Su *Advances in Difference Equations* (2020) 2020:194
https://doi.org/10.1186/s13662-020-02656-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Degenerate Hopf bifurcation in a Leslie–Gower predator–prey model with predator harvest

Juan Su[1*]

*Correspondence:
sujuanmath@163.com
[1]Department of Mathematics, Chengdu Normal University, Chengdu, China

Abstract

In this paper, we investigate the degenerate Hopf bifurcation of a Leslie–Gower predator–prey system with predator harvest. The known work discussed the Hopf bifurcation of this system when the first Lyapunov number does not vanish and gave an example of a stable weak focus with order 2. However, the thorough discussion of center-type equilibrium for all possible parameters has not been completed. In this paper, by computing the first two focal values, we decompose the variety with resultant and prove that the center-type equilibrium is a weak focus with order at most 2 for all the possible parameter values. Moreover, numerical simulations are employed to show the appearance of two limit cycles from degenerate Hopf bifurcation. Our results finish the study of Hopf bifurcation in this system.

MSC: 34C25; 92D25

Keywords: Leslie–Gower predator–prey model; Hopf bifurcation; Focal value; Resultant elimination

1 Introduction

The shortage of natural resources will be a severe problem of the social development in the near future. So how to exploit these resources in a reasonable way is of great importance for the sustainable development. Harvest is commonly practiced in fishery, forestry, and wildlife management. The maximum sustainable yield provides insight into the coexistence of all interacting species. Recently, much attention has been paid to the dynamics of predator–prey models with harvesting, and researchers have been trying to find how harvest affects the population dynamics (see, e.g., [2–4, 6, 16, 17]).

Lotka–Volterra predator–prey model is a pioneer work, which gives much insight into the description of interaction between predators and prey by an ordinary differential system. Since the model neglects some real situation, a number of studies have described the interaction in a more actual way. One of such works is Leslie–Gower predator–prey model [12, 13], which takes the form

$$\begin{cases} \dot{x} = r_1 x(1 - \frac{x}{K}) - axy, \\ \dot{y} = r_2 y(1 - \frac{y}{bx}), \end{cases} \tag{1.1}$$

where $x(t)$, $y(t)$ denote the densities of prey and predators at time t, respectively. It assumes that the prey grows with intrinsic growth rate r_1 and carrying capacity K in the absence of predation, and ax describes the feeding rate of prey consumption by predators. The predator grows with intrinsic growth rate r_2 and carrying capacity bx proportional to the population of prey, where b is a measure of the food quality of prey for conversion into predator births. Moreover, r_1, r_2, a, b, and K are all positive constants. Hsu and Huang [9] investigated the global stability of system (1.1).

To describe the interaction of krill (prey) and baleen whale (predator) in Southern Ocean, May *et al.* [14] considered the harvest on system (1.1), i.e.,

$$\begin{cases} \dot{x} = r_1 x(1 - \frac{x}{K}) - axy - H_1(x), \\ \dot{y} = r_2 y(1 - \frac{y}{bx}) - H_2(y), \end{cases} \tag{1.2}$$

where H_1 and H_2 represent the harvests on prey and predators, respectively. When both prey and predators are harvested with constant-effort, i.e., $H_i(s) = ks$ $(i = 1, 2)$, the maximum sustainable yield was analyzed in [2, 14]. As shown in [10], system (1.2) in this case can be written as the non-harvest form, i.e., system (1.1), so the dynamics of the two systems are similar. When prey are harvested at constant-yield and predators are harvested with constant-effort, i.e., $H_1(s) = h_1$ and $H_2(s) = ks$, respectively, Beddington and Cooke [1] studied the effects of predator harvest on the maximum sustainable yield for prey.

Moreover, the dynamics of system (1.2) are also investigated to see how harvest affects the interaction of prey and predators. When only prey are harvested at constant-yield, i.e., $H_1(s) = h_1$ and $H_2(s) = 0$, system (1.2) exhibits rich bifurcation phenomena such as saddle-node bifurcation, degenerate Hopf bifurcations, and Bogdanov–Takens bifurcation [8, 15, 19]. When only predators are harvested at constant-yield, i.e., $H_1(s) = 0$ and $H_2(s) = h_2$, Huang *et al.* [10, 11] investigated the local bifurcations of system (1.2), including saddle-node bifurcation, Hopf bifurcation, and Bogdanov–Takens bifurcation of codimension 2 and 3. Moreover, degenerate Hopf bifurcation was just studied by an example of a stable weak focus with order 2 (see [10, Theorem 3.6]). Thus, the dynamics of the center-type equilibrium is not studied thoroughly for all the parameters.

In this paper, we extend the study of the center-type equilibrium in [10] for all possible parameters and investigate the degenerate Hopf bifurcation of system (1.2) with $H_1(s) = 0$ and $H_2(s) = h_2$, i.e.,

$$\begin{cases} \dot{x} = r_1 x(1 - \frac{x}{K}) - axy, \\ \dot{y} = r_2 y(1 - \frac{y}{bx}) - h_2. \end{cases} \tag{1.3}$$

The rest of the paper is organized as follows. In Sect. 2, we recall the dynamic properties of equilibria in [10] and compute the first two focal values at the center-type equilibrium. In Sect. 3, for all possible parameters, we prove that the center-type equilibrium is a weak focus of order at most 2 by the resultant elimination [7]. Moreover, we give the parameter condition of each order. In Sect. 4, we employ numerical simulation to show the appearance of two limit cycles from degenerate Hopf bifurcation and give a brief conclusion.

2 Preliminary results

With rescaling $x \to \frac{x}{K}$, $y \to \frac{ay}{r_1}$, and $t \to r_1 t$, system (1.3) reads

$$\begin{cases} \dot{x} = x(1-x) - xy, \\ \dot{y} = y(\delta - \frac{\beta y}{x}) - h, \end{cases} \tag{2.1}$$

where $\beta = \frac{r_2}{abK}$, $\delta = \frac{r_2}{r_1}$, and $h = \frac{ah_2}{r_1}$ are all positive. It is clear that system (2.1) is orbitally equivalent to system (1.3). By the biological sense, we discuss system (2.1) in the region

$$\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

Let $E(\overline{x},\overline{y})$ be an equilibrium of system (2.1). Then $\overline{y} = 1 - \overline{x}$, and $\overline{x}$ is a positive zero of the function

$$f(x) := (\beta + \delta)x^2 - (2\beta + \delta - h)x + \beta$$

in the interval $(0,1)$. By setting

$$h_1 := 2\beta + \delta - 2\sqrt{\beta^2 + \beta\delta}, \tag{2.2}$$

Lemma 2.1 in [10] gives the following statement of the equilibria of system (2.1).

(i) If $h > h_1$, system (2.1) has no equilibrium.
(ii) If $h = h_1$, system (2.1) has only one equilibrium $E_0(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})$.
(iii) If $h < h_1$, system (2.1) has two equilibria $E_1(x_1,y_1)$ and $E_2(x_2,y_2)$, where $x_{1,2} = \frac{2\beta+\delta-h\mp\sqrt{(\delta-h)^2-4\beta h}}{2(\beta+\delta)}$ and $y_{1,2} = 1 - x_{1,2}$.

Moreover, Theorems 2.2 and 3.3 in [10] indicate that E_0 is either a saddle-node or a cusp of codimension at most 3, and Bogdanov–Takens bifurcations of codimension 2 and 3 are investigated in [10, Theorem 3.4] and [11, Theorem 2.3], respectively. For E_1 and E_2, the dynamics are given in the following lemma.

Lemma 2.1 ([10, Theorem 2.3]) *Let $\beta, \delta > 0$ and $0 < h < h_1$. Then E_1 is an anti-saddle and E_2 is a saddle. More exactly, E_1 is a stable (resp. an unstable) node or focus if $h < h_2$ (resp. $h > h_2$) and center-type if $h = h_2$, where*

$$h_2 = \frac{1}{4}\{6\beta - 4\beta^2 + 3\delta - 6\beta\delta - 2\delta^2 + (-1 + 2\beta + 2\delta)\sqrt{-8\beta + 4\beta^2 + 4\beta\delta + \delta^2}\}.$$

Huang, Gong, and Ruan [10] studied E_1 by computing the first Lyapunov number and discussed the supercritical and subcritical Hopf bifurcations when the first Lyapunov number does not vanish. Furthermore, they studied the degenerate Hopf bifurcation for $(\alpha,\beta,\delta) = (\frac{-4+\sqrt{51}}{100}, \frac{-3+2\sqrt{51}}{25}, \frac{4(1+\sqrt{51})}{125})$ and found that E_1 is a weak focus of order 2.

In order to study the center-type equilibrium for all parameters, the following preparations are made to compute the first two focal values. The time scaling $t \to xt$ changes system (2.1) into the polynomial system

$$\begin{cases} \dot{x} = x^2(1 - x - y), \\ \dot{y} = y(\delta x - \beta y) - hx, \end{cases} \tag{2.3}$$

which is orbitally equivalent to system (2.1) in $\mathbb{R}^2_+$. The Jacobian matrix of system (2.3) at $E_1(x_1, y_1)$ is given by

$$J|_{E_1} = \begin{bmatrix} -x_1^2 & -x_1^2 \\ (1 - x_1)\delta - h & -2\beta(1 - x_1) + \delta x_1 \end{bmatrix},$$

where $y_1 = 1 - x_1$ is used. Let D and T be the determinant and the trace of $J|_{E_1}$, respectively. Then

$$D = x_1^2\bigl(-2(\beta + \delta)x_1 + 2\beta + \delta - h\bigr), \qquad T = 2\beta x_1 + \delta x_1 - x_1^2 - 2\beta.$$

Since $x_1 < \frac{2\beta+\delta-h}{2(\beta+\delta)}$, it follows that $D > 0$. Moreover, substituting $x_1 = x_1(\beta, \delta, h) := \frac{2\beta+\delta-h-\sqrt{(\delta-h)^2-4\beta h}}{2(\beta+\delta)}$ into $T = 0$, we obtain $h = h_2$. Let

$$\mathcal{T} := \bigl\{(\beta, \delta, h) \in \mathbb{R}^3 : \beta > 0, \delta > 0, 0 < h < h_1, T = 0\bigr\}.$$

For $(\beta, \delta, h) \in \mathcal{T}$, the Jacobian matrix has a pair of pure imaginary eigenvalues $\pm \mathbf{i}w$, where

$$w = x_1\sqrt{-2(\beta + \delta)x_1 + 2\beta + \delta - h} > 0$$

because of $D > 0$.

