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Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials

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Abstract

The polyexponential functions were introduced by Hardy and rediscovered by Kim, as inverses to the polylogarithm functions. Recently, the type 2 poly-Bernoulli numbers and polynomials were defined by means of the polyexponential functions. In this paper, we introduce the degenerate polyexponential functions and the degenerate type 2 poly-Bernoulli numbers and polynomials, as degenerate versions of such functions and numbers and polynomials. We derive several explicit expressions and some identities for those numbers and polynomials.

MSC: 11B83; 05A19

Keywords: Degenerate polyexponential functions; Type 2 degenerate poly-Bernoulli polynomials; Type 2 degenerate poly-Bernoulli numbers

1 Introduction

For $k \in \mathbb{Z}$, the polyexponential function is defined by

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k} \quad (\text{see [10]}). \tag{1}$$

By (1), we see that $\mathrm{Ei}_1(x) = e^x - 1$.

The polyexponential function was first introduced by Hardy and is given by

$$e(x,a|s) = \sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!} \quad \bigl(\mathrm{Re}(a) > 0\bigr).$$

We note here that $e(x,1|k) = \frac{1}{x}\mathrm{Ei}_k(x)$.

In [10], the type 2 poly-Bernoulli polynomials are defined by

$$\frac{1}{e^t-1}\mathrm{Ei}_k\bigl(\log(1+t)\bigr)e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{2}$$

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called type 2 poly-Bernoulli numbers.

From (1) and (2), we note that $B_n^{(1)}(x) = B_n(x)$ $(n \geq 0)$, where $B_n(x)$ are ordinary Bernoulli polynomials given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{(see [1–18, 20–24]).}$$

In particular, $B_n = B_n(0)$ $(n \geq 0)$ are called Bernoulli numbers.

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined as

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = e_\lambda^1(t) = (1 + \lambda t)^{\frac{1}{\lambda}} \quad \text{(see [11–15, 17, 19]).} \tag{3}$$

In [2, 3], Carlitz considered the degenerate Bernoulli polynomials which are given by

$$\frac{t}{e_\lambda(t) - 1} e_\lambda^x(t) = \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \tag{4}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called degenerate Bernoulli numbers.

Recently, the degenerate polylogarithm function was defined by Kim–Kim as

$$l_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)! n^k} x^n \quad \big(k \in \mathbb{Z}, |x| < 1\big) \text{ (see [17]),} \tag{5}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x - \lambda) \cdots (x - (n-1)\lambda)$ $(n \geq 1)$.

Note that $\lim_{\lambda \to 0} l_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x)$ is the polylogarithm of index k.

For $k \in \mathbb{Z}$, the degenerate poly-Bernoulli numbers are defined by

$$\frac{1}{x} l_{k,\lambda}(x)\big|_{x=1-e_\lambda(-t)} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} \quad \text{(see [17]).} \tag{6}$$

In [17], the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \quad (n \geq 0). \tag{7}$$

As an inversion formula of (7), the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n \geq 0) \text{ (see [23]).} \tag{8}$$

From (7) and (8), we note that

$$\frac{1}{k!}\big(e_\lambda(t) - 1\big)^k = \sum_{n=k}^{\infty} S_{2,l}(n,k) \frac{t^n}{n!} \tag{9}$$

and

$$\frac{1}{k!}\big(\log_\lambda(1 + t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!} \quad (k \geq 0) \text{ (see [17]),} \tag{10}$$

where $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

Kaneko defined the poly-Bernoulli numbers by making use of the polylogarithm functions and Kim–Kim–Kim–Jang studied degenerate poly-Bernoulli numbers and polynomials by using polyexponential function [18]. The polyexponential functions were first introduced by Hardy and rediscovered recently by Kim–Kim [10], as inverses to the polylogarithm functions. In addition, the type 2 poly-Bernoulli numbers and polynomials were defined by means of the polyexponential functions. In this paper, we study the degenerate polyexponential functions and the degenerate type 2 poly-Bernoulli polynomials and numbers, as degenerate versions of such functions and numbers and polynomials. We derive several explicit expressions and some identities for those numbers and polynomials.

