Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Persistence and extinction of a stochastic predator–prey model with modified Leslie–Gower and Holling-type II schemes

Dengxia Zhou[1], Meng Liu[2] and Zhijun Liu[1*]

*Correspondence:
zjliu@hbmzu.edu.cn;
zhijun_liu47@hotmail.com
[1]Department of Mathematics,
Hubei Minzu University, Hubei, P.R.
China
Full list of author information is
available at the end of the article

Abstract

In this paper, we use an Ornstein–Uhlenbeck process to describe the environmental stochasticity and propose a stochastic predator–prey model with modified Leslie–Gower and Holling-type II schemes. For each species, sharp sufficient conditions for persistence in the mean and extinction are respectively obtained. The results demonstrate that the persistence and extinction of the species have close relationships with the environmental stochasticity. In addition, the theoretical results are numerically illustrated by some simulations.

Keywords: Persistence in the mean; Extinction; Itô's formula; Intensity of noise

1 Introduction

The well-known predator–prey framework with modified Leslie–Gower and Holling-type II schemes (PPFMLHS) formulated by Aziz-Alaoui and Okiye [1] can be illustrated as follows:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(r_1 - ax(t) - \frac{cy(t)}{h+x(t)}), \\ \frac{dy(t)}{dt} = y(t)(r_2 - \frac{fy(t)}{h+x(t)}), \end{cases} \tag{1}$$

where a, c, f and h are assumed to be positive constants. a means the intraspecific competition strength, c measures the per capita reduction rate, h characterises the safeguard of the environment and f possesses the like signification of c. In the past two decades, model (1) and its generalisations have been subjected to intensive research, and a mass of attractive features have been provided. For example, Aziz-Alaoui and Okiye [1] tested the boundedness and global stability of model (1); Guo and Song [2] dissected model (1) perturbed by the impulse; Abid et al. [3] probed into the optimal control of model (1); see [3–15] for more related outcomes.

The parameters in model (1) are hypothesised to be deterministic, which neglects the environmental perturbations, and hence model (1) cannot accurately depict the real situations. A mass of scholars (see [16–23]) introduced stochasticity into deterministic systems to dissect the functions of stochasticity on population dynamics. Particularly, under the

Springer

hypothesis that the growth rates in model (1) are disturbed by the random perturbations with $r_i \to r_i + \sigma_i \frac{dB_i(t)}{dt}$, several authors (see [16–18, 20]) tested the following stochastic PPFMLHS:

$$
\begin{cases}
dx(t) = x(t)(r_1 - ax(t) - \frac{cy(t)}{h+x(t)}) \, dt + \sigma_1 x(t) \, dB_1(t), \\
dy(t) = y(t)(r_2 - \frac{fy(t)}{h+x(t)}) \, dt + \sigma_2 y(t) \, dB_2(t),
\end{cases}
\tag{2}
$$

where σ_i^2 means the intensity of white noise, $B_i(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}_t, P)$, a given complete probability space. Ji et al. [17, 18] probed into several dynamical characteristics of system (2) and offered extinct and persistent conditions for the system. Liu et al. [20] examined the persistence and extinction of model (2) with impulsive toxicant input.

Model (2) hypothesises that the growth rate in the random environments is linear with respect to the Gaussian white noise

$$
\tilde{r}_i(t) = r_i + \sigma_i \frac{dB_i(t)}{dt}, \quad i = 1, 2.
$$

Integrating on the interval $[0, T]$ results in

$$
\bar{r}_i = \frac{1}{T} \int_0^T \tilde{r}_i(t) \, dt \to r_i + \sigma_i \frac{B_i(T)}{T} \sim N\left(r_i, \sigma_i^2/T\right).
$$

Therefore, the variance of the average per capita growth rate $\bar{r}_i$ over an interval of length T tends to ∞ as $T \to 0$. This is insufficient to describe the real situation. Several authors (see [24, 25]) have claimed that using the mean-reverting Ornstein–Uhlenbeck process is a more appropriate approach to incorporate the environmental perturbations. On account of this approach, one has

$$
d\tilde{r}_i(t) = \alpha_i\left(r_i - \tilde{r}_i(t)\right) dt + \xi_i \, dB_i(t), \quad i = 1, 2,
$$

i.e.

$$
\tilde{r}_i(t) = r_i + (r_{i0} - r_i)e^{-\alpha_i t} + \xi_i \int_0^t e^{-\alpha_i(t-s)} \, dB_i(s)
$$

$$
= r_i + (r_{i0} - r_i)e^{-\alpha_i t} + \sigma_i(t)\frac{dB_i(t)}{dt}, \quad i = 1, 2,
$$

where $r_{i0} = \tilde{r}_i(0)$, $\sigma_i(t) = \frac{\xi_i}{\sqrt{2\alpha_i}}\sqrt{1 - e^{-2\alpha_i t}}$, $\alpha_i > 0$ characterises the speed of reversion, ξ_i^2 means the intensity of stochastic perturbations. We then derive the following stochastic PPFMLHS:

$$
\begin{cases}
dx(t) = x(t)(r_1 + (r_{10} - r_1)e^{-\alpha_1 t} - ax(t) - \frac{cy(t)}{h+x(t)}) \, dt \\
\qquad + \sigma_1(t)x(t) \, dB_1(t), \\
dy(t) = y(t)(r_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{fy(t)}{h+x(t)}) \, dt \\
\qquad + \sigma_2(t)y(t) \, dB_2(t).
\end{cases}
\tag{3}
$$

As far as we know, little research has been conducted to explore model (3). For this reason, we delve into the properties of model (3).

The arrangement of this paper is as follows. In Sect. 2, the persistence and extinction threshold for each population are proffered. In Sect. 3, some numerical simulations are performed to evidence the theoretical outcomes. In Sect. 4, a number of concluding remarks are put forward.

2 Main results

Define

$$R_+^2 = \left\{ z \in R^2 | z_i > 0, i = 1, 2 \right\}, \qquad b_i(t) = r_i - \frac{\xi_i^2}{4\alpha_i} + \frac{\xi_i^2}{4\alpha_i} e^{-2\alpha_i t},$$

$$\bar{b}_i = \lim_{t \to +\infty} t^{-1} \int_0^t b_i(s)\,\mathrm{d}s = r_i - \frac{\xi_i^2}{4\alpha_i}, \quad i = 1, 2.$$

Lemma 1 *For arbitrary $(x(0), y(0)) \in R_+^2$, model (3) possesses a unique solution $(x(t), y(t)) \in R_+^2$ for all $t \geq 0$ a.s. (almost surely).*

Proof Pay attention to the following system:

$$\begin{cases} \mathrm{d}u(t) = (b_1(t) + (r_{10} - r_1)e^{-\alpha_1 t} - ae^{u(t)} - \frac{ce^{v(t)}}{h + e^{u(t)}})\,\mathrm{d}t \\ \qquad\quad + \sigma_1(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}v(t) = (b_2(t) + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{fe^{v(t)}}{h + e^{u(t)}})\,\mathrm{d}t \\ \qquad\quad + \sigma_2(t)\,\mathrm{d}B_2(t), \end{cases} \tag{4}$$

and $u(0) = \ln x(0)$, $v(0) = \ln y(0)$. Due to the fact that the coefficients of system (4) adhere to the Lipschitz condition, system (4) possesses a unique local solution $(u(t), v(t))$ on $[0, \tau_*)$ (see Theorems 3.15–3.17 in [26]), where τ_* means the explosion time. Then we can deduce from Itô's formula that on $[0, \tau_*)$ model (3) possesses a unique solution $(x(t), y(t)) = (e^{u(t)}, e^{v(t)})$ which is positive. Now we validate $\tau_* = +\infty$. Focus on the following systems:

$$\mathrm{d}\Phi(t) = \Phi(t)\big(r_1 + (r_{10} - r_1)e^{-\alpha_1 t} - a\Phi(t)\big)\,\mathrm{d}t$$
$$\qquad\quad + \sigma_1(t)\Phi(t)\,\mathrm{d}B_1(t), \quad \Phi(0) = x(0); \tag{5}$$

$$\mathrm{d}\psi(t) = \psi(t)\left(r_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{f}{h}\psi(t)\right)\mathrm{d}t$$
$$\qquad\quad + \sigma_2(t)\psi(t)\,\mathrm{d}B_2(t), \quad \psi(0) = y(0); \tag{6}$$

$$\mathrm{d}\varphi(t) = \varphi(t)\left(r_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{f}{h + \Phi(t)}\varphi(t)\right)\mathrm{d}t$$
$$\qquad\quad + \sigma_2(t)\varphi(t)\,\mathrm{d}B_2(t), \quad \Phi(0) = x(0). \tag{7}$$