Let $u = x - x_1$, $v = y - (1 - x_1)$. Then system (2.3) becomes

$$\begin{cases} \dot{u} = -x_1^2 u - x_1^2 v - 2x_1 u^2 - 2x_1 uv - u^3 - u^2 v, \\ \dot{v} = (-\delta x_1 + \delta - h)u + (2\beta x_1 + \delta x_1 - 2\beta)v + \delta uv - \beta v^2. \end{cases} \tag{2.4}$$

With the transformation $u = 2z_1$, $v = -(4\beta x_1 + 2\delta x_1 - 4\beta)x_1^{-2}z_1 + 2wx_1^{-2}z_2$ and time rescaling $t \to wt$, system (2.4) is normalized as

$$\begin{cases} \dot{z_1} = -z_2 - \frac{4}{x_1}z_1 z_2 - \frac{4}{2\beta x_1+\delta x_1-2\beta}z_1^2 z_2, \\ \dot{z_2} = z_1 - \frac{4\beta^2 x_1+2\beta\delta x_1-8\beta x_1^2-\delta x_1^2+4x_1^3-4\beta^2+8\beta x_1}{w^2}z_1^2 \\ \qquad + \frac{2(4\beta^2 x_1+2\beta\delta x_1-4\beta x_1^2-\delta x_1^2-4\beta^2+4\beta x_1)}{wx_1^2}z_1 z_2 - \frac{2\beta}{x_1^2}z_2^2 - \frac{4}{w}z_1^2 z_2. \end{cases} \tag{2.5}$$

In the coordination $z_1 = r\cos\theta$, $z_2 = r\sin\theta$, system (2.5) becomes

$$\frac{dr}{d\theta} = R_2(\theta)r^2 + R_3(\theta)r^3 + R_4(\theta)r^4 + R_5(\theta)r^5 + O\bigl(r^6\bigr), \tag{2.6}$$

where $R_i(\theta)$ ($i = 2, 3, 4, 5$) is polynomials of $(\sin\theta, \cos\theta)$, and their coefficients are decided by those of system (2.5). To compute focal values, we consider the solutions of (2.5) in the formal series $r(\theta, r_0) = \sum_{k=1}^{\infty} r_k(\theta) r_0^k$ together with the initial condition

$$r(0, r_0) = r_0, \tag{2.7}$$

where r_0 is sufficiently small. Substituting this formal series into (2.6) and comparing the coefficients, we get

$$\begin{aligned} &\dot{r}_2 = R_2, \qquad \dot{r}_3 = R_3 + 2R_2 r_2, \dot{r}_4 = R_4 + 3R_3 r_2 + R_2\left(r_2^2 + 2r_3\right), \\ &\dot{r}_5 = R_5 + 4R_4 r_2 + 3R_3\left(r_2^2 + r_3\right) + 2R_2(r_2 r_3 + r_4). \end{aligned} \tag{2.8}$$

Initial condition (2.7) is equivalent to

$$r_1(0) = 1, \qquad r_2(0) = r_3(0) = r_4(0) = r_5(0) = 0. \tag{2.9}$$

By (2.8) and (2.9), the first two focal values at E_1 are given by

$$L_1 := \frac{1}{2\pi} r_3(2\pi) = \frac{-f_1(\beta, \delta, h, x_1)}{2x_1^4 w^{\frac{5}{2}}}, \qquad L_2 := \frac{1}{2\pi} r_5(2\pi) = \frac{f_2(\beta, \delta, h, x_1)}{36 x_1^{10} w^{\frac{7}{2}}}, \tag{2.10}$$

where f_1 and f_2 are given in the Appendix, and $x_1 = x_1(\beta, \delta, h)$.

3 Degenerate Hopf bifurcation at E_1

Having the first two focal values, we can obtain the dynamics of E_1 when it is of center type.

Theorem 3.1 *For $(\beta, \delta, h) \in \mathcal{T}$, $E_1(x_1, y_1)$ is a weak focus with order at most 2. More exactly, it is order 1 if $(\beta, \delta, h) \in \mathcal{T}_1 := \{(\beta, \delta, h) \in \mathcal{T} : \widetilde{f}_1(\beta, \delta, h) \neq 0\}$; otherwise, it is order 2; where $\widetilde{f}_1(\beta, \delta, h) := g_1(\beta, \delta, h)\sqrt{(\delta - h)^2 - 4\beta h} + g_2(\beta, \delta, h)$, and g_1 and g_2 are given in the Appendix.*

To prove Theorem 3.1, we need to show that the first two focal values have no common zeros for $(\beta, \delta, h) \in \mathcal{T}$. For this purpose, we first give the proof of the following lemma.

Lemma 3.1 *Let $V(f_0, f_1, f_2, T) := \{(\beta, \delta, h, x_1) \in \mathbb{R}^4 : f_0 = f_1 = f_2 = T = 0\}$. Then*

$$V(f_0, f_1, f_2, T) \cap S = \emptyset,$$

where $f_0 := (\beta + \delta)x_1^2 - (2\beta + \delta - h)x_1 + \beta$ and $S = \{(\beta, \delta, h, x_1) \in \mathbb{R}^4 : (\beta, \delta, h) \in \mathcal{T}, 0 < x_1 < \frac{2\beta+\delta-h}{2(\beta+\delta)}\}$.

Proof Taking the order $x_1 \prec \delta \prec \beta \prec h$ in elimination, we calculate the following resultants [7] by software Maple:

$$\begin{aligned} &r_{12} = T, \qquad r_{13} = \operatorname{res}(f_0, f_1, h) = x_1 \widetilde{r}_{13}, \qquad r_{14} = \operatorname{res}(f_0, f_2, h) = x_1^3 \widetilde{r}_{14}, \\ &r_{23} = \operatorname{res}(r_{12}, r_{13}, \beta) = -8x_1^6 (x_1 - 1)^4 (\delta - x_1) \widetilde{r}_{23}, \\ &r_{24} = \operatorname{res}(r_{12}, r_{14}, \beta) = 256 x_1^{16} (x_1 - 1)^9 (\delta - x_1) \widetilde{r}_{24}, \\ &r_{34} = \operatorname{res}(r_{23}, r_{24}, \delta) = 0, \end{aligned} \tag{3.1}$$

where $\operatorname{res}(f,g,x)$ denotes the resultant of polynomials f and g with respect to the variable x, and $\widetilde{r}_{13}$ and $\widetilde{r}_{14}$ are given in the Appendix,

$$\begin{aligned}
\widetilde{r}_{23} &= \delta x_1^2 + x_1^3 + \delta^2 + \delta x_1 + 2x_1^2 - 2x_1,\\
\widetilde{r}_{24} &= 21\delta^5 x_1^4 + 39\delta^4 x_1^5 + 15\delta^3 x_1^6 - 5\delta^2 x_1^7 - 4\delta x_1^8 - 2x_1^9 + 21\delta^6 x_1^2 + 15\delta^5 x_1^3 + 87\delta^4 x_1^4\\
&\quad - 45\delta^3 x_1^5 - 200\delta^2 x_1^6 + 30\delta x_1^7 + 92x_1^8 - 42\delta^6 x_1 - 45\delta^5 x_1^2 - 342\delta^4 x_1^3 + 136\delta^3 x_1^4\\
&\quad + 14\delta^2 x_1^5 - 167\delta x_1^6 + 190x_1^7 + 21\delta^6 - 32\delta^5 x_1 + 313\delta^4 x_1^2 - 410\delta^3 x_1^3 + 781\delta^2 x_1^4\\
&\quad - 70\delta x_1^5 - 347x_1^6 + 41\delta^5 - 122\delta^4 x_3 + 443\delta^3 x_1^2 - 971\delta^2 x_1^3 + 756\delta x_1^4 - 211x_1^5\\
&\quad + 33\delta^4 - 139\delta^3 x_1 + 455\delta^2 x_1^2 - 725\delta x_1^3 + 376x_1^4 - 90\delta^2 x_1 + 180\delta x_1^2 - 90x_1^3.
\end{aligned}$$

Let $\operatorname{lcoeff}(\xi,x)$ denote the leading coefficient of ξ with respect to x. Then from $(\beta,\delta,h,x_1)\in S$ it follows that

$$\begin{aligned}
&\operatorname{lcoeff}(f_0,h) = x_1 > 0, \qquad \operatorname{lcoeff}(r_{12},\beta) = 2(x_1-1) < 0,\\
&\operatorname{lcoeff}(r_{23},\delta) = -8x_1^5(x_1-1)^4 > 0.
\end{aligned} \tag{3.2}$$

By Theorem 1 in [5], we have the decomposition

$$\begin{aligned}
V(f_0,T,f_1,f_2) &= V\big(f_0,T,f_1,f_2,\operatorname{lcoeff}(f_0,h)\big)\\
&\quad \cup V\left(\frac{f_0,T,f_1,f_2,r_{12},r_{13},r_{14},\operatorname{lcoeff}(r_{12},\beta)}{\operatorname{lcoeff}(f_0,h)}\right)\\
&\quad \cup V\left(\frac{f_0,T,f_1,f_2,r_{12},r_{13},r_{14},r_{23},r_{24},\operatorname{lcoeff}(r_{23},\delta)}{\operatorname{lcoeff}(f_0,h),\operatorname{lcoeff}(r_{12},\beta)}\right)\\
&\quad \cup V\left(\frac{f_0,T,f_1,f_2,r_{12},r_{13},r_{14},r_{23},r_{24},r_{34}}{\operatorname{lcoeff}(f_0,h),\operatorname{lcoeff}(r_{12},\beta),\operatorname{lcoeff}(r_{23},\delta)}\right),
\end{aligned} \tag{3.3}$$

where $V(\frac{\xi_1,\xi_2,\ldots,\xi_n}{\eta_1,\eta_2,\ldots,\eta_m}) = V(\xi_1,\ldots,\xi_n)\backslash(\bigcup_{k=1}^m V(\eta_k))$ as used in [5].