2 Type 2 degenerate poly-Bernoulli numbers and polynomials

The degenerate polyexponential function is defined in [15]. In the light of (1), we now consider the degenerate modified polyexponential function given by

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{(n-1)! n^k} \quad \bigl(k \in \mathbb{Z}, |x| < 1\bigr). \tag{11}$$

Note that $\mathrm{Ei}_{1,\lambda}(x) = e_\lambda(x) - 1$.

From (11), we note that

$$\frac{d}{dx} \mathrm{Ei}_{k,\lambda}(x) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{(n-1)! n^{k-1}} = \frac{1}{x} \mathrm{Ei}_{k-1,\lambda}(x). \tag{12}$$

For $k \geq 2$, by (12), we have

$$\begin{aligned} \mathrm{Ei}_{k,\lambda}(x) &= \underbrace{\int_0^x \frac{1}{t} \int_0^t \frac{1}{t} \int_0^t \cdots \frac{1}{t} \int_0^t}_{(k-2)\text{-times}} \frac{1}{t} \mathrm{Ei}_{1,\lambda}(t)\, dt \cdots dt \\ &= \underbrace{\int_0^x \frac{1}{t} \int_0^t \frac{1}{t} \int_0^t \cdots \frac{1}{t} \int_0^t}_{(k-2)\text{-times}} \frac{1}{t} \bigl(e_\lambda(t) - 1\bigr)\, dt \cdots dt. \end{aligned} \tag{13}$$

In view of (2) and using the degenerate modified polyexponential function, we define the type 2 degenerate poly-Bernoulli polynomials by

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}). \tag{14}$$

When $x = 0$, $B_{n,\lambda}^{(k)} = B_{n,\lambda}^{(k)}(0)$ are called type 2 degenerate poly-Bernoulli numbers.

It is well known that the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log_\lambda(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!} \quad (\text{see [17]}). \tag{15}$$

When $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ $(n \geq 0)$, are called degenerate Bernoulli numbers of the second kind.

Note that $\lim_{\lambda\to 0} b_{n,\lambda} = b_n$ $(n \geq 0)$. Here b_n are the Bernoulli numbers of the second kind, according to Roman [23], given by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \quad \text{(see [6, 22, 23])}. \tag{16}$$

From (12), we note that

$$\begin{aligned}
\frac{d}{dx}\operatorname{Ei}_{k,\lambda}\bigl(\log_\lambda(1+x)\bigr) &= \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}(\log_\lambda(1+x))^n}{(n-1)!n^k} \\
&= \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)}\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}(\log_\lambda(1+x))^n}{(n-1)!n^{k-1}} \\
&= \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)}\operatorname{Ei}_{k-1,\lambda}\bigl(\log_\lambda(1+x)\bigr).
\end{aligned} \tag{17}$$

By (17), for $k \geq 2$, we have

$$\operatorname{Ei}_{k,\lambda}\bigl(\log_\lambda(1+x)\bigr) = \int_0^x \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t \cdots \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t}_{(k-2)\text{-times}} \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} t\,dt \cdots dt. \tag{18}$$

Thus, from (14) and (18), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}\frac{x^n}{n!} &= \frac{1}{e_\lambda(x)-1}\operatorname{Ei}_{k,\lambda}\bigl(\log_\lambda(1+x)\bigr) \\
&= \frac{1}{e_\lambda(x)-1}\int_0^x \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t \cdots \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t}_{(k-2)\text{-times}} \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} t\,dt \cdots dt \\
&= \frac{x}{e_\lambda(x)-1}\sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \\
&\quad\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\frac{x^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \\
&\quad\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\beta_{n-m,\lambda}\frac{x^n}{n!}.
\end{aligned} \tag{19}$$

