Ansari et al. *Advances in Difference Equations* (2020) 2020:192
https://doi.org/10.1186/s13662-020-02645-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Approximation by modified Kantorovich–Szász type operators involving Charlier polynomials

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Abstract

In this paper, we give some direct approximation results by modified Kantorovich–Szász type operators involving Charlier polynomials. Further, approximation results are also developed in polynomial weighted spaces. Moreover, for the functions of bounded variation, approximation results are proved. Finally, some graphical examples are provided to show comparisons of convergence between old and modified operators towards a function under different parameters and conditions.

MSC: 41A10; 41A25; 41A36

Keywords: Szász operator; Charlier polynomials; Modulus of continuity; Rate of convergence

1 Introduction

In 1950, Szász [20] introduced positive linear operators in the sense of exponential growth on nonnegative semiaxes and exhaustively investigated them. These operators later became known as Szász operators. The Szász type operators involving Charlier polynomials were defined in [21] as

$$L_n(f;y,b) = e^{-1}\left(1-\frac{1}{b}\right)^{(b-1)ny}\sum_{l=0}^{\infty}\frac{C_l^{(b)}(-(b-1)ny)}{l!}f\left(\frac{l}{n}\right), \tag{1.1}$$

where $b>1$ and $y\geq 0$, having the generating functions [5] of the form

$$e^t\left(1-\frac{t}{b}\right)^u = \sum_{l=0}^{\infty}C_l^{(b)}(u)\frac{t^l}{l!}, \quad |t|<b, \tag{1.2}$$

where $C_l^{(b)}(u)=\sum_{r=0}^{l}\binom{l}{r}(-u)_r(\frac{1}{b})^r$ and $(k)_0=1$, $(k)_m=k(k+1)\cdots(k+m-1)$, for $m\geq 1$.

Motivated by the work done in [9], we define the Kantorovich generalization [10] of (1.2) as follows:

$$\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) = \nu_n e^{-1}\left(1-\frac{1}{b}\right)^{(b-1)\mu_n y}\sum_{l=0}^{\infty}\frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!}\int_{l/\nu_n}^{l+1/\nu_n} f(t)\,dt \tag{1.3}$$

where μ_n and ν_n are sequences of positive numbers which are increasing and unbounded such that

$$\lim_{n\to\infty}\frac{1}{\nu_n} = 0, \qquad \frac{\mu_n}{\nu_n} = 1 + O\left(\frac{1}{\nu_n}\right). \tag{1.4}$$

If we take $\mu_n = \nu_n = n$, we will have the operators defined in [9].

For some recent and interesting results on the various generalizations and corresponding approximation results, we refer to [1, 3, 6–8, 14–17, 22].

2 Auxiliary results

We first present some auxiliary results.

Lemma 2.1 *Let $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}$ be defined by (1.3). Then, we have*

1. $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(1;y) = 1$,
2. $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t;y) = \frac{\mu_n}{\nu_n}y + \frac{3}{2\nu_n}$
3. $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t^2;y) = \frac{\mu_n^2}{\nu_n^2}y^2 + \frac{\mu_n}{\nu_n^2}(4+\frac{1}{b-1})y + \frac{10}{3\nu_n^2}$.
4. $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t^3;y) = \frac{\mu_n^3}{\nu_n^3}y^3 + \frac{\mu_n^2}{\nu_n^3}(\frac{15}{2}+\frac{3}{b-1})y^2 + \frac{\mu_n}{\nu_n^3}(\frac{31}{2}+\frac{15}{2(b-1)}+\frac{2}{(b-1)^2})y + \frac{37}{4\nu_n^3}$.
5. $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t^4;y) =$
$\frac{\mu_n^4}{\nu_n^4}y^4 + \frac{\mu_n^3}{\nu_n^4}(12+\frac{6}{b-1})y^3 + \frac{\mu_n^2}{\nu_n^4}(46+\frac{36}{b-1}+\frac{11}{(b-1)^2})y^2 + \frac{\mu_n}{\nu_n^4}(64+\frac{46}{b-1}+\frac{24}{(b-1)^2}+\frac{6}{(b-1)^3})y + \frac{151}{5\nu_n^4}$.

Proof With the help of the Charlier polynomials' generating function given by (1.2), after some simple calculations, we obtain

$$\sum_{l=0}^{\infty}\frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!} = e\left(1-\frac{1}{b}\right)^{-(b-1)\mu_n y},$$

$$\sum_{l=0}^{\infty}\frac{lC_l^{(b)}(-(b-1)\mu_n y)}{l!} = e\left(1-\frac{1}{b}\right)^{-(b-1)\mu_n y}(1+\mu_n y),$$

$$\sum_{l=0}^{\infty}\frac{l^2C_l^{(b)}(-(b-1)\mu_n y)}{l!} = e\left(1-\frac{1}{b}\right)^{-(b-1)\mu_n y}\left(\mu_n^2y^2 + \mu_n y\left(3+\frac{1}{b-1}\right)+2\right),$$

$$\sum_{l=0}^{\infty}\frac{l^3C_l^{(b)}(-(b-1)\mu_n y)}{l!} = e\left(1-\frac{1}{b}\right)^{-(b-1)\mu_n y}\left(\mu_n^3y^3 + \mu_n^2\left(6+\frac{3}{b-1}\right)y^2 + \mu_n\left(10+\frac{6}{b-1}+\frac{2}{(b-1)^2}\right)y+5\right),$$

$$\sum_{l=0}^{\infty}\frac{l^4C_l^{(b)}(-(b-1)\mu_n y)}{l!}=e\left(1-\frac{1}{b}\right)^{-(b-1)\mu_n y}\left(\mu_n^4y^4+\mu_n^3\left(10+\frac{6}{b-1}\right)y^3\right.$$
$$+\mu_n^2\left(32+\frac{30}{b-1}+\frac{11}{(b-1)^2}\right)y^2$$
$$\left.+\mu_n\left(37+\frac{32}{b-1}+\frac{20}{(b-1)^2}+\frac{6}{(b-1)^3}\right)y+15\right).$$

From the above equalities, the claims of the lemma can be obtained. □

Lemma 2.2 *For the operator* $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}$ *given by* (1.3), *we have the following equalities*:

$$(i)\quad \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t-y;y)=\left(\frac{\mu_n}{\nu_n}-1\right)y+\frac{1}{\nu_n},$$

$$(ii)\quad \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\big((t-y)^2;y\big)=\left(\frac{\mu_n}{\nu_n}-1\right)^2y^2+\left(\frac{\mu_n}{\nu_n^2}\left(3+\frac{1}{b-1}\right)-\frac{2}{\nu_n}\right)y+\frac{2}{\nu_n^2},$$

$$(iii)\quad \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\big((t-y)^4;y\big)$$
$$=\left(\frac{\mu_n}{\nu_n}-1\right)^4y^4$$
$$+2\left(\frac{\mu_n^3}{\nu_n^4}\left(5+\frac{3}{b-1}\right)-\frac{6\mu_n^2}{\nu_n^3}\left(2+\frac{1}{b-1}\right)+\frac{3\mu_n}{\nu_n^2}\left(3+\frac{1}{b-1}\right)-\frac{2}{\nu_n}\right)y^3$$
$$+\left(\frac{\mu_n^2}{\nu_n^4}\left(32+\frac{30}{b-1}+\frac{11}{(b-1)^2}\right)-\frac{4\mu_n}{\nu_n^3}\left(10+\frac{6}{b-1}+\frac{2}{(b-1)^2}\right)+\frac{12}{\nu_n^2}\right)y^2$$
$$+\left(\frac{\mu_n}{\nu_n^4}\left(37+\frac{32}{b-1}+\frac{20}{(b-1)^2}+\frac{6}{(b-1)^3}\right)-\frac{20}{\nu_n^3}\right)y+\frac{15}{\nu_n^4}.$$

3 Local approximation results

In what follows, let $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t-y;y)=\chi_{\mu_n,\nu_n}(y)$ and $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}((t-y)^2;y)=\xi_{\mu_n,\nu_n}(y)$. We will now give two theorems on the uniform convergence and the order of approximation.