By (3.1)–(3.3), we can deduce that

$$\begin{aligned}
V(f_0,T,f_1,f_2)\cap S &= V(f_0,T,f_1,f_2,r_{12},r_{13},r_{14},r_{23},r_{24},r_{34})\cap S,\\
&= V(f_0,T,f_1,f_2,r_{13},r_{14},r_{23},r_{24})\cap S.
\end{aligned} \tag{3.4}$$

Moreover, by computing the resultant

$$\begin{aligned}
\operatorname{res}(\widetilde{r}_{23},\widetilde{r}_{24},\delta) &= 1152x_1^4(x_1-1)^4(x_1-2-\sqrt{3})(x_1+\sqrt{2}+1)\left(x_1+\frac{\sqrt{22}+4}{3}\right)\\
&\quad \times\left(x_1-\frac{\sqrt{22}-4}{3}\right)(x_1-\sqrt{2}+1)(x_1-2+\sqrt{3}),
\end{aligned} \tag{3.5}$$

we see that the first five factors of the right-hand side are all positive because of $0 < x_1 < 1$. By (3.1) and (3.5), (3.4) can be written as

$$V(f_0,T,f_1,f_2)\cap S = V_1\cup V_2\cup V_3\cup V_4,$$

where

$$V_1 = V(f_0, T, f_1, f_2, \widetilde{r}_{13}, \widetilde{r}_{14}, x_1 = \delta) \cap S,$$

$$V_2 = V\left(f_0, T, f_1, f_2, \widetilde{r}_{13}, \widetilde{r}_{14}, \widetilde{r}_{23}, \widetilde{r}_{24}, x_1 = \frac{\sqrt{22}-4}{3}\right) \cap S,$$

$$V_3 = V(f_0, T, f_1, f_2, r_{13}, r_{14}, \widetilde{r}_{23}, \widetilde{r}_{24}, x_1 = \sqrt{2}-1) \cap S,$$

$$V_4 = V(f_0, T, f_1, f_2, r_{13}, r_{14}, \widetilde{r}_{23}, \widetilde{r}_{24}, x_1 = 2-\sqrt{3}) \cap S.$$

It suffices to prove that $V_i = \emptyset$, $i = 1, 2, 3, 4$. In fact, substituting $\delta = x_1$ into T, we see that

$$T|_{\delta=x_1} = 2\beta(x_1 - 1) < 0,$$

showing that $V_1 \subseteq V(T, \delta = x_1) \cap S = \emptyset$. In order to obtain that $V_2 = \emptyset$, we substitute $x_1 = \frac{\sqrt{22}-4}{3}$ into $\widetilde{r}_{23}$ and $\widetilde{r}_{24}$, and obtain

$$\widetilde{r}_{23}|_{x_1=\frac{\sqrt{22}-4}{3}} = -\frac{1}{27}(9\delta + 2 + \sqrt{22})(-3\delta - 8 + 2\sqrt{22}),$$

$$\begin{aligned}\widetilde{r}_{24}|_{x_1=\frac{\sqrt{22}-4}{3}} = {} & \frac{-71+14\sqrt{22}}{19{,}683}(9\delta + 2 + \sqrt{22})\big(675\sqrt{22}\delta^4 - 5103\delta^5 + 16{,}647\sqrt{22}\delta^3 \\ & - 12{,}285\delta^4 + 76{,}746\sqrt{22}\delta^2 - 78{,}519\delta^3 - 28{,}254\sqrt{22}\delta - 354{,}246\delta^2 \\ & + 130{,}842\delta + 705{,}280 - 150{,}340\sqrt{22}\big).\end{aligned}$$

Simple computation shows that

$$\{\delta \in \mathbb{R}_+ : \widetilde{r}_{23}|_{x_1=\frac{\sqrt{22}-4}{3}} = \widetilde{r}_{24}|_{x_1=\frac{\sqrt{22}-4}{3}} = 0\} = \emptyset.$$

Thus it follows that $V_2 \subseteq V(\widetilde{r}_{23}, \widetilde{r}_{24}, x_1 = \frac{\sqrt{22}-4}{3}) \cap S = \emptyset$. Similarly, we can prove that $V_3 = \emptyset$. To obtain $V_4 = \emptyset$, we substitute $x_1 = 2 - \sqrt{3}$ into the equations $\widetilde{r}_{23} = \widetilde{r}_{23} = T = f_0 = 0$, and get $\delta = 2\sqrt{3} - 3$, $\beta = \frac{7}{2} - 2\sqrt{3}$, and $h = 2 - \sqrt{3}$. For $(\beta, \delta, h) = (\frac{7}{2} - 2\sqrt{3}, 2\sqrt{3} - 3, 2 - \sqrt{3})$, it immediately follows from (2.2) that $h_1 = 2 - \sqrt{3}$. So $h = h_1$, and $(\frac{7}{2} - 2\sqrt{3}, 2\sqrt{3} - 3, 2 - \sqrt{3}, 2 - \sqrt{3}) \notin S$. Therefore, $V_4 \subseteq V(\widetilde{r}_{23}, \widetilde{r}_{23}, T, f_0, x_1 = 2 - \sqrt{3}) \cap S = \emptyset$.

Since $V_1 = V_2 = V_3 = V_4 = \emptyset$, we see that $V(f_0, f_1, f_2, T) \cap S = \emptyset$. □

Now we can give the proof of Theorem 3.1.

Proof of Theorem 3.1 Note that the zeros of L_i are decided by the zeros of f_i $(i = 1, 2)$. By Lemma 3.1, we can see that f_1 and f_2 have no common zeros when $T = 0$. Therefore, L_1 and L_2 have no common zeros for $(\beta, \delta, h) \in \mathcal{T}$, showing that E_1 is a weak focus with order at most 2. More exactly, it is order 1 if and only if $f_1 \neq 0$. So substituting $x_1 = \frac{2\beta+\delta-h-\sqrt{(\delta-h)^2-4\beta h}}{2(\beta+\delta)}$ into $f_1 = 0$, we get $\widetilde{f}_1(\beta, \delta, h) = g_1(\beta, \delta, h)\sqrt{(\delta - h)^2 - 4\beta h} + g_2(\beta, \delta, h) = 0$. Therefore, E_1 is a weak focus of order 1 if and only if $(\beta, \delta, h) \in \mathcal{T}_1 = \{(\beta, \delta, h) \in \mathcal{T} : \widetilde{f}_1(\beta, \delta, h) \neq 0\}$.

To prove that E_1 is a weak focus of order 2, we need to show that $\{(\beta, \delta, h) \in \mathcal{T}, \widetilde{f}_1(\beta, \delta, h) = 0\} \neq \emptyset$. In fact, it can be checked that if

$$(\beta, \delta, h) = \left(\frac{-4+\sqrt{51}}{125}, \frac{2\sqrt{51}-3}{25}, \frac{4(\sqrt{51}+1)}{125}\right),$$

then $x_1 = \frac{1}{4}$, $T = 0$, and $\widetilde{f}_1 = 0$, showing that $(\frac{-4+\sqrt{51}}{125}, \frac{2\sqrt{51}-3}{25}, \frac{4(\sqrt{51}+1)}{125}) \in \mathcal{T}_1$. Therefore, E_1 is a weak focus with order 2 if $(\beta, \delta, h) \in \mathcal{T}_1$. □

4 Simulation and conclusions

Let $(\beta_*, \delta_*, h_*) = (\frac{\sqrt{51}-4}{100}, \frac{2\sqrt{51}-3}{25}, \frac{4(\sqrt{51}+1)}{125})$. Then it can be checked that $E_1(0.2, 0.8)$ is a stable weak focus of order 2 if $(\beta, \delta, h) = (\beta_*, \delta_*, h_*)$. To display two limit cycles from degenerate Hopf bifurcation, we plot three orbits of system (2.1) with $(\beta, \delta, h) = (\beta_* - 1.428429 \times 10^{-5}, \delta_*, h_* - 2.605207 \times 10^{-4})$. In Fig. 1, the orbits from $P_1 = (0.1990005, 0.801)$, $P_2 = (0.199005, 0.801)$, and $P_3 = (0.19905, 0.801)$ are plotted, separately. It is clear that the orbits from P_1 and P_2 spiral inward and outward, respectively. So by the Poincaré–Bendixson theorem [18], there is an unstable limit cycle between the orbits from P_1 and P_2. Moreover, since the orbit from P_3 spirals inward, there is a stable limit cycle between the orbits from P_2 and P_3. Therefore, there are two limit cycles surrounding $E_1(0.199, 0.801)$, and the inner one is unstable and the outer one is stable.

In this paper, we extend the study of the center-type equilibrium in system (2.1) and obtain that it is a weak focus of order at most 2. In the previous work [10, 11], Huang *et al.* investigated various bifurcations including saddle-node bifurcation, subcritical and supercritical Hopf bifurcation, and Bogdanov–Takens bifurcation in this system. Moreover, degenerate Hopf bifurcation was just studied for a fixed parameter value in [10] and by Bogdanov–Takens bifurcation with codimension 3 in [11]. However, these results just show the occurrence of degenerate Hopf bifurcation in the neighborhood of the fixed parameter values, not for all possible parameters. For all parameters, our study proves that the center-type equilibrium in system (2.1) is a weak focus with order at most 2. Thus

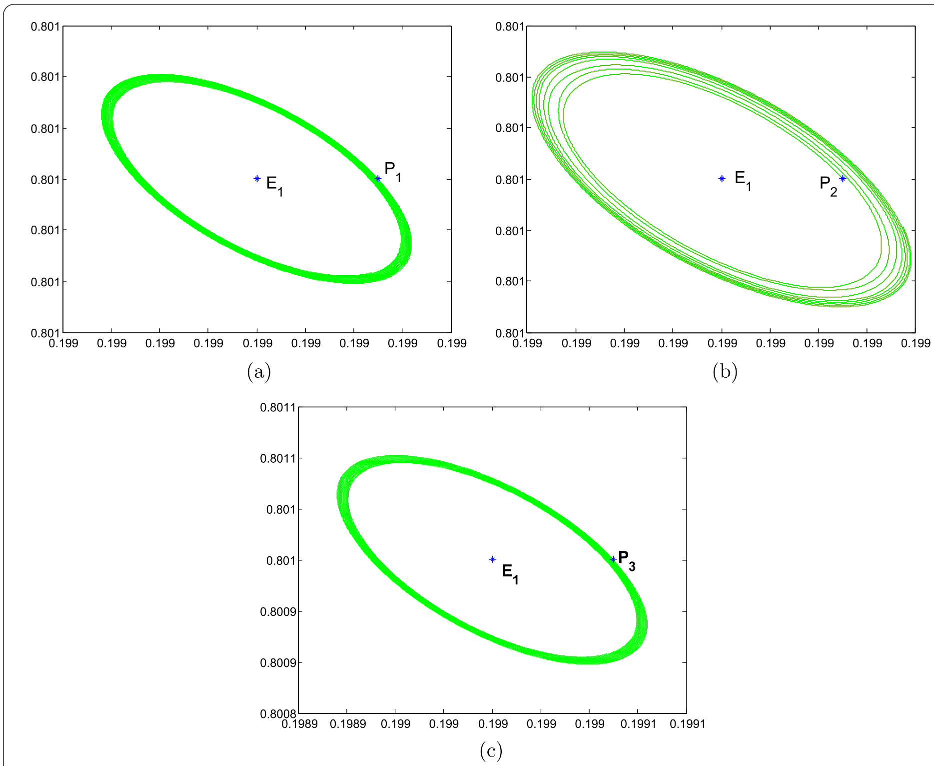

Figure 1 Orbits from P_1, P_2 and P_3

there are at most two limit cycles arising from degenerate Hopf bifurcation. The results in [10, 11] and this paper reveal that the codimension of local bifurcations in system (2.1) is at most 3.