Therefore, by (19), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$\begin{aligned} B_{n,\lambda}^{(k)} &= \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1,\ldots,m_{k-1}} \\ &\quad \times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1} \beta_{n-m,\lambda}. \end{aligned} \tag{20}$$

From (14), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} &= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-1} e_\lambda^x(t) \\ &= \sum_{l=0}^{\infty} B_{l,\lambda}^{(k)} \frac{t^l}{l!} \sum_{m=0}^{\infty} (x)_{m,\lambda} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} (x)_{n-l,\lambda} B_{l,\lambda}^{(k)} \right) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

Therefore, by comparing the coefficients on both sides of (21), we obtain the following theorem.

Theorem 2 *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} (x)_{n-l,\lambda} B_{l,\lambda}^{(k)}.$$

Now, we observe that

$$\begin{aligned} \frac{1}{e_\lambda(t)-1} \mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr) &= \frac{1}{e_\lambda(t)-1} \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k} \\ &= \frac{1}{e_\lambda(t)-1} \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}}{m^{k-1}} \frac{1}{m!} \bigl(\log_\lambda(1+t)\bigr)^m \\ &= \frac{1}{e_\lambda(t)-1} \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}}{m^{k-1}} \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\ &= \frac{1}{e_\lambda(t)-1} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{(1)_{m,\lambda}}{m^{k-1}} S_{1,\lambda}(n,m) \right) \frac{t^n}{n!} \\ &= \frac{t}{e_\lambda(t)-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\sum_{m=1}^{n+1} S_{1,\lambda}(n+1,m) \frac{(1)_{m,\lambda}}{m^{k-1}} \right) \frac{t^n}{n!}. \end{aligned} \tag{22}$$

Thus, by (4) and (22), we get

$$\begin{aligned} \frac{1}{e_\lambda(t)-1} \mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr) &= \sum_{j=0}^{\infty} \beta_{j,\lambda} \frac{t^j}{j!} \sum_{l=0}^{\infty} \frac{1}{l+1} \left(\sum_{m=1}^{l+1} \frac{S_{1,\lambda}(l+1,m)(1)_{m,\lambda}}{m^{k-1}} \right) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{\binom{n}{l}}{l+1} \sum_{m=1}^{l+1} \frac{S_{1,\lambda}(l+1,m)}{m^{k-1}} (1)_{m,\lambda} \beta_{n-l,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{23}$$

On the other hand,

$$\frac{1}{e_\lambda(t)-1}\operatorname{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}\frac{t^n}{n!}. \tag{24}$$

Therefore, by (23) and (24), we obtain the following theorem.

Theorem 3 *For $n\geq 0$, we have*

$$B_{n,\lambda}^{(k)}=\sum_{l=0}^{n}\frac{\binom{n}{l}}{l+1}\sum_{m=1}^{l+1}\frac{S_{1,\lambda}(l+1,m)}{m^{k-1}}(1)_{m,\lambda}\beta_{n-l,\lambda}.$$

By letting $\lambda\to 0$, we get

$$B_n^{(k)}=\sum_{l=0}^{n}\frac{\binom{n}{l}}{l+1}\sum_{m=1}^{l+1}\frac{S_1(l+1,m)}{m^{k-1}}B_{n-l}\quad(n\geq 0).$$

From (14), we note that

$$\begin{aligned}\operatorname{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)&=\bigl(e_\lambda(t)-1\bigr)\sum_{l=0}^{\infty}B_{l,\lambda}^{(k)}\frac{t^l}{l!}\\&=\left(\sum_{m=0}^{\infty}\frac{(1)_{m,\lambda}}{m!}t^m-1\right)\sum_{l=0}^{\infty}B_{l,\lambda}^{(k)}\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}B_{m,\lambda}^{(k)}-B_{n,\lambda}^{(k)}\right)\frac{t^n}{n!}\\&=\sum_{n=1}^{\infty}\bigl(B_{n,\lambda}^{(k)}(1)-B_{n,\lambda}^{(k)}\bigr)\frac{t^n}{n!}.\end{aligned}\tag{25}$$

On the other hand,

$$\begin{aligned}\operatorname{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)&=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k}\\&=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\frac{1}{m!}\bigl(\log_\lambda(1+t)\bigr)^m\\&=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!}\\&=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)}{m^{k-1}}\right)\frac{t^n}{n!}.\end{aligned}\tag{26}$$

Therefore, by (25) and (26), we obtain the following theorem.