Theorem 3.1 *Let* $f\in C[0,\infty)\cap G$. *Then* $\lim_{n\to\infty}\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)=f(y)$, *the sequence of operators Eq.* (1.3) *converges uniformly in each compact subset of* $[0,\infty)$, *where*

$$G:=\left\{f:\mathbb{R}_0^+\to\mathbb{R},\left|F(y)\right|=\left|\int_0^y f(s)\,ds\right|\le Ke^{By},K\in\mathbb{R}^+\text{ and }B\in\mathbb{R}\right\}.$$

Proof From Lemma 2.1(1)–(3), we get

$$\lim_{n\to\infty}\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\big(s^k;y\big)=y^k,\quad k=0,1,2.$$

The proof of the theorem is established by taking advantage of the above uniform convergence in each compact subset of $[0,\infty)$ and the famous Korovkin's theorem. □

Suppose $f\in\tilde{C}[0,\infty)$, i.e., f belongs to the space of uniformly continuous functions on $[0,\infty)$. If $\delta>0$, then the modulus of continuity $\omega(f,\delta)$ is defined by

$$\omega(f,\delta):=\sup_{\substack{\sigma,\varsigma\in[0,\infty)\\|\sigma-\varsigma|\le\delta}}\left|f(\sigma)-f(\varsigma)\right|.$$

Theorem 3.2 *Let $f \in \tilde{C}[0,\infty) \cap E$. For the operators $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)$ given by (1.3) the following estimate holds*:

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \\
&\quad \leq \left\{1 + \sqrt{(\mu_n - \nu_n)^2 y^2 + \left(\left(4 + \frac{1}{b-1}\right)\mu_n - 3\nu_n\right)y + \frac{10}{3}}\right\}\omega\left(f, \frac{1}{\nu_n}\right).
\end{aligned} \tag{3.1}$$

Proof From (1.3) and the property of modulus of continuity, the left-hand side of (3.1) leads to

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \\
&\quad \leq \nu_n e^{-1}\left(1 - \frac{1}{b}\right)^{(b-1)\mu_n y} \sum_{l=0}^{\infty} \frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!} \int_{l/\nu_n}^{l+1/\nu_n} \left|f(t) - f(y)\right| dt \\
&\quad \leq \left\{1 + \frac{1}{\delta}\nu_n e^{-1}\left(1 - \frac{1}{b}\right)^{(b-1)\mu_n y} \sum_{l=0}^{\infty} \frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!} \int_{l/\nu_n}^{l+1/\nu_n} |t-y|\, dt\right\}\omega(f,\delta).
\end{aligned}$$

Using Cauchy–Schwarz inequality for the integral, we get

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \\
&\quad \leq \left\{1 + \frac{1}{\delta}e^{-1}\left(1 - \frac{1}{b}\right)^{(b-1)\mu_n y}\right. \\
&\qquad \left. \times \sum_{l=0}^{\infty} \frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!}\left(\nu_n \int_{l/\nu_n}^{l+1/\nu_n} (t-y)^2\, dt\right)^{1/2}\right\}\omega(f,\delta).
\end{aligned} \tag{3.2}$$

In the above sum, we apply Cauchy–Schwarz inequality, and then in view of Lemma 2.1, (3.2) becomes

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \\
&\quad \leq \left\{1 + \frac{1}{\delta}\left(e^{-1}\left(1 - \frac{1}{b}\right)^{(b-1)\mu_n y} \sum_{l=0}^{\infty} \frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!}\right)^{1/2}\right. \\
&\qquad \left. \times \left(\nu_n e^{-1}\left(1 - \frac{1}{b}\right)^{(b-1)\mu_n y} \sum_{l=0}^{\infty} \frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!} \int_{l/\nu_n}^{l+1/\nu_n} (t-y)^2\, dt\right)^{1/2}\right\}\omega(f,\delta) \\
&\quad \leq \left\{1 + \frac{1}{\delta}\left(\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2; y, b\right)\right)^{1/2}\right\}\omega(f,\delta) \\
&\quad \leq \left\{1 + \frac{1}{\delta}\frac{1}{\nu_n}\sqrt{(\mu_n - \nu_n)^2 y^2 + \left(\left(4 + \frac{1}{b-1}\right)\mu_n - 3\nu_n\right)y + \frac{10}{3}}\right\}\omega\left(f, \frac{1}{\delta}\right),
\end{aligned}$$

where, taking $\delta = \frac{1}{\nu_n}$, we get (3.1). □

Let $a_1, a_2 > 0$ be fixed. We now consider the following space of Lipschitz type (see [18]):

$$\mathrm{Lip}_M^{(a_1,a_2)}(r) := \left\{ f \in C[0,\infty) : \left|f(\kappa) - f(\lambda)\right| \le M \frac{|\kappa - \lambda|^r}{(\kappa + a_1\lambda^2 + a_2\lambda)^{r/2}}; \lambda, \kappa > 0 \right\}, \tag{3.3}$$

where M is a positive constant and $0 < r \le 1$.

Theorem 3.3 *Let $f \in \mathrm{Lip}_M^{(a_1,a_2)}(r)$ and $r \in (0,1]$, then $\forall y > 0$, we have*

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \le M \left(\frac{\xi_{\mu_n,\nu_n}(y)}{(a_1y^2 + a_2y)} \right)^{r/2}.$$

Proof Since

$$\left|f(t) - f(y)\right| \le M \frac{|t-y|^r}{(t + a_1y^2 + a_2y)^{r/2}},$$

one has

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \le M \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)} \left(\frac{|t-y|^r}{(t + a_1y^2 + a_2y)^{r/2}}; y \right).$$

Applying Hölder's inequality with $p = \frac{2}{r}$ and $\frac{2}{2-r}$, we find that

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \le M \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)} \left(\frac{(t-y)^2}{t + a_1y^2 + a_2y}; y \right)^{r/2}.$$

Since $f \in \mathrm{Lip}_M^{(a_1,a_2)}(r)$ and $\frac{1}{t+a_1y^2+a_2y} < \frac{1}{a_1y^2+a_2y}$, $\forall y \in (0,\infty)$, we have

$$\begin{aligned}
\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| &\le M \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)} \left(\frac{(t-y)^2}{a_1y^2 + a_2y}; y \right)^{r/2} \\
&\le \frac{M}{(a_1y^2 + a_2y)^{r/2}} \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)} \left((t-y)^2; y \right)^{r/2} \\
&= M \left(\frac{\xi_{\mu_n,\nu_n}(y)}{(a_1y^2 + a_2y)} \right)^{r/2}.
\end{aligned}$$

Our proof is now completed. □

We denote the space of all functions h on $[0,1)$ which are real-valued, uniformly continuous, as well as bounded by $\tilde{C}_B[0,\infty)$ and endow it with the norm $\|h\|_\infty = \sup_{y\in[0,1)} |h(y)|$. Further, we obtain a local direct estimate for the operators (1.3), using the Lipschitz maximal function of order r introduced by Lenze [13] as:

$$\tilde{\omega}_r(h,y) = \sup_{t\neq y, t\in[0,\infty)} \frac{|h(t) - h(y)|}{|t-y|^r}, \tag{3.4}$$

where $y \in [0,1)$ and $r \in (0,1]$.