On the basis of the comparison theorem [27], for $t \in [0, \tau_*)$,

$$x(t) \leq \Phi(t), \qquad \psi(t) \leq y(t) \leq \varphi(t), \quad \text{a.s.} \tag{8}$$

In accordance to Theorem 2.2 in [22],

$$\Phi(t) = \frac{e^{\int_0^t b_1(s)\,ds - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)\,dB_1(s)}}{x^{-1}(0) + a\int_0^t e^{\int_0^s b_1(\tau)\,d\tau - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 s}-1)+\int_0^s \sigma_1(\tau)\,dB_1(\tau)}\,ds},\tag{9}$$

$$\psi(t) = \frac{e^{\int_0^t b_2(s)\,ds - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 t}-1)+\int_0^t \sigma_2(s)\,dB_2(s)}}{y^{-1}(0) + \frac{f}{h}\int_0^t e^{\int_0^s b_2(\tau)\,d\tau - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 s}-1)+\int_0^s \sigma_2(\tau)\,dB_2(\tau)}\,ds}\tag{10}$$

and

$$\varphi(t) = \frac{e^{\int_0^t b_2(s)\,ds - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 t}-1)+\int_0^t \sigma_2(s)\,dB_2(s)}}{y^{-1}(0) + \int_0^t \frac{f}{h+\Phi(s)}e^{\int_0^s b_2(\tau)\,d\tau - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 s}-1)+\int_0^s \sigma_2(\tau)\,dB_2(\tau)}\,ds}.\tag{11}$$

Due to the fact that $\Phi(t)$, $\psi(t)$, $\varphi(t)$ are global, we can deduce that $\tau_* = +\infty$. □

Lemma 2 ([28]) *Let $\Gamma(t) \in C(\Omega \times [0,+\infty), [0,+\infty))$.*

(I) *If one can find out three positive constants κ, μ and μ_0 such that, for all $t \geq \kappa$,*
$\ln \Gamma(t) \leq \mu t - \mu_0 \int_0^t \Gamma(s)\,ds + F(t)$, *where $F(t)/t \to 0$ as $t \to +\infty$, then*
$\limsup_{t\to+\infty} t^{-1}\int_0^t \Gamma(s)\,ds \leq \frac{\mu}{\mu_0}$ *a.s.*

(II) *If one can find out three positive constants κ, μ and μ_0 such that, for all $t \geq \kappa$,*
$\ln \Gamma(t) \geq \mu t - \mu_0 \int_0^t \Gamma(s)\,ds + F(t)$, *where $F(t)/t \to 0$ as $t \to +\infty$, then*
$\liminf_{t\to+\infty} t^{-1}\int_0^t \Gamma(s)\,ds \geq \frac{\mu}{\mu_0}$ *a.s.*

Lemma 3 *If $\bar{b}_1 > 0$ and $\bar{b}_2 > 0$, then $\lim_{t\to+\infty} t^{-1}\ln y(t) = 0$ a.s.*

Proof Choose sufficiently large T which fulfils that, for $t \geq T$,

$$(\bar{b}_i - \varepsilon)t \leq \int_0^t b_i(s)\,ds \leq (\bar{b}_i + \varepsilon)t, \qquad e^{(\bar{b}_i - \varepsilon)t} \geq 2e^{(\bar{b}_i - \varepsilon)T}.$$

For $t \geq T$, one can deduce from (9) that

$$\Phi(t) = \frac{e^{\int_0^t b_1(s)\,ds - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)\,dB_1(s)}}{x^{-1}(0) + a\int_0^t e^{\int_0^s b_1(\tau)\,d\tau - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 s}-1)+\int_0^s \sigma_1(\tau)\,dB_1(\tau)}\,ds}$$

$$\leq \frac{e^{\int_0^t b_1(s)\,ds - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)\,dB_1(s)}}{a\int_0^t e^{\int_0^s b_1(\tau)\,d\tau - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 s}-1)+\int_0^s \sigma_1(\tau)\,dB_1(\tau)}\,ds}$$

$$\leq \frac{e^{(\bar{b}_1+\varepsilon)t - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)\,dB_1(s)}}{ae^{\min_{0\leq v\leq t}\{\int_0^v \sigma_1(\tau)\,dB_1(\tau) - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 v}-1)\}}\int_T^t e^{(\bar{b}_1-\varepsilon)s}\,ds}$$

$$= \frac{(\bar{b}_1 - \varepsilon)e^{(\bar{b}_1+\varepsilon)t - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)\,dB_1(s)}}{a(e^{(\bar{b}_1-\varepsilon)t} - e^{(\bar{b}_1-\varepsilon)T})e^{\min_{0\leq v\leq t}\{\int_0^v \sigma_1(\tau)\,dB_1(\tau) - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 v}-1)\}}}$$

$$\leq \frac{2(\bar{b}_1 - \varepsilon)e^{(\bar{b}_1+\varepsilon)t - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)\,dB_1(s)}}{ae^{(\bar{b}_1-\varepsilon)t}e^{\min_{0\leq v\leq t}\{\int_0^v \sigma_1(\tau)\,dB_1(\tau) - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 v}-1)\}}}$$

$$= \frac{2(\bar{b}_1 - \varepsilon)}{a}e^{2\varepsilon t}L_1(t),$$

where

$$L_1(t) = \frac{e^{\int_0^t \sigma_1(s)\,dB_1(s) - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 t}-1)}}{e^{\min_{0\le v\le t}\{\int_0^v \sigma_1(\tau)\,dB_1(\tau) - \frac{r_{10}-r_1}{\alpha_1}(e^{-\alpha_1 v}-1)\}}}.$$

It then follows from $L_1(t) \ge 1$ that

$$\int_T^t \frac{f}{h+\Phi(s)} e^{\int_0^s b_2(\tau)\,d\tau - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 s}-1) + \int_0^s \sigma_2(\tau)\,dB_2(\tau)}\,ds$$

$$\ge \int_T^t \frac{f e^{(\bar{b}_2-\varepsilon)s - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 s}-1) + \int_0^s \sigma_2(\tau)\,dB_2(\tau)}}{h + \frac{2(\bar{b}_1-\varepsilon)}{a}e^{2\varepsilon s}L_1(s)}\,ds$$

$$\ge \int_T^t \frac{f e^{(\bar{b}_2-\varepsilon)s - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 s}-1) + \int_0^s \sigma_2(\tau)\,dB_2(\tau)}}{(h + \frac{2(\bar{b}_1-\varepsilon)}{a})e^{2\varepsilon s}L_1(s)}\,ds$$

$$= \frac{af}{ah + 2(\bar{b}_1-\varepsilon)} \int_T^t e^{(\bar{b}_2-3\varepsilon)s + \int_0^s \sigma_2(\tau)\,dB_2(\tau) - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 s}-1)} L_1^{-1}(s)\,ds$$

$$\ge \frac{af}{ah + 2(\bar{b}_1-\varepsilon)} \frac{1}{\bar{b}_2-3\varepsilon}\left(e^{(\bar{b}_2-3\varepsilon)t} - e^{(\bar{b}_2-3\varepsilon)T}\right) \min_{0\le v\le t}\{L_2(v)\}$$

$$= L_3(t)\left(e^{(\bar{b}_2-3\varepsilon)t} - e^{(\bar{b}_2-3\varepsilon)T}\right),$$

where

$$L_2(t) = L_1^{-1}(t) e^{\int_0^t \sigma_2(\tau)\,dB_2(\tau) - \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2 t}-1)},$$

$$L_3(t) = \frac{af}{ah + 2(\bar{b}_1-\varepsilon)} \frac{1}{\bar{b}_2-3\varepsilon} \min_{0\le v\le t}\{L_2(v)\}.$$

