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On fractional hybrid and non-hybrid multi-term integro-differential inclusions with three-point integral hybrid boundary conditions

Sina Etemad[1], Shahram Rezapour[1,3*] and Mohammad Esmael Samei[2]

*Correspondence:
rezapourshahram@yahoo.ca;
sh.rezapour@azaruniv.ac.ir
[1]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
[3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the existence of solutions for two nonlinear fractional multi-term integro-differential inclusions in two hybrid and non-hybrid versions. The boundary value conditions are in the form of three-point integral hybrid conditions. In this way, we define a new operator based on the integral solution of the given boundary value inclusion problem and then we use assumptions of a Dhage's fixed point result for this fractional operator in the hybrid case. Also, the approximate endpoint property is applied for the corresponding set-valued maps in the non-hybrid case. Finally, we provide two examples to illustrate our main results.

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Keywords: Approximate endpoint property; Boundary value problem; Fractional hybrid integro-differential inclusion; Dhage fixed point theorem

1 Introduction

To design powerful computing software, we need strong mathematical tools. In other words, if one can make exact patterns for natural phenomena and processes by using new mathematical formulas and operators, then more flexible algorithms can be written in the software programming based on such relations and formulas. This results in accurate computer calculations with the least error in the shortest time. In this way, many researchers are currently studying various types of advanced mathematical models using fractional differential equations and related inclusion versions with more general boundary value conditions [1–8]. Indeed, they try to model the processes such that they cover many general cases and in this situation; mathematicians would like to solve a wide range of these boundary value problems with advanced and complicated boundary conditions. Recently, many papers have been published on the existence of solutions for different fractional boundary value problems (see, for example, [9–34]). In the last few decades, fractional hybrid differential equations and inclusions with hybrid or non-hybrid boundary value conditions have received a great deal of interest and attention of many researchers (see, for example, [35–42]).

The starting point for this field is related to a joint work of Dhage and Lakshmikantham in 2010. They introduced a new category of nonlinear differential equations, called ordinary hybrid differential equations, and studied the existence of extremal solutions for this boundary value problem by establishing some fundamental differential inequalities [43]. In 2012, Zhao et al. provided an extension for the Dhage's work to fractional order and considered a boundary value problem of fractional hybrid differential equations [44]. Later, many papers have been published by researchers, in which authors studied different properties of solutions for fractional hybrid boundary value problems. In 2016, Ahmad et al. studied the existence of solutions for the nonlocal boundary value problem of fractional hybrid integro-differential inclusion

$$\begin{cases} {}^{c}\mathcal{D}_{0}^{\alpha}(\frac{k(t)-\sum_{i=1}^{m}\mathcal{I}_{0}^{\beta_i}h_i(t,k(t))}{g(t,k(t))}) \in \mathcal{G}(t,k(t)) = 0, \quad t \in [0,1], \\ k(0) = \mu(x), \qquad k(1) = A \in \mathbb{R}, \end{cases}$$

where ${}^{c}\mathcal{D}_{0}^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (1,2]$ and $\mathcal{I}_{0}^{\phi}$ is the Riemann–Liouville fractional integral of order $\phi > 0$ with $\phi \in \{\beta_1, \beta_2, \dots, \beta_m\}$ [45]. Next, Baleanu et al. derived some existence results and a theorem on the dimension of the solution set for the fractional hybrid inclusion problem

$${}^{c}\mathcal{D}_{0}^{\nu}\left(\frac{k(t)}{\Lambda(t,k(t),\mathcal{I}^{\alpha_1}k(t),\dots,\mathcal{I}^{\alpha_n}k(t))}\right) \in \Psi\big(t,k(t),\mathcal{I}^{\beta_1}k(t),\dots,\mathcal{I}^{\beta_m}k(t)\big),$$

for $t \in [0,1]$, supplemented with boundary value conditions $k(0) = k_0^*$ and $k(1) = k_1^*$, where $\nu \in (1,2]$, ${}^{c}\mathcal{D}^{\nu}$ and $\mathcal{I}^{\gamma}$ denote the Caputo derivative operators of fractional order ν and the Riemann–Liouville integral operator of fractional order $\gamma \in \{\alpha_i, \beta_j\} \subset (0,\infty)$ for $i = 1,\dots,n$ and $j = 1,\dots,m$, respectively [46]. In 2019, Samei et al. discussed the existence of solutions for the fractional hybrid Caputo–Hadamard differential inclusion

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\alpha}(\frac{k(t)-f(t,k(t),\mathcal{I}^{\beta_1}h_1(t,k(t)),\dots,\mathcal{I}^{\beta_n}h_n(t,k(t)))}{g(t,k(t),\mathcal{I}^{\gamma_1}k(t),\dots,\mathcal{I}^{\gamma_m}k(t))}) \in K(t,k(t)), \\ k(1) = \mu(t), \qquad k(e) = \eta(t), \end{cases}$$

for $t \in [1,e]$, where $\alpha \in (1,2]$, $n,m \in \mathbb{N}$, $\beta_i > 0$ for $i = 1,2,\dots,n$, $\gamma_i > 0$ for $i = 1,2,\dots,m$, the functions $g : J \times \mathbb{R}^{m+1} \to \mathbb{R} \setminus \{0\}$, $f : J \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $h_i : J \times \mathbb{R} \to \mathbb{R}$ are continuous, $\mu,\eta \in C(J,\mathbb{R})$, $K : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map with certain conditions, and the operators ${}^{\mathrm{CH}}\mathcal{D}_{1^+}^{(\cdot)}$ and $\mathcal{I}^{(\cdot)}$ denote the fractional Caputo–Hadamard derivative and the fractional Hadamard integral of order $(\cdot)$, respectively [40].

By mixing ideas of the above works, we investigate the fractional hybrid multi-term Caputo integro-differential inclusion

$${}^{c}\mathcal{D}_{0}^{\omega}\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\dots,\varphi_n(k(t)))}\right) \in \mathcal{K}\big(t,k(t),\phi_1\big(k(t)\big),\dots,\phi_m\big(k(t)\big)\big), \tag{1}$$

with three-point integral hybrid boundary value conditions

$$\begin{cases} \lambda_1\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))}\right)\big|_{t=0} \\ \quad + \lambda_2\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))}\right)\big|_{t=1} = a, \\ \lambda_3{}^c\mathcal{D}_0^{\beta}\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))}\right)\big|_{t=\eta} \\ \quad + \lambda_4\int_0^1 {}^c\mathcal{D}_0^{\beta}\left(\frac{k(s)}{\xi(s,k(s),\varphi_1(k(s)),\ldots,\varphi_n(k(s)))}\right)\,\mathrm{d}s = b, \end{cases} \tag{2}$$

where $t \in J = [0,1]$, $\omega \in (1,2]$, $\beta \in (0,1]$, $\eta \in (0,1)$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, a, b \in \mathbb{R}^+$ and ${}^c\mathcal{D}_0^{\gamma}$ denotes the fractional Caputo derivative of order $\gamma \in \{\omega, \beta\}$. Also $\xi : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R} \setminus \{0\}$ is a continuous function and $\mathcal{K} : [0,1] \times \mathbb{R}^{m+1} \to \mathcal{P}(\mathbb{R})$ is a set-valued map via some certain properties. For $i = 1, 2, \ldots, m$, let

$$\phi_i\bigl(k(t)\bigr) = \mathcal{I}_0^{\varpi_i}k(t) = \int_0^t \frac{(t-s)^{\varpi_i - 1}}{\Gamma(\varpi_i)}k(s)\,\mathrm{d}s$$

and, for $i = 1, 2, \ldots, n$, let

$$\varphi_i\bigl(k(t)\bigr) = \mathcal{I}_0^{\varrho_i}k(t) = \int_0^t \frac{(t-s)^{\varrho_i - 1}}{\Gamma(\varrho_i)}k(s)\,\mathrm{d}s,$$

with $\varpi_i, \varrho_i > 0$ and $m, n \in \mathbb{N}$. If we put $\xi(t, k(t), \varphi_1(k(t)), \ldots, \varphi_n(k(t))) = 1$ and $\mathcal{K}(t, k(t), \phi_1(k(t)), \ldots, \phi_m(k(t))) = \mathcal{S}(t, k(t))$, then the fractional hybrid multi-term integro-differential inclusion (1)–(2) reduces to the fractional non-hybrid inclusion problem

$$\begin{cases} {}^c\mathcal{D}_0^{\omega}k(t) \in \mathcal{S}(t, k(t)) \quad (t \in J), \\ \lambda_1 k(0) + \lambda_2 k(1) = a, \qquad \lambda_3{}^c\mathcal{D}_0^{\beta}k(\eta) + \lambda_4\int_0^1 {}^c\mathcal{D}_0^{\beta}k(s)\,\mathrm{d}s = b. \end{cases} \tag{3}$$

We review the existence of solutions for two given fractional hybrid and non-hybrid inclusion problems. It is noted that the fractional hybrid multi-term integro-differential inclusion presented in this paper is new in the sense that the boundary value conditions are stated as three-point mixed Caputo integro-derivative hybrid conditions. Also, this hybrid boundary value problem is general and it involves many fractional dynamical systems as special cases. In this way, we use the Dhage fixed point theorem for the hybrid case and the approximate endpoint property for the non-hybrid case.

The paper is organized as follows: In the next Sect. 2, some basic definitions and applied results are presented. In Sect. 3, we state our main existence results and used techniques in this direction. Finally, two illustrative examples about the corresponding existence results are given in the last Sect. 4.

2 Preliminaries

In this section, we recall some definitions and theorems needed in the sequel. Let $\omega > 0$. The fractional Riemann–Liouville integral of a function $k : [a,b] \to \mathbb{R}$ is defined by

$$\mathcal{I}_0^{\omega}k(t) = \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}k(s)\,\mathrm{d}s,$$

provided that the right-hand side integral exists [47, 48]. Now, let $n-1<\omega<n$ and $n=[\omega]+1$. The fractional Caputo derivative of a function $k\in C^{(n)}([a,b],\mathbb{R})$ is defined by

$$ {}^{c}\mathcal{D}_0^{\omega}k(t)=\int_0^t\frac{(t-s)^{n-\omega-1}}{\Gamma(n-\omega)}k^{(n)}(s)\,\mathrm{d}s, $$

provided that the right-hand side integral exists [47, 48]. It has been proved that the general solution for the homogeneous fractional differential equation ${}^{c}\mathcal{D}_0^{\omega}k(t)=0$ is in the form $k(t)=m_0^*+m_1^*t+m_2^*t^2+\cdots+m_{n-1}^*t^{n-1}$ and we have

$$ \mathcal{I}_0^{\omega}\big({}^{c}\mathcal{D}_0^{\omega}k(t)\big)=k(t)+\sum_{j=0}^{n-1}m_j^*t^j=k(t)+m_0^*+m_1^*t+m_2^*t^2+\cdots+m_{n-1}^*t^{n-1}, $$

where $m_0^*,\dots,m_{n-1}^*$ are some real constants and $n=[\omega]+1$ [49].