Theorem 3.4 *Let* $f \in \tilde{C}_B[0,\infty)$ *and* $0 < r \leq 1$, *then* $\forall y \in [0,\infty)$

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \leq \tilde{\omega}_r(f,y)\left(\xi_{\mu_n,\nu_n}(y)\right)^r.$$

Proof By equation (3.4),

$$\left|f(t) - f(y)\right| \leq \tilde{\omega}_r(f,y)|t-y|^r.$$

Applying $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}$ on both sides of the above inequality, then using Lemma 2.1, as well as Hölder's inequality with $p = 2/r$, $q = 2/(2-r)$, we obtain

$$\begin{aligned}\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| &\leq \tilde{\omega}_r(f,y)\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(|t-y|^r;y\right)\\ &\leq \tilde{\omega}_r(f,y)\left(\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2;y\right)\right)^{r/2}\\ &= \tilde{\omega}_r(f,y)\left(\xi_{\mu_n,\nu_n}(y)\right)^{r/2}.\end{aligned}$$

Thus, we have our desired result. □

The Peetre's K-functional is given by

$$K(g,\delta) = \inf_{h\in\tilde{C}_B^2[0,\infty)}\left\{\|g-h\|_\infty + \delta\|h\|_{\tilde{C}_B^2}\right\},$$

where $\tilde{C}_B^2[0,\infty) = \{h \in \tilde{C}_B[0,\infty) : h', h'' \in \tilde{C}_B[0,\infty)\}$ with the norm $\|h\|_{\tilde{C}_B^2} = \|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty$. Also, the inequality

$$K(g,\delta) \leq M\left(\omega_2(g,\sqrt{\delta}) + \min\{1,\delta\}\|g\|_\infty\right)$$

holds for all $\delta > 0$, where ω_2 is the second-order modulus of smoothness of $g \in \tilde{C}_B[0,\infty)$, which is defined by

$$\omega_2(g,\delta) = \sup_{0<|h|\leq\delta}\ \sup_{y,y+2h\in[0,\infty)}\left|g(y+2h) - 2g(y+h) + g(y)\right|.$$

Theorem 3.5 *If* $f \in \tilde{C}_B[0,\infty)$, *then*

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \leq 4K(f,\zeta_{\mu_n,\nu_n}(y)) + \omega(f,\chi_{\mu_n,\nu_n}(y)),$$

where $\zeta_{\mu_n,\nu_n}(y) = (\xi_{\mu_n,\nu_n}(y) + \chi^2_{\mu_n,\nu_n}(y))/4$. *Furthermore*,

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \leq M\left(\omega_2\left(f,\sqrt{\zeta_{\mu_n,\nu_n}(y)}\right) + \min\{1,\zeta_{\mu_n,\nu_n}(y)\}\|f\|_\infty\right) + \omega\left(f,\left|\chi_{\mu_n,\nu_n}(y)\right|\right).$$

Proof For $f \in \tilde{C}_B[0,\infty)$, we define the auxiliary operator as follows:

$$\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(f;y) = \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f\left(\frac{\mu_n}{\nu_n}y + \frac{3}{2\nu_n}\right) + f(y). \tag{3.5}$$

After taking the absolute value of both sides,

$$\begin{aligned}\left|\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(f;y)\right| &\le \left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)\right| + \left|f\left(\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}\right)\right| + \left|f(y)\right| \\ &\le \|f\|_\infty\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(1;y,b)\right| + \|f\|_\infty + \|f\|_\infty \\ &\le 3\|f\|_\infty. \end{aligned} \tag{3.6}$$

By Lemma 2.1, we have $\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(t;y) = y$, and therefore $\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(t-y;y) = 0$.

Let $g \in \tilde{C}_B^2[0,\infty)$, using Taylor's theorem, we can write

$$g(t) = g(y) + g'(y)(t-y) + \int_y^t (t-u)g''(u)\,du.$$

Applying operator $\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}$ to the above equation, we get

$$\begin{aligned}&\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(g;y) - g(y) \\ &\quad= \tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}\left(\int_y^t (t-u)g''(u)\,du;y\right) \\ &\quad= \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(\int_y^t (t-u)g''(u)\,du;y\right) - \int_y^{\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}}\left(\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}-u\right)g''(u)\,du.\end{aligned}$$

Now taking the absolute value of both sides, we obtain

$$\begin{aligned}&\left|\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(g;y) - g(y)\right| \\ &\quad\le \left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(\int_y^t (t-u)g''(u)\,du;y\right)\right| + \left|\int_y^{\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}}\left(\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}-u\right)g''(u)\,du\right| \\ &\quad\le \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(\left|\int_y^t \left|(t-u)\right|\left|g''(u)\right|du\right|;y\right) + \left|\left(\int_y^{\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}}\left|\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}-u\right|\left|g''(u)\right|du\right)\right| \\ &\quad\le \|g''\|_\infty\left\{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(\left|\int_y^t \left|(t-u)\right|du\right|;y\right) + \left|\left(\int_y^{\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}}\left|\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}-u\right|du\right)\right|\right\}.\end{aligned}$$

Therefore, by using the norm on g, we have

$$\begin{aligned}\left|\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(g;y) - g(y)\right| &\le \|g\|_{\tilde{C}_B^2}\left\{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2;y\right) + \left(\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}-y\right)^2\right\} \\ &\le \|g\|_{\tilde{C}_B^2}\left\{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2;y\right) + \left(\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t-y;y)\right)^2\right\} \\ &\le \|g\|_{\tilde{C}_B^2}\left\{\xi_{\mu_n,\nu_n}(y) + \chi_{\mu_n,\nu_n}^2(y)\right\}.\end{aligned} \tag{3.7}$$

Now, using the definition of auxiliary operators (3.5), we get

$$\begin{aligned}&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y)\right| \\ &\quad\le \left|\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(f;y) - f(y) + f\left(\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}\right) - f(y)\right|\end{aligned}$$

$$\begin{aligned}
&\leq \left|\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(f-g;y)\right| + \left|\tilde{\mathcal{Q}}_{n,b}^{(\mu_n,\nu_n)}(g;y)-g(y)\right| \\
&\quad + \left|g(y)-f(y)\right| + \left|f\left(\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}\right)-f(y)\right|.
\end{aligned}$$

Combining (3.6) and (3.7) with the above equation, we get

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| \\
&\quad \leq 3\|f-g\|_\infty + \|g\|_{\tilde{C}_B^2}\left\{\xi_{\mu_n,\nu_n}(y)+\chi_{\mu_n,\nu_n}^2(y)\right\} + \|f-g\|_\infty + \omega\left(f,\left|\frac{\mu_n}{\nu_n}y+\frac{3}{2\nu_n}-y\right|\right) \\
&\quad \leq 4\|f-g\|_\infty + \|g\|_{\tilde{C}_B^2}4\zeta_{\mu_n,\nu_n}(y) + \omega\left(f,\left|\chi_{\mu_n,\nu_n}(y)\right|\right),
\end{aligned}$$

and after taking the infimum on the right-hand side over all $g \in \tilde{C}_B^2$, we have

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| \\
&\quad \leq 4K\left(f,\zeta_{\mu_n,\nu_n}(y)\right) + \omega\left(f,\left|\chi_{\mu_n,\nu_n}(y)\right|\right) \\
&\quad \leq M\left(\omega_2\left(f,\sqrt{\zeta_{\mu_n,\nu_n}(y)}\right) + \min\left\{1,\zeta_{\mu_n,\nu_n}(y)\right\}\|f\|_\infty\right) + \omega\left(f,\left|\chi_{\mu_n,\nu_n}(y)\right|\right).
\end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.6 *Let* $f \in \tilde{C}_B^1[0,\infty)$, *then* $\forall y \geq 0$ *and* $\delta > 0$,

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| \leq \left\{\left|f'(y)\right| + 2\omega\left(f',\delta_n(y)\right)\right\}.$$

Proof Since $f \in \tilde{C}_B^1[0,\infty)$, we can write

$$f(t)-f(y) = f'(y)(t-y) + \int_y^t \left(f'(u)-f'(y)\right)du. \tag{3.8}$$

Now, using the well-known property of the modulus of continuity for $\delta > 0$ and $f \in \tilde{C}_B^1[0,\infty)$,

$$\left|f'(u)-f'(y)\right| \leq \left(\frac{|u-y|}{\delta}+1\right)\omega\left(f',\delta\right),$$

hence

$$\left|\int_y^t \left(f'(u)-f'(y)\right)du\right| \leq \left(\frac{(t-y)^2}{\delta}+|t-y|\right)\omega\left(f',\delta\right)$$