Thus (11) implies that

$$\frac{1}{\varphi(t)} \ge e^{-\int_T^t b_2(s)\,ds + \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2(t-T)}-1) - \int_T^t \sigma_2(s)\,dB_2(s)}$$

$$\times L_3(t)\left(e^{(\bar{b}_2-3\varepsilon)t} - e^{(\bar{b}_2-3\varepsilon)T}\right)$$

$$\ge e^{-\int_T^t b_2(s)\,ds + \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2(t-T)}-1) - \int_T^t \sigma_2(s)\,dB_2(s)} \times \frac{1}{2}L_3(t)e^{(\bar{b}_2-3\varepsilon)t}$$

$$\ge L_4(t) \times e^{-4\varepsilon t},$$

where

$$L_4(t) = \frac{1}{2}L_3(t)e^{\int_0^T b_2(s)\,ds + \frac{r_{20}-r_2}{\alpha_2}(e^{-\alpha_2(t-T)}-1) - \int_T^t \sigma_2(s)\,dB_2(s)}.$$

For this reason,

$$t^{-1}\ln\varphi(t) < -t^{-1}\ln L_4(t) + 4\varepsilon. \tag{12}$$

We then deduce from $\lim_{t\to+\infty} t^{-1}\int_0^t \sigma_i(s)\,dB_i(s) = 0$ ($i = 1, 2$) that if $\bar{b}_2 > 0$,

$$\lim_{t\to+\infty} t^{-1}\ln L_4(t) = 0 \quad \text{a.s.}$$

This, together with (12), indicates

$$\limsup_{t \to +\infty} t^{-1} \ln y(t) \le \limsup_{t \to +\infty} t^{-1} \ln \varphi(t) \le 0 \quad \text{a.s.}$$

Taking advantage of Itô's formula to (6) deduces

$$d \ln \psi(t) = \left(b_2(t) + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{f}{h}\psi(t) \right) dt + \sigma_2(t)\,dB_2(t),$$

namely,

$$t^{-1} \ln \psi(t) = t^{-1} \ln y(0) + t^{-1} \int_0^t b_2(s)\,ds - \frac{(r_{20} - r_2)}{\alpha_2 t}\left(e^{-\alpha_2 t} - 1 \right)$$
$$- \frac{f}{h}t^{-1} \int_0^t \psi(s)\,ds + t^{-1} \int_0^t \sigma_2(s)\,dB_2(s). \tag{13}$$

For arbitrary $\varepsilon > 0$, one can find out $T > 0$ such that, for $t \ge T$,

$$\bar{b}_2 - \varepsilon/2 \le t^{-1}\left[\ln y(0) - \frac{(r_{20} - r_2)}{\alpha_2}\left(e^{-\alpha_2 t} - 1 \right) + \int_0^t b_2(s)\,ds \right] \le \bar{b}_2 + \varepsilon/2.$$

We then deduce from (13) that, for $t \ge T$,

$$t^{-1} \ln \psi(t) \le \bar{b}_2 + \varepsilon - \frac{f}{h}t^{-1} \int_0^t \psi(s)\,ds + t^{-1} \int_0^t \sigma_2(s)\,dB_2(s), \tag{14}$$

$$t^{-1} \ln \psi(t) \ge \bar{b}_2 - \varepsilon - \frac{f}{h}t^{-1} \int_0^t \psi(s)\,ds + t^{-1} \int_0^t \sigma_2(s)\,dB_2(s). \tag{15}$$

Choose $0 < \varepsilon < \bar{b}_2$. Making use of (I) and (II) in Lemma 2 yields that

$$\frac{h(\bar{b}_2 - \varepsilon)}{f} \le \liminf_{t \to +\infty} t^{-1} \int_0^t \psi(s)\,ds \le \limsup_{t \to +\infty} t^{-1} \int_0^t \psi(s)\,ds$$
$$\le \frac{h(\bar{b}_2 + \varepsilon)}{f} \quad \text{a.s.}$$

We then deduce from the arbitrariness of ε that

$$\lim_{t \to \infty} t^{-1} \int_0^t \psi(s)\,ds = \frac{h\bar{b}_2}{f} \quad \text{a.s.,} \tag{16}$$

which indicates that $\lim_{t \to +\infty} t^{-1} \ln \psi(t) = 0$ a.s. In accordance to (8),

$$\liminf_{t \to +\infty} t^{-1} \ln y(t) \ge \lim_{t \to +\infty} t^{-1} \ln \psi(t) = 0 \quad \text{a.s.} \tag{17}$$

$\square$

Theorem 1 ([28]) *For model* (3), *the following conclusions hold*:
  (i) *If* $\bar{b}_1 < 0$, $\bar{b}_2 < 0$, *then both* x *and* y *become extinct, i.e.* $\lim_{t \to +\infty} x(t) = 0$,
      $\lim_{t \to +\infty} y(t) = 0$ *a.s.*

(ii) *If $\bar{b}_1 < 0$, $\bar{b}_2 > 0$, then x becomes extinct and y is persistent in the mean, i.e.*
$\lim_{t \to +\infty} t^{-1} \int_0^t y(s)\, ds = h\bar{b}_2/f$ *a.s.*

(iii) *If $\bar{b}_1 > 0$, $\bar{b}_2 < 0$, then y becomes extinct and x is persistent in the mean, i.e.*
$\lim_{t \to +\infty} t^{-1} \int_0^t x(s)\, ds = \bar{b}_1/a$ *a.s.*

(iv) *When $\bar{b}_1 > 0$, $\bar{b}_2 > 0$, (a) if $\bar{b}_1 < c\bar{b}_2/f$, then x becomes extinct and y is persistent in the mean, i.e. $\lim_{t \to +\infty} t^{-1} \int_0^t y(s)\, ds = h\bar{b}_2/f$ a.s.; (b) if $\bar{b}_1 > c\bar{b}_2/f$, then*
$\lim_{t \to +\infty} t^{-1} \int_0^t x(s)\, ds = \bar{b}_1/a - c\bar{b}_2/(af)$, $\lim_{t \to +\infty} t^{-1} \int_0^t \frac{y(s)}{h+x(s)}\, ds = \bar{b}_2/f$ *a.s.*

Proof (i). Taking advantage of Itô's formula to (3) results in

$$d \ln x(t) = \left(b_1(t) + (r_{10} - r_1)e^{-\alpha_1 t} - ax(t) - \frac{cy(t)}{h+x(t)} \right) dt + \sigma_1(t)\, dB_1(t),$$

$$d \ln y(t) = \left(b_2(t) + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{fy(t)}{h+x(t)} \right) dt + \sigma_2(t)\, dB_2(t).$$