Theorem 4 *For $n\in\mathbb{N}$ and $k\in\mathbb{Z}$, we have*

$$B_{n,\lambda}^{(k)}(1)-B_{n,\lambda}^{(k)}=\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)}{m^{k-1}}.$$

From (11), we note that

$$\mathrm{Ei}_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n!} x^n = e_\lambda(x) - 1. \tag{27}$$

By (27), we get

$$\begin{aligned}\mathrm{Ei}_{1,\lambda}\big(\log_\lambda(1+t)\big) &= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}}{m!} \big(\log_\lambda(1+t)\big)^m \\ &= \sum_{m=1}^{\infty} (1)_{m,\lambda} \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} (1)_{m,\lambda} S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{28}$$

On the other hand,

$$\mathrm{Ei}_{1,\lambda}\big(\log_\lambda(1+t)\big) = t. \tag{29}$$

Therefore, by (28) and (29), we obtain the following theorem.

Theorem 5 *For $n \in \mathbb{N}$, we have*

$$\sum_{m=1}^{n} (1)_{m,\lambda} S_{1,\lambda}(n,m) = \delta_{n,1},$$

where $\delta_{n,k}$ is the Kronecker delta.

Note that

$$\lim_{\lambda \to 0} \big(B_{n,\lambda}^{(1)}(1) - B_{n,\lambda}^{(1)} \big) = B_n(1) - B_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \tag{30}$$

Thus, by Theorems 4 and 5, we get

$$\beta_{n,\lambda}(1) - \beta_{n,\lambda} = \sum_{m=1}^{n} (1)_{m,\lambda} S_{1,\lambda}(n,m) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \tag{31}$$

From (14), we note that

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)} \frac{t^n}{n!}.$$

By replacing t by $e_\lambda(t)-1$, we get

$$\begin{aligned}
&\sum_{m=0}^{\infty} B_{m,\lambda}^{(k)} \frac{1}{m!}\bigl(e_\lambda(t)-1\bigr)^m \\
&\quad = \frac{\mathrm{Ei}_{k,\lambda}(t)}{e_\lambda(e_\lambda(t)-1)-1} = \frac{e_\lambda(t)-1}{e_\lambda(e_\lambda(t)-1)-1}\frac{t}{e_\lambda(t)-1}\frac{1}{t}\mathrm{Ei}_{k,\lambda}(t) \\
&\quad = \sum_{i_1=0}^{\infty} \beta_{i_1,\lambda}\frac{1}{i_1!}\bigl(e_\lambda(t)-1\bigr)^{i_1}\sum_{j=0}^{\infty}\beta_{j,\lambda}\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)^k}\frac{t^m}{m!} \\
&\quad = \sum_{i_1=0}^{\infty} \beta_{i_1,\lambda}\sum_{i_2=i_1}^{\infty} S_{2,\lambda}(i_2,i_1)\frac{t^{i_2}}{i_2!}\sum_{j=0}^{\infty}\beta_{j,\lambda}\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)^k}\frac{t^m}{m!} \\
&\quad = \sum_{i_2=0}^{\infty}\sum_{i_1=0}^{i_2} \beta_{i_1,\lambda}S_{2,\lambda}(i_2,i_1)\frac{t^{i_2}}{i_2!}\sum_{j=0}^{\infty}\beta_{j,\lambda}\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)^k}\frac{t^m}{m!} \\
&\quad = \sum_{i_3=0}^{\infty}\left(\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}\binom{i_3}{i_2}\beta_{i_1,\lambda}S_{2,\lambda}(i_2,i_1)\beta_{i_3-i_2,\lambda}\right)\frac{t^{i_3}}{i_3!}\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)^k}\frac{t^m}{m!} \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{i_3=0}^{n}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}\binom{n}{i_3}\binom{i_3}{i_2}\beta_{i_1,\lambda}S_{2,\lambda}(i_2,i_1)\beta_{i_3-i_2,\lambda}\frac{(1)_{n-i_3+1,\lambda}}{(n-i_3+1)^k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{32}$$