Appendix

f_1 and f_2 in (2.10) are given by

$$\begin{aligned} f_1 :={}& 8x_1^7\beta + x_1^7\delta - 28x_1^6\beta^2 - 16x_1^6\beta\delta - 2x_1^6\delta^2 + 24x_1^5\beta^3 + 14x_1^5\beta^2\delta + 2x_1^5\beta\delta^2 - 24x_1^4\beta^4 \\ & - 24x_1^4\beta^3\delta - 8x_1^4\beta^2\delta^2 - x_1^4\beta\delta^3 + 16x_1^3\beta^5 + 24x_1{}^3\beta^4\delta + 12x_1^3\beta^3\delta^2 + 2x_1^3\beta^2\delta^3 \\ & - hx_1^6 - 4hx_1^4\beta^2 - hx_1^4\beta\delta + 4hx_1^3\beta^3 + 2hx_1^3\beta^2\delta - 8x_1^6\beta + x_1^6\delta + 48x_1^5\beta^2 + 12x_1^5\beta\delta \\ & - 48x_1^4\beta^3 - 10x_1^4\beta^2\delta + x_1^4\beta\delta^2 + 64x_1^3\beta^4 + 40x_1^3\beta^3\delta + 6x_1^3\beta^2\delta^2 - 48x_1^2\beta^5 - 48x_1^2\beta^4\delta \\ & - 12x_1^2\beta^3\delta^2 + 4hx_1^3\beta^2 - 4hx_1^2\beta^3 - 20x_1^4\beta^2 + 24x_1^3\beta^3 - 4x_1^3\beta^2\delta - 56x_1^2\beta^4 - 16x_1^2\beta^3\delta \\ & + 48x_1\beta^5 + 24x_1\beta^4\delta + 16x_1\beta^4 - 16\beta^5, \\ f_2 :={}& - 4992x_1^{18}\beta^2 - 3120x_1^{18}\beta\delta - 312x_1^{18}\delta^2 + 37{,}440x_1^{17}\beta^3 + 41{,}136x_1^{17}\beta^2\delta \\ & + 13{,}746x_1^{17}\beta\delta^2 + 1296x_1^{17}\delta^3 - 95{,}040x_1^{16}\beta^4 - 133{,}680x_1^{16}\beta^3\delta - 69{,}624x_1^{16}\beta^2\delta^2 \\ & - 15{,}774x_1^{16}\beta\delta^3 - 1242x_1^{16}\delta^4 + 104{,}736x_1^{15}\beta^5 + 125{,}776x_1^{15}\beta^4\delta + 43{,}308x_1^{15}\beta^3\delta^2 \\ & - 644x_1^{15}\beta^2\delta^3 - 2616x_1{}^{15}\beta\delta^4 - 323x_1^{15}\delta^5 - 101{,}312x_1^{14}\beta^6 - 99{,}120x_1^{14}\beta^5\delta \\ & + 24{,}656x_1^{14}\beta^4\delta^2 + 69{,}768x_1^{14}\beta^3\delta^3 + 34{,}822x_1^{14}\beta^2\delta^4 + 7110x_1^{14}\beta\delta^5 + 515x_1^{14}\delta^6 \\ & + 102{,}656x_1^{13}\beta^7 + 184{,}960x_1^{13}\beta^6\delta + 150{,}496x_1^{13}\beta^5\delta^2 + 77{,}520x_1^{13}\beta^4\delta^3 \\ & + 28{,}788x_1^{13}\beta^3\delta^4 + 7590x_1^{13}\beta^2\delta^5 + 1246x_1{}^{13}\beta\delta^6 + 94x_1^{13}\delta^7 + 12{,}416x_1^{12}\beta^8 \\ & + 158{,}496x_1^{12}\beta^7\delta + 323{,}136x_1^{12}\beta^6\delta^2 + 293{,}736x_1^{12}\beta^5\delta^3 + 143{,}952x_1^{12}\beta^4\delta^4 \\ & + 39{,}480x_1^{12}\beta^3\delta^5 + 5722x_1^{12}\beta^2\delta^6 + 345x_1^{12}\beta\delta^7 + 2176x_1{}^{11}\beta^9 - 51{,}520x_1^{11}\beta^8\delta \\ & - 132{,}512x_1^{11}\beta^7\delta^2 - 138{,}128x_1^{11}\beta^6\delta^3 - 76{,}368x_1^{11}\beta^5\delta^4 - 23{,}256x_1^{11}\beta^4\delta^5 \\ & - 3440x_1^{11}\beta^3\delta^6 - 92x_1^{11}\beta^2\delta^7 + 23x_1^{11}\beta\delta^8 + 11{,}264x_1^{10}\beta^{10} + 65{,}920x_1^{10}\beta^9\delta \\ & + 126{,}208x_1^{10}\beta^8\delta^2 + 121{,}952x_1^{10}\beta^7\delta^3 + 69{,}120x_1^{10}\beta^6\delta^4 + 24{,}392x_1^{10}\beta^5\delta^5 \\ & + 5360x_1^{10}\beta^4\delta^6 + 674x_1^{10}\beta^3\delta^7 + 36x_1^{10}\beta^2\delta^8 - 17{,}920x_1^9\beta^{11} - 98{,}560x_1^9\beta^{10}\delta \\ & - 210{,}048x_1^9\beta^9\delta^2 - 238{,}144x_1^9\beta^8\delta^3 - 160{,}480x_1^9\beta^7\delta^4 - 66{,}480x_1^9\beta^6\delta^5 \\ & - 16{,}600x_1{}^9\beta^5\delta^6 - 2284x_1^9\beta^4\delta^7 - 132x_1^9\beta^3\delta^8 - 16{,}384x_1^8\beta^{12} - 65{,}536x_1^8\beta^{11}\delta \\ & - 114{,}688x_1^8\beta^{10}\delta^2 - 114{,}688x_1^8\beta^9\delta^3 - 71{,}680x_1^8\beta^8\delta^4 - 28{,}672x_1^8\beta^7\delta^5 \\ & - 7168x_1^8\beta^6\delta^6 - 1024x_1^8\beta^5\delta^7 - 64x_1^8\beta^4\delta^8 + 624hx_1^{17}\beta + 312hx_1{}^{17}\delta + 2448hx_1^{16}\beta^2 \\ & + 696hx_1^{16}\beta\delta - 201hx_1^{16}\delta^2 - 19{,}776hx_1^{15}\beta^3 - 24{,}768hx_1^{15}\beta^2\delta - 9366hx_1{}^{15}\beta\delta^2 \\ & - 945hx_1^{15}\delta^3 + 17{,}056hx_1{}^{14}\beta^4 + 33{,}292hx_1^{14}\beta^3\delta + 23{,}612hx_1^{14}\beta^2\delta^2 + 7004hx_1^{14}\beta\delta^3 \\ & + 690hx_1^{14}\delta^4 + 2880hx_1{}^{13}\beta^5 + 8376hx_1^{13}\beta^4\delta + 15{,}152hx_1^{13}\beta^3\delta^2 + 10{,}140hx_1^{13}\beta^2\delta^3 \\ & + 2623hx_1^{13}\beta\delta^4 + 237hx_1{}^{13}\delta^5 + 31{,}392hx_1^{12}\beta^6 + 100{,}160hx_1^{12}\beta^5\delta \end{aligned}$$