As a consequence,

$$
\begin{aligned}
\ln x(t) - \ln x(0) &= \int_0^t b_1(s)\, ds - \frac{r_{10} - r_1}{\alpha_1} \left(e^{-\alpha_1 t} - 1 \right) - a \int_0^t x(s)\, ds \\
&\quad - c \int_0^t \frac{y(s)}{h+x(s)}\, ds + \int_0^t \sigma_1(s)\, dB_1(s),
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\ln y(t) - \ln y(0) &= \int_0^t b_2(s)\, ds - \frac{r_{20} - r_2}{\alpha_2} \left(e^{-\alpha_2 t} - 1 \right) \\
&\quad - f \int_0^t \frac{y(s)}{h+x(s)}\, ds + \int_0^t \sigma_2(s)\, dB_2(s).
\end{aligned}
\tag{19}
$$

We then deduce from (18) that, for sufficiently large t,

$$t^{-1} \ln \frac{x(t)}{x(0)} \leq \bar{b}_1 + \varepsilon + t^{-1} \int_0^t \sigma_1(s)\, dB_1(s) - \frac{r_{10} - r_1}{t\alpha_1} \left(e^{-\alpha_1 t} - 1 \right). \tag{20}$$

In accordance to $\lim_{t \to +\infty} t^{-1} \int_0^t \sigma_1(s)\, dB_1(s) = 0$ and $\bar{b}_1 + \varepsilon < 0$, we derive $\lim_{t \to +\infty} x(t) = 0$ a.s. Analogously, $\bar{b}_2 < 0$ means that $\lim_{t \to +\infty} y(t) = 0$ a.s.

(ii). Note that $\bar{b}_1 < 0$, hence (i) indicates that $\lim_{t \to +\infty} x(t) = 0$. As a result, for sufficiently large t,

$$\ln y(t) - \ln y(0) \leq (\bar{b}_2 + \varepsilon)t - \frac{f}{h+\varepsilon} \int_0^t y(s)\, ds + \int_0^t \sigma_2(s)\, dB_2(s), \tag{21}$$

$$\ln y(t) - \ln y(0) \geq (\bar{b}_2 - \varepsilon)t - \frac{f}{h-\varepsilon} \int_0^t y(s)\, ds + \int_0^t \sigma_2(s)\, dB_2(s). \tag{22}$$

Making use of Lemma 2 to (21) and (22) results in

$$\limsup_{t \to +\infty} t^{-1} \int_0^t y(s)\, ds \leq \frac{(h+\varepsilon)(\bar{b}_2 + \varepsilon)}{f}, \qquad \liminf_{t \to +\infty} t^{-1} \int_0^t y(s)\, ds \geq \frac{(h-\varepsilon)(\bar{b}_2 - \varepsilon)}{f}.$$

We then deduce from the arbitrariness of ε that $\lim_{t \to +\infty} t^{-1} \int_0^t y(s)\, ds = h\bar{b}_2/f$ a.s.

(iii). Since $\bar{b}_2 < 0$, then analogous to the proof of (i), one can validate that $\lim_{t \to +\infty} y(t) = 0$. The proof of (iii) is analogous to that of (ii), thus is left out.

(iv). (a). Compute that $(18) \times f - (19) \times c$,

$$
\begin{aligned}
t^{-1} f \ln \frac{x(t)}{x(0)} = {} & c t^{-1} \ln \frac{y(t)}{y(0)} + f t^{-1} \int_0^t b_1(s)\,ds - c t^{-1} \int_0^t b_2(s)\,ds \\
& - \frac{r_{10} - r_1}{\alpha_1 t}\left(e^{-\alpha_1 t} - 1\right)f + \frac{r_{20} - r_2}{\alpha_2 t}\left(e^{-\alpha_2 t} - 1\right)c \\
& - a f t^{-1} \int_0^t x(s)\,ds + f t^{-1} \int_0^t \sigma_1(s)\,dB_1(s) \\
& - c t^{-1} \int_0^t \sigma_2(s)\,dB_2(s).
\end{aligned}
$$

On the basis of Lemma 3, for arbitrary $\varepsilon > 0$, we can find out $T > 0$ such that, for $t \geq T$, $c t^{-1} \ln \frac{y(t)}{y(0)} \leq \varepsilon/5$. For this reason,

$$
t^{-1} f \ln x(0) \leq \varepsilon/5,
$$

$$
\frac{r_{20} - r_2}{\alpha_2 t}\left(e^{-\alpha_2 t} - 1\right)c - \frac{r_{10} - r_1}{\alpha_1 t}\left(e^{-\alpha_1 t} - 1\right)f \leq \varepsilon/5,
$$

$$
f t^{-1} \int_0^t \sigma_1(s)\,dB_1(s) - c t^{-1} \int_0^t \sigma_2(s)\,dB_2(s) \leq \varepsilon/5,
$$

$$
f t^{-1} \int_0^t b_1(s)\,ds - c t^{-1} \int_0^t b_2(s)\,ds \leq f\bar{b}_1 - c\bar{b}_2 + \varepsilon/5.
$$

As a result, for $t \geq T$,

$$
t^{-1} f \ln x(t) \leq \varepsilon + f\bar{b}_1 - c\bar{b}_2. \tag{23}
$$

Choose $0 < \varepsilon < c\bar{b}_2 - f\bar{b}_1$. Consequently, $\lim_{t \to +\infty} x(t) = 0$. The proof of

$$
\lim_{t \to +\infty} t^{-1} \int_0^t y(s)\,ds = h\bar{b}_2/f
$$

is analogous to that of (ii) and thereby is left out.

(b). On the basis of (19),

$$
\begin{aligned}
t^{-1} \ln y(t) - t^{-1} \ln y(0) = {} & t^{-1} \int_0^t b_2(s)\,ds - \frac{r_{20} - r_2}{\alpha_2 t}\left(e^{-\alpha_2 t} - 1\right) \\
& - f t^{-1} \int_0^t \frac{y(s)}{h + x(s)}\,ds \\
& + t^{-1} \int_0^t \sigma_2(s)\,dB_2(s). \tag{24}
\end{aligned}
$$

We then deduce from Lemma 3 and $\lim_{t \to +\infty} t^{-1} \int_0^t \sigma_2(s)\,dB_2(s) = 0$ that

$$
\lim_{t \to +\infty} t^{-1} \int_0^t \frac{y(s)}{h + x(s)}\,ds = \frac{\bar{b}_2}{f}. \tag{25}
$$

As a consequence, for any $\varepsilon > 0$, we can find out $T > 0$ such that, for $t \geq T$,

$$
\begin{aligned}
-\frac{c\bar{b}_2}{f} - \varepsilon &\leq -\frac{r_{10} - r_1}{\alpha_1 t}\left(e^{-\alpha_1 t} - 1\right) - ct^{-1}\int_0^t \frac{y(s)}{h + x(s)}\,ds + t^{-1}\ln x(0) \\
&\leq -\frac{c\bar{b}_2}{f} + \varepsilon.
\end{aligned}
\tag{26}
$$

Applying (26) to (18) gives that, for $t \geq T$,

$$
t^{-1}\ln x(t) \geq \bar{b}_1 - \frac{c\bar{b}_2}{f} - 2\varepsilon - at^{-1}\int_0^t x(s)\,ds + t^{-1}\int_0^t \sigma_1(s)\,dB_1(s),
$$

$$
t^{-1}\ln x(t) \leq \bar{b}_1 - \frac{c\bar{b}_2}{f} + 2\varepsilon - at^{-1}\int_0^t x(s)\,ds + t^{-1}\int_0^t \sigma_1(s)\,dB_1(s).
$$