Assume that $(\mathcal{X},\|\cdot\|_{\mathcal{X}})$ is a normed space. The set of all subsets of $\mathcal{X}$, the set of all closed subsets of $\mathcal{X}$, the set of all bounded subsets of $\mathcal{X}$, the set of all compact subsets of $\mathcal{X}$, and the set of all convex subsets of $\mathcal{X}$ are represented by $\mathcal{P}(\mathcal{X})$, $\mathcal{P}_{cl}(\mathcal{X})$, $\mathcal{P}_b(\mathcal{X})$, $\mathcal{P}_{cp}(\mathcal{X})$, and $\mathcal{P}_{cv}(\mathcal{X})$, respectively. We say that $k^*\in\mathcal{X}$ is a fixed point for the set-valued map $\mathcal{K}:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ if $k^*\in\mathcal{K}(k^*)$ [50]. The set of all fixed points of the set-valued map $\mathcal{K}$ is denoted by $\mathcal{FIX}(\mathcal{K})$ [50]. The Pompeiu–Hausdorff metric $\mathrm{PH}_d:\mathcal{P}(\mathcal{X})\times\mathcal{P}(\mathcal{X})\to\mathbb{R}\cup\{\infty\}$ is defined by

$$ \mathrm{PH}_{d_{\mathcal{X}}}(A_1,A_2)=\max\Big\{\sup_{a_1\in A_1}d_{\mathcal{X}}(a_1,A_2),\sup_{a_2\in A_2}d_{\mathcal{X}}(A_1,a_2)\Big\}, $$

where $d_{\mathcal{X}}(A_1,a_2)=\inf_{a_1\in A_1}d_{\mathcal{X}}(a_1,a_2)$ and $d_{\mathcal{X}}(a_1,A_2)=\inf_{a_2\in A_2}d_{\mathcal{X}}(a_1,a_2)$ [50]. A set-valued map $\mathcal{K}:\mathcal{X}\to\mathcal{P}_{cl}(\mathcal{X})$ is said to be Lipschitz with a Lipschitz constant $\lambda^*>0$ whenever we have $\mathrm{PH}_{d_{\mathcal{X}}}(\mathcal{K}(k_1),\mathcal{K}(k_2))\le\lambda^*d_{\mathcal{X}}(k_1,k_2)$ for all $k_1,k_2\in\mathcal{X}$. A Lipschitz map $\mathcal{K}$ is called a contraction whenever $\lambda^*\in(0,1)$ [50]. We say that the set-valued map $\mathcal{K}$ is completely continuous whenever the set $\mathcal{K}(W)$ is relatively compact for every $W\in\mathcal{P}_b(\mathcal{X})$. A set-valued map $\mathcal{K}:[0,1]\to\mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if the function $t\mapsto d_{\mathcal{X}}(\upsilon,\mathcal{K}(t))$ is measurable for all $\upsilon\in\mathbb{R}$ [50, 51]. We say that the set-valued map $\mathcal{K}$ is an upper semi-continuous (u.s.c.) whenever for each $k^*\in\mathcal{X}$, the set $\mathcal{K}(k^*)$ belongs to $\mathcal{P}_{cl}(\mathcal{X})$ and for every open set $\mathcal{V}$ containing $\mathcal{K}(k^*)$, there exists an open neighborhood $\mathcal{U}_0^*$ of k^* such that $\mathcal{K}(\mathcal{U}_0^*)\subseteq\mathcal{V}$ [50]. A real-valued function $k:\mathbb{R}\to\mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty}k(a_n)\le k(a)$ for all sequences $\{a_n\}_{n\ge1}$ with $a_n\to a$ [50].

The graph of the set-valued map $\mathcal{K}:\mathcal{X}\to\mathcal{P}_{cl}(\mathcal{Y})$ is defined by

$$ \mathrm{Graph}(\mathcal{K})=\big\{(k,s)\in\mathcal{X}\times\mathcal{Y}:s\in\mathcal{K}(k)\big\}. $$

We say that the graph of $\mathcal{K}$ is a closed set if for each sequence $\{k_n\}_{n\ge1}$ in $\mathcal{X}$ and $\{s_n\}_{n\ge1}$ in $\mathcal{Y}$, $k_n\to k_0$, $s_n\to s_0$ and $s_n\in\mathcal{K}(k_n)$, we have $s_0\in\mathcal{K}(k_0)$ [50, 51]. Suppose that the set-valued map $\mathcal{K}:\mathcal{X}\to\mathcal{P}_{cl}(\mathcal{Y})$ is upper semi-continuous. Then $\mathrm{Graph}(\mathcal{K})$ is a subset of the product space $\mathcal{X}\times\mathcal{Y}$ which is a closed set. Conversely, if the set-valued map $\mathcal{K}$ is completely continuous and has a closed graph, then $\mathcal{K}$ is upper semi-continuous [50, Proposition 2.1]. A set-valued map $\mathcal{K}$ is convex-valued if $\mathcal{K}(k)$ is a convex set for each element $k\in\mathcal{X}$. A set of selections of a set-valued map $\mathcal{K}$ at point $k \in C([0,1],\mathbb{R})$ is defined by

$$(\mathcal{SEL})_{\mathcal{K},k} := \{\vartheta \in \mathcal{L}^1([0,1],\mathbb{R}) : \vartheta(t) \in \mathcal{K}(t,k(t))\},$$

for almost all $t \in [0,1]$ [50, 51]. If $\mathcal{K}$ is an arbitrary set-valued map, then for each function $k \in C([0,1],\mathcal{X})$, we have $(\mathcal{SEL})_{\mathcal{K},k} \neq \emptyset$ whenever $\dim \mathcal{X} < \infty$ [50]. A set-valued map $\mathcal{K} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Carathéodory whenever $t \mapsto \mathcal{K}(t,k)$ is a measurable mapping for each function $k \in \mathbb{R}$ and $k \mapsto \mathcal{K}(t,k)$ is an upper semi-continuous mapping for almost all $t \in [0,1]$ [50, 51]. Moreover, a Carathéodory set-valued map $\mathcal{K} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be $\mathcal{L}^1$-Carathéodory whenever for each constant $\mu > 0$ there exists function $\phi_\mu \in \mathcal{L}^1([0,1],\mathbb{R}^+)$ such that

$$\|\mathcal{K}(t,k)\| = \sup_{t\in[0,1]} \{|q| : q \in \mathcal{K}(t,k)\} \le \phi_\mu(t),$$

for all $|k| \le \mu$ and for almost all $t \in [0,1]$ [50, 51]. We say that $u \in \mathcal{X}$ is an endpoint of the set-valued map $\mathcal{K} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ whenever we have $\mathcal{K}(k) = \{k\}$ [52]. Also, the set-valued map $\mathcal{K}$ has an approximate endpoint property whenever $\inf_{k\in\mathcal{X}} \sup_{k^*\in\mathcal{K}(k)} d_{\mathcal{X}}(k,k^*) = 0$ [52]. We need the following results.

Theorem 1 ([53]) *Suppose that $\mathcal{X}$ is a separable Banach space, $\mathcal{K} : [0,1] \times \mathcal{X} \to \mathcal{P}_{cp,cv}(\mathcal{X})$ is an $\mathcal{L}^1$-Carathéodory set-valued map and $\Xi : \mathcal{L}^1([0,1],\mathcal{X}) \to C([0,1],\mathcal{X})$ is a linear continuous mapping. Then the composition $\Xi \circ (\mathcal{SEL})_{\mathcal{K}} : C([0,1],\mathcal{X}) \to \mathcal{P}_{cp,cv}(C([0,1],\mathcal{X}))$ is an operator in the product space $C([0,1],\mathcal{X}) \times C([0,1],\mathcal{X})$ with action $k \mapsto (\Xi \circ (\mathcal{SEL})_{\mathcal{K}})(k) = \Xi((\mathcal{SEL})_{\mathcal{K},k})$ having the closed graph property.*

Theorem 2 ([54]) *Let $\mathcal{X}$ be a Banach algebra. Assume that there exist a single-valued map $\Phi : \mathcal{X} \to \mathcal{X}$ and a set-valued map $\Upsilon : \mathcal{X} \to \mathcal{P}_{cp,cv}(\mathcal{X})$ such that*
- (i) *Φ is a Lipschitz operator with a Lipschitz constant δ^*,*
- (ii) *Υ is an upper semi-continuous operator with the compactness property,*
- (iii) *$2\delta^*\hat{\Delta} < 1$ is such that $\hat{\Delta} = \|\Upsilon(\mathcal{X})\|$.*

Then either there is a solution in $\mathcal{X}$ for the operator inclusion $k \in (\Phi k)(\Upsilon k)$ or the set $\mathcal{O}^* = \{v^* \in \mathcal{X} | \lambda_* v^* \in (\Phi v^*)(\Upsilon v^*), \lambda_* > 1\}$ is unbounded.

Theorem 3 ([52]) *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a complete metric space and $\psi : [0,\infty) \to [0,\infty)$ be an upper semi-continuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ for all $t > 0$. Suppose that $\mathcal{K} : \mathcal{X} \to \mathcal{P}_{cl,b}(\mathcal{X})$ is a set-valued map so that $\mathrm{PH}_{d_{\mathcal{X}}}(\mathcal{K}k_1, \mathcal{K}k_2) \le \psi(d_{\mathcal{X}}(k_1,k_2))$ for all $k_1, k_2 \in \mathcal{X}$. Then $\mathcal{K}$ has a unique endpoint if and only if $\mathcal{K}$ has the approximate endpoint property.*

3 Main results

Now, we investigate the existence of solutions for the fractional hybrid and non-hybrid multi-term integro-differential inclusion problems (1)–(2) and (3). Consider the Banach space

$$\mathcal{X} = \{k(t) : k(t) \in C_{\mathbb{R}}([0,1])\},$$

with the norm $\|k\|_{\mathcal{X}} = \sup_{t\in[0,1]} |k(t)|$.

