho}{r^{\frac{1}{\alpha}}(t)\mu^{\alpha+1}(t)}\\ &\quad-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}\left(\omega(t)-\frac{\rho(t)}{\mu^\alpha(t)}\right)^{\frac{\alpha+1}{\alpha}}.\end{aligned}$$

Using Lemma 1.1, with

$$C=\frac{\rho'(t)}{\rho(t)}, D=\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}, \text{and} V=\frac{\rho(t)}{\mu^\alpha(t)},$$

we obtain

$$\begin{aligned}\omega'(t) &\leq -\kappa\rho(t)p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^\beta\eta(t)+\frac{\rho'(t)}{\mu^\alpha(t)}+\frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(t)}\\ &\quad+\frac{\alpha\rho}{r^{\frac{1}{\alpha}}(t)\mu^{\alpha+1}(t)}\\ &\leq -\kappa\rho(t)p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^\beta\eta(t)+\left(\frac{\rho(t)}{\mu^\alpha(t)}\right)'\\ &\quad+\frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(t)}.\end{aligned}\tag{3.20}$$

We can write inequality (2.5) in the form

$$-\frac{\rho(t)}{\mu^{\alpha}(t)} \le \rho(t)\frac{r(t)(u'(t))^{\alpha}}{u^{\alpha}(t)} \le 0. \tag{3.21}$$

Integrating (3.20) from t_2 to t, we get

$$\begin{aligned}&\int_{t_2}^{t}\left(\kappa\rho(t)p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^{\beta}\eta(t)-\frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}\right)\mathrm{d}s\\ &\quad\le \omega(t_2)-\omega(t)+\frac{\rho(s)}{\mu^{\alpha}(s)}+\frac{\rho(t_2)}{\mu^{\alpha}(t_2)}.\end{aligned}$$

In view of the definition of $\omega(t)$, we get

$$\begin{aligned}&\int_{t_2}^{t}\left(\kappa\rho(t)p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^{\beta}\eta(t)-\frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}\right)\mathrm{d}s\\ &\quad\le \rho(t_2)\frac{r(t_2)(u'(t_2))^{\alpha}}{u^{\alpha}(t_2)}-\rho(t)\frac{r(t)(u'(t))^{\alpha}}{u^{\alpha}(t)}.\end{aligned} \tag{3.22}$$

Using inequality (3.21) into (3.22), we are led to

$$\frac{\mu^{\alpha}(t)}{\rho(t)}\int_{t_2}^{t}\left(\kappa\rho(t)p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^{\beta}\eta(t)-\frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}\right)\mathrm{d}s\le 1.$$

Taking the limit superior of both sides of the inequality, we get a contradiction. This completes the proof. □

4 Discussion and examples

By using Lemma 2.5, we further improve the established oscillation criteria in Theorems 3.2, 3.3, and 3.5.

Corollary 4.1 *Assume that there exist constants γ and δ such that $\gamma+\delta\in[0,1)$ and (2.10) and (2.11) hold. If*

$$\limsup_{t\to\infty}\mu^{\gamma}(t)\mu^{1-\gamma-\delta}\big(g(s)\big)^{\alpha}\int_{t_1}^{t}\mu^{\delta\alpha}\big(g(s)\big)p(s)\eta\big(g(s)\big)\,\mathrm{d}s>\frac{(1-\gamma)^{\alpha}}{\kappa},$$

then (1.1) is oscillatory.

Proof Suppose to the contrary of the corollary's statement that equation (1.1) has a nonoscillatory solution u on $[t_0,\infty)$. Without loss of generality, we may assume that $u(t)>0$, $u(g(t))>0$ for $t\ge t_1\ge t_0$. From (H_2), as $t\to\infty$, we get

$$\mu^{\gamma}(t)\mu^{1-\gamma-\delta}\big(g(t)\big)\le\mu(t)^{1-\delta}\to 0.$$

Then, we see that $\int_{t_1}^{t}\mu^{\delta\alpha}(g(t))p(s)\,\mathrm{d}s$ and $\int_{t_0}^{t}p(s)\,\mathrm{d}s$ are unbounded. Hence, (2.1) is necessary for (3.1) to be valid. From Lemma 2.1, $(\mathbf{P}_1)$ is satisfied for $t\ge t_1$. By (1.1), we obtain

$$-r(t)\big(u'(t)\big)^{\alpha}\ge -r(t_1)\big(u'(t_1)\big)^{\alpha}+\int_{t_1}^{t}\kappa p(s)u^{\beta}\big(g(s)\big)\,\mathrm{d}s.$$