Choose $0 < \varepsilon < (\bar{b}_1 - c\bar{b}_2/f)/2$. On the basis of Lemma 2,

$$
\begin{aligned}
\frac{\bar{b}_1}{a} - \frac{c\bar{b}_2}{af} - \frac{2\varepsilon}{a} &\leq \liminf_{t\to+\infty} t^{-1}\int_0^t x(s)\,ds \leq \limsup_{t\to+\infty} t^{-1}\int_0^t x(s)\,ds \\
&\leq \frac{\bar{b}_1}{a} - \frac{c\bar{b}_2}{af} + \frac{2\varepsilon}{a}.
\end{aligned}
$$

We then deduce from the arbitrariness of ε that $\lim_{t\to+\infty} t^{-1}\int_0^t x(s)\,ds = \bar{b}_1/a - c\bar{b}_2/(af)$. □

3 Discussions and numerical simulations

Now we test the functions of the mean-reverting Ornstein–Uhlenbeck process on the persistence and extinction of model (3). There are two key parameters in the Ornstein–Uhlenbeck process: the speed of reversion α_i and the intensity of the perturbation ξ_i. In light of Theorem 1, the persistence and extinction of system (3) are entirely dominated by the signs of $\bar{b}_1$, $\bar{b}_2$ and $\bar{b}_1 - c\bar{b}_2/f$. Clearly,

$$
\frac{\partial \bar{b}_i}{\partial \alpha_i} > 0, \qquad \frac{\partial(\bar{b}_1 - c\bar{b}_2/f)}{\partial \alpha_1} > 0, \qquad \frac{\partial(\bar{b}_1 - c\bar{b}_2/f)}{\partial \alpha_2} < 0,
$$

$$
\frac{\partial \bar{b}_i}{\partial(\xi_i^2)} < 0, \qquad \frac{\partial(\bar{b}_1 - c\bar{b}_2/f)}{\partial(\xi_1^2)} < 0, \qquad \frac{\partial(\bar{b}_1 - c\bar{b}_2/f)}{\partial(\xi_2^2)} > 0.
$$

For this reason, with the rise of α_i (respectively, ξ_i), species i tends to be persistent (respectively, extinct), $i = 1, 2$. Furthermore, due to the fact that $\frac{\partial(\bar{b}_1 - c\bar{b}_2/f)}{\partial \alpha_2} < 0$ (respectively, $\frac{\partial(\bar{b}_1 - c\bar{b}_2/f)}{\partial(\xi_2^2)} > 0$), thus sufficiently large α_2 (respectively, ξ_2) could make the prey population extinct (respectively, persistent) provided $\bar{b}_1 > 0$ and $\bar{b}_2 > 0$.

Now we numerically validate the above outcomes (here we only provide the functions of α_i since the functions of ξ_i can be proffered analogously). On the basis of the Milstein
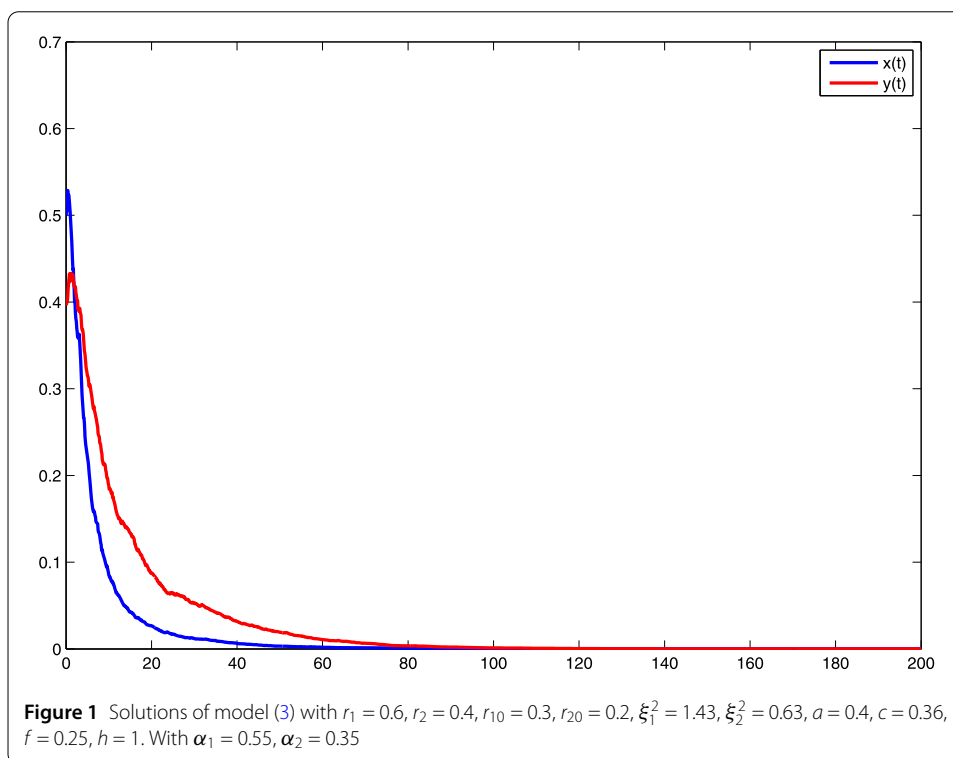
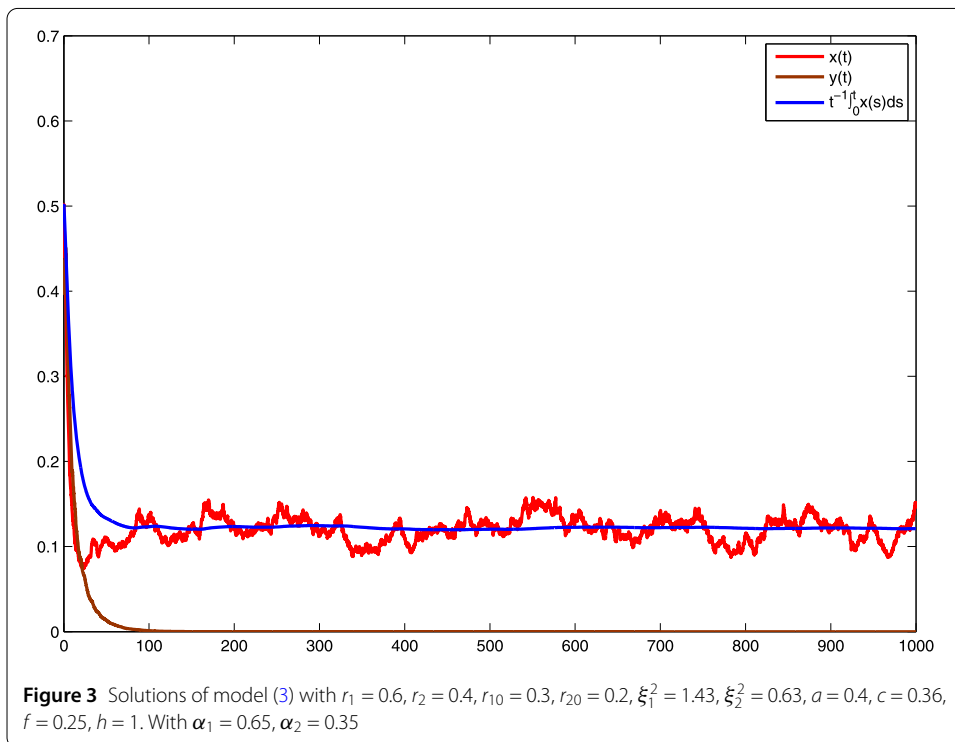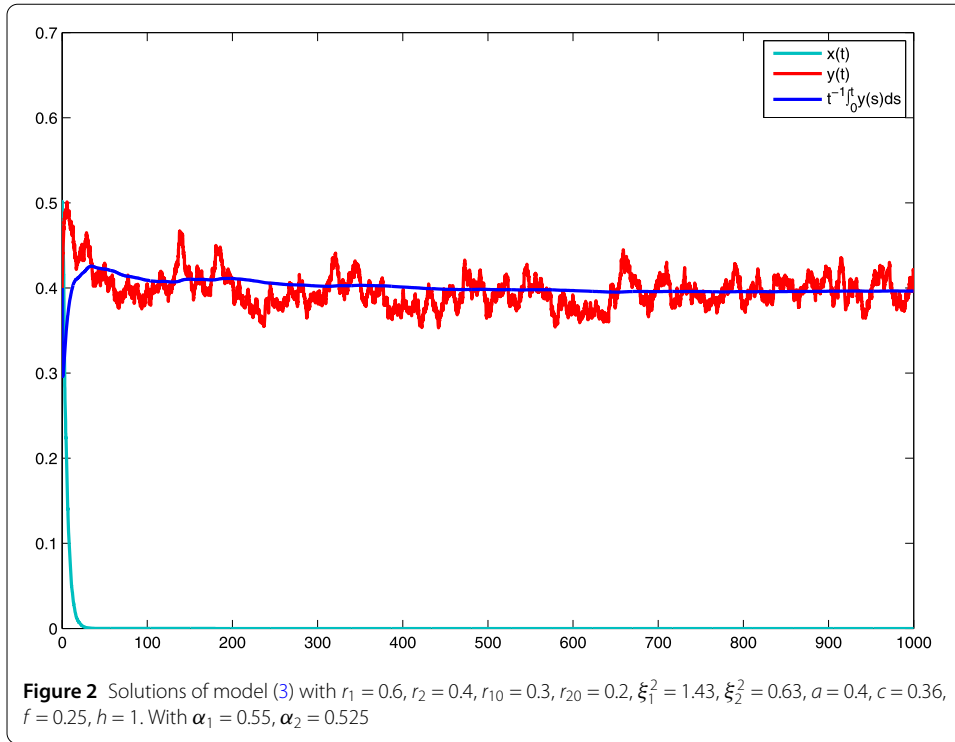method offered in [29], model (3) can be discretized as follows:

$$
\begin{cases}
x^{k+1} = x^k + x^k \left(r_1 + (r_{10} - r_1)e^{-\alpha_1(k\Delta t)} - ax^k - \frac{cy^k}{h+x^k} \right)\Delta t \\
\quad + \frac{\xi_1}{\sqrt{2\alpha_1}}\sqrt{1 - e^{-2\alpha_1(k\Delta t)}}x^k\sqrt{\Delta t}\xi^k \\
\quad + \frac{\xi_1^2}{4\alpha_1}(1 - e^{-2\alpha_1(k\Delta t)})x^k((\xi^k)^2\Delta t - \Delta t), \\
y^{k+1} = y^k + y^k \left(r_2 + (r_{20} - r_2)e^{-\alpha_2(k\Delta t)} - \frac{cy^k}{h+x^k} \right)\Delta t \\
\quad + \frac{\xi_2}{\sqrt{2\alpha_2}}\sqrt{1 - e^{-2\alpha_2(k\Delta t)}}y^k\sqrt{\Delta t}\eta^k \\
\quad + \frac{\xi_2^2}{4\alpha_2}(1 - e^{-2\alpha_2(k\Delta t)})y^k((\eta^k)^2\Delta t - \Delta t),
\end{cases}
$$

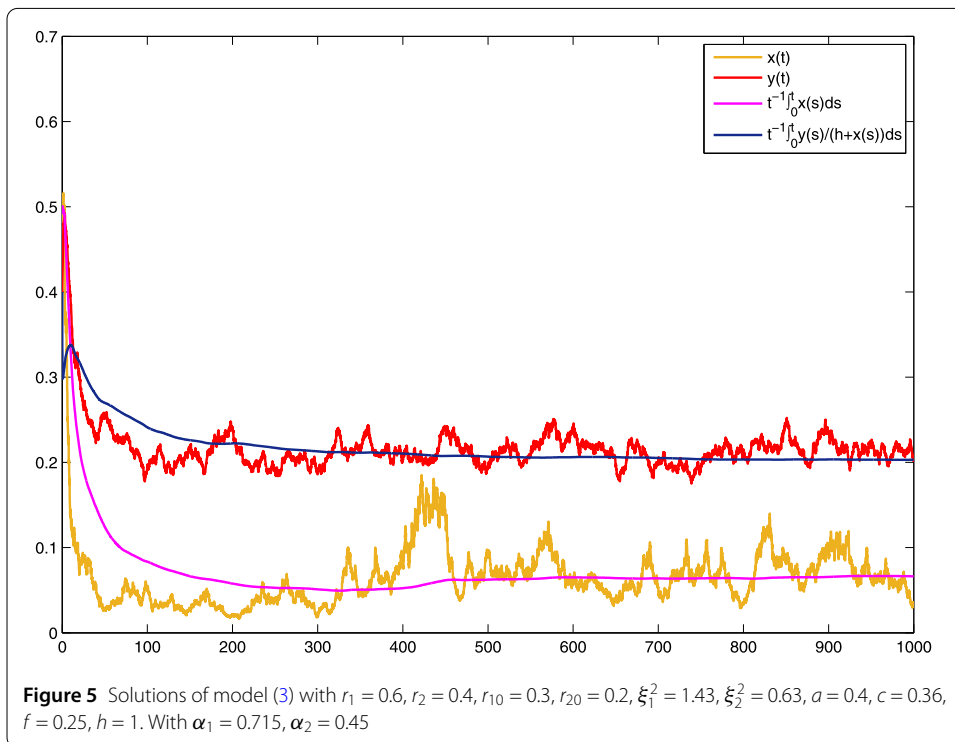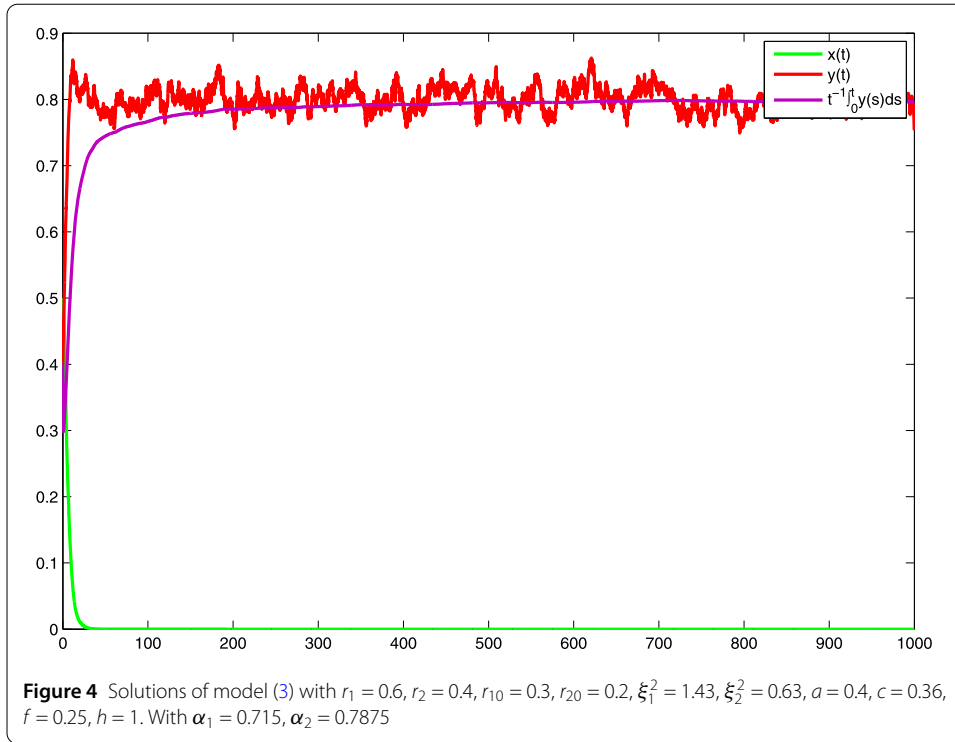where ξ^k, η^k, $k = 1, 2, \ldots, K$, mean independent Gaussian random variables.