Lemma 4 *Let $z \in \mathcal{X}$. A function k_0^* is a solution for the fractional hybrid equation*

$$ {}^c\mathcal{D}_0^{\omega}\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))}\right) = z(t), \quad \bigl(t \in [0,1], \omega \in (1,2]\bigr), \tag{4} $$

with four-point integral hybrid boundary value conditions

$$ \begin{cases} \lambda_1\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))}\right)|_{t=0} \\ \quad + \lambda_2\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))}\right)|_{t=1} = a, \\ \lambda_3 {}^c\mathcal{D}_0^{\beta}\left(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))}\right)|_{t=\eta} \\ \quad + \lambda_4 \int_0^1 {}^c\mathcal{D}_0^{\beta}\left(\frac{k(s)}{\xi(s,k(s),\varphi_1(k(s)),\ldots,\varphi_n(k(s)))}\right) \mathrm{d}s = b, \end{cases} \tag{5} $$

if and only if k_0^* is a solution for the fractional integral equation

$$ \begin{aligned} k(t) = {} & \xi\bigl(t,k(t),\varphi_1\bigl(k(t)\bigr),\ldots,\varphi_n\bigl(k(t)\bigr)\bigr) \\ & \times \left[\frac{a\Delta_1 + b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1} + \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}z(s)\,\mathrm{d}s \right. \\ & - \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}z(s)\,\mathrm{d}s \\ & - \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^{\eta} (\eta-s)^{\omega-\beta-1}z(s)\,\mathrm{d}s \\ & \left. - \frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta-1}z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right], \end{aligned} \tag{6} $$

where Δ_1 and Δ_2 are positive real constants defined as follows:

$$ \Delta_1 := \lambda_3\eta^{1-\beta}(3-\beta) + \lambda_4\Gamma(2-\beta) \neq 0, \qquad \Delta_2 := (3-\beta)\Gamma(2-\beta). \tag{7} $$

Proof Assume that k_0^* is a solution for the hybrid differential equation (4). Then, there exist constants $m_0^*, m_1^* \in \mathbb{R}$ such that

$$ \frac{k_0^*(t)}{\xi(t,k_0^*(t),\varphi_1(k_0^*(t)),\ldots,\varphi_n(k_0^*(t)))} = \mathcal{I}_0^{\omega}z(t) + m_0^* + m_1^*t. $$

Then,

$$ k_0^*(t) = \xi\bigl(t,k_0^*(t),\varphi_1\bigl(k_0^*(t)\bigr),\ldots,\varphi_n\bigl(k_0^*(t)\bigr)\bigr)\left[\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}z(s)\,\mathrm{d}s + m_0^* + m_1^*t\right], \tag{8} $$

and so, for each $\beta \in (0,1]$, we get

$$ \begin{aligned} & {}^c\mathcal{D}_0^{\beta}\left(\frac{k_0^*(t)}{\xi(t,k_0^*(t),\varphi_1(k_0^*(t)),\ldots,\varphi_n(k_0^*(t)))}\right) \\ & \quad = \int_0^t \frac{(t-s)^{\omega-\beta-1}}{\Gamma(\omega-\beta)}z(s)\,\mathrm{d}s + m_1^*\frac{t^{1-\beta}}{\Gamma(2-\beta)}, \end{aligned} $$

$$\begin{aligned}&\int_0^1 {}^c\mathcal{D}_0^\beta\left(\frac{k_0^*(s)}{\xi(s,k_0^*(s),\varphi_1(k_0^*(s)),\ldots,\varphi_n(k_0^*(s)))}\right)\mathrm{d}s\\ &\quad=\int_0^1\int_0^s\frac{(s-\tau)^{\omega-\beta-1}}{\Gamma(\omega-\beta)}z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s+m_1^*\frac{1}{(3-\beta)}.\end{aligned}$$

By using three-point integral hybrid boundary value conditions, we obtain

$$\begin{aligned}m_0^*={}&\frac{a}{(\lambda_1+\lambda_2)}-\frac{\lambda_2 b\Delta_2}{(\lambda_1+\lambda_2)\Delta_1}-\frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1(1-s)^{\omega-1}z(s)\,\mathrm{d}s\\ &+\frac{\lambda_2\lambda_3\Delta_2}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta(\eta-s)^{\omega-\beta-1}z(s)\,\mathrm{d}s\\ &+\frac{\lambda_2\lambda_4\Delta_2}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s(s-\tau)^{\omega-\beta-1}z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}m_1^*={}&\frac{b\Delta_2}{\Delta_1}-\frac{\lambda_3\Delta_2}{\Delta_1\Gamma(\omega-\beta)}\int_0^\eta(\eta-s)^{\omega-\beta-1}z(s)\,\mathrm{d}s\\ &-\frac{\lambda_4\Delta_2}{\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s(s-\tau)^{\omega-\beta-1}z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

By substituting the values of m_0^* and m_1^* into (8), we get

$$\begin{aligned}k_0^*(t)={}&\xi\bigl(t,k_0^*(t),\varphi_1\bigl(k_0^*(t)\bigr),\ldots,\varphi_n\bigl(k_0^*(t)\bigr)\bigr)\\ &\times\left[\frac{a\Delta_1+b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1}+\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}z(s)\,\mathrm{d}s\right.\\ &-\frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1(1-s)^{\omega-1}z(s)\,\mathrm{d}s\\ &-\frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta(\eta-s)^{\omega-\beta-1}z(s)\,\mathrm{d}s\\ &\left.-\frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s(s-\tau)^{\omega-\beta-1}z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right].\end{aligned}$$

This shows that the function k_0^* is a solution for the fractional integral equation (6). Conversely, one can easily prove that k_0^* is a solution for the boundary value problem (4)–(5) whenever k_0^* is a solution function for the fractional integral equation (6). □

Definition 5 An absolutely continuous function $k:[0,1]\to\mathbb{R}$ is a solution for the fractional hybrid multi-term inclusion problem (1)–(2) whenever there exists an integrable function $\vartheta\in\mathcal{L}^1([0,1],\mathbb{R})$ with $\vartheta(t)\in\mathcal{K}(t,k(t),\phi_1(k(t)),\ldots,\phi_m(k(t)))$ for almost all $t\in[0,1]$ satisfying three-point integral hybrid boundary value conditions

$$\begin{cases}\lambda_1(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))})|_{t=0}\\ \quad+\lambda_2(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))})|_{t=1}=a,\\ \lambda_3{}^c\mathcal{D}_0^\beta(\frac{k(t)}{\xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t)))})|_{t=\eta}\\ \quad+\lambda_4\int_0^1{}^c\mathcal{D}_0^\beta(\frac{k(s)}{\xi(s,k(s),\varphi_1(k(s)),\ldots,\varphi_n(k(s)))})\,\mathrm{d}s=b\end{cases}$$

and

$$\begin{aligned} k(t) &= \xi\big(t, k(t), \varphi_1\big(k(t)\big), \dots, \varphi_n\big(k(t)\big)\big) \\ &\quad\times \Bigg[\frac{a\Delta_1 + b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1} + \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1} z(s)\,\mathrm{d}s \\ &\quad- \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1} z(s)\,\mathrm{d}s \\ &\quad- \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1} z(s)\,\mathrm{d}s \\ &\quad- \frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta-1} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \Bigg], \end{aligned}$$

for all $t \in [0,1]$.

Now, we prove our first result about the inclusion problem (1)–(2).

Theorem 6 *Suppose that $\mathcal{K} : [0,1] \times \mathbb{R}^{m+1} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is a set-valued mapping and $\xi : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R} \setminus \{0\}$ is a continuous function. Assume that*

($\mathcal{C}1$) *there is a bounded mapping $A : [0,1] \to \mathbb{R}^+$ so that for each $k_1, \dots, k_{n+1}, k_1', \dots, k_{n+1}' \in \mathbb{R}$ and for all $t \in [0,1]$, we have*

$$\big|\xi\big(t, k_1(t), \dots, k_{n+1}(t)\big) - \xi\big(t, k_1'(t), \dots, k_{n+1}'(t)\big)\big| \le A(t)\sum_{i=1}^{n+1}\big|k_i(t) - k_i'(t)\big|,$$

($\mathcal{C}2$) *the set-valued map $\mathcal{K} : [0,1] \times \mathbb{R}^{m+1} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ has the $\mathcal{L}^1$-Carathéodory property,*

($\mathcal{C}3$) *there is a positive mapping $\Theta(t) \in \mathcal{L}^1([0,1], \mathbb{R}^+)$ such that*

$$\begin{aligned} \big\|\mathcal{K}(t, k_1, k_2, \dots, k_{m+1})\big\| &= \sup\big\{|\vartheta| : \vartheta \in \mathcal{K}\big(t, k_1(t), k_2(t), \dots, k_{m+1}(t)\big)\big\} \\ &\le \Theta(t), \end{aligned}$$

for all $k_1, \dots, k_{m+1} \in \mathbb{R}$ and for almost all $t \in [0,1]$,

($\mathcal{C}4$) *there is a positive real number $\tilde{r} \in \mathbb{R}$ such that*

$$\tilde{r} > \frac{\xi^* \Lambda^* \|\Theta\|_{\mathcal{L}^1}}{1 - A^*[1 + \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Gamma(\varrho_2+1)} + \dots + \frac{1}{\Gamma(\varrho_n+1)}]\Lambda^*\|\Theta\|_{\mathcal{L}^1}}, \tag{9}$$

where $\|\Theta\|_{\mathcal{L}^1} = \int_0^1 |\Theta(s)|\,\mathrm{d}s$,

$$\xi^* = \sup_{t\in[0,1]}\big|\xi\big(t, \overbrace{0, 0, \dots, 0}^{(n+1)}\big)\big|,$$