Therefore, from (3.8) and the above equation, we have

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| \\
&\quad \leq \left|f'(y)\right|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(|t-y|;y\right) + \left(\frac{1}{\delta}\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2;y\right) + \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(|t-y|;y\right)\right)\omega\left(f',\delta\right).
\end{aligned}$$

After applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| \\
&\quad\le \left(\left|f'(y)\right|+\omega\left(f',\delta\right)\right)\sqrt{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2;y\right)}\sqrt{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(1;y)} \\
&\qquad+\left(\frac{1}{\delta}\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2;y\right)\right)\omega\left(f',\delta\right) \\
&\quad=\left(\left|f'(y)\right|+\omega\left(f',\delta\right)\right)\delta_n(y)+\left(\frac{\delta_n^2(y)}{\delta}\right)\omega\left(f',\delta\right).
\end{aligned}$$

Choosing $\delta=\delta_n(y)$, we get our desired result. □

For $f\in\tilde{C}_B[0,\infty)$, the Ditzian–Totik modulus of smoothness [4] of the first order is given by

$$\omega_\varphi(f,\delta)=\sup_{0<h\le\delta}\left\{\left|f\left(y+\frac{h\varphi(y)}{2}\right)-f\left(y-\frac{h\varphi(y)}{2}\right)\right|;y\pm\frac{h\varphi(y)}{2}\in[0,\infty)\right\},$$

and an appropriate Peetre's K-functional is defined by

$$K_\varphi(f,\delta)=\inf_{g\in W_\varphi[0,\infty)}\left\{\|f-g\|_\infty+\delta\|\varphi g'\|_\infty\right\},\quad \delta>0,$$

where $W_\varphi[0,\infty):=\{g:g\in \mathrm{AC}_{\mathrm{loc}}[0,\infty),\|\varphi g'\|_\infty<\infty\}$ where $g\in \mathrm{AC}_{\mathrm{loc}}[0,\infty)$ means g is absolutely continuous on every compact subset $[a,b]$ of $[0,\infty)$. It is known from [4] that there exists a constant M such that

$$M^{-1}\omega_\varphi(f,\delta)\le K_\varphi(f,\delta)\le M\omega_\varphi(f,\delta). \tag{3.9}$$

Now, we find the order of approximation of the sequence of operators (1.3) by means of Ditzian–Totik modulus of smoothness.

Theorem 3.7 *For any $f\in\tilde{C}_B[0,\infty)$ and $y\in[0,\infty)$,*

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right|\le M\omega_\varphi\left(f,\frac{\delta_n(y)}{\sqrt{y}}\right).$$

Proof Let $\varphi(y)=\sqrt{y}$, then by Taylor's theorem, for any $g\in W_\varphi[0,\infty)$, we get

$$g(t)=g(y)+\int_y^t g'(u)\,du=g(y)+\int_y^t\frac{g'(u)\varphi(u)}{\varphi(u)}\,du,$$

therefore,

$$\begin{aligned}
\left|g(t)-g(y)\right|&=\left\|\varphi g'\right\|_\infty\left|\int_y^t\frac{1}{\varphi(u)\,du}\right| \\
&=2\left\|\varphi g'\right\|_\infty|\sqrt{t}-\sqrt{y}| \\
&=2\left\|\varphi g'\right\|_\infty\frac{|t-y|}{\sqrt{t}+\sqrt{y}},
\end{aligned}$$

which gives

$$\left|g(t)-g(y)\right| \leq 2\left\|\varphi g'\right\|_{\infty}\frac{|t-y|}{\sqrt{y}} = 2\left\|\varphi g'\right\|_{\infty}\frac{|t-y|}{\varphi(y)}.$$

Using Lemma 2.1 and the above equation, for any $g \in W_{\varphi}[0,\infty)$, we get

$$\begin{aligned}\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| &\leq \left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f-g;y)\right| + \left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(g;y)-g(y)\right| + \left|g(y)-f(y)\right| \\ &\leq 2\|f-g\|_{\infty} + \frac{2\|\varphi g'\|_{\infty}}{\varphi(y)}\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\big(|t-y|;y\big).\end{aligned}$$

Applying the Cauchy–Schwarz inequality yields

$$\begin{aligned}\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| &\leq 2\|f-g\|_{\infty} + \frac{2\|\varphi g'\|_{\infty}}{\varphi(y)}\sqrt{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\big((t-y)^2;y\big)} \\ &= 2\|f-g\|_{\infty} + \frac{2\|\varphi g'\|_{\infty}}{\varphi(y)}\delta_n(y).\end{aligned}$$

Taking infimum on the right-hand side over all $g \in W_{\varphi}[0,\infty)$, we get

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| \leq 2K_{\varphi}\left(f,\frac{\delta_n(y)}{\sqrt{y}}\right),$$

which leads to the required result with the help of the relation between Peetre's K-functional and Ditzian–Totik modulus of smoothness as given by the relation (3.9). □

4 Approximation results in weighted spaces

Let $\nu > 0$. We denote $C_{\nu}[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq M_f(1+t^{\nu}), \forall t \geq 0\}$ equipped with the norm

$$\|f\|_{\nu} = \sup_{t\in[0,\infty)}\frac{|f(t)|}{1+t^{\nu}}. \tag{4.1}$$

Further, let $C_2^*[0,\infty)$ be the subspace of $C_2[0,\infty)$ consisting of functions f such that $\lim_{t\to\infty}\frac{f(t)}{1+t^2}$ exists.

Theorem 4.1 *For each $f \in C_2^*[0,\infty)$ and $r > 0$, the following relation holds*:

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)|}{(1+y^2)^{1+r}} = 0.$$

Proof Let $y_0 > 0$ be arbitrary but fixed, then by (4.1), we can write

$$\begin{aligned}&\sup_{y\in[0,\infty)}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)|}{(1+y^2)^{1+r}} \\ &\quad\leq \sup_{y\leq y_0}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)|}{(1+y^2)^{1+r}} + \sup_{y>y_0}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)|}{(1+y^2)^{1+r}}\end{aligned}$$

$$\leq \sup_{y\leq y_0}\left\{\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right|\right\}+\sup_{y>y_0}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)+f(y)|}{(1+y^2)^{1+r}}.$$

Since $|f(t)|\leq \|f\|_2(1+y^2)$, we get

$$\sup_{y\in[0,\infty)}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)|}{(1+y^2)^{1+r}} \tag{4.2}$$

$$\leq \left\|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right\|_{C[0,y_0]}+\|f\|_2\sup_{y>y_0}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(1+t^2;y)|}{(1+y^2)^{1+r}}+\sup_{y>y_0}\frac{\|f(y)\|}{(1+y^2)^{1+r}}$$

$$=I_1+I_2+I_3 \quad \text{(say)}. \tag{4.3}$$

By Korovkin's theorem, we can see that the sequence of operators $\{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)\}$ converges uniformly to the function f on every closed interval $[0,a]$ as $n\to\infty$, (cf. [12, p. 149]). Therefore, for a given $\epsilon>0$, $\exists n_1\in\mathbb{N}$ such that

$$I_1=\left\|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right\|_{C[0,y_0]}<\frac{\epsilon}{3},\quad \forall n\geq n_1. \tag{4.4}$$

By using Lemma 2.1, we can find $n_2\in\mathbb{N}$ such that

$$\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(1+t^2;y\right)-\left(1+y^2\right)\right|<\frac{\epsilon}{3\|f\|_2},\quad \forall n\geq n_2,$$

$$\text{or}\quad \mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left(1+t^2;y\right)<\left(1+y^2\right)+\frac{\epsilon}{3\|f\|_2},\quad \forall n\geq n_2.$$