Choose $r_1 = 0.6$, $r_2 = 0.4$, $r_{10} = 0.3$, $r_{20} = 0.2$, $\xi_1^2 = 1.43$, $\xi_2^2 = 0.63$, $a = 0.4$, $c = 0.36$, $f = 0.25$, $h = 1$ (these and the following parameter values are hypothesised). Now, we let α_1 and α_2 vary.

- Choose $\alpha_1 = 0.55$, $\alpha_2 = 0.35$. Then $\bar{b}_1 = -0.05$, $\bar{b}_2 = -0.05$. On the basis of (i) in Theorem 1, both x and y become extinct. See Fig. 1.
- Choose $\alpha_1 = 0.55$, $\alpha_2 = 0.525$. Then $\bar{b}_1 = -0.05$, $\bar{b}_2 = 0.1$. On the basis of (ii) in Theorem 1, x becomes extinct and $\lim_{t\to+\infty} t^{-1} \int_0^t y(s)\,ds = h\bar{b}_1/f = 0.4$. See Fig. 2.
- Choose $\alpha_1 = 0.65$, $\alpha_2 = 0.35$. Then $\bar{b}_1 = 0.05$, $\bar{b}_2 = -0.05$. On the basis of (iii) in Theorem 1, y becomes extinct and $\lim_{t\to+\infty} t^{-1} \int_0^t x(s)\,ds = \bar{b}_1/a = 0.125$. See Fig. 3. Comparing Fig. 1 with Fig. 2, one could perceive that with the rise of α_2, the predator population inclines to be persistent. Analogously, comparing Fig. 1 with Fig. 3, one could perceive that with the rise of α_1, the prey population inclines to be persistent.

Figure 1 Solutions of model (3) with $r_1 = 0.6$, $r_2 = 0.4$, $r_{10} = 0.3$, $r_{20} = 0.2$, $\xi_1^2 = 1.43$, $\xi_2^2 = 0.63$, $a = 0.4$, $c = 0.36$, $f = 0.25$, $h = 1$. With $\alpha_1 = 0.55$, $\alpha_2 = 0.35$

Figure 2 Solutions of model (3) with $r_1 = 0.6$, $r_2 = 0.4$, $r_{10} = 0.3$, $r_{20} = 0.2$, $\xi_1^2 = 1.43$, $\xi_2^2 = 0.63$, $a = 0.4$, $c = 0.36$, $f = 0.25$, $h = 1$. With $\alpha_1 = 0.55$, $\alpha_2 = 0.525$

Figure 3 Solutions of model (3) with $r_1 = 0.6$, $r_2 = 0.4$, $r_{10} = 0.3$, $r_{20} = 0.2$, $\xi_1^2 = 1.43$, $\xi_2^2 = 0.63$, $a = 0.4$, $c = 0.36$, $f = 0.25$, $h = 1$. With $\alpha_1 = 0.65$, $\alpha_2 = 0.35$

- Choose $\alpha_1 = 0.715$, $\alpha_2 = 0.7875$. Then $\bar{b}_1 = 0.1$, $\bar{b}_1 = 0.2$, $\bar{b}_1 < c\bar{b}_2/f = 0.288$. On the basis of (a) in Theorem 1, x becomes extinct and $\lim_{t \to +\infty} t^{-1} \int_0^t y(s)\,ds = h\bar{b}_2/f = 0.8$. See Fig. 4.

Figure 4 Solutions of model (3) with $r_1 = 0.6$, $r_2 = 0.4$, $r_{10} = 0.3$, $r_{20} = 0.2$, $\xi_1^2 = 1.43$, $\xi_2^2 = 0.63$, $a = 0.4$, $c = 0.36$, $f = 0.25$, $h = 1$. With $\alpha_1 = 0.715$, $\alpha_2 = 0.7875$

Figure 5 Solutions of model (3) with $r_1 = 0.6$, $r_2 = 0.4$, $r_{10} = 0.3$, $r_{20} = 0.2$, $\xi_1^2 = 1.43$, $\xi_2^2 = 0.63$, $a = 0.4$, $c = 0.36$, $f = 0.25$, $h = 1$. With $\alpha_1 = 0.715$, $\alpha_2 = 0.45$

- Choose $\alpha_1 = 0.715$, $\alpha_2 = 0.45$. Then $\bar{b}_1 = 0.1$, $\bar{b}_2 = 0.05$, $\bar{b}_1 > c\bar{b}_2/f = 0.072$. On the basis of (b) in Theorem 1, $\lim_{t \to +\infty} t^{-1} \int_0^t x(s) \, ds = \bar{b}_1/a - c\bar{b}_2/(af) = 0.07$. $\lim_{t \to +\infty} t^{-1} \int_0^t \frac{y(s)}{h+x(s)} \, ds = \bar{b}_2/f = 0.2$. See Fig. 5. In comparison with Fig. 4, one could perceive that with the rise of α_2, the prey population inclines to become extinct.

4 Concluding remarks

In the present article, we took advantage of a mean-reverting Ornstein–Uhlenbeck process to portray the random perturbations in the environment and formulated a stochastic PPFMLHS which might be more appropriate than model (2). We offered the persistence-and-extinction threshold of the model and uncovered some significant functions of Ornstein–Uhlenbeck process: sufficiently large α_i (the speed of reversion) could make species i persistent; furthermore, in some cases sufficiently large α_2 could make species 1 (the prey population) become extinct.

Several problems remain to be solved. First, the present article tests the predator–prey framework, it would be interesting to dissect the food-chain framework (see [30]). Second, the present article probed into the white noises, one could examine other random noises such as the telephone noise (see [31]), the Lévy noise (see [32]) etc. When the telephone noise is considered, model (3) is replaced with

$$
\begin{cases}
\mathrm{d}x(t) = x(t)(r_1(\lambda(t)) + (r_{10}(\lambda(t)) - r_1(\lambda(t)))e^{-\alpha_1(\lambda(t))t} - a(\lambda(t))x(t) \\
\qquad - \frac{c(\lambda(t))y(t)}{h(\lambda(t))+x(t)})\,\mathrm{d}t + \sigma_1(t,\lambda(t))x(t)\,\mathrm{d}B_1(t), \\
\mathrm{d}y(t) = y(t)(r_2(\lambda(t)) + (r_{20}(\lambda(t)) - r_2(\lambda(t)))e^{-\alpha_2(\lambda(t))t} - \frac{f(\lambda(t))y(t)}{h(\lambda(t))+x(t)})\,\mathrm{d}t \\
\qquad + \sigma_2(t,\lambda(t))y(t)\,\mathrm{d}B_2(t),
\end{cases}
$$

where $\lambda(t)$ is a Markov chain with finite states. When the Lévy noise is considered, model (3) is replaced with

$$
\begin{cases}
\mathrm{d}x(t) = x(t^-)(r_1 + (r_{10} - r_1)e^{-\alpha_1 t^-} - ax(t^-) - \frac{cy(t^-)}{h+x(t^-)})\,\mathrm{d}t \\
\qquad + \sigma_1(t^-)x(t^-)\,\mathrm{d}B_1(t) + \int_{\Xi} \eta_2(\xi)\widetilde{\Gamma}(\mathrm{d}t,\mathrm{d}\xi), \\
\mathrm{d}y(t) = y(t^-)(r_2 + (r_{20} - r_2)e^{-\alpha_2 t^-} - \frac{fy(t^-)}{h+x(t^-)})\,\mathrm{d}t \\
\qquad + \sigma_2(t^-)y(t^-)\,\mathrm{d}B_2(t) + \int_{\Xi} \eta_2(\xi)\widetilde{\Gamma}(\mathrm{d}t,\mathrm{d}\xi),
\end{cases}
$$

where $\mu(t^-)$ means the left limit of $\mu(t)$, $\widetilde{\Gamma}$ is the compensating measure of a Poisson measure Γ, $\Xi \subseteq (0, +\infty)$ adheres to $\gamma(\Xi) < +\infty$, where γ is the characteristic measure of $\widetilde{\Gamma}$. Finally, one could take the intraspecific competition of the predator population into account, which was neglected in model (3).