$$+ 101{,}928hx_1^{12}\beta^4\delta^2 + 45{,}860hx_1^{12}\beta^3\delta^3 + 9454hx_1^{12}\beta^2\delta^4 + 724hx_1^{12}\beta\delta^5$$
$$+ 14{,}976hx_1^{11}\beta^7 + 13{,}056hx_1^{11}\beta^6\delta + 416hx_1^{11}\beta^5\delta^2 - 2344hx_1^{11}\beta^4\delta^3 - 228hx_1^{11}\beta^3\delta^4$$
$$+ 340hx_1^{11}\beta^2\delta^5 + 82hx_1^{11}\beta\delta^6 - 6016hx_1^{10}\beta^8 + 3840hx_1{}^{10}\beta^7\delta + 16{,}672hx_1^{10}\beta^6\delta^2$$
$$+ 12{,}976hx_1^{10}\beta^5\delta^3 + 4392hx_1^{10}\beta^4\delta^4 + 668hx_1^{10}\beta^3\delta^5 + 30hx_1^{10}\beta^2\delta^6 + 1280hx_1^9\beta^9$$
$$- 17{,}280hx_1^9\beta^8\delta - 41{,}600hx_1^9\beta^7\delta^2 - 36{,}800hx_1^9\beta^6\delta^3 - 15{,}600hx_1^9\beta^5\delta^4$$
$$- 3160hx_1^9\beta^4\delta^5 - 240hx_1^9\beta^3\delta^6 - 18{,}944hx_1^8\beta^{10} - 56{,}832hx_1^8\beta^9\delta - 71{,}040hx_1^8\beta^8\delta^2$$
$$- 47{,}360hx_1^8\beta^7\delta^3 - 17{,}760hx_1^8\beta^6\delta^4 - 3552hx_1^8\beta^5\delta^5 - 296hx_1^8\beta^4\delta^6 + 9984x_1^{17}\beta^2$$
$$+ 2496x_1^{17}\beta\delta - 312x_1^{17}\delta^2 - 107{,}328x_1^{16}\beta^3 - 79{,}728x_1^{16}\beta^2\delta - 13{,}194x_1^{16}\beta\delta^2$$
$$+ 201x_1^{16}\delta^3 + 335{,}232x_1^{15}\beta^4 + 366{,}720x_1^{15}\beta^3\delta + 141{,}600x_1^{15}\beta^2\delta^2 + 21{,}840x_1{}^{15}\beta\delta^3$$
$$+ 945x_1^{15}\delta^4 - 365{,}664x_1{}^{14}\beta^5 - 298{,}448x_1^{14}\beta^4\delta - 49{,}660x_1^{14}\beta^3\delta^2 + 3152x_1^{14}\beta^2\delta^3$$
$$- 2198x_1^{14}\beta\delta^4 - 690x_1{}^{14}\delta^5 + 312{,}768x_1^{13}\beta^6 + 10{,}176x_1{}^{13}\beta^5\delta - 409{,}984x_1^{13}\beta^4\delta^2$$
$$- 310{,}980x_1^{13}\beta^3\delta^3 - 89{,}286x_1{}^{13}\beta^2\delta^4 - 9789x_1^{13}\beta\delta^5 - 237x_1^{13}\delta^6 - 390{,}272x_1^{12}\beta^7$$
$$- 551{,}232x_1^{12}\beta^6\delta - 419{,}520x_1^{12}\beta^5\delta^2 - 249{,}896x_1^{12}\beta^4\delta^3 - 102{,}636x_1^{12}\beta^3\delta^4$$
$$- 23{,}146x_1^{12}\beta^2\delta^5 - 2124x_1^{12}\beta\delta^6 - 325{,}760x_1^{11}\beta^8 - 1{,}445{,}760x_1^{11}\beta^7\delta$$
$$- 2{,}018{,}624x_1^{11}\beta^6\delta^2 - 1{,}335{,}504x_3^{11}\beta^5\delta^3 - 463{,}424x_1^{11}\beta^4\delta^4 - 82{,}156x_1^{11}\beta^3\delta^5$$
$$- 6208x_3^{11}\beta^2\delta^6 - 82x_1^{11}\beta\delta^7 + 139{,}392x_1^{10}\beta^9 + 688{,}640x_1^{10}\beta^8\delta + 1{,}034{,}016x_1^{10}\beta^7\delta^2$$
$$+ 731{,}296x_1^{10}\beta^6\delta^3 + 267{,}168x_1^{10}\beta^5\delta^4 + 45{,}408x_1^{10}\beta^4\delta^5 + 1256x_1^{10}\beta^3\delta^6$$
$$- 392x_1^{10}\beta^2\delta^7 - 178{,}688x_1^9\beta^{10} - 708{,}864x_1^9\beta^9\delta - 1{,}044{,}736x_1^9\beta^8\delta^2$$
$$- 788{,}160x_1^9\beta^7\delta^3 - 339{,}360x_1^9\beta^6\delta^4 - 85{,}424x_1^9\beta^5\delta^5 - 11{,}872x_1{}^9\beta^4\delta^6$$
$$- 708x_1^9\beta^3\delta^7 + 177{,}664x_1^8\beta^{11} + 809{,}984x_1^8\beta^{10}\delta + 1{,}440{,}384x_1^8\beta^9\delta^2$$
$$+ 1{,}340{,}800x_1^8\beta^8\delta^3 + 716{,}960x_1^8\beta^7\delta^4 + 221{,}952x_1^8\beta^6\delta^5 + 36{,}952x_1^8\beta^5\delta^6$$
$$+ 2552x_1^8\beta^4\delta^7 + 131{,}072x_1^7\beta^{12} + 458{,}752x_1^7\beta^{11}\delta + 688{,}128x_1^7\beta^{10}\delta^2$$
$$+ 573{,}440x_1^7\beta^9\delta^3 + 286{,}720x_1{}^7\beta^8\delta^4 + 86{,}016x_1^7\beta^7\delta^5 + 14{,}336x_1^7\beta^6\delta^6$$
$$+ 1024x_1^7\beta^5\delta^7 - 570h^2x_1^{15}\beta - 240h^2x_1^{15}\delta - 288h^2x_1^{14}\beta^2 + 180h^2x_1^{14}\beta\delta$$
$$+ 171h^2x_1^{14}\delta^2 + 1144h^2x_1^{13}\beta^3 + 2788h^2x_1^{13}\beta^2\delta + 1475h^2x_1^{13}\beta\delta^2 + 179h^2x_1^{13}\delta^3$$
$$+ 5656h^2x_1{}^{12}\beta^4 + 8972h^2x_1^{12}\beta^3\delta + 4028h^2x_1^{12}\beta^2\delta^2 + 478h^2x_1{}^{12}\beta\delta^3$$
$$+ 6080h^2x_1^{11}\beta^5 + 7376h^2x_1^{11}\beta^4\delta + 3336h^2x_1^{11}\beta^3\delta^2 + 774h^2x_1^{11}\beta^2\delta^3 + 95h^2x_1^{11}\beta\delta^4$$
$$- 2400h^2x_1^{10}\beta^6 - 1080h^2x_1^{10}\beta^5\delta + 24h^2x_1^{10}\beta^4\delta^2 - 114h^2x_1^{10}\beta^3\delta^3 - 48h^2x_1^{10}\beta^2\delta^4$$
$$+ 3168h^2x_1^9\beta^7 + 2064h^2x_1^9\beta^6\delta - 648h^2x_1^9\beta^5\delta^2 - 612h^2x_1^9\beta^4\delta^3 - 84h^2x_1^9\beta^3\delta^4$$
$$- 6400h^2x_1^8\beta^8 - 12{,}800h^2x_1^8\beta^7\delta - 9600h^2x_1^8\beta^6\delta^2 - 3200h^2x_1^8\beta^5\delta^3 - 400h^2x_1^8\beta^4\delta^4$$
$$- 624hx_1^{16}\beta - 4896hx_1^{15}\beta^2 + 444hx_1^{15}\beta\delta + 480hx_1^{15}\delta^2 + 61{,}392hx_1^{14}\beta^3$$
$$+ 49{,}584hx_1^{14}\beta^2\delta + 8226hx_1^{14}\beta\delta^2 - 342hx_1^{14}\delta^3 - 81{,}904hx_1^{13}\beta^4 - 113{,}384hx_1^{13}\beta^3\delta$$
$$- 53{,}736hx_1^{13}\beta^2\delta^2 - 9318hx_1^{13}\beta\delta^3 - 358hx_1^{13}\delta^4 + 6624hx_1^{12}\beta^5 - 30{,}264hx_1^{12}\beta^4\delta$$

$$- 60{,}108hx_1^{12}\beta^3\delta^2 - 27{,}738hx_1^{12}\beta^2\delta^3 - 3598hx_1^{12}\beta\delta^4 - 201{,}984hx_1^{11}\beta^6$$
$$- 454{,}384hx_1^{11}\beta^5\delta - 327{,}136hx_1^{11}\beta^4\delta^2 - 95{,}752hx_1^{11}\beta^3\delta^3 - 10{,}416hx_1^{11}\beta^2\delta^4$$
$$- 190hx_1^{11}\beta\delta^5 - 29{,}376hx_1^{10}\beta^7 + 15{,}712hx_1^{10}\beta^6\delta + 28{,}704hx_1^{10}\beta^5\delta^2$$
$$+ 4464hx_1^{10}\beta^4\delta^3 - 3292hx_1^{10}\beta^3\delta^4 - 948hx_1^{10}\beta^2\delta^5 - 13{,}312hx_1^9\beta^8$$
$$- 117{,}312hx_1^9\beta^7\delta - 142{,}368hx_1^9\beta^6\delta^2 - 67{,}040hx_1^9\beta^5\delta^3 - 13{,}752hx_1^9\beta^4\delta^4$$
$$- 996hx_1^9\beta^3\delta^5 + 12{,}800hx_1^8\beta^9 + 137{,}600hx_1^8\beta^8\delta + 217{,}600hx_1^8\beta^7\delta^2$$
$$+ 136{,}000hx_1^8\beta^6\delta^3 + 37{,}600hx_1^8\beta^5\delta^4 + 3800hx_1^8\beta^4\delta^5 + 113{,}664hx_1^7\beta^{10}$$
$$+ 284{,}160hx_1^7\beta^9\delta + 284{,}160hx_1^7\beta^8\delta^2 + 142{,}080hx_1^7\beta^7\delta^3 + 35{,}520hx_1^7\beta^6\delta^4$$
$$+ 3552hx_1^7\beta^5\delta^5 - 4992x_1^{16}\beta^2 + 624x_1^{16}\beta\delta + 102{,}336x_1^{15}\beta^3 + 41{,}040x_1^{15}\beta^2\delta$$
$$+ 126x_1^{15}\beta\delta^2 - 240x_1^{15}\delta^3 - 435{,}456x_1^{14}\beta^4 - 354{,}240x_1^{14}\beta^3\delta - 96{,}504x_1^{14}\beta^2\delta^2$$
$$- 8406x_1^{14}\beta\delta^3 + 171x_1^{14}\delta^4 + 430{,}272x_1^{13}\beta^5 + 188{,}128x_1^{13}\beta^4\delta + 20{,}524x_1^{13}\beta^3\delta^2$$
$$+ 25{,}260x_1^{13}\beta^2\delta^3 + 7843x_1^{13}\beta\delta^4 + 179x_1^{13}\delta^5 - 142{,}080x_1^{12}\beta^6 + 782{,}208x_1^{12}\beta^5\delta$$
$$+ 993{,}824x_1^{12}\beta^4\delta^2 + 410{,}460x_1^{12}\beta^3\delta^3 + 65{,}790x_1^{12}\beta^2\delta^4 + 3120x_1^{12}\beta\delta^5$$
$$+ 428{,}544x_1^{11}\beta^7 + 475{,}008x_1^{11}\beta^6\delta + 566{,}064x_1^{11}\beta^5\delta^2 + 438{,}704x_1^{11}\beta^4\delta^3$$
$$+ 149{,}388x_1^{11}\beta^3\delta^4 + 18{,}326x_1^{11}\beta^2\delta^5 + 95x_1^{11}\beta\delta^6 + 1{,}612{,}800x_1^{10}\beta^8$$
$$+ 4{,}658{,}048x_1^{10}\beta^7\delta + 4{,}670{,}528x_1^{10}\beta^6\delta^2 + 2{,}177{,}712x_1^{10}\beta^5\delta^3 + 489{,}232x_1^{10}\beta^4\delta^4$$
$$+ 46{,}030x_1^{10}\beta^3\delta^5 + 996x_1^{10}\beta^2\delta^6 - 885{,}248x_1^9\beta^9 - 2{,}637{,}696x_1^9\beta^8\delta$$
$$- 2{,}716{,}800x_1^9\beta^7\delta^2 - 1{,}264{,}704x_1^9\beta^6\delta^3 - 249{,}576x_1^9\beta^5\delta^4 - 4356x_1^9\beta^4\delta^5$$
$$+ 3416x_1^9\beta^3\delta^6 + 912{,}384x_1^8\beta^{10} + 2{,}799{,}744x_1^8\beta^9\delta + 3{,}244{,}160x_1^8\beta^8\delta^2$$
$$+ 1{,}875{,}520x_1^8\beta^7\delta^3 + 581{,}760x_1^8\beta^6\delta^4 + 93{,}352x_1^8\beta^5\delta^5 + 6152x_1^8\beta^4\delta^6$$
$$- 741{,}888x_1^7\beta^{11} - 2{,}790{,}144x_1^7\beta^{10}\delta - 4{,}051{,}200x_1^7\beta^9\delta^2 - 2{,}981{,}760x_1^7\beta^8\delta^3$$
$$- 1{,}188{,}000x_1^7\beta^7\delta^4 - 244{,}464x_1^7\beta^6\delta^5 - 20{,}352x_1^7\beta^5\delta^6 - 458{,}752x_1^6\beta^{12}$$
$$- 1{,}376{,}256x_1^6\beta^{11}\delta - 1{,}720{,}320x_1^6\beta^{10}\delta^2 - 1{,}146{,}880x_1^6\beta^9\delta^3 - 430{,}080x_1^6\beta^8\delta^4$$
$$- 86{,}016x_1^6\beta^7\delta^5 - 7168x_1^6\beta^6\delta^6 + 9h^3x_1^{14} + 72h^3x_1^{13}\beta + 36h^3x_1^{13}\delta + 192h^3x_1^{12}\beta^2$$
$$+ 99h^3x_1^{12}\beta\delta + 540h^3x_1^{11}\beta^3 + 342h^3x_1^{11}\beta^2\delta + 36h^3x_1^{11}\beta\delta^2 - 48h^3x_1^{10}\beta^4$$
$$- 108h^3x_1^{10}\beta^3\delta - 42h^3x_1^{10}\beta^2\delta^2 + 432h^3x_1^9\beta^5 + 264h^3x_1^9\beta^4\delta + 24h^3x_1^9\beta^3\delta^2$$
$$- 672h^3x_1^8\beta^6 - 672h^3x_1^8\beta^5\delta - 168h^3x_1^8\beta^4\delta^2 + 570h^2x_1^{14}\beta - 27h^2x_1^{14}\delta$$
$$- 684h^2x_1^{13}\beta\delta - 108h^2x_1^{13}\delta^2 - 4656h^2x_1^{12}\beta^3 - 6356h^2x_1^{12}\beta^2\delta - 1559h^2x_1^{12}\beta\delta^2$$
$$- 17{,}664h^2x_1^{11}\beta^4 - 18{,}760h^2x_1^{11}\beta^3\delta - 4478h^2x_1^{11}\beta^2\delta^2 - 108h^2x_1^{11}\beta\delta^3$$
$$- 17{,}056h^2x_1^{10}\beta^5 - 15{,}600h^2x_1^{10}\beta^4\delta - 5220h^2x_1^{10}\beta^3\delta^2 - 772h^2x_1^{10}\beta^2\delta^3$$
$$+ 2784h^2x_1^9\beta^6 - 4752h^2x_1^9\beta^5\delta - 2928h^2x_1^9\beta^4\delta^2 - 180h^2x_1^9\beta^3\delta^3 - 9696h^2x_1^8\beta^7$$
$$- 1728h^2x_1^8\beta^6\delta + 3528h^2x_1^8\beta^5\delta^2 + 984h^2x_1^8\beta^4\delta^3 + 25{,}600h^2x_1^7\beta^8$$
$$+ 38{,}400h^2x_1^7\beta^7\delta + 19{,}200h^2x_1^7\beta^6\delta^2 + 3200h^2x_1^7\beta^5\delta^3 + 2448hx_1^{14}\beta^2$$