Hence

$$\begin{aligned}
I_2&=\|f\|_2\sup_{y>y_0}\frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(1+t^2;y)|}{(1+y^2)^{1+r}}\\
&<\|f\|_2\sup_{y>y_0}\frac{1}{(1+y^2)^{1+r}}\left(\left(1+y^2\right)+\frac{\epsilon}{3\|f\|_2}\right)\\
&<\|f\|_2\sup_{y>y_0}\left(\frac{1}{(1+y^2)^r}\right)+\frac{\epsilon}{3}\\
&<\frac{\|f\|_2}{(1+y_0^2)^r}+\frac{\epsilon}{3},\quad \forall n\geq n_2.
\end{aligned} \tag{4.5}$$

Now, using (4.1),

$$I_3=\sup_{y>y_0}\frac{|f(y)|}{(1+y^2)^{1+r}}\leq\frac{\|f\|_2}{(1+y_0^2)^r}. \tag{4.6}$$

Let us denote $n_0=\max\{n_1,n_2\}$, then by (4.4), (4.5), and (4.6), we get

$$I_1+I_2+I_3<2\frac{\|f\|_2}{(1+y_0^2)^r}+\frac{2\epsilon}{3},\quad \forall n\geq n_0. \tag{4.7}$$

Choose y_0 so large that

$$2\frac{\|f\|_2}{(1+y_0^2)^r}<\frac{\epsilon}{3}. \tag{4.8}$$

Then, combining (4.3), (4.7), and (4.8), we obtain

$$\sup_{y\in[0,\infty)} \frac{|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)|}{(1+y^2)^{1+r}} < \epsilon, \quad \forall n \geq n_0.$$

Hence, the proof is completed. □

Now, we will obtain the rate of convergence of the operators $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)$ defined by (1.3) for the functions having derivatives of bounded variation. Let $\mathrm{DBV}[0,\infty)$ be the space of functions in $C_2[0,\infty)$, which have the derivative of bounded variation on every finite subinterval of $[0,\infty)$. Here, we show at the point y, where $f'(y+)$ and $f'(y-)$ exist, the operators $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)$ converge to the function $f(y)$. A function $f \in \mathrm{DBV}[0,\infty)$ can be represented as

$$f(y) = \int_0^y g(t)\,dt + f(0),$$

where g denotes a function of bounded variation on every finite subinterval $[0,\infty)$. Many researchers studied in this direction and their work pertaining to this area is described in the papers [2, 11, 19], etc.

In order to study the order of convergence of the operators $\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)$ for the functions having a derivative of bounded variation, we rewrite the operator (1.3) as follows:

$$\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y) = \int_0^\infty W(t,y)f(t)\,dt, \tag{4.9}$$

where $W(t,y)$ is a kernel given by

$$W(t,y) = \nu_n e^{-1}\left(1-\frac{1}{b}\right)^{(b-1)\mu_n y} \sum_{l=0}^{\infty} \frac{C_l^{(b)}(-(b-1)\mu_n y)}{l!}\chi_I(t),$$

$\chi_I(t)$ being the characteristic function of $I = [\frac{l}{\nu_n}, \frac{l+1}{\nu_n}]$.

Lemma 4.2 *Let for all $x > 0$ and sufficiently large n,*

(1) $\lambda_{\mu_n,\nu_n}(t,y) = \int_0^t W(u,y)\,du \leq \frac{\xi^2_{\mu_n,\nu_n}(x)}{(y-t)^2}$, $0 \leq t < y$,

(2) $1-\lambda_{\mu_n,\nu_n}(t,y) = \int_t^\infty W(u,y)\,du \leq \frac{\xi^2_{\mu_n,\nu_n}(y)}{(t-y)^2}$, $y \leq t < \infty$.

Proof Using Lemma 2.1 and the definition of the kernel, we get

$$\begin{aligned}
\lambda_{\mu_n,\nu_n}(t,y) &= \int_0^t W(u,y)\,du \\
&\leq \int_0^t \left(\frac{y-u}{y-t}\right)^2 W(u,y)\,du \\
&\leq \frac{1}{(y-t)^2}\int_0^t (u-y)^2 W(u,y)\,du.
\end{aligned}$$

Hence, we have

$$\begin{aligned}\lambda_{\mu_n,\nu_n}(t,y) &\le \frac{1}{(y-t)^2}\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\big((u-y)^2;y,b\big)\\ &\le \frac{1}{(y-t)^2}\xi_{\mu_n,\nu_n}^2(y).\end{aligned}$$

In the same fashion, we can prove the other inequality, therefore, we omit the details. □

Let $\bigvee_a^b f$ be the total variation of f on $[a,b]$, i.e.,

$$\bigvee_a^b f = V\big(f;[a,b]\big) = \sup_{P\in\mathbb{P}}\left(\sum_{i=1}^n \big|f(y_i)-f(y_{i-1})\big|\right), \tag{4.10}$$

where $\mathbb{P}$ is the set of all partitions $P=\{a=y_0,y_1,\dots,y_n=b\}$ of $[a,b]$, whick also has the property

$$\bigvee_a^b f = \bigvee_a^c f + \bigvee_c^b f.$$

Let

$$f_y'(t) = \begin{cases} f'(t)-f'(y-), & 0\le t<y,\\ 0, & t=y,\\ f'(t)-f'(y+), & y<t<\infty.\end{cases} \tag{4.11}$$

Theorem 4.3 *Let $f\in \mathrm{DBV}[0,\infty)$, $y>0$, and n be sufficiently large, then we get*

$$\begin{aligned}&\big|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\big|\\ &\quad\le \left|\frac{1}{2}\big(f'(y+)+f'(y-)\big)\right|\big|\psi_{\mu_n,\nu_n}(y)\big| + \frac{\xi_{\mu_n,\nu_n}^2(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{y-y/k}^{y+y/k} f_y'\right) + \frac{y}{\sqrt{n}}\left(\bigvee_{y-y/k}^{y+y/k} f_y'\right)\\ &\qquad + \frac{\xi_{\mu_n,\nu_n}^2(y)}{y^2}\big|f(2y)-f(y)-yf'(y+)\big| + \left(\frac{M_f+|f(y)|}{y^2}+4M_f\right)\xi_{\mu_n,\nu_n}^2(y)\\ &\qquad + \big|f'(y+)\big|\big|\psi_{\mu_n,\nu_n}(y)\big| + \left|\frac{1}{2}\big(f'(y+)-f'(y-)\big)\right|\xi_{\mu_n,\nu_n}(y).\end{aligned}$$

Proof By (4.11), we obtain

$$\begin{aligned}f'(t) &= \frac{1}{2}\big(f'(y+)+f'(y-)\big)+f_y'(t)+\frac{1}{2}\big(f'(y+)-f'(y-)\big)\operatorname{sgn}(t-y)\\ &\quad + \delta_y(t)\left(f'(t)-\frac{1}{2}\big(f'(y+)+f'(y-)\big)\right),\end{aligned} \tag{4.12}$$

where

$$\delta_y(t) = \begin{cases} 1, & t=y,\\ 0, & t\ne y.\end{cases}$$

Now using Lemma 2.1, equations (4.9) and (4.12), we get

$$\begin{aligned}
&\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\\
&\quad=\int_0^\infty \bigl(f(t)-f(y)\bigr)W(t,y)\,dt\\
&\quad=\int_0^\infty \left(\int_y^t f'(u)\,du\right)W(t,y)\,dt\\
&\quad=\int_0^\infty \left[\int_y^t \left\{\frac{1}{2}\bigl(f'(y+)+f'(y-)\bigr)+f_y'(u)+\frac{1}{2}\bigl(f'(y+)-f'(y-)\bigr)\operatorname{sgn}(u-y)\right.\right.\\
&\qquad\left.\left.+\delta_y(u)\left(f'(u)-\frac{1}{2}\bigl(f'(y+)+f'(y-)\bigr)\right)\right\}du\right]W(t,y)\,dt.
\end{aligned}$$