On the other hand,

$$\begin{aligned}
\sum_{m=0}^{\infty} B_{m,\lambda}^{(k)} \frac{1}{m!}\bigl(e_\lambda(t)-1\bigr)^m &= \sum_{m=0}^{\infty} B_{m,\lambda}^{(k)}\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_{m,\lambda}^{(k)}S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{33}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 6

$$\sum_{m=0}^{n} B_{m,\lambda}^{(k)}S_{2,\lambda}(n,m) = \sum_{i_3=0}^{n}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}\binom{n}{i_3}\binom{i_3}{i_2}\beta_{i_1,\lambda}S_{2,\lambda}(i_2,i_1)\beta_{i_3-i_2,\lambda}\frac{(1)_{n-i_3+1,\lambda}}{(n-i_3+1)^k}.$$

3 Further remark

The higher-order degenerate Bernoulli polynomials are defined by Carlitz and given by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [2, 3])}, \tag{34}$$

where r is a positive integer.

When $x=0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called higher-order degenerate Bernoulli numbers.

We observe that

$$\begin{aligned}
\frac{1}{t^r}\bigl(\log_\lambda(1+t)\bigr)^r &= \frac{r!}{t^r}\frac{1}{r!}\bigl(\log_\lambda(1+t)\bigr)^r \\
&= \frac{r!}{t^r}\sum_{n=r}^{\infty} S_{1,\lambda}(n,r)\frac{t^n}{n!} \\
&= \frac{r!}{t^r}\sum_{n=0}^{\infty} S_{1,\lambda}(n+r,r)\frac{n!}{(n+r)!}\frac{t^{n+r}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{S_{1,\lambda}(n+r,r)}{\binom{n+r}{r}}\frac{t^n}{n!}.
\end{aligned} \tag{35}$$

Replacing t by $\log_\lambda(1+t)$ in (34), we get

$$\begin{aligned}
\left(\frac{\log_\lambda(1+t)}{t}\right)^r &= \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(r)}\frac{1}{m!}\bigl(\log_\lambda(1+t)\bigr)^m \\
&= \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(r)}\sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \beta_{m,\lambda}^{(r)} S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{36}$$

Therefore, by (35) and (36), we obtain the following theorem.

Theorem 7 *For $n \geq 0$, we have*

$$S_{1,\lambda}(n+r,r) = \binom{n+r}{r}\sum_{m=0}^{n} \beta_{m,\lambda}^{(r)} S_{1,\lambda}(n,m).$$

Now, we consider the inversion formula of Theorem 7. Replacing t by $e_\lambda(t)-1$ in (35), we get

$$\begin{aligned}
\left(\frac{t}{e_\lambda(t)-1}\right)^r &= \sum_{m=0}^{\infty} \frac{S_1(m+r,r)}{\binom{m+r}{r}}\frac{1}{m!}\bigl(e_\lambda(t)-1\bigr)^m \\
&= \sum_{m=0}^{\infty} \frac{S_1(m+r,r)}{\binom{m+r}{r}}\sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \frac{S_{1,\lambda}(m+r,r)}{\binom{m+r}{r}} S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{37}$$

On the other hand,

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}\frac{t^n}{n!} \quad \text{(see [2, 3])}. \tag{38}$$

Therefore, by (37) and (38), we obtain the following theorem.