$$
\begin{aligned}
&- 1140hx_1^{14}\beta\delta + 27hx_1^{14}\delta^2 - 63{,}456hx_1^{13}\beta^3 - 24{,}240hx_1^{13}\beta^2\delta + 1152hx_1^{13}\beta\delta^2\\
&+ 108hx_1^{13}\delta^3 + 137{,}616hx_1^{12}\beta^4 + 125{,}868hx_1^{12}\beta^3\delta + 35{,}244hx_1^{12}\beta^2\delta^2\\
&+ 2821hx_1^{12}\beta\delta^3 - 28{,}320hx_1^{11}\beta^5 + 72{,}360hx_1^{11}\beta^4\delta + 87{,}216hx_1^{11}\beta^3\delta^2\\
&+ 19{,}458hx_1^{11}\beta^2\delta^3 + 108hx_1^{11}\beta\delta^4 + 482{,}496hx_1^{10}\beta^6 + 748{,}496hx_1^{10}\beta^5\delta\\
&+ 347{,}640hx_1^{10}\beta^4\delta^2 + 54{,}032hx_1^{10}\beta^3\delta^3 + 1670hx_1^{10}\beta^2\delta^4 - 31{,}744h{x_1}^9\beta^7\\
&- 108{,}576hx_1^9\beta^6\delta - 30{,}384hx_1^9\beta^5\delta^2 + 15{,}712hx_1^9\beta^4\delta^3 + 5496hx_1^9\beta^3\delta^4\\
&+ 151{,}040hx_1^8\beta^8 + 419{,}328hx_1^8\beta^7\delta + 318{,}432hx_1^8\beta^6\delta^2 + 92{,}272hx_1^8\beta^5\delta^3\\
&+ 9000hx_1^8\beta^4\delta^4 - 83{,}200hx_1^7\beta^9 - 377{,}600hx_1^7\beta^8\delta - 403{,}200hx_1^7\beta^7\delta^2\\
&- 161{,}600hx_1^7\beta^6\delta^3 - 22{,}000hx_1^7\beta^5\delta^4 - 284{,}160hx_1^6\beta^{10} - 568{,}320hx_1^6\beta^9\delta\\
&- 426{,}240hx_1^6\beta^8\delta^2 - 142{,}080hx_1^6\beta^7\delta^3 - 17{,}760hx_1^6\beta^6\delta^4 - 32{,}448x_1^{14}\beta^3\\
&- 2448x_1^{14}\beta^2\delta + 570x_1^{14}\beta\delta^2 - 9x_1^{14}\delta^3 + 245{,}376x_1^{13}\beta^4 + 143{,}040x_1^{13}\beta^3\delta\\
&+ 24{,}240x_1^{13}\beta^2\delta^2 - 540x_1^{13}\beta\delta^3 - 36x_1^{13}\delta^4 - 144{,}192x_1^{12}\beta^5 - 8992x_1^{12}\beta^4\delta\\
&- 56{,}436x_1^{12}\beta^3\delta^2 - 29{,}080x_1^{12}\beta^2\delta^3 - 1361x_1^{12}\beta\delta^4 - 533{,}120x_1^{11}\beta^6\\
&- 1{,}425{,}312x_1^{11}\beta^5\delta - 886{,}928x_1^{11}\beta^4\delta^2 - 202{,}260x_1^{11}\beta^3\delta^3 - 15{,}322x_1^{11}\beta^2\delta^4\\
&- 36x_1^{11}\beta\delta^5 + 133{,}376x_1^{10}\beta^7 - 56{,}960x_1^{10}\beta^6\delta - 665{,}360x_1^{10}\beta^5\delta^2 - 434{,}232x_1^{10}\beta^4\delta^3\\
&- 77{,}448x_1^{10}\beta^3\delta^4 - 856x_1^{10}\beta^2\delta^5 - 3{,}612{,}160x_1^9\beta^8 - 7{,}345{,}024x_1^9\beta^7\delta\\
&- 5{,}143{,}680{x_1}^9\beta^6\delta^2 - 1{,}545{,}648x_1^9\beta^5\delta^3 - 184{,}952x_1^9\beta^4\delta^4 - 5340x_1^9\beta^3\delta^5\\
&+ 2{,}267{,}392x_1^8\beta^9 + 4{,}674{,}560x_1^8\beta^8\delta + 3{,}189{,}760x_1^8\beta^7\delta^2 + 793{,}856x_1^8\beta^6\delta^3\\
&+ 4680x_1^8\beta^5\delta^4 - 17{,}640x_1^8\beta^4\delta^5 - 2{,}352{,}640x_1^7\beta^{10} - 5{,}672{,}960x_1^7\beta^9\delta\\
&- 5{,}053{,}440x_1^7\beta^8\delta^2 - 2{,}117{,}120x_1^7\beta^7\delta^3 - 421{,}920x_1^7\beta^6\delta^4 - 32{,}320x_1^7\beta^5\delta^5\\
&+ 1{,}723{,}904x_1^6\beta^{11} + 5{,}250{,}560x_1^6\beta^{10}\delta + 6{,}001{,}920x_1^6\beta^9\delta^2 + 3{,}281{,}920x_1^6\beta^8\delta^3\\
&+ 867{,}040x_1^6\beta^7\delta^4 + 88{,}992x_1^6\beta^6\delta^5 + 917{,}504x_1^5\beta^{12} + 2{,}293{,}760x_1^5\beta^{11}\delta\\
&+ 2{,}293{,}760x_1^5\beta^{10}\delta^2 + 1{,}146{,}880x_1^5\beta^9\delta^3 + 286{,}720x_1^5\beta^8\delta^4 + 28{,}672x_1^5\beta^7\delta^5\\
&- 72h^3x_1^{12}\beta - 192h^3x_1^{11}\beta^2 - 828h^3x_1^{10}\beta^3 - 216h^3x_1^{10}\beta^2\delta + 96h^3x_1^9\beta^4\\
&+ 108h^3x_1^9\beta^3\delta - 864h^3x_1^8\beta^5 - 264h^3x_1^8\beta^4\delta + 1344h^3x_1^7\beta^6 + 672h^3x_1^7\beta^5\delta\\
&+ 288h^2x_1^{12}\beta^2 + 216h^2x_1^{12}\beta\delta + 5880h^2x_1^{11}\beta^3 + 3568h^2x_1^{11}\beta^2\delta + 18{,}360h^2x_1^{10}\beta^4\\
&+ 10{,}652h^2x_1^{10}\beta^3\delta + 648h^2x_1^{10}\beta^2\delta^2 + 17{,}664h^2x_1^9\beta^5 + 12{,}048h^2x_1^9\beta^4\delta\\
&+ 2628h^2x_1^9\beta^3\delta^2 + 6048h^2x_1^8\beta^6 + 12{,}744h^2x_1^8\beta^5\delta + 2904h^2x_1^8\beta^4\delta^2 + 10{,}080h^2x_1^7\beta^7\\
&- 2736h^2x_1^7\beta^6\delta - 2880h^2x_1^7\beta^5\delta^2 - 38{,}400h^2x_1^6\beta^8 - 38{,}400h^2x_1^6\beta^7\delta\\
&- 9600h^2x_1^6\beta^6\delta^2 + 21{,}840h{x_1}^{12}\beta^3 - 576hx_1^{12}\beta^2\delta - 216hx_1^{12}\beta\delta^2 - 97{,}744hx_1^{11}\beta^4\\
&- 50{,}512hx_1^{11}\beta^3\delta - 6560hx_1^{11}\beta^2\delta^2 + 16{,}416hx_1^{10}\beta^5 - 87{,}432hx_1^{10}\beta^4\delta\\
&- 43{,}124hx_1^{10}\beta^3\delta^2 - 648hx_1^{10}\beta^2\delta^3 - 549{,}504hx_1^9\beta^6 - 538{,}064hx_1^9\beta^5\delta\\
&- 130{,}224hx_1^9\beta^4\delta^2 - 5580hx_1^9\beta^3\delta^3 + 121{,}216hx_1^8\beta^7 + 100{,}896hx_1^8\beta^6\delta
\end{aligned}
$$