Using Lemma 2.5, we get

$$\begin{aligned} -r(t)\bigl(u'(t)\bigr)^{\alpha} &\geq \left(\frac{u(g(s))}{\mu^{\delta}(g(s))}\right)^{\alpha}\int_{t_1}^{t} k\mu^{\delta\alpha}\bigl(g(s)\bigr)p(s)\eta\bigl(g(s)\bigr)\,\mathrm{d}s \\ &= \left(\frac{u(t)\mu^{1-\gamma-\delta}(g(s))}{\mu^{1-\gamma}(t)}\right)^{\alpha}\int_{t_1}^{t} k\mu^{\delta\alpha}\bigl(g(s)\bigr)p(s)\eta\bigl(g(s)\bigr)\,\mathrm{d}s \\ &= \left(\frac{u(t)\mu^{1-\gamma-\delta}(g(s))}{\mu^{1-\gamma}(t)}\right)^{\alpha}\int_{t_1}^{t} k\mu^{\delta\alpha}\bigl(g(s)\bigr)p(s)\eta\bigl(g(s)\bigr)\,\mathrm{d}s. \end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned} \frac{u(t)}{\mu^{1-\gamma}(t)} &\geq \frac{-(r(t))^{\frac{1}{\alpha}}u'(t)\mu(t)}{(1-\gamma)\mu^{(1-\gamma)}(t)} \\ &\geq \frac{-r^{\frac{1}{\alpha}}(t)u'(t)\mu(t)}{(1-\gamma)}. \end{aligned}$$

Therefore, (4.1) becomes

$$\begin{aligned} -r(t)\bigl(u'(t)\bigr)^{\alpha} &\geq \left(\frac{-r^{\frac{1}{\alpha}}(t)u'(t)\mu^{\gamma}(t)\mu^{1-\gamma-\delta}(g(s))}{(1-\gamma)}\right)^{\alpha}\int_{t_1}^{t} k\mu^{\delta\alpha}\bigl(g(s)\bigr)p(s)\eta\bigl(g(s)\bigr)\,\mathrm{d}s \\ &\geq -r(t)\bigl(u'(t)\bigr)^{\alpha}\left(\frac{\mu^{\gamma}(t)\mu^{1-\gamma-\delta}(g(s))}{(1-\gamma)}\right)^{\alpha}\int_{t_1}^{t} k\mu^{\delta\alpha}\bigl(g(s)\bigr)p(s)\eta\bigl(g(s)\bigr)\,\mathrm{d}s. \end{aligned}$$

Then

$$\left(\frac{\mu^{\gamma}(t)\mu^{1-\gamma-\delta}(g(s))}{(1-\gamma)}\right)^{\alpha}\int_{t_1}^{t} k\mu^{\delta\alpha}\bigl(g(s)\bigr)p(s)\eta\bigl(g(s)\bigr)\,\mathrm{d}s \leq 1.$$

This completes the proof. □

Corollary 4.2 *Assume that γ is a constant satisfying (3.4) and $0 \leq \gamma < 1$. If $\alpha = \beta$ and*

$$\liminf_{t\to\infty}\int_{t}^{g(t)} p(s)\widehat{\mu}^{\alpha}\bigl(g(s)\bigr)\,\mathrm{d}s > \frac{(1-\gamma)^{\beta}}{\kappa\,\mathrm{e}},$$

then (1.1) is oscillatory.

Corollary 4.3 *Assume that (2.1) holds. If there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\left\{\frac{\mu^{\alpha}(t)}{\rho(t)}\int_{T}^{t}\left(\kappa\rho(t)\eta(s)p(s) - \frac{r(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right)\mathrm{d}s\right\} > 1,$$

where

$$p(s) = p(t)\left(\frac{\mu(g(t))}{\mu(t)}\right)^{\beta(1-\gamma)}, \tag{4.2}$$

then (1.1) is oscillatory.

For an appropriate choice of the function ρ (1 or $\mu(t)$, or $\mu^{\alpha}(t)$), Theorem 3.5 and Corollary 4.1 can be used to study the oscillation of (1.1) in a wide range of applications. Hence, by choosing $\rho(t) = \mu^{\alpha}(t)$, we get the following results:

Corollary 4.4 *Assume that* (2.1) *holds. If*

$$\limsup_{t\to\infty} \int_T^t \left(\kappa p(s)\eta(s) \frac{\mu^{\beta}(g(s))}{\mu^{\beta-\alpha}(s)} - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1} r^{1/\alpha}(s)\mu(s)} \right) \mathrm{d}s > 1,$$

for any $T \in [t_0, \infty)$, *then* (1.1) *is oscillatory.*

Corollary 4.5 *Assume that* (2.1) *holds and* p *is defined as in* (4.2). *If*

$$\limsup_{t\to\infty} \int_T^t \left(\kappa \eta(s)\mu^{\alpha}(t)p(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1} r^{1/\alpha}(s)\mu(s)} \right) \mathrm{d}s > 1,$$

then (1.1) *is oscillatory.*

Example 4.1 Consider the equation

$$\left(t^{2\alpha}\left(u'\right)^{\alpha}\right)' + p_0 t^{\upsilon-1} u^{\beta}(\lambda t) = 0, \tag{4.3}$$

where $\lambda \geq 1$ and

$$\upsilon = \begin{cases} \alpha + 1, & \text{for } \alpha > \beta; \\ \beta, & \text{for } \alpha \leq \beta. \end{cases}$$

We note that $\kappa = 1$, $r(t) := t^{2\alpha}$, $p(t) := p_0 t^{\upsilon-1}$, $g(t) := \lambda t$, and $f(v) := v^{\beta}$. Thus, we see that $\mu(t) = 1/t$ and

$$\int_{t_0}^{\infty} \left(\frac{1}{v^{2\alpha}} \int_{t_0}^{v} s^{\upsilon-1} \,\mathrm{d}s \right)^{1/\alpha} \mathrm{d}v = \infty \quad \text{(i.e., (2.6) holds).}$$

First, let $\alpha < \beta$. We see that (3.1) is not satisfied and therefore, Theorem 3.1 does not apply. Also, since (3.4), namely $\kappa a_2 p_0 > \beta\lambda^{\beta}$, for any a_2, Theorem 3.2 does not apply in this example. On the other hand, by Theorem 3.3, we see that

$$\lim_{t\to\infty} \frac{p_0}{\lambda^{\alpha}} \int_{t_0}^{t} s^{\beta-\alpha-1} \,\mathrm{d}s = \infty,$$

and thus (4.3) is oscillatory.

Assume that $\alpha > \beta$. Using Theorem 3.2, we get that (4.3) is oscillatory.

Finally, let $\alpha = \beta$. From Lemma 2.3 and 2.4, any positive solution u of (4.3) satisfies $\lim_{t\to\infty} u(t) = \infty$ and there exist positive constants δ_1 and δ_2 and $t_\delta \geq t_1$ such that

$$\frac{\delta_1}{t} \leq u(t) \leq \frac{\delta_2}{t^{\eta}},$$

where

$$\eta = \left(\frac{1}{\lambda}\right)^{\frac{\beta}{\alpha}} \left(\frac{p_0}{\alpha}\right)^{1/\alpha},$$

for $t \geq t_\delta$. The following list shows the conditions that have resulted from our theorems:

$$\begin{aligned}
&\text{Theorem 3.1}: && \text{cannot be applied;} \\
&\text{Theorem 3.3}: && (\mathbf{c}_1) \quad \frac{p_0}{\lambda^\alpha}\ln\lambda > \frac{1}{\mathrm{e}}; \\
&\text{Theorem 3.4}: && (\mathbf{c}_2) \quad p_0\left(1 + \frac{p_0}{\alpha}\lambda^{(1-\alpha)}\right)^\alpha \ln\lambda > \frac{\lambda^\alpha}{\mathrm{e}}; \\
&\text{Corollary 4.1}: && (\mathbf{c}_3) \quad p_0(\lambda)^{\alpha(\gamma-1)} > \alpha(1-\gamma)^\alpha(1-\delta); \\
&\text{Corollary 4.2}: && (\mathbf{c}_4) \quad p_0\left(1 + \frac{p_0}{\alpha}\lambda^{(1-\alpha)}\right)^\alpha \ln\lambda > \frac{\lambda^\alpha(1-\gamma)^\alpha}{\mathrm{e}}; \\
&\text{Corollary 4.4}: && (\mathbf{c}_5) \quad \frac{p_0}{\lambda^\alpha} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}; \\
&\text{Corollary 4.5}: && (\mathbf{c}_6) \quad \frac{p_0}{\lambda^{\alpha(1-\gamma)}} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},
\end{aligned}$$

where $\gamma := p_0\lambda^{-\alpha}$ and $\delta := p_0^{1/\alpha}/(\lambda\alpha^{1/\alpha})$. We note that Theorem 3.4 improves Theorem 3.3, Corollary 4.2 improves Theorem 3.4, and Corollary 4.5 improves Corollary 4.4.

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Author details
[1]Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), Athens, Greece. [2]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt. [3]School of Control Science and Engineering, Shandong University, Jinan, P.R. China. [4]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, P.R. China. [5]School of Informatics, Linyi University, Linyi, P.R. China. [6]Department of Mathematics, Faculty of Science, Saadah University, Saadah, Yemen.

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