Since $\int_y^t \delta_y(u)\,du=0$, we have

$$\begin{aligned}
&\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\\
&\quad=\frac{1}{2}\bigl(f'(y+)+f'(y-)\bigr)\int_0^\infty (t-y)W(t,y)\,dt+\int_0^\infty \left(\int_y^t f_y'(u)\,du\right)W(t,y)\,dt\\
&\qquad+\frac{1}{2}\bigl(f'(y+)-f'(y-)\bigr)\int_0^\infty |t-y|W(t,y)\,dt.
\end{aligned}\tag{4.13}$$

Now, we break the second term on the right-hand side of the above equation as follows:

$$\begin{aligned}
&\int_0^\infty \left(\int_y^t f_y'(u)\,du\right)W(t,y)\,dt\\
&\quad=-\int_0^y \left(\int_t^y f_y'(u)\,du\right)W(t,y)\,dt+\int_y^\infty \left(\int_y^t f_y'(u)\,du\right)W(t,y)\,dt\\
&\quad=-I_1+I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1&=\int_0^y \left(\int_t^y f_y'(u)\,du\right)W(t,y)\,dt,\\
I_2&=\int_y^\infty \left(\int_y^t f_y'(u)\,du\right)W(t,y)\,dt.
\end{aligned}$$

Taking the absolute value on both sides of (4.13), we have

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right|\\
&\quad\le\left|\frac{1}{2}\bigl(f'(y+)+f'(y-)\bigr)\right|\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t-y;y)\right|+|I_1|+|I_2|\\
&\qquad+\left|\frac{1}{2}\bigl(f'(y+)-f'(y-)\bigr)\right|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\bigl(|t-y|;y\bigr).
\end{aligned}$$

After applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(f;y)-f(y)\right| \\
&\quad\le\left|\frac{1}{2}\left(f'(y+)+f'(y-)\right)\right|\left|\psi_{\mu_n,\nu_n}(y)\right|+|I_1|+|I_2| \\
&\qquad+\left|\frac{1}{2}\left(f'(y+)-f'(y-)\right)\right|\sqrt{\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\left((t-y)^2;y\right)} \\
&\quad=\left|\frac{1}{2}\left(f'(y+)+f'(y-)\right)\right|\left|\psi_{\mu_n,\nu_n}(y)\right|+|I_1|+|I_2|+\left|\frac{1}{2}\left(f'(y+)-f'(y-)\right)\right|\xi_{\mu_n,\nu_n}(y). \qquad (4.14)
\end{aligned}$$

Now applying Lemma 4.2 and integration by parts, I_1 can be written as

$$\begin{aligned}
I_1&=\int_0^y\left(\int_t^y f_y'(u)\,du\right)W(t,y)\,dt \\
&=\int_0^y\left(\int_t^y f_y'(u)\,du\right)\frac{\partial}{\partial t}\lambda_{\mu_n,\nu_n}(t,y)\,dt \\
&=\int_0^y f_y'(t)\lambda_{\mu_n,\nu_n}(t,y)\,dt.
\end{aligned}$$

On taking the absolute value of I_1, we have

$$\begin{aligned}
|I_1|&=\int_0^y\left|f_y'(t)\right|\lambda_{\mu_n,\nu_n}(t,y)\,dt \\
&\le\int_0^{y-y/\sqrt{n}}\left|f_y'(t)\right|\lambda_{\mu_n,\nu_n}(t,y)\,dt+\int_{y-y/\sqrt{n}}^y\left|f_y'(t)\right|\lambda_{\mu_n,\nu_n}(t,y)\,dt \\
&=K_1+K_2,\quad \text{say.}
\end{aligned}$$

Since $f_y'(y)=0$, by (4.11), we have

$$K_1=\int_0^{y-y/\sqrt{n}}\left|f_y'(t)-f_y'(y)\right|\lambda_{\mu_n,\nu_n}(t,y)\,dt.$$

Now, using Lemma 4.2,

$$K_1\le\xi_{\mu_n,\nu_n}^2(y)\int_0^{y-y/\sqrt{n}}\left|f_y'(t)-f_y'(y)\right|\frac{dt}{(y-t)^2}.$$

By the definition of total variation (4.10) and taking $t=y-y/u$, we obtain

$$\begin{aligned}
K_1&\le\xi_{\mu_n,\nu_n}^2(y)\int_0^{y-y/\sqrt{n}}\left(\bigvee_t^y f_y'\right)\frac{dt}{(y-t)^2} \\
&=\xi_{\mu_n,\nu_n}^2(y)\int_1^{\sqrt{n}}\left(\bigvee_{y-y/u}^y f_y'\right)\frac{du}{y}.
\end{aligned}$$

Now, after breaking the integral into a sum, we have

$$\begin{aligned}
K_1 &\le \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\left(\bigvee_{y-y/u}^{y} f'_y\right)du \\
&\le \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{y-y/k}^{y} f'_y\right)\left(\int_k^{k+1} du\right) \\
&= \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{y-y/k}^{y} f'_y\right).
\end{aligned}$$

Since by Lemma 4.2, $\lambda_{\mu_n,\nu_n}(t,y) \le 1$ and using (4.11), we get

$$\begin{aligned}
K_2 &= \int_{y-y/n}^{y}\left(\bigvee_{t}^{y} f'_y\right)dt \\
&\le \left(\bigvee_{y-y/n}^{y} f'_y\right)\int_{y-y/n}^{y} dt \\
&= \frac{y}{\sqrt{n}}\left(\bigvee_{y-y/n}^{y} f'_y\right).
\end{aligned}$$

Thus, we get

$$|I_1| \le \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{y-y/k}^{y} f'_y\right) + \frac{y}{\sqrt{n}}\left(\bigvee_{y-y/n}^{y} f'_y\right). \tag{4.15}$$

Using Lemma 4.2, we can write

$$\begin{aligned}
|I_2| &= \left|\int_y^\infty\left(\int_y^t f'_y(u)\,du\right)W(t,y)\,dt\right| \\
&\le \left|\int_y^{2y}\left(\int_y^t f'_y(u)\,du\right)\frac{\partial}{\partial t}\bigl(1-\lambda_{\mu_n,\nu_n}(t,y)\bigr)\,dt\right| + \left|\int_{2y}^\infty\left(\int_y^t f'_y(u)\,du\right)W(t,y)\,dt\right|.
\end{aligned}$$

Now, applying integration by parts and (4.11), we get

$$\begin{aligned}
|I_2| &= \left|\int_y^{2y} f'_y(u)\,du\bigl(1-\lambda_{\mu_n,\nu_n}(2y,y)\bigr) - \int_y^{2y} f'_y(t)\bigl(1-\lambda_{\mu_n,\nu_n}(t,y)\bigr)\,dt\right| \\
&\quad + \left|\int_{2y}^\infty\left(\int_y^t \bigl(f'(u)-f'(y+)\bigr)\,du\right)W(t,y)\,dt\right| \\
&\le \left|\int_y^{2y} f'_y(u)\,du\right|.\frac{\xi^2_{\mu_n,\nu_n}(y)}{y^2} + \int_y^{2y}\bigl|f'_y(t)\bigr|(1-\lambda_{\mu_n,\nu_n})(t,y)\,dt \\
&\quad + \left|\int_{2y}^\infty \bigl(f(t)-f(y)\bigr)W(t,y)\,dt\right| + \bigl|f'(y+)\bigr|\left|\int_{2y}^\infty (t-y)W(t,y)\,dt\right| \\
&= P_1 + P_2 + P_3 + P_4, \quad \text{say.}
\end{aligned}$$