Author details
[1] Department of Mathematics, Hubei Minzu University, Hubei, P.R. China. [2] School of Mathematical Science, Huaiyin Normal University, Huaian, P.R. China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
1. Aziz-Alaoui, M.A., Okiye, M.D.: Boundedness and global stability for a predator–prey model with modified Leslie–Gower and Holling-type II schemes. Appl. Math. Lett. **16**, 1069–1075 (2003)
2. Guo, H.J., Song, X.Y.: An impulsive predator–prey system with modified Leslie–Gower and Holling type II schemes. Chaos Solitons Fractals **36**, 1320–1331 (2008)
3. Abid, W., Yafia, R., Aziz-Alaoui, M.A., Aghriche, A.: Dynamics analysis and optimality in selective harvesting predator–prey model with modified Leslie–Gower and Holling-type II. Nonauton. Dyn. Syst. **6**, 1–17 (2019)
4. Banerjee, M., Banerjee, S.: Turing instabilities and spatio-temporal chaos in ratio-dependent Holling–Tanner model. Math. Biosci. **236**, 64–76 (2012)
5. Chen, F.D., Chen, L.J., Xie, X.D.: On a Leslie–Gower predator–prey model incorporating a prey refuge. Nonlinear Anal., Real World Appl. **10**, 2905–2908 (2009)
6. Tian, Y.L., Weng, P.X.: Stability analysis of diffusive predator–prey model with modified Leslie–Gower and Holling-type III schemes. Appl. Math. Comput. **218**, 3733–3745 (2011)
7. Song, X.Y., Li, Y.F.: Dynamic behaviors of the periodic predator–prey model with modified Leslie–Gower Holling-type II schemes and impulsive effect. Nonlinear Anal., Real World Appl. **9**, 64–79 (2008)
8. Yafia, R., Adnani, F., Alaoui, H.T.: Limit cycle and numerical simulations for small and large delays in a predator–prey model with modified Leslie–Gower and Holling-type II schemes. Nonlinear Anal., Real World Appl. **9**, 2055–2067 (2008)
9. Nie, L.F., Teng, Z.D., Hu, L., Peng, J.G.: Qualitative analysis of a modified Leslie–Gower and Holling-type II predator–prey model with state dependent impulsive effects. Nonlinear Anal., Real World Appl. **11**, 1364–1373 (2010)
10. Nindjin, A.F., Aziz-Alaoui, M.A., Cadivel, M.: Analysis of a predator–prey model with modified Leslie–Gower and Holling-type II schemes with time delay. Nonlinear Anal., Real World Appl. **7**, 1104–1118 (2006)
11. Ali, N., Jazar, M.: Global dynamics of a modified Leslie–Gower predator–prey model with Crowley–Martin functional responses. Appl. Math. Comput. **43**, 271–293 (2013)
12. Ghaziania, R.K., Alidousti, J., Eshkaftaki, A.B.: Stability and dynamics of a fractional order Leslie–Gower prey-predator model. Appl. Math. Model. **40**, 2075–2086 (2016)
13. Zhou, J., Kim, C., Shi, J.G.: Positive steady state solutions of a diffusive Leslie–Gower predator–prey model with Holling type II functional response and cross-diffusion. Dyn. Syst. **34**, 3875–3899 (2014)
14. Cao, J.Z., Yuan, R.: Bifurcation analysis in a modified Leslie–Gower model with Holling type II functional response and delay. Nonlinear Dyn. **84**, 1341–1352 (2016)
15. Guan, X.N., Wang, W.M., Cai, Y.L.: Spatiotemporal dynamics of a Leslie–Gower predator–prey model incorporating a prey refuge. Nonlinear Anal., Real World Appl. **12**, 2385–2395 (2011)
16. Xu, Y., Liu, M., Yang, Y.: Analysis of a stochastic two-predators one-prey system with modified Leslie–Gower and Holling-type II schemes. J. Appl. Anal. Comput. **7**, 713–727 (2017)
17. Ji, C.Y., Jiang, D.Q., Shi, N.Z.: Analysis of a predator–prey model with modified Leslie–Gower and Holling type II schemes with stochastic perturbation. J. Math. Anal. Appl. **359**, 482–498 (2009)
18. Ji, C.Y., Jiang, D.Q., Shi, N.Z.: A note on a predator–prey model with modified Leslie–Gower and Holling-type II schemes with stochastic perturbation. J. Math. Anal. Appl. **377**, 435–440 (2011)
19. Zou, X.L., Fan, D.J., Wang, K.: Stationary distribution and stochastic Hopf bifurcation for a predator–prey system with noises. Discrete Contin. Dyn. Syst., Ser. B **18**, 1507–1519 (2013)
20. Liu, M., Du, C.X., Deng, M.L.: Persistence and extinction of a modified Leslie–Gower Holling-type II stochastic predator–prey model with impulsive toxicant input in polluted environments. Nonlinear Anal. Hybrid Syst. **27**, 177–190 (2018)
21. Beddington, J.R., May, R.M.: Harvesting natural populations in a randomly fluctuating environment. Science **197**, 463–465 (1977)
22. Jiang, D.Q., Shi, N.Z.: A note on non-autonomous logistic equation with random perturbation. J. Math. Anal. Appl. **303**, 164–172 (2005)
23. Li, X.Y., Mao, X.R.: Population dynamical behavior of non-autonomous Lotka–Volterra competitive system with random perturbation. Discrete Contin. Dyn. Syst. **24**, 523–545 (2009)
24. Zhao, Y., Yuan, S.L., Ma, J.L.: Survival and stationary distribution analysis of a stochastic competitive model of three species in a polluted environment. Bull. Math. Biol. **77**, 1285–1326 (2015)
25. Cai, Y.L., Jiao, J.J., Gui, Z.J., Liu, Y.T., Wang, W.M.: Environmental variability in a stochastic epidemic model. Appl. Math. Comput. **329**, 210–226 (2018)
26. Mao, X.R., Yuan, C.G.: Stochastic Differential Equations with Markovian Switching. Imperial College Press, London (2006)
27. Ikeda, N., Wantanabe, S.: Stochastic Differential Equations and Diffusion Processes. North-Holland, Amsterdam (1981)
28. Liu, M., Wang, K., Wu, Q.: Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle. Bull. Math. Biol. **73**, 1969–2012 (2011)
29. Higham, D.J.: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. **43**, 525–546 (2011)
30. Liu, M., Bai, C.Z.: Analysis of a stochastic tri-trophic food-chain model with harvesting. J. Math. Biol. **73**, 597–625 (2016)

31. Liu, M.: Dynamics of a stochastic regime-switching predator–prey model with modified Leslie–Gower Holling-type II schemes and prey harvesting. Nonlinear Dyn. **96**, 417–442 (2019)
32. Bao, J.H., Mao, X.R., Yin, G., Yuan, C.G.: Competitive Lotka–Volterra population dynamics with jumps. Nonlinear Anal., Real World Appl. **74**, 6601–6616 (2011)