$A^* = \sup_{t\in[0,1]}|A(t)|$, $\varrho_i > 0$ for $i = 1, 2, \dots, n$ and

$$\begin{aligned} \Lambda^* &= \frac{1}{\Gamma(\omega+1)} + \frac{|\lambda_2|}{(|\lambda_1|+|\lambda_2|)\Gamma(\omega+1)} \\ &\quad+ \frac{|\lambda_3||\Delta_2|[|\lambda_1| + 2|\lambda_2|]\eta^{\omega-\beta}}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+1)} \end{aligned}$$

$$+\frac{|\lambda_4||\Delta_2|[|\lambda_1|+2|\lambda_2|]}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+2)} + \frac{(|a||\Delta_1|+|b||\Delta_2|)[|\lambda_1|+2|\lambda_2|]}{(|\lambda_1|+|\lambda_2|)|\Delta_1|}. \tag{10}$$

If

$$A^*\left[1+\frac{1}{\Gamma(\varrho_1+1)}+\frac{1}{\Gamma(\varrho_2+1)}+\cdots+\frac{1}{\Gamma(\varrho_n+1)}\right]\Lambda^*\|\Theta\|_{\mathcal{L}^1}<\frac{1}{2},$$

then the hybrid multi-term inclusion problem (1)–(2) *has a solution.*

Proof For every $k\in\mathcal{X}$, define the set of selections of the operator $\mathcal{K}$ by

$$(\mathcal{SEL})_{\mathcal{K},k}=\bigl\{\vartheta\in\mathcal{L}^1\bigl([0,1]\bigr):\vartheta(t)\in\mathcal{K}\bigl(t,k(t),\phi_1\bigl(k(t)\bigr),\ldots,\phi_m\bigl(k(t)\bigr)\bigr)\bigr\}$$

for almost all $t\in[0,1]$. Consider the set-valued map $\mathcal{H}:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ defined by

$$\mathcal{H}(k)=\left\{\begin{array}{l} h\in\mathcal{X}: \\ h(t)=\left\{\begin{array}{l} \xi(t,k(t),\varphi_1(k(t)),\ldots,\varphi_n(k(t))) \\ \quad\times[\frac{a\Delta_1+b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1} \\ \quad+\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ \quad-\frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1(1-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ \quad-\frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)} \\ \quad\times\int_0^\eta(\eta-s)^{\omega-\beta-1}\vartheta(s)\,\mathrm{d}s \\ \quad-\frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)} \\ \quad\times\int_0^1\int_0^s(s-\tau)^{\omega-\beta-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s], \quad \vartheta\in(\mathcal{SEL})_{\mathcal{K},k} \end{array}\right. \end{array}\right\},$$

for all $t\in[0,1]$. It is obvious that the function h_0 is a solution for the hybrid multi-term inclusion problem (1)–(2) if and only if h_0 is a fixed point of the operator $\mathcal{H}$. Now, define the single-valued mapping $\Phi:\mathcal{X}\to\mathcal{X}$ by

$$(\Phi k)(t)=\xi\bigl(t,k(t),\varphi_1\bigl(k(t)\bigr),\ldots,\varphi_n\bigl(k(t)\bigr)\bigr)$$

and the set-valued map $\Upsilon:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ by

$$(\Upsilon k)(t)=\left\{\begin{array}{l} \zeta\in\mathcal{X}: \\ \zeta(t)=\left\{\begin{array}{l} \frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ \quad-\frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1(1-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ \quad-\frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta(\eta-s)^{\omega-\beta-1}\vartheta(s)\,\mathrm{d}s \\ \quad-\frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)} \\ \quad\times\int_0^1\int_0^s(s-\tau)^{\omega-\beta-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ \quad+\frac{a\Delta_1+b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1}, \qquad \vartheta\in(\mathcal{SEL})_{\mathcal{K},k} \end{array}\right. \end{array}\right\},$$

for all $t \in [0,1]$. Then, we obtain $\mathcal{H}(k) = (\Phi k)(\Upsilon k)$. We show that both operators Φ and Υ satisfy assumptions of Theorem 2. We first prove that the operator Φ is Lipschitz. Let $k_1, k_2 \in \mathcal{X}$. Since for $i = 1, 2, \ldots, n$, we have $\varphi_i(k(t)) = \int_0^t \frac{(t-s)^{\varrho_i - 1}}{\Gamma(\varrho_i)} k(s)\,\mathrm{d}s$, thus assumption $(\mathcal{C}1)$ implies that

$$\begin{aligned}
\bigl|(\Phi k_1)(t) - (\Phi k_2)(t)\bigr| &= \bigl|\xi\bigl(t, k_1(t), \varphi_1\bigl(k_1(t)\bigr), \ldots, \varphi_n\bigl(k_1(t)\bigr)\bigr) \\
&\quad - \xi\bigl(t, k_2(t), \varphi_1\bigl(k_2(t)\bigr), \ldots, \varphi_n\bigl(k_2(t)\bigr)\bigr)\bigr| \\
&= \bigl|\xi\bigl(t, k_1(t), \mathcal{I}_0^{\varrho_1} k_1(t), \ldots, \mathcal{I}_0^{\varrho_n} k_1(t)\bigr) \\
&\quad - \xi\bigl(t, k_2(t), \mathcal{I}_0^{\varrho_1} k_2(t), \ldots, \mathcal{I}_0^{\varrho_n} k_2(t)\bigr)\bigr| \\
&\leq A(t)\biggl[1 + \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Gamma(\varrho_2+1)} + \cdots + \frac{1}{\Gamma(\varrho_n+1)}\biggr] \\
&\quad \times \bigl|k_1(t) - k_2(t)\bigr|
\end{aligned}$$

for all $t \in [0,1]$. Hence, we get

$$\|\Phi k_1 - \Phi k_2\|_{\mathcal{X}} \leq A^*\biggl[1 + \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Gamma(\varrho_2+1)} + \cdots + \frac{1}{\Gamma(\varrho_n+1)}\biggr]\|k_1 - k_2\|_{\mathcal{X}}$$

for all $k_1, k_2 \in \mathcal{X}$. This shows that operator Φ is Lipschitz with a Lipschitz constant

$$A^*\biggl[1 + \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Gamma(\varrho_2+1)} + \cdots + \frac{1}{\Gamma(\varrho_n+1)}\biggr] > 0.$$

In this step, we prove that the set-valued map Υ has convex values. Let $k_1, k_2 \in \Upsilon k$. Choose $\vartheta_1, \vartheta_2 \in (\mathcal{SEL})_{\mathcal{K},k}$ such that

$$\begin{aligned}
k_j(t) &= \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta_j(s)\,\mathrm{d}s \\
&\quad - \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}\vartheta_j(s)\,\mathrm{d}s \\
&\quad - \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1}\vartheta_j(s)\,\mathrm{d}s \\
&\quad - \frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta-1}\vartheta_j(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \frac{a\Delta_1 + b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1},
\end{aligned}$$

for $j = 1, 2$ and for almost all $t \in [0,1]$. Let $\mu \in (0,1)$. Then, we have

$$\begin{aligned}
\mu k_1(t) + (1-\mu)k_2(t) &= \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\bigl[\mu\vartheta_1(s) + (1-\mu)\vartheta_2(s)\bigr]\,\mathrm{d}s \\
&\quad - \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)} \\
&\quad \times \int_0^1 (1-s)^{\omega-1}\bigl[\mu\vartheta_1(s) + (1-\mu)\vartheta_2(s)\bigr]\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \\
&\times \int_0^\eta (\eta - s)^{\omega-\beta-1} \big[\mu \vartheta_1(s) + (1-\mu)\vartheta_2(s)\big]\, \mathrm{d}s \\
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \\
&\times \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \big[\mu \vartheta_1(\tau) + (1-\mu)\vartheta_2(\tau)\big]\, \mathrm{d}\tau\, \mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for almost all $t \in [0,1]$. Since $\mathcal{K}$ has convex values, $(\mathcal{SEL})_{\mathcal{K},k}$ is convex-valued. This follows that $\mu\vartheta_1(t) + (1-\mu)\vartheta_2(t) \in (\mathcal{SEL})_{\mathcal{K},k}$ for all $t \in [0,1]$ and so Υk is a convex set for all $k \in \mathcal{X}$. Now, we prove that the operator Υ is completely continuous. In order to do this, we have to prove two equicontinuity and uniform boundedness properties for the set $\Upsilon(\mathcal{X})$. First, we show that Υ maps all bounded sets into bounded subsets of $\mathcal{X}$. For a positive number $\varepsilon^* \in \mathbb{R}$, consider the bounded ball $\mathcal{V}_{\varepsilon^*} = \{k \in \mathcal{X} : \|k\|_{\mathcal{X}} \leq \varepsilon^*\}$. For every $k \in \mathcal{V}_{\varepsilon^*}$ and $\zeta \in \Upsilon k$, there exists a function $\vartheta \in (\mathcal{SEL})_{\mathcal{K},k}$ such that

$$
\begin{aligned}
\zeta(t) = {}& \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \vartheta(s)\, \mathrm{d}s \\
&- \frac{\lambda_2}{(\lambda_1 + \lambda_2)\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \vartheta(s)\, \mathrm{d}s \\
&- \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \vartheta(s)\, \mathrm{d}s \\
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \vartheta(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$. Then, we have

$$
\begin{aligned}
\big|\zeta(t)\big| \leq {}& \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \big|\vartheta(s)\big|\, \mathrm{d}s \\
&+ \frac{|\lambda_2|}{|(\lambda_1 + \lambda_2)|\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \big|\vartheta(s)\big|\, \mathrm{d}s \\
&+ \frac{|\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]|}{|(\lambda_1 + \lambda_2)\Delta_1|\Gamma(\omega - \beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \big|\vartheta(s)\big|\, \mathrm{d}s \\
&+ \frac{|\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]|}{|(\lambda_1 + \lambda_2)\Delta_1|\Gamma(\omega - \beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \big|\vartheta(\tau)\big|\, \mathrm{d}\tau\, \mathrm{d}s \\
&+ \frac{|a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]|}{|(\lambda_1 + \lambda_2)\Delta_1|} \\
\leq {}& \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \Theta(s)\, \mathrm{d}s \\
&+ \frac{|\lambda_2|}{|(\lambda_1 + \lambda_2)|\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \Theta(s)\, \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\lambda_3 \Delta_2|[|(\lambda_1+\lambda_2)|+|\lambda_2|]}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^\eta (\eta-s)^{\omega-\beta-1}\Theta(s)\,\mathrm{d}s \\
&+ \frac{|\lambda_4 \Delta_2|[|(\lambda_1+\lambda_2)|+|\lambda_2|]}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1}\Theta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ \frac{|a\Delta_1|+|b\Delta_2|[|(\lambda_1+\lambda_2)|+|\lambda_2|]}{|(\lambda_1+\lambda_2)\Delta_1|} \\
\le{}& \Bigg[\frac{1}{\Gamma(\omega+1)} + \frac{|\lambda_2|}{(|\lambda_1|+|\lambda_2|)\Gamma(\omega+1)} \\
&+ \frac{|\lambda_3||\Delta_2|[|\lambda_1|+2|\lambda_2|]\eta^{\omega-\beta}}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+1)} \\
&+ \frac{|\lambda_4||\Delta_2|[|\lambda_1|+2|\lambda_2|]}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+2)} \\
&+ \frac{(|a||\Delta_1|+|b||\Delta_2|)[|\lambda_1|+2|\lambda_2|]}{(|\lambda_1|+|\lambda_2|)|\Delta_1|} \Bigg] \|\Theta\|_{\mathcal{L}^1} \\
={}& \Lambda^* \|\Theta\|_{\mathcal{L}^1},
\end{aligned}
$$