Now, by (4.11), we get

$$\begin{aligned} P_1 &= \frac{\xi^2_{\mu_n,\nu_n}(y)}{y^2}\left|\int_y^{2y} f_y'(u)\,du\right| \\ &= \frac{\xi^2_{\mu_n,\nu_n}(y)}{y^2}\left|\int_y^{2y} \bigl(f'(u)-f'(y+)\bigr)\,du\right| \\ &\le \frac{\xi^2_{\mu_n,\nu_n}(y)}{y^2}\bigl|f(2y)-f(y)-yf'(y+)\bigr| \end{aligned}$$

and

$$\begin{aligned} P_2 &= \int_y^{2y} \bigl|f_y'(t)\bigr|.\bigl(1-\lambda_{\mu_n,\nu_n}(t,y)\bigr)\,dt \\ &= \int_y^{y+y/\sqrt{n}} \bigl|f_y'(t)\bigr|.\bigl(1-\lambda_{\mu_n,\nu_n}(t,y)\bigr)\,dt \\ &\quad + \int_{y+y/\sqrt{n}}^{2y} \bigl|f_y'(t)\bigr|.\bigl(1-\lambda_{\mu_n,\nu_n}(t,y)\bigr)\,dt \\ &= J_1 + J_2, \quad \text{say.} \end{aligned}$$

Using Lemma 4.2, $1-\lambda_{\mu_n,\nu_n}(t,y)\le 1$ and (4.11), we get

$$\begin{aligned} J_1 &= \int_y^{y+y/\sqrt{n}} \bigl|f_y'(t)\bigr|.\bigl(1-\lambda_{\mu_n,\nu_n}(t,y)\bigr)\,dt \\ &\le \int_y^{y+y/\sqrt{n}} \bigl|f_y'(t)-f_y'(y)\bigr|\,dt \\ &\le \int_y^{y+y/\sqrt{n}} \left(\bigvee_y^t f_y'\right)dt \\ &\le \left(\bigvee_y^{y+y/\sqrt{n}} f_y'\right)\int_y^{y+y/\sqrt{n}} dt \\ &\le \frac{y}{\sqrt{n}}\left(\bigvee_y^{y+y/\sqrt{n}} f_y'\right). \end{aligned}$$

Now, again with the help of Lemma 4.2 and (4.11), we obtain

$$\begin{aligned} J_2 &= \int_{y+y/\sqrt{n}}^{2y} \bigl|f_y'(t)\bigr|\bigl(1-\lambda_{\mu_n,\nu_n}(t,y)\bigr)\,dt \\ &\le \xi^2_{\mu_n,\nu_n}(y)\int_{y+y/\sqrt{n}}^{2y} \bigl|f_y'(t)-f_y'(y)\bigr|\frac{dt}{(t-y)^2}. \end{aligned}$$

By using (4.10) and $t=y+y/u$, we get

$$J_2 \le \xi^2_{\mu_n,\nu_n}(y)\int_{y+y/\sqrt{n}}^{2y} \left(\bigvee_y^t f_y'\right)\frac{dt}{(t-y)^2}$$

$$
\begin{aligned}
&= \xi^2_{\mu_n,\nu_n}(y)\int_1^{\sqrt{n}}\left(\bigvee_y^{y+y/u} f'_y\right)\frac{du}{y} \\
&\le \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\left(\bigvee_y^{y+y/u} f'_y\right)du \\
&\le \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_y^{y+y/k} f'_y\right)\left(\int_k^{k+1} du\right) \\
&= \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_y^{y+y/k} f'_y\right).
\end{aligned}
$$

Hence, we derive

$$
P_2 \le \frac{y}{\sqrt{n}}\left(\bigvee_y^{y+y/\sqrt{n}} f'_y\right) + \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_y^{y+y/k} f'_y\right).
$$

Now, we estimate P_3. As $t \ge 2y$, then using $2(t-y) \ge t$ and $t-y \ge y$, we get

$$
\begin{aligned}
P_3 &= \left|\int_{2y}^{\infty}\bigl(f(t)-f(y)\bigr)W(t,y)\,dt\right| \\
&\le \int_{2y}^{\infty}\bigl|f(t)\bigr|W(t,y)\,dt + \int_{2y}^{\infty}\bigl|f(y)\bigr|W(t,y)\,dt \\
&\le M_f\int_{2y}^{\infty}\bigl(1+t^2\bigr)W(t,y)\,dt + \bigl|f(y)\bigr|\int_{2y}^{\infty}W(t,y)\,dt \\
&= \bigl(M_f+\bigl|f(y)\bigr|\bigr)\int_{2y}^{\infty}W(t,y)\,dt + M_f\int_{2y}^{\infty}t^2W(t,y)\,dt \\
&\le \bigl(M_f+\bigl|f(y)\bigr|\bigr)\int_{2y}^{\infty}\frac{(t-y)^2}{y^2}W(t,y)\,dt + M_f\int_{2y}^{\infty}4(t-y)^2W(t,y)\,dt \\
&\le \left(\frac{M_f+|f(y)|}{y^2}+4M_f\right)\int_{2y}^{\infty}(t-y)^2W(t,y)\,dt \\
&\le \left(\frac{M_f+|f(y)|}{y^2}+4M_f\right)\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}\bigl((t-y)^2;y\bigr) \\
&= \left(\frac{M_f+|f(y)|}{y^2}+4M_f\right)\xi^2_{\mu_n,\nu_n}(y).
\end{aligned}
$$

We can compute P_4 as follows:

$$
\begin{aligned}
P_4 &= \bigl|f'(y+)\bigr|\left|\int_{2y}^{\infty}(t-y)W(t,y)\,dt\right| \\
&\le \bigl|f'(y+)\bigr|\left|\int_{0}^{\infty}(t-y)W(t,y)\,dt\right| \\
&= \bigl|f'(y+)\bigr|\bigl|\mathcal{Q}_{n,b}^{(\mu_n,\nu_n)}(t-y;y)\bigr| \\
&= \bigl|f'(y+)\bigr|\bigl|\psi_{\mu_n,\nu_n}(y)\bigr|.
\end{aligned}
$$

Hence, we get

$$\begin{aligned}|I_2| \leq{}& \frac{\xi^2_{\mu_n,\nu_n}(y)}{y^2}\left|f(2y)-f(y)-yf'(y+)\right| \\ &+ \frac{y}{\sqrt{n}}\left(\bigvee_{y}^{y+y/\sqrt{n}} f'_y\right) + \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{y}^{y+y/k} f'_y\right) \\ &+ \left(\frac{M_f+|f(y)|}{y^2}+4M_f\right)\xi^2_{\mu_n,\nu_n}(y) + \left|f'(y+)\right|\left|\psi_{\mu_n,\nu_n}(y)\right|. \end{aligned} \tag{4.16}$$

Now, from (4.14)–(4.16), we obtain

$$\begin{aligned}\left|\mathcal{Q}^{(\mu_n,\nu_n)}_{n,b}(f;y)-f(y)\right| \leq{}& \left|\frac{1}{2}\left(f'(y+)+f'(y-)\right)\right|\left|\psi_{\mu_n,\nu_n}(y)\right| + |I_1| + |I_2| \\ &+ \left|\frac{1}{2}\left(f'(y+)-f'(y-)\right)\right|\xi_{\mu_n,\nu_n}(y) \\ \leq{}& \left|\frac{1}{2}\left(f'(y+)+f'(y-)\right)\right|\left|\psi_{\mu_n,\nu_n}(y)\right| \\ &+ \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{y-y/k}^{y} f'_y\right) + \frac{y}{\sqrt{n}}\left(\bigvee_{y-y/\sqrt{n}}^{y} f'_y\right) \\ &+ \frac{\xi^2_{\mu_n,\nu_n}(y)}{y^2}\left|f(2y)-f(y)-yf'(y+)\right| \\ &+ \frac{y}{\sqrt{n}}\left(\bigvee_{y}^{y+y/\sqrt{n}} f'_y\right) + \frac{\xi^2_{\mu_n,\nu_n}(y)}{y}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{y}^{y+y/k} f'_y\right) \\ &+ \left(\frac{M_f+|f(y)|}{y^2}+4M_f\right)\xi^2_{\mu_n,\nu_n}(y) + \left|f'(y+)\right|\left|\psi_{\mu_n,\nu_n}(y)\right| \\ &+ \left|\frac{1}{2}\left(f'(y+)-f'(y-)\right)\right|\xi_{\mu_n,\nu_n}(y), \end{aligned}$$

which gives the desired result. □

5 Graphical examples

Example 5.1 Let us take $f(x) = 5x^4 - 11x^3 + 2x^2$. The convergence of the sequence of operators defined by Eq. (1.3) when $\mu_n = \nu_n = n$ towards the function $f(x)$ (cyan) is shown for $n = 10, 50, 100$, respectively, in Figs. 1–3 taking $b = 2$ (blue), $b = 6$ (black), and $b = 15$ (red). Figures 4–6 illustrate the convergence of the sequence of operators defined by Eq. (1.3) taking $\mu_n = n + \sqrt{n+1}$, $\nu_n = n + 12$ towards the function $f(x)$ (cyan) for $n = 10, 50, 100$, keeping the value of b the same.