Theorem 8 *For $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(r)} = \sum_{m=0}^{n} \frac{S_{1,\lambda}(m+r,m)}{\binom{m+r}{r}} S_{2,\lambda}(n,m).$$

Replacing t by $\log_{\lambda}(1+t)$ in (9) and making use of (10), we get

$$\begin{aligned}\frac{1}{k!}t^k &= \sum_{m=k}^{\infty} S_{2,\lambda}(m,k)\frac{1}{m!}\bigl(\log_{\lambda}(1+t)\bigr)^m \\ &= \sum_{m=k}^{\infty} S_{2,\lambda}(m,k)\sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n} S_{2,\lambda}(m,k)S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{39}$$

Therefore, by comparing the coefficient on both sides of (39), we get

$$\sum_{m=k}^{n} S_{1,\lambda}(n,m)S_{2,\lambda}(m,k) = \delta_{n,k} \quad (0 \leq k \leq n), \tag{40}$$

where $\delta_{n,k}$ is Kronecker's delta.

The degenerate Bernoulli numbers of the second kind of order r are given by

$$\left(\frac{t}{\log_{\lambda}(1+t)}\right)^r = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}\frac{t^n}{n!}. \tag{41}$$

Note that $\lim_{\lambda\to 0} b_{n,\lambda}^{(r)} = b_n^{(r)}$ are the Bernoulli numbers of the second kind of order r.

Replacing t by $e_{\lambda}(t) - 1$, we get

$$\begin{aligned}\frac{1}{t^r}\bigl(e_{\lambda}(t)-1\bigr)^r &= \sum_{m=0}^{\infty} b_{m,\lambda}^{(r)}\frac{1}{m!}\bigl(e_{\lambda}(t)-1\bigr)^m \\ &= \sum_{m=0}^{\infty} b_{m,\lambda}^{(r)}\sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} b_{m,\lambda}^{(r)} S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{42}$$

On the other hand,

$$\begin{aligned}\frac{1}{t^r}\bigl(e_{\lambda}(t)-1\bigr)^r &= \frac{r!}{t^r}\frac{1}{r!}\bigl(e_{\lambda}(t)-1\bigr)^r = \frac{r!}{t^r}\sum_{n=r}^{\infty} S_{2,\lambda}(n,r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{S_{2,\lambda}(n+r,r)}{\binom{n+r}{r}}\frac{t^n}{n!}.\end{aligned} \tag{43}$$

Thus, by (42) and (43), we get

$$S_{2,\lambda}(n+r,r)=\binom{n+r}{r}\sum_{m=0}^{n}b_{m,\lambda}^{(r)}S_{2,\lambda}(n,m). \tag{44}$$

By the same method as in the above, the inversion formula of (44) is given by

$$b_n^{(r)}=\sum_{m=0}^{n}\frac{S_{2,\lambda}(m+r,r)}{\binom{m+r}{r}}S_{1,\lambda}(n,m)\quad (n\geq 0). \tag{45}$$

4 Conclusion

Recently, Kim–Kim introduced degenerate polyexponential functions and degenerate Bell polynomials [15] and they studied degenerate poly-Bernoulli numbers and polynomials from degenerate polylogarithm function. In [10], Kim–Kim also studied polyexponential functions as an inverse to the polylogarithm functions, constructed type 2 poly-Bernoulli polynomials by using this and derived various properties of type 2 poly-Bernoulli numbers. In addition, they investigated unipoly functions attached to each suitable arithmetic function as a universal concept which includes the polylogarithm and polyexponential functions as special cases. As the degenerate version of the type 2 poly-Bernoulli polynomials, we study the degenerate polyexponential functions and the degenerate type 2 poly-Bernoulli numbers and polynomials. Finally, we derive several explicit expressions and some identities for those numbers and polynomials. Proof techniques and results developed in this research paper are expected to be of great help to researchers in this field in the future.

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Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; JK and HL checked the results of the paper and typed the paper; DSK and TK completed the revision of the article. All authors have read and agreed with the published version of the manuscript.

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