where Λ^* is given in (10). Thus, $\|\zeta\| \le \Lambda^* \|\Theta\|_{\mathcal{L}^1}$ and this means that the set $\Upsilon(\mathcal{X})$ is uniformly bounded. Now, we prove that the operator Υ maps bounded sets into equicontinuous sets. Let $k \in V_{\varepsilon^*}$ and $\zeta \in \Upsilon k$. Choose $\vartheta \in (\mathcal{SEL})_{\mathcal{K},k}$ such that

$$
\begin{aligned}
\zeta(t) ={}& \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)} \int_0^\eta (\eta-s)^{\omega-\beta-1}\vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$. Assume that $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
\big|\zeta(t_2)-\zeta(t_1)\big| \le{}& \frac{1}{\Gamma(\omega)} \int_0^{t_1} \big[(t_2-s)^{\omega-1}-(t_1-s)^{\omega-1}\big]\big|\vartheta(s)\big|\,\mathrm{d}s \\
&+ \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t_2-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s \\
&+ \frac{|\lambda_3\Delta_2(t_2-t_1)(\lambda_1+\lambda_2)|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^\eta (\eta-s)^{\omega-\beta-1}\big|\vartheta(s)\big|\,\mathrm{d}s \\
&+ \frac{|\lambda_4\Delta_2(t_2-t_1)(\lambda_1+\lambda_2)|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1}\big|\vartheta(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ \frac{|b\Delta_2(t_2-t_1)(\lambda_1+\lambda_2)|}{|(\lambda_1+\lambda_2)\Delta_1|} \\
\le{}& \frac{1}{\Gamma(\omega)} \int_0^{t_1} \big[(t_2-s)^{\omega-1}-(t_1-s)^{\omega-1}\big]\Theta(s)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t_2 - s)^{\omega-1} \Theta(s)\,\mathrm{d}s \\
&+ \frac{|\lambda_3||\Delta_2|(t_2 - t_1)(|\lambda_1| + |\lambda_2|)}{(|\lambda_1| + |\lambda_2|)|\Delta_1|\Gamma(\omega-\beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \Theta(s)\,\mathrm{d}s \\
&+ \frac{|\lambda_4||\Delta_2|(t_2 - t_1)(|\lambda_1| + |\lambda_2|)}{(|\lambda_1| + |\lambda_2|)|\Delta_1|\Gamma(\omega-\beta)} \int_0^1 \int_0^s (s - \tau)^{\omega-\beta-1} \Theta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ \frac{|b||\Delta_2|(t_2 - t_1)(|\lambda_1| + |\lambda_2|)}{(|\lambda_1| + |\lambda_2|)|\Delta_1|}.
\end{aligned}
$$

Notice that the right-hand side tends to zero independently of $k \in \mathcal{V}_{\varepsilon^*}$ as $t_2 \to t_1$. By using the Arzelà–Ascoli theorem, the complete continuity of $\Upsilon : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is deduced. In the following, we prove that Υ has a closed graph and this implies the upper semi-continuity of the operator Υ. Assume that $k_n \in \mathcal{V}_{\varepsilon^*}$ and $\zeta_n \in (\Upsilon k_n)$ with $k_n \to k^*$ and $\zeta_n \to \zeta^*$. We claim that $\zeta^* \in (\Upsilon k^*)$. For every $n \geq 1$ and $\zeta_n \in (\Upsilon k_n)$, choose $\vartheta_n \in (\mathcal{SEL})_{\mathcal{K},k_n}$ such that

$$
\begin{aligned}
\zeta_n(t) = {}& \frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega-1} \vartheta_n(s)\,\mathrm{d}s \\
&- \frac{\lambda_2}{(\lambda_1 + \lambda_2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} \vartheta_n(s)\,\mathrm{d}s \\
&- \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega-\beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \vartheta_n(s)\,\mathrm{d}s \\
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega-\beta)} \int_0^1 \int_0^s (s - \tau)^{\omega-\beta-1} \vartheta_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$. It is sufficient to show that there exists a function $\vartheta^* \in (\mathcal{SEL})_{\mathcal{K},k^*}$ such that

$$
\begin{aligned}
\zeta^*(t) = {}& \frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega-1} \vartheta^*(s)\,\mathrm{d}s \\
&- \frac{\lambda_2}{(\lambda_1 + \lambda_2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} \vartheta^*(s)\,\mathrm{d}s \\
&- \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega-\beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \vartheta^*(s)\,\mathrm{d}s \\
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega-\beta)} \int_0^1 \int_0^s (s - \tau)^{\omega-\beta-1} \vartheta^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$. Define the continuous linear operator $\Xi : \mathcal{L}^1([0,1],\mathbb{R}) \to \mathcal{X} = C([0,1],\mathbb{R})$ by

$$
\begin{aligned}
\Xi(\vartheta)(t) = k(t) = {}& \frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega-1} \vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_2}{(\lambda_1 + \lambda_2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} \vartheta(s)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^1 \int_0^s (s - \tau)^{\omega-\beta-1} \vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$. Hence,

$$
\begin{aligned}
\|\zeta_n(t) - \zeta^*(t)\| = {} & \Bigg\| \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \big(\vartheta_n(s) - \vartheta^*(s)\big)\,\mathrm{d}s \\
& - \frac{\lambda_2}{(\lambda_1 + \lambda_2)\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \big(\vartheta_n(s) - \vartheta^*(s)\big)\,\mathrm{d}s \\
& - \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \\
& \times \int_0^\eta (\eta - s)^{\omega-\beta-1} \big(\vartheta_n(s) - \vartheta^*(s)\big)\,\mathrm{d}s \\
& - \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \\
& \times \int_0^1 \int_0^s (s - \tau)^{\omega-\beta-1} \big(\vartheta_n(\tau) - \vartheta^*(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \Bigg\| \\
\to {} & 0.
\end{aligned}
$$

Thus, by using Theorem 1, it is deduced that the operator $\Xi \circ (\mathcal{SEL})_{\mathcal{S}}$ has a closed graph. Also, since $\zeta_n \in \Xi((\mathcal{SEL})_{\mathcal{K},k_n})$ and $k_n \to k^*$, so there exists $\vartheta^* \in (\mathcal{SEL})_{\mathcal{K},k^*}$ such that

$$
\begin{aligned}
\zeta^*(t) = {} & \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \vartheta^*(s)\,\mathrm{d}s \\
& - \frac{\lambda_2}{(\lambda_1 + \lambda_2)\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \vartheta^*(s)\,\mathrm{d}s \\
& - \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \vartheta^*(s)\,\mathrm{d}s \\
& - \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^1 \int_0^s (s - \tau)^{\omega-\beta-1} \vartheta^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
& + \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$. Hence, $\zeta^* \in (\Upsilon k^*)$ and so Υ has a closed graph. This means that the operator Υ is upper semi-continuous. On the other hand, since the operator Υ has compact values, so Υ is a compact and upper semi-continuous operator. By using assumption $(\mathcal{C}3)$, we have

$$
\begin{aligned}
\hat{\Delta} = \|\Upsilon(\mathcal{X})\| &= \sup_{t\in[0,1]} \big\{ |\Upsilon k| : k \in \mathcal{X} \big\} \\
&= \Bigg[\frac{1}{\Gamma(\omega + 1)} + \frac{|\lambda_2|}{(|\lambda_1| + |\lambda_2|)\Gamma(\omega + 1)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\lambda_3||\Delta_2|[|\lambda_1| + 2|\lambda_2|]\eta^{\omega-\beta}}{(|\lambda_1| + |\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+1)} \\
&+ \frac{|\lambda_4||\Delta_2|[|\lambda_1| + 2|\lambda_2|]}{(|\lambda_1| + |\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+2)} \\
&+ \frac{(|a||\Delta_1| + |b||\Delta_2|)[|\lambda_1| + 2|\lambda_2|]}{(|\lambda_1| + |\lambda_2|)|\Delta_1|}\Bigg]\|\Theta\|_{\mathcal{L}^1} \\
&= \Lambda^*\|\Theta\|_{\mathcal{L}^1}.
\end{aligned}
$$

Put

$$
\delta^* = \Lambda^*\left[1 + \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Gamma(\varrho_2+1)} + \cdots + \frac{1}{\Gamma(\varrho_n+1)}\right].
$$