$$
\begin{aligned}
&- 27{,}008hx_1^8\beta^5\delta^2 - 17{,}832hx_1^8\beta^4\delta^3 - 350{,}720hx_1^7\beta^8 - 592{,}512hx_1^7\beta^7\delta \\
&- 276{,}448hx_1^7\beta^6\delta^2 - 38{,}208hx_1^7\beta^5\delta^3 + 179{,}200hx_1^6\beta^9 + 480{,}000hx_1^6\beta^8\delta \\
&+ 320{,}000hx_1^6\beta^7\delta^2 + 62{,}400hx_1^6\beta^6\delta^3 + 378{,}880hx_1^5\beta^{10} + 568{,}320hx_1{}^5\beta^9\delta \\
&+ 284{,}160hx_1^5\beta^8\delta^2 + 47{,}360hx_1^5\beta^7\delta^3 - 50{,}112x_1{}^{12}\beta^4 - 21{,}840x_1^{12}\beta^3\delta \\
&+ 288x_1^{12}\beta^2\delta^2 + 72x_1^{12}\beta\delta^3 - 63{,}456x_1^{11}\beta^5 + 18{,}512x_1^{11}\beta^4\delta + 44{,}632x_1^{11}\beta^3\delta^2 \\
&+ 3184x_1{}^{11}\beta^2\delta^3 + 843{,}840x_1^{10}\beta^6 + 967{,}728x_1^{10}\beta^5\delta + 309{,}040x_1^{10}\beta^4\delta^2 \\
&+ 33{,}300x_1^{10}\beta^3\delta^3 + 216x_1^{10}\beta^2\delta^4 - 606{,}464x_1^9\beta^7 + 58{,}752x_1^9\beta^6\delta + 559{,}920x_1^9\beta^5\delta^2 \\
&+ 171{,}824x_1^9\beta^4\delta^3 + 2844x_1^9\beta^3\delta^4 + 4{,}401{,}280x_1^8\beta^8 + 6{,}167{,}456x_1^8\beta^7\delta \\
&+ 2{,}747{,}072x_1{}^8\beta^6\delta^2 + 431{,}496x_1^8\beta^5\delta^3 + 15{,}192x_1^8\beta^4\delta^4 - 3{,}089{,}280x_1^7\beta^9 \\
&- 4{,}290{,}880x_1^7\beta^8\delta - 1{,}620{,}960x_1{}^7\beta^7\delta^2 - 28{,}144x_1^7\beta^6\delta^3 + 54{,}096x_1^7\beta^5\delta^4 \\
&+ 3{,}543{,}040x_1^6\beta^{10} + 6{,}564{,}480x_1^6\beta^9\delta + 4{,}249{,}600x_1^6\beta^8\delta^2 + 1{,}150{,}560x_1^6\beta^7\delta^3 \\
&+ 110{,}400x_1^6\beta^6\delta^4 - 2{,}455{,}040x_1^5\beta^{11} - 5{,}850{,}880x_1^5\beta^{10}\delta - 4{,}951{,}680x_1^5\beta^9\delta^2 \\
&- 1{,}791{,}040x_1^5\beta^8\delta^3 - 235{,}520x_1^5\beta^7\delta^4 - 1{,}146{,}880x_1^4\beta^{12} - 2{,}293{,}760x_1^4\beta^{11}\delta \\
&- 1{,}720{,}320x_1^4\beta^{10}\delta^2 - 573{,}440x_1^4\beta^9\delta^3 - 71{,}680x_1^4\beta^8\delta^4 + 288h^3x_1^9\beta^3 - 48h^3x_1^8\beta^4 \\
&+ 432h^3x_1^7\beta^5 - 672h^3x_1^6\beta^6 - 2368h^2x_1^{10}\beta^3 - 6352h^2x_1^9\beta^4 - 864h^2x_1^9\beta^3\delta \\
&- 8480h^2x_1^8\beta^5 - 3824h^2x_1^8\beta^4\delta - 10{,}848h^2x_1^7\beta^6 - 6912h^2x_1^7\beta^5\delta - 3744h^2x_1^6\beta^7 \\
&+ 2400h^2x_1^6\beta^6\delta + 25{,}600h^2x_1^5\beta^8 + 12{,}800h^2x_1^5\beta^7\delta + 24{,}976hx_1^{10}\beta^4 + 4736hx_1^{10}\beta^3\delta \\
&+ 11{,}232hx_1^9\beta^5 + 36{,}960hx_1^9\beta^4\delta + 864hx_1^9\beta^3\delta^2 + 302{,}496hx_1^8\beta^6 + 147{,}376hx_1^8\beta^5\delta \\
&+ 7792hx_1^8\beta^4\delta^2 - 104{,}064hx_1^7\beta^7 - 4192hx_1^7\beta^6\delta + 28{,}272hx_1^7\beta^5\delta^2 + 369{,}280hx_1^6\beta^8 \\
&+ 377{,}088hx_1^6\beta^7\delta + 83{,}712hx_1^6\beta^6\delta^2 - 185{,}600hx_1^5\beta^9 - 291{,}200hx_1^5\beta^8\delta \\
&- 92{,}800hx_1^5\beta^7\delta^2 - 284{,}160hx_1^4\beta^{10} - 284{,}160hx_1^4\beta^9\delta - 71{,}040hx_1^4\beta^8\delta^2 \\
&+ 38{,}304x_1^{10}\beta^5 - 24{,}976x_1^{10}\beta^4\delta - 2368x_1^{10}\beta^3\delta^2 - 475{,}968x_1^9\beta^6 - 244{,}512x_1{}^9\beta^5\delta \\
&- 30{,}608x_1^9\beta^4\delta^2 - 288x_1^9\beta^3\delta^3 + 432{,}000x_1^8\beta^7 - 200{,}256x_1^8\beta^6\delta - 193{,}392x_1{}^8\beta^5\delta^2 \\
&- 3920x_1^8\beta^4\delta^3 - 3{,}033{,}216x_1^7\beta^8 - 2{,}654{,}720x_1^7\beta^7\delta - 590{,}912x_1^7\beta^6\delta^2 \\
&- 21{,}792x_1{}^7\beta^5\delta^3 + 2{,}387{,}584x_1^6\beta^9 + 1{,}954{,}560x_1^6\beta^8\delta + 164{,}832x_1^6\beta^7\delta^2 \\
&- 94{,}176x_1^6\beta^6\delta^3 - 3{,}270{,}144x_1^5\beta^{10} - 4{,}404{,}480x_1^5\beta^9\delta - 1{,}849{,}088x_1^5\beta^8\delta^2 \\
&- 242{,}752x_1^5\beta^7\delta^3 + 2{,}204{,}160x_1^4\beta^{11} + 3{,}870{,}720x_1{}^4\beta^{10}\delta + 2{,}160{,}768x_1^4\beta^9\delta^2 \\
&+ 388{,}224x_1^4\beta^8\delta^3 + 917{,}504x_1^3\beta^{12} + 1{,}376{,}256x_1^3\beta^{11}\delta + 688{,}128x_1^3\beta^{10}\delta^2 \\
&+ 114{,}688x_1^3\beta^9\delta^3 + 1792h^2x_1^7\beta^5 + 4416h^2x_1^6\beta^6 + 192h^2x_1^5\beta^7 - 6400h^2x_1^4\beta^8 \\
&- 8832hx_1^8\beta^5 - 64{,}896hx_1^7\beta^6 - 3584hx_1^7\beta^5\delta + 28{,}480hx_1^6\beta^7 - 16{,}896hx_1^6\beta^6\delta \\
&- 187{,}904hx_1^5\beta^8 - 90{,}432hx_1^5\beta^7\delta + 94{,}720hx_1^4\beta^9 + 68{,}480hx_1^4\beta^8\delta \\
&+ 113{,}664hx_1^3\beta^{10} + 56{,}832hx_1^3\beta^9\delta + 95{,}872x_1^8\beta^6 + 8832x_1{}^8\beta^5\delta - 97{,}792x_1^7\beta^7 \\
&+ 89{,}728x_1^7\beta^6\delta + 1792x_1^7\beta^5\delta^2 + 1{,}114{,}880x_1^6\beta^8 + 462{,}016x_1^6\beta^7\delta + 12{,}480x_1^6\beta^6\delta^2
\end{aligned}
$$

$$
\begin{aligned}
&- 1{,}010{,}432x_1^5\beta^9 - 317{,}440x_1^5\beta^8\delta + 81{,}664x_1^5\beta^7\delta^2 + 1{,}829{,}888x_1^4\beta^{10} \\
&+ 1{,}600{,}384x_1^4\beta^9\delta + 327{,}296{x_1}^4\beta^8\delta^2 - 1{,}222{,}144x_1^3\beta^{11} - 1{,}410{,}304x_1^3\beta^{10}\delta \\
&- 390{,}144x_1^3\beta^9\delta^2 - 458{,}752x_1^2\beta^{12} - 458{,}752x_1^2\beta^{11}\delta - 114{,}688x_1^2\beta^{10}\delta^2 + 512hx_1^5\beta^7 \\
&+ 37{,}632hx_1^4\beta^8 - 19{,}200hx_1^3\beta^9 - 18{,}944hx_1^2\beta^{10} - 2048x_1^6\beta^7 - 170{,}240x_1^5\beta^8 \\
&- 512x_1^5\beta^7\delta + 196{,}608x_1^4\beta^9 - 20{,}224x_1^4\beta^8\delta - 571{,}904x_1^3\beta^{10} - 244{,}224x_1^3\beta^9\delta \\
&+ 383{,}488x_1^2\beta^{11} + 218{,}624x_1^2\beta^{10}\delta + 131{,}072x_1\beta^{12} + 65{,}536x_1\beta^{11}\delta - 8192x_1^3\beta^9 \\
&+ 76{,}800x_1^2\beta^{10} - 52{,}224x_1\beta^{11} - 16{,}384\beta^{12}.
\end{aligned}
$$