Also, a direct comparison between the convergence of the old operator applied to f (when $\mu_n = \nu_n = n$ discussed in [9]) (blue) and the new operator (red) defined in Eq. (1.3) towards $f(x)$ (cyan) is shown in Figs. 7–9, respectively, for $n = 10, 50, 100$, and $b = 10$. It is clear that the new operator exhibits faster convergence towards the limit than the old operator. Also, the new operator is giving flexibility in choosing parameters in the form of the sequences μ_n and ν_n.

Figure 1 Convergence of the operators when $\mu_n = \nu_n = n$ and $n = 10$

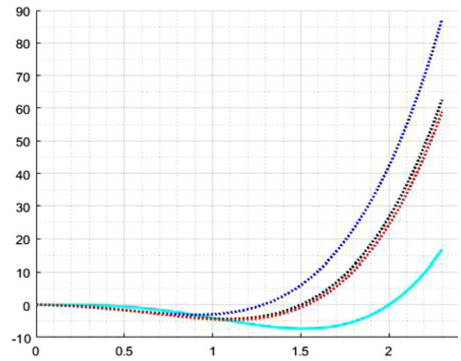


Figure 2 Convergence of the operators when $\mu_n = \nu_n = n$ and $n = 50$

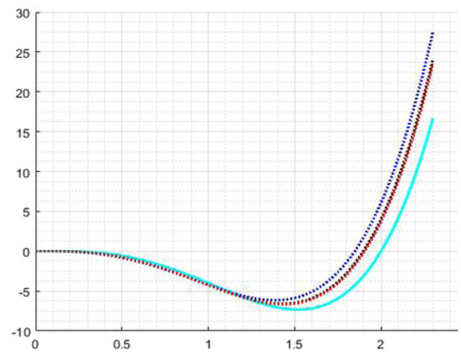


Figure 3 Convergence of the operators when $\mu_n = \nu_n = n$ and $n = 100$

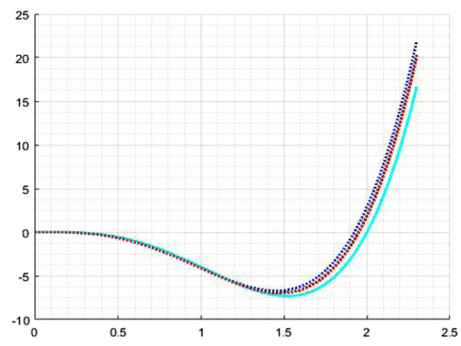


Figure 4 Convergence of the operators when $\mu_n = n + \sqrt{n+1}$, $\nu_n = n + 12$ and $n = 10$

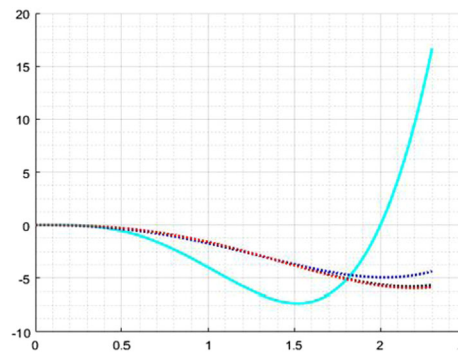


Figure 5 Convergence of the operators when $\mu_n = n + \sqrt{n+1}$, $\nu_n = n + 12$ and $n = 50$

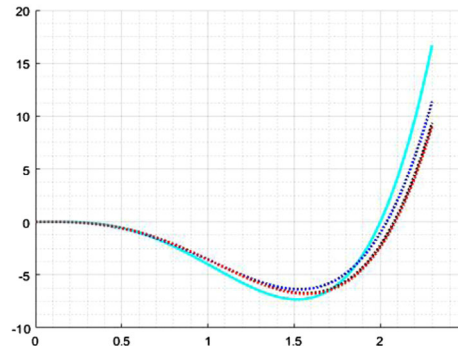


Figure 6 Convergence of the operators when $\mu_n = n + \sqrt{n+1}$, $\nu_n = n + 12$ and $n = 100$

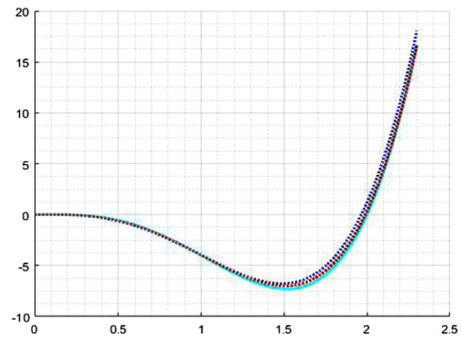


Figure 7 Comparison between the operators when $\mu_n = \nu_n = n$, $b = 10$ and $n = 10$

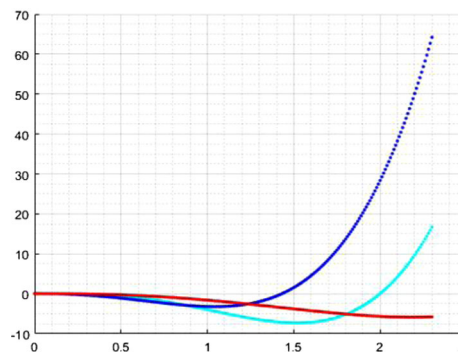
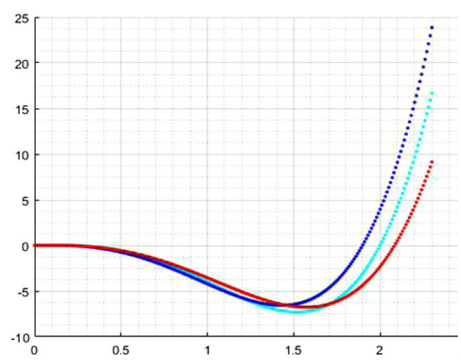


Figure 8 Comparison between the operators when $\mu_n = \nu_n = n$, $b = 10$ and $n = 50$



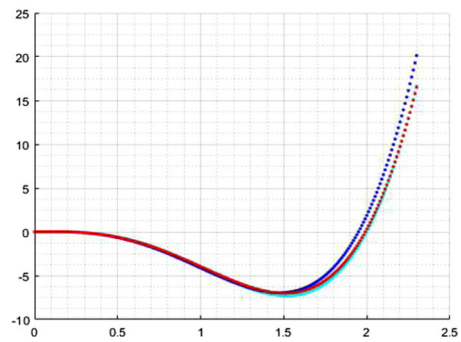


Figure 9 Comparison between the operators when $\mu_n = \nu_n = n$, $b = 10$ and $n = 100$

6 Conclusions

We have modified the sequence of operators discussed in [9] and developed many approximation properties such as direct theorems, rate of convergence in weighted spaces, and approximation for functions of bounded variation. Moreover, we have also shown the convergence of old and modified new operators graphically.

Acknowledgements
The author (K.J. Ansari) extends his appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under grant number G.R.P-93-41.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 12 February 2020 Accepted: 16 April 2020 Published online: 01 May 2020

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