Then, $\hat{\Delta}\delta^* < \frac{1}{2}$. Now by using Theorem 2 for Υ, we get that one of the conditions, (a) or (b), holds. We first investigate condition (b). By considering Theorem 2 and assumption $(\mathcal{C}4)$, assume that k is an arbitrary member of $\mathcal{O}^*$ with $\|k\| = \tilde{r}$. Then, $\lambda_* k(t) \in (\Phi k)(t)(\Upsilon k)(t)$ for all $\lambda_* > 1$. Choose the related function $\vartheta \in (\mathcal{SEL})_{\mathcal{K},k}$. Then for each $\lambda_* > 1$, we have

$$
\begin{aligned}
k(t) = {}& \frac{1}{\lambda_*}\xi\big(t, k(t), \varphi_1\big(k(t)\big), \ldots, \varphi_n\big(k(t)\big)\big) \\
&\times\left[\frac{a\Delta_1 + b\Delta_2[t(\lambda_1+\lambda_2) - \lambda_2]}{(\lambda_1+\lambda_2)\Delta_1} + \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s\right. \\
&- \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2) - \lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1}\vartheta(s)\,\mathrm{d}s \\
&\left.- \frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2) - \lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right]
\end{aligned}
$$

for all $t \in [0,1]$. Thus, one can write

$$
\begin{aligned}
\big|k(t)\big| = {}& \frac{1}{\lambda_*}\big|\xi\big(t, k(t), \varphi_1\big(k(t)\big), \ldots, \varphi_n\big(k(t)\big)\big)\big| \\
&\times\left[\frac{|a\Delta_1 + b\Delta_2[t(\lambda_1+\lambda_2) - \lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|} + \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s\right. \\
&+ \frac{|\lambda_2|}{|(\lambda_1+\lambda_2)|\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s \\
&+ \frac{|\lambda_3\Delta_2[t(\lambda_1+\lambda_2) - \lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1}\big|\vartheta(s)\big|\,\mathrm{d}s \\
&\left.+ \frac{|\lambda_4\Delta_2[t(\lambda_1+\lambda_2) - \lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta-1}\big|\vartheta(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}s\right] \\
= {}& \frac{1}{\lambda_*}\Big[\big|\xi\big(t, k(t), \varphi_1\big(k(t)\big), \ldots, \varphi_n\big(k(t)\big)\big) - \xi(t,0,0,\ldots,0)\big| \\
&+ \big|\xi(t,0,0,\ldots,0)\big|\Big] \\
&\times\left[\frac{|a\Delta_1 + b\Delta_2[t(\lambda_1+\lambda_2) - \lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|} + \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\lambda_2|}{|(\lambda_1+\lambda_2)|\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s \\
&+ \frac{|\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^\eta (\eta-s)^{\omega-\beta-1}\big|\vartheta(s)\big|\,\mathrm{d}s \\
&+ \frac{|\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1}\big|\vartheta(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}s\Bigg] \\
\le{}& \left[A^*\left[1+\frac{1}{\Gamma(\varrho_1+1)}+\frac{1}{\Gamma(\varrho_2+1)}+\cdots+\frac{1}{\Gamma(\varrho_n+1)}\right]\|k\|+\xi^*\right] \\
&\times\Bigg[\frac{|a\Delta_1+b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|}+\frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\Theta(s)\,\mathrm{d}s \\
&+ \frac{|\lambda_2|}{|(\lambda_1+\lambda_2)|\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1}\Theta(s)\,\mathrm{d}s \\
&+ \frac{|\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^\eta (\eta-s)^{\omega-\beta-1}\Theta(s)\,\mathrm{d}s \\
&+ \frac{|\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]|}{|(\lambda_1+\lambda_2)\Delta_1|\Gamma(\omega-\beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1}\Theta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg] \\
\le{}& \left[A^*\left[1+\frac{1}{\Gamma(\varrho_1+1)}+\frac{1}{\Gamma(\varrho_2+1)}+\cdots+\frac{1}{\Gamma(\varrho_n+1)}\right]\tilde{r}+\xi^*\right]\Lambda^*\|\Theta\|_{\mathcal{L}^1},
\end{aligned}
$$

for all $t \in [0,1]$. Hence, we get

$$
\tilde{r} \le \frac{\xi^*\Lambda^*\|\Theta\|_{\mathcal{L}^1}}{1-A^*[1+\frac{1}{\Gamma(\varrho_1+1)}+\frac{1}{\Gamma(\varrho_2+1)}+\cdots+\frac{1}{\Gamma(\varrho_n+1)}]\Lambda^*\|\Theta\|_{\mathcal{L}^1}}.
$$

According to condition (9), we see that condition (b) of Theorem 2 is impossible. Thus, $k \in (\Phi k)(\Upsilon k)$. Hence, the operator $\mathcal{H}$ has a fixed point and so the hybrid multi-term inclusion problem (1)–(2) has a solution. □

Here, we investigate the existence of solutions for the non-hybrid inclusion problem (3).

Definition 7 An absolutely continuous function $k : [0,1] \to \mathbb{R}$ is a solution for the fractional inclusion problem (3) whenever there exists an integrable function $\vartheta \in \mathcal{L}^1([0,1],\mathbb{R})$ with $\vartheta(t) \in \mathcal{S}(t,k(t))$ for almost all $t \in [0,1]$ satisfying three-point integral boundary value conditions

$$
\lambda_1 k(0)+\lambda_2 k(1)=a, \qquad \lambda_3 {}^c\mathcal{D}_0^\beta k(\eta)+\lambda_4\int_0^1 {}^c\mathcal{D}_0^\beta k(s)\,\mathrm{d}s=b
$$

and

$$
\begin{aligned}
k(t) ={}& \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\
&- \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1}\vartheta(s)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \vartheta(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$.

For $k \in \mathcal{X}$, the set of selections of $\mathcal{S}$ is defined by

$$
(\mathcal{SEL})_{\mathcal{S},k} = \{ \vartheta \in \mathcal{L}^1([0,1]) : \vartheta(t) \in \mathcal{S}(t, k(t)) \},
$$

for almost all $t \in [0,1]$. Define the operator $\mathcal{N} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$
\mathcal{N}(k) = \left\{ \begin{array}{l} h \in \mathcal{X} : \\ h(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \vartheta(s)\, \mathrm{d}s \\ \quad - \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \vartheta(s)\, \mathrm{d}s \\ \quad - \frac{\lambda_3 \Delta_2 [t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1 \Gamma(\omega-\beta)} \\ \quad \times \int_0^\eta (\eta - s)^{\omega-\beta-1} \vartheta(s)\, \mathrm{d}s \\ \quad - \frac{\lambda_4 \Delta_2 [t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1 \Gamma(\omega-\beta)} \\ \quad \times \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \vartheta(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\ \quad + \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1}, \quad \vartheta \in (\mathcal{SEL})_{\mathcal{S},k} \end{array} \right. \end{array} \right\}. \tag{11}
$$

Now, we prove next result by using the approximate endpoint property for the set-valued map $\mathcal{N}$ which is defined in (11).

Theorem 8 *Let $\mathcal{S} : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be a set-valued map. Assume that*

($\mathcal{C}5$) *The nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ is upper semi-continuous such that $\liminf_{t\to\infty}(t - \psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$.*

($\mathcal{C}6$) *The operator $\mathcal{S} : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is an integrable bounded set-valued map so that $\mathcal{S}(\cdot, k) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for all $k \in \mathbb{R}$.*

($\mathcal{C}7$) *There exists a non-negative function $\sigma \in C([0,1],[0,\infty))$ such that*

$$
\mathrm{PH}_{d_{\mathcal{X}}} (\mathcal{S}(t, k_1(t)), \mathcal{S}(t, k_1'(t))) \le \sigma(t) \psi (|k_1(t) - k_1'(t)|) \frac{1}{\Lambda^{**}},
$$

for all $t \in [0,1]$ and $k_1, k_1' \in \mathbb{R}$, where $\sup_{t\in[0,1]} |\sigma(t)| = \|\sigma\|$ and

$$
\begin{aligned}
\Lambda^{**} = \|\sigma\| \bigg[& \frac{1}{\Gamma(\omega+1)} + \frac{|\lambda_2|}{(|\lambda_1| + |\lambda_2|)\Gamma(\omega+1)} \\
& + \frac{|\lambda_3| |\Delta_2| [|\lambda_1| + 2|\lambda_2|] \eta^{\omega-\beta}}{(|\lambda_1| + |\lambda_2|) |\Delta_1| \Gamma(\omega - \beta + 1)} \\
& + \frac{|\lambda_4| |\Delta_2| [|\lambda_1| + 2|\lambda_2|]}{(|\lambda_1| + |\lambda_2|) |\Delta_1| \Gamma(\omega - \beta + 2)} \bigg].
\end{aligned} \tag{12}
$$

($\mathcal{C}8$) *The operator $\mathcal{N}$ has the approximate endpoint property, where $\mathcal{N}$ is given in (11). Then the fractional non-hybrid inclusion problem (3) has a solution.*

Proof We show that the set-valued map $\mathcal{N}:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ has an endpoint. In this way, we first prove that $\mathcal{N}(k)$ is a closed set for all $k\in\mathcal{X}$. First of all, by using assumption $(\mathcal{C}6)$, the set-valued map $t\mapsto\mathcal{S}(t,k(t))$ is measurable and it has closed values for all $k\in\mathcal{X}$. Thus, $\mathcal{S}$ has measurable selection and the set $(\mathcal{SEL})_{\mathcal{S},k}$ is nonempty. Now, we show that $\mathcal{N}(k)$ is a closed subset of $\mathcal{X}$ for all $k\in\mathcal{X}$. Let $\{k_n\}_{n\geq 1}$ be a sequence in $\mathcal{N}(k)$ with $k_n\to u$. For each n, there exists $\vartheta_n\in(\mathcal{SEL})_{\mathcal{S},k}$ such that

$$\begin{aligned} k_n(t) ={}& \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta_n(s)\,\mathrm{d}s \\ &- \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}\vartheta_n(s)\,\mathrm{d}s \\ &- \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1}\vartheta_n(s)\,\mathrm{d}s \\ &- \frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta-1}\vartheta_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &+ \frac{a\Delta_1+b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1}, \end{aligned}$$

for almost all $t\in[0,1]$. Since $\mathcal{S}$ is compact set-valued map, we pass into a subsequence (if necessary) to obtain that a subsequence $\{\vartheta_n\}_{n\geq 1}$ converges to some $\vartheta\in\mathcal{L}^1([0,1])$. Hence, we have $\vartheta\in(\mathcal{SEL})_{\mathcal{S},k}$ and

$$\begin{aligned} \lim_{n\to\infty} k_n(t) ={}& \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ &- \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ &- \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1}\vartheta(s)\,\mathrm{d}s \\ &- \frac{\lambda_4\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &+ \frac{a\Delta_1+b\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1} \\ ={}& k(t), \end{aligned}$$

for all $t\in[0,1]$. This implies that $k\in\mathcal{N}(k)$, and so operator $\mathcal{N}$ is closed-valued. Also, $\mathcal{N}(k)$ is a bounded set for all $k\in\mathcal{X}$ because $\mathcal{S}$ is a compact set-valued map. Finally, we are going to prove that $\mathrm{PH}_{d_{\mathcal{X}}}(\mathcal{N}(k),\mathcal{N}(k'))\leq\psi(\|k-k'\|)$ holds. Let $k,k'\in\mathcal{X}$ and $z_1\in\mathcal{N}(k')$. Choose $\vartheta_1\in(\mathcal{SEL})_{\mathcal{S},k'}$ such that

$$\begin{aligned} z_1(t) ={}& \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta_1(s)\,\mathrm{d}s \\ &- \frac{\lambda_2}{(\lambda_1+\lambda_2)\Gamma(\omega)}\int_0^1 (1-s)^{\omega-1}\vartheta_1(s)\,\mathrm{d}s \\ &- \frac{\lambda_3\Delta_2[t(\lambda_1+\lambda_2)-\lambda_2]}{(\lambda_1+\lambda_2)\Delta_1\Gamma(\omega-\beta)}\int_0^\eta (\eta-s)^{\omega-\beta-1}\vartheta_1(s)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \vartheta_1(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for almost all $t \in [0,1]$. Since

$$
\mathrm{PH}_{d_{\mathcal{X}}}\big(\mathcal{S}(t,k(t)), \mathcal{S}(t,k'(t))\big) \le \sigma(t)\psi\big(\big|k(t) - k'(t)\big|\big)\frac{1}{\Lambda^{**}},
$$

for all $t \in [0,1]$, there exists $h^* \in \mathcal{S}(t,k(t))$ such that

$$
\big|\vartheta_1(t) - h^*\big| \le \sigma(t)\psi\big(\big|k(t) - k'(t)\big|\big)\frac{1}{\Lambda^{**}},
$$

for all $t \in [0,1]$. Consider a set-valued map $\mathcal{B} : [0,1] \to \mathcal{P}(\mathcal{X})$ which is given by

$$
\mathcal{B}(t) = \left\{ h^* \in \mathcal{X} : \big|\vartheta_1(t) - h^*\big| \le \sigma(t)\psi\big(\big|k(t) - k'(t)\big|\big)\frac{1}{\Lambda^{**}} \right\}.
$$

Since ϑ_1 and $\varrho = \sigma\psi(|k - k'|)\frac{1}{\Lambda^{**}}$ are measurable, one can easily check that the set-valued map $\mathcal{B}(\cdot) \cap \mathcal{S}(\cdot, k(\cdot))$ is measurable. Now, choose $\vartheta_2(t) \in \mathcal{K}(t,k(t))$ such that

$$
\big|\vartheta_1(t) - \vartheta_2(t)\big| \le \sigma(t)\psi\big(\big|k(t) - k'(t)\big|\big)\frac{1}{\Lambda^{**}}
$$

for all $t \in [0,1]$. We select $z_2 \in \mathcal{N}(k)$ such that

$$
\begin{aligned}
z_2(t) = {}& \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \vartheta_2(s)\, \mathrm{d}s \\
&- \frac{\lambda_2}{(\lambda_1 + \lambda_2)\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \vartheta_2(s)\, \mathrm{d}s \\
&- \frac{\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^\eta (\eta - s)^{\omega-\beta-1} \vartheta_2(s)\, \mathrm{d}s \\
&- \frac{\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1 \Gamma(\omega - \beta)} \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \vartheta_2(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\
&+ \frac{a\Delta_1 + b\Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]}{(\lambda_1 + \lambda_2)\Delta_1},
\end{aligned}
$$

for all $t \in [0,1]$. Hence, one can get

$$
\begin{aligned}
\big|z_1(t) - z_2(t)\big| \le {}& \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \big|\vartheta_1(s) - \vartheta_2(s)\big|\, \mathrm{d}s \\
&+ \frac{|\lambda_2|}{|(\lambda_1 + \lambda_2)|\Gamma(\omega)} \int_0^1 (1-s)^{\omega-1} \big|\vartheta_1(s) - \vartheta_2(s)\big|\, \mathrm{d}s \\
&+ \frac{|\lambda_3 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]|}{|(\lambda_1 + \lambda_2)\Delta_1|\Gamma(\omega - \beta)} \\
&\times \int_0^\eta (\eta - s)^{\omega-\beta-1} \big|\vartheta_1(s) - \vartheta_2(s)\big|\, \mathrm{d}s \\
&+ \frac{|\lambda_4 \Delta_2 [t(\lambda_1 + \lambda_2) - \lambda_2]|}{|(\lambda_1 + \lambda_2)\Delta_1|\Gamma(\omega - \beta)}
\end{aligned}
$$

$$\begin{aligned}
&\times \int_0^1 \int_0^s (s-\tau)^{\omega-\beta-1} \bigl|\vartheta_1(\tau)-\vartheta_2(\tau)\bigr| \,\mathrm{d}\tau \,\mathrm{d}s \\
\le{}& \frac{1}{\Gamma(\omega+1)} \|\sigma\| \psi\bigl(\|k-k'\|\bigr) \frac{1}{\Lambda^{**}} \\
&+ \frac{|\lambda_2|}{(|\lambda_1|+|\lambda_2|)\Gamma(\omega+1)} \|\sigma\| \psi\bigl(\|k-k'\|\bigr) \frac{1}{\Lambda^{**}} \\
&+ \frac{|\lambda_3||\Delta_2|[|\lambda_1|+2|\lambda_2|]\eta^{\omega-\beta}}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+1)} \|\sigma\| \psi\bigl(\|k-k'\|\bigr) \frac{1}{\Lambda^{**}} \\
&+ \frac{|\lambda_4||\Delta_2|[|\lambda_1|+2|\lambda_2|]}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+2)} \|\sigma\| \psi\bigl(\|k-k'\|\bigr) \frac{1}{\Lambda^{**}} \\
={}& \biggl[\frac{1}{\Gamma(\omega+1)} + \frac{|\lambda_2|}{(|\lambda_1|+|\lambda_2|)\Gamma(\omega+1)} \\
&+ \frac{|\lambda_3||\Delta_2|[|\lambda_1|+2|\lambda_2|]\eta^{\omega-\beta}}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+1)} \\
&+ \frac{|\lambda_4||\Delta_2|[|\lambda_1|+2|\lambda_2|]}{(|\lambda_1|+|\lambda_2|)|\Delta_1|\Gamma(\omega-\beta+2)}\biggr] \|\sigma\| \psi\bigl(\|k-k'\|\bigr) \frac{1}{\Lambda^{**}} \\
={}& \Lambda^{**} \psi\bigl(\|k-k'\|\bigr) \frac{1}{\Lambda^{**}} = \psi\bigl(\|k-k'\|\bigr).
\end{aligned}$$

Thus, we obtain $\|z_1 - z_2\| \le \psi(\|k-k'\|)$. It follows that $\mathrm{PH}_{d_{\mathcal{X}}}(\mathcal{N}(k), \mathcal{N}(k')) \le \psi(\|k-k'\|)$ holds for all $k, k' \in \mathcal{X}$. Since the set-valued map $\mathcal{N}$ has approximate endpoint property $(\mathcal{C}8)$, Theorem 3 implies that operator $\mathcal{N}$ has a unique endpoint, that is, there exists $k^* \in \mathcal{X}$ such that $\mathcal{N}(k^*) = \{k^*\}$. Thus, k^* is a solution for the fractional inclusion problem (3). □

4 Examples

Here, we provide two examples to illustrate our main results.

Example 1 Consider the fractional hybrid multi-term inclusion problem

$$\begin{aligned}
&{}^{c}\mathcal{D}_0^{1.64}\biggl(\frac{k(t)}{\frac{t}{2097}(k(t) + \frac{|\mathcal{I}^{0.71}k(t)|}{1+|\mathcal{I}^{0.71}k(t)|} + \arctan(\mathcal{I}^{0.65}k(t)) + \sin(\mathcal{I}^{0.32}k(t))) + 0.002}\biggr) \\
&\quad\in \biggl[-5, \bigl(t^3+3\bigr)\sin k(t) + \cos\bigl(\mathcal{I}^{0.15}k(t)\bigr) \\
&\qquad + \frac{2}{5}\sin\bigl(\mathcal{I}^{0.26}k(t)\bigr) + \sin^2\bigl(\mathcal{I}^{0.31}k(t)\bigr) + \frac{27}{5}\biggr],
\end{aligned} \tag{13}$$

with the integral hybrid boundary value conditions

$$\begin{cases}
0.11(\frac{k(t)}{\frac{t}{2097}(\Sigma(t))+0.002})|_{t=0} \\
\quad + 0.17(\frac{k(t)}{\frac{t}{2097}(\Sigma(t))+0.002})|_{t=1} = 0.23, \\
0.82\,{}^{c}\mathcal{D}_0^{0.27}(\frac{k(t)}{\frac{t}{2097}(\Sigma(t))+0.002})|_{t=0.59} \\
\quad + 0.54\int_0^1 {}^{c}\mathcal{D}_0^{0.27}(\frac{k(s)}{\frac{s}{2097}(\Sigma(s))+0.002})\,\mathrm{d}s = 0.19,
\end{cases} \tag{14}$$

for $t \in [0,1]$ where

$$\Sigma(t) = k(t) + \frac{|\mathcal{I}^{0.71}k(t)|}{1 + |\mathcal{I}^{0.71}k(t)|} + \arctan\bigl(\mathcal{I}^{0.65}k(t)\bigr) + \sin\bigl(\mathcal{I}^{0.32}k(t)\bigr).$$

Put $\omega = 1.64$, $\beta = 0.27$, $\eta = 0.59$, $a = 0.23$, $b = 0.19$, $\lambda_1 = 0.11$, $\lambda_2 = 0.17$, $\lambda_3 = 0.82$, and $\lambda_4 = 0.54$. For $n = 3$, consider a continuous map $\xi : [0,1] \times \mathbb{R}^{3+1} \to \mathbb{R} \setminus \{0\}$ defined by

$$\xi\bigl(t, k(t), \mathcal{I}^{\varrho_1}k(t), \mathcal{I}^{\varrho_2}k(t), \mathcal{I}^{\varrho_3}k(t)\bigr) = \left(\frac{k(t)}{\frac{t}{2097}(\Sigma(t)) + 0.002}\right),$$

where for $i = 1,2,3$, we put $\varphi_i(k(t)) = \mathcal{I}^{\varrho_i}k(t)$ with $\varrho_1 = 0.71$, $\varrho_2 = 0.65$, and $\varrho_3 = 0.32$. Note that $\xi^* = \sup_{t\in[0,1]} |\xi(t,0,0,0,0)| = 0.002$. Since the single-valued map ξ is Lipschitz, for each $k, k' \in \mathbb{R}$, we have

$$\begin{aligned}
&\bigl|\xi\bigl(t, k(t), \mathcal{I}^{\varrho_1}k(t), \mathcal{I}^{\varrho_2}k(t), \mathcal{I}^{\varrho_3}k(t)\bigr) \\
&\quad - \xi\bigl(t, k'(t), \mathcal{I}^{\varrho_1}k'(t), \mathcal{I}^{\varrho_2}k'(t), \mathcal{I}^{\varrho_3}k'(t)\bigr)\bigr| \\
&\quad \le A(t)\left[1 + \frac{1}{\Gamma(\varrho_1 + 1)} + \frac{1}{\Gamma(\varrho_2 + 1)} + \frac{1}{\Gamma(\varrho_3 + 1)}\right]\bigl|k(t) - k'(t)\bigr| \\
&\quad = \frac{t}{2097}\left[1 + \frac{1}{\Gamma(1.71)} + \frac{1}{\Gamma(1.65)} + \frac{1}{\Gamma(1.32)}\right]\bigl|k(t) - k'(t)\bigr| \\
&\quad \simeq \frac{4.3269}{2097} t\bigl|k(t) - k'(t)\bigr|.
\end{aligned}$$

Note that

$$A^*\left[1 + \frac{1}{\Gamma(\varrho_1 + 1)} + \frac{1}{\Gamma(\varrho_2 + 1)} + \frac{1}{\Gamma(\varrho_3 + 1)}\right] \simeq 0.00206.$$

For $m = 3$, consider a set-valued map $\mathcal{K} : [0,1] \times \mathbb{R}^{3+1} \to \mathcal{P}(\mathbb{R})$ by

$$\begin{aligned}
&\mathcal{K}\bigl(t, k(t), \mathcal{I}^{\varpi_1}k(t), \mathcal{I}^{\varpi_2}k(t), \mathcal{I}^{\varpi_3}k(t)\bigr) \\
&\quad = \biggl[-5, \bigl(t^3 + 3\bigr)\sin k(t) + \cos\bigl(\mathcal{I}^{0.15}k(t)\bigr) \\
&\qquad + \frac{2}{5}\sin\bigl(\mathcal{I}^{0.26}k(t)\bigr) + \sin^2\bigl(\mathcal{I}^{0.31}k(t)\bigr) + \frac{27}{5}\biggr],
\end{aligned}$$

where, for $i = 1,2,3$, we put $\phi_i(k(t)) = \mathcal{I}^{\varpi_i}k(t)$ with $\varpi_1 = 0.15$, $\varpi_2 = 0.26$, and $\varpi_3 = 0.31$. Since

$$\begin{aligned}
|\zeta| &\le \max\biggl[-5, \bigl(t^3 + 3\bigr)\sin k(t) + \cos\bigl(\mathcal{I}^{0.15}k(t)\bigr) \\
&\quad + \frac{2}{5}\sin\bigl(\mathcal{I}^{0.26}k(t)\bigr) + \sin^2\bigl(\mathcal{I}^{0.31}k(t)\bigr) + \frac{27}{5}\biggr] \\
&\le t^3 + 10.8,
\end{aligned}$$

for all $\zeta \in \mathcal{K}(t, k(t), \mathcal{I}^{\varpi_1}k(t), \mathcal{I}^{\varpi_2}k(t), \mathcal{I}^{\varpi_3}k(t))$, we find

$$\begin{aligned}&\bigl\|\mathcal{K}\bigl(t, k(t), \mathcal{I}^{\varpi_1}k(t), \mathcal{I}^{\varpi_2}k(t), \mathcal{I}^{\varpi_3}k(t)\bigr)\bigr\| \\ &\quad = \sup\bigl\{|\vartheta| : \vartheta \in \mathcal{K}\bigl(t, k(t), \mathcal{I}^{\varpi_1}k(t), \mathcal{I}^{\varpi_2}k(t), \mathcal{I}^{\varpi_3}k(t)\bigr)\bigr\} \\ &\quad \leq \Theta(t) = t^3 + 10.8.\end{aligned}$$

Put $\Theta(t) = t^3 + 10.8$ for all $t \in [0,1]$. Then, $\|\Theta\|_{\mathcal{L}^1} = \int_0^1 |\Theta(s)|\,ds \simeq 11.05$. Hence, we get $\Lambda^* \simeq 2.86$. Now, choose $\tilde{r} > 0$ with $\tilde{r} > 0.0676074$. Then, we have

$$\Lambda^*\left[1 + \frac{1}{\Gamma(\varrho_1 + 1)} + \frac{1}{\Gamma(\varrho_2 + 1)} + \frac{1}{\Gamma(\varrho_3 + 1)}\right]\Lambda^*\|\Theta\|_{\mathcal{L}^1} \simeq 0.06510218 < \frac{1}{2}.$$

Now by using Theorem 6, the hybrid multi-term inclusion problem (13)–(14) has a solution.

Example 2 Consider the fractional non-hybrid differential inclusion

$$^{c}\mathcal{D}_0^{1.37}k(t) \in \left[0, \frac{0.0007}{2(0.8 + t)}\left(\frac{k^2(t) + |k(t)|}{1 + |k(t)|}\right)\right], \tag{15}$$

for $t \in [0,1]$, with three-point integral boundary value conditions

$$\begin{aligned}&0.8k(0) + 0.08k(1) = 0.21, \\ &0.76\,{}^{c}\mathcal{D}_0^{0.54}k(0.81) + 0.26\int_0^1 {}^{c}\mathcal{D}_0^{0.54}k(s)\,ds = 0.18,\end{aligned} \tag{16}$$

where $^{c}\mathcal{D}_0^{1.37}$ denotes the fractional Caputo derivative of order $\omega = 1.37$. Put $\beta = 0.54$, $\eta = 0.81$, $a = 0.21$, $b = 0.18$, $\lambda_1 = 0.8$, $\lambda_2 = 0.08$, $\lambda_3 = 0.76$, and $\lambda_4 = 0.26$. Consider the Banach space $\mathcal{X} = \{k(t) : k(t) \in C([0,1], \mathbb{R})\}$ with the norm $\|k\|_{\mathcal{X}} = \sup_{t\in[0,1]} |k(t)|$. Also, define a set-valued map $\mathcal{S} : [0,1] \times \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$\mathcal{S}\bigl(t, k(t)\bigr) = \left[0, \frac{0.0007}{2(0.8 + t)}\left(\frac{k^2(t) + |k(t)|}{1 + |k(t)|}\right)\right]$$

for all $t \in [0,1]$. Consider a function $\sigma \in C([0,1], [0,\infty))$ defined by $\sigma(t) = \frac{0.0007}{0.8+t}$ for all t with $\|\sigma\| = \frac{0.0007}{0.8} = 0.000875$. Also, consider a non-negative and nondecreasing upper semi-continuous function $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = \frac{t}{2}$ for all $t > 0$. It is clear that $\liminf_{t\to\infty}(t - \psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$. Now, for each $k_1, k_1' \in \mathcal{X}$, we have

$$\begin{aligned}\mathrm{PH}_{d_{\mathcal{X}}}\bigl(\mathcal{S}\bigl(t, k_1(t)\bigr), \mathcal{S}\bigl(t, k_1'(t)\bigr)\bigr) &\leq \frac{0.0007}{0.8 + t}\cdot\frac{1}{2}\bigl(\bigl|k_1 - k_1'\bigr|\bigr) \\ &= \frac{0.0007}{0.8 + t}\psi\bigl(\bigl|k_1 - k_1'\bigr|\bigr) \\ &\leq \sigma(t)\psi\bigl(\bigl|k_1 - k_1'\bigr|\bigr)\frac{1}{\Lambda^{**}},\end{aligned}$$

where $\Lambda^{**} \simeq 0.00145$. Finally, consider an operator $\mathcal{N} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined by

$$\mathcal{N}(k) = \bigl\{h \in \mathcal{X} : \text{ there exists } \vartheta \in (\mathcal{SEL})_{\mathcal{S},k} \text{ such that } h(t) = z(t) \text{ for all } t \in [0,1]\bigr\},$$

where

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(1.37)} \int_0^t (t-s)^{0.37} \vartheta(s)\,\mathrm{d}s \\ &\quad - \frac{0.08}{0.88\Gamma(1.37)} \int_0^1 (1-s)^{0.37} \vartheta(s)\,\mathrm{d}s \\ &\quad - \frac{0.76 \times 2.1785[0.88t - 0.08]}{0.88 \times 2.5824\Gamma(0.83)} \int_0^{0.81} (\eta - s)^{0.83-1} \vartheta(s)\,\mathrm{d}s \\ &\quad - \frac{0.26 \times 2.1785[0.88t - 0.08]}{0.88 \times 2.5824\Gamma(0.83)} \int_0^1 \int_0^s (s-\tau)^{0.83-1} \vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &\quad + \frac{(0.21 \times 2.5824) + (0.18 \times 2.1758)[0.88t - 0.08]}{0.88 \times 2.5824}. \end{aligned}$$

Now by using Theorem 8, the fractional non-hybrid inclusion problem (15) with three-point integral boundary value conditions (16) has a solution.

5 Conclusion

It is known that a lot of natural phenomena and processes in the world are modeled by different types of fractional differential equations and inclusions. This diversity factor in studying complicate fractional differential equations and inclusions increases our ability for exact modeling of more phenomena. This is useful in making modern software which allow for more cost-free testing and less material consumption. In this work, we investigate the existence of solutions for two fractional hybrid and non-hybrid multi-term integro-differential inclusions with integral hybrid boundary value conditions. In this work, we investigate the existence of solutions for two fractional hybrid and non-hybrid multi-term integro-differential inclusions with integral hybrid boundary value conditions. It is noted that the fractional hybrid multi-term integro-differential inclusion presented in this paper is new in the sense that the boundary value conditions are stated as three-point mixed Caputo integro-derivative hybrid conditions. Also, this hybrid boundary value problem is general and it involves many fractional dynamical systems as special cases. In this way, we use the Dhage's fixed point result and approximate endpoint property for a set-valued map in our proofs. Finally, we give two examples to illustrate our main results. This topic can be used in mathematical modeling of applied problems in science, engineering, and the real-world phenomena [4, 28].

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The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. [2]Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran. [3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

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