g_1 and g_2 in Theorem 3.1 are given by

$$
\begin{aligned}
g_1 = {}& 256h^2\beta^9 - 1408h^2\beta^8\delta - 3264h^2\beta^7\delta^2 - 4128h^2\beta^6\delta^3 - 3072h^2\beta^5\delta^4 - 1344h^2\beta^4\delta^5 \\
&- 320h^2\beta^3\delta^6 - 32h^2\beta^2\delta^7 + 1024h\beta^10 + 6144h\beta^9\delta + 15{,}872h\beta^8\delta^2 + 23{,}040h\beta^7\delta^3 \\
&+ 20{,}544h\beta^6\delta^4 + 11{,}520h\beta^5\delta^5 + 3968h\beta^4\delta^6 + 768h\beta^3\delta^7 + 64h\beta^2\delta^8 - 256\beta^9\delta^2 \\
&- 1408\beta^8\delta^3 - 3456\beta^7\delta^4 - 4992\beta^6\delta^5 - 4608\beta^5\delta^6 - 2688\beta^4\delta^7 - 896\beta^3\delta^8 \\
&- 128\beta^2\delta^9 - 64h^3\beta^7 - 288h^3\beta^6\delta - 512h^3\beta^5\delta^2 - 448h^3\beta^4\delta^3 - 192h^3\beta^3\delta^4 \\
&- 32h^3\beta^2\delta^5 + 768h^2\beta^8 + 3456h^2\beta^7\delta + 6624h^2\beta^6\delta^2 + 7104h^2\beta^5\delta^3 + 4640h^2\beta^4\delta^4 \\
&+ 1824h^2\beta^3\delta^5 + 384h^2\beta^2\delta^6 + 32h^2\beta\delta^7 - 2304h\beta^9 - 11{,}136h\beta^8\delta - 23{,}808h\beta^7\delta^2 \\
&- 30{,}144h\beta^6\delta^3 - 25{,}088h\beta^5\delta^4 - 13{,}888h\beta^4\delta^5 - 4800h\beta^3\delta^6 - 896h\beta^2\delta^7 \\
&- 64h\beta\delta^8 + 896\beta^8\delta^2 + 3712\beta^7\delta^3 + 6656\beta^6\delta^4 + 7040\beta^5\delta^5 + 4928\beta^4\delta^6 + 2240\beta^3\delta^7 \\
&+ 576\beta^2\delta^8 + 64\beta\delta^9 - 96h^4\beta^5 - 248h^4\beta^4\delta - 216h^4\beta^3\delta^2 - 72h^4\beta^2\delta^3 - 8h^4\beta\delta^4 \\
&+ 960h^3\beta^6 + 3072h^3\beta^5\delta + 3712h^3\beta^4\delta^2 + 2112h^3\beta^3\delta^3 + 576h^3\beta^2\delta^4 + 64h^3\beta\delta^5 \\
&- 3456h^2\beta^7 - 12{,}544h^2\beta^6\delta - 18{,}944h^2\beta^5\delta^2 - 15{,}584h^2\beta^4\delta^3 - 7456h^2\beta^3\delta^4 \\
&- 1952h^2\beta^2\delta^5 - 224h^2\beta\delta^6 + 4864h\beta^8 + 18{,}688h\beta^7\delta + 32{,}640h\beta^6\delta^2 \\
&+ 35{,}200h\beta^5\delta^3 + 25{,}152h\beta^4\delta^4 + 11{,}328h\beta^3\delta^5 + 2880h\beta^2\delta^6 + 320h\beta\delta^7 \\
&+ 384\beta^8\delta - 384\beta^7\delta^2 - 3840\beta^6\delta^3 - 6912\beta^5\delta^4 - 6720\beta^4\delta^5 - 4032\beta^3\delta^6 - 1344\beta^2\delta^7 \\
&- 192\beta\delta^8 + 480h^4\beta^4 + 960h^4\beta^3\delta + 672h^4\beta^2\delta^2 + 216h^4\beta\delta^3 + 24h^4\delta^4 - 3840h^3\beta^5 \\
&- 9600h^3\beta^4\delta - 9216h^3\beta^3\delta^2 - 4416h^3\beta^2\delta^3 - 1056h^3\beta\delta^4 - 96h^3\delta^5 + 10{,}240h^2\beta^6 \\
&+ 30{,}080h^2\beta^5\delta + 37{,}632h^2\beta^4\delta^2 + 26{,}368h^2\beta^3\delta^3 + 10{,}752h^2\beta^2\delta^4 + 2400h^2\beta\delta^5 \\
&+ 224h^2\delta^6 - 10{,}240h\beta^7 - 33{,}280h\beta^6\delta - 52{,}480h\beta^5\delta^2 - 52{,}480h\beta^4\delta^3 \\
&- 33{,}536h\beta^3\delta^4 - 13{,}056h\beta^2\delta^5 - 2816h\beta\delta^6 - 256h\delta^7 + 512\beta^8 + 1536\beta^7\delta \\
&+ 5632\beta^6\delta^2 + 11{,}648\beta^5\delta^3 + 13{,}440\beta^4\delta^4 + 9856\beta^3\delta^5 + 4480\beta^2\delta^6 + 1152\beta\delta^7 \\
&+ 128\delta^8 - 8h^6\beta - h^6\delta + 120h^5\beta^2 + 96h^5\beta\delta + 18h^5\delta^2 - 1056h^4\beta^3 - 1308h^4\beta^2\delta \\
&- 600h^4\beta\delta^2 - 96h^4\delta^3 + 5056h^3\beta^4 + 8448h^3\beta^3\delta + 5808h^3\beta^2\delta^2 + 1952h^3\beta\delta^3 \\
&+ 264h^3\delta^4 - 10{,}880h^2\beta^5 - 22{,}832h^2\beta^4\delta - 21{,}760h^2\beta^3\delta^2 - 11{,}488h^2\beta^2\delta^3
\end{aligned}
$$

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\begin{aligned}
& - 3328h^2\beta\delta^4 - 416h^2\delta^5 + 8064h\beta^6 + 18{,}944h\beta^5\delta + 24{,}480h\beta^4\delta^2 + 20{,}096h\beta^3\delta^3 \\
& + 10{,}144h\beta^2\delta^4 + 2880h\beta\delta^5 + 352h\delta^6 - 512\beta^7 - 832\beta^6\delta - 2688\beta^5\delta^2 - 4480\beta^4\delta^3 \\
& - 4480\beta^3\delta^4 - 2688\beta^2\delta^5 - 896\beta\delta^6 - 128\delta^7, \\
g_2 = {} & 384h^2\beta^8\delta^2 + 1920h^2\beta^7\delta^3 + 3936h^2\beta^6\delta^4 + 4224h^2\beta^5\delta^5 + 2496h^2\beta^4\delta^6 + 768h^2\beta^3\delta^7 \\
& + 96h^2\beta^2\delta^8 - 1536h\beta^9\delta^2 - 8448h\beta^8\delta^3 - 19{,}584h\beta^7\delta^4 - 24{,}768h\beta^6\delta^5 \\
& - 18{,}432h\beta^5\delta^6 - 8064h\beta^4\delta^7 - 1920h\beta^3\delta^8 - 192h\beta^2\delta^9 + 384\beta^8\delta^4 + 1920\beta^7\delta^5 \\
& + 3968\beta^6\delta^6 + 4352\beta^5\delta^7 + 2688\beta^4\delta^8 + 896\beta^3\delta^9 + 128\beta^2\delta^10 - 192h^4\beta^7 \\
& - 768h^4\beta^6\delta - 1216h^4\beta^5\delta^2 - 968h^4\beta^4\delta^3 - 408h^4\beta^3\delta^4 - 88h^4\beta^2\delta^5 - 8h^4\beta\delta^6 \\
& + 1792h^3\beta^8 + 8192h^3\beta^7\delta + 15{,}456h^3\beta^6\delta^2 + 15{,}552h^3\beta^5\delta^3 + 8992h^3\beta^4\delta^4 \\
& + 2976h^3\beta^3\delta^5 + 512h^3\beta^2\delta^6 + 32h^3\beta\delta^7 - 4352h^2\beta^9 - 22{,}016h^2\beta^8\delta \\
& - 48{,}320h^2\beta^7\delta^2 - 60{,}384h^2\beta^6\delta^3 - 47{,}104h^2\beta^5\delta^4 - 23{,}456h^2\beta^4\delta^5 - 7264h^2\beta^3\delta^6 \\
& - 1280h^2\beta^2\delta^7 - 96h^2\beta\delta^8 + 1024h\beta^10 + 4608h\beta^9\delta + 12{,}928h\beta^8\delta^2 + 28{,}288h\beta^7\delta^3 \\
& + 42{,}176h\beta^6\delta^4 + 39{,}680h\beta^5\delta^5 + 23{,}168h\beta^4\delta^6 + 8192h\beta^3\delta^7 + 1600h\beta^2\delta^8 \\
& + 128h\beta\delta^9 - 256\beta^9\delta^2 - 1024\beta^8\delta^3 - 2560\beta^7\delta^4 - 4992\beta^6\delta^5 - 6400\beta^5\delta^6 \\
& - 4928\beta^4\delta^7 - 2240\beta^3\delta^8 - 576\beta^2\delta^9 - 64\beta\delta^10 - 32h^5\beta^5 - 104h^5\beta^4\delta - 120h^5\beta^3\delta^2 \\
& - 56h^5\beta^2\delta^3 - 8h^5\beta\delta^4 + 832h^4\beta^6 + 2688h^4\beta^5\delta + 3392h^4\beta^4\delta^2 + 2128h^4\beta^3\delta^3 \\
& + 672h^4\beta^2\delta^4 + 80h^4\beta\delta^5 - 5504h^3\beta^7 - 19{,}840h^3\beta^6\delta - 29{,}312h^3\beta^5\delta^2 \\
& - 23{,}264h^3\beta^4\delta^3 - 10{,}720h^3\beta^3\delta^4 - 2720h^3\beta^2\delta^5 - 288h^3\beta\delta^6 + 11{,}520h^2\beta^8 \\
& + 46{,}336h^2\beta^7\delta + 81{,}088h^2\beta^6\delta^2 + 81{,}984h^2\beta^5\delta^3 + 52{,}448h^2\beta^4\delta^4 + 21{,}280h^2\beta^3\delta^5 \\
& + 5088h^2\beta^2\delta^6 + 544h^2\beta\delta^7 - 3072h\beta^9 - 11{,}648h\beta^8\delta - 27{,}264h\beta^7\delta^2 \\
& - 48{,}000h\beta^6\delta^3 - 55{,}808h\beta^5\delta^4 - 40{,}448h\beta^4\delta^5 - 18{,}048h\beta^3\delta^6 - 4608h\beta^2\delta^7 \\
& - 512h\beta\delta^8 + 768\beta^8\delta^2 + 2304\beta^7\delta^3 + 4224\beta^6\delta^4 + 6528\beta^5\delta^5 + 6720\beta^4\delta^6 \\
& + 4032\beta^3\delta^7 + 1344\beta^2\delta^8 + 192\beta\delta^9 - 56h^6\beta^3 - 88h^6\beta^2\delta - 36h^6\beta\delta^2 - 4h^6\delta^3 \\
& + 672h^5\beta^4 + 1392h^5\beta^3\delta + 960h^5\beta^2\delta^2 + 264h^5\beta\delta^3 + 24h^5\delta^4 - 4288h^4\beta^5 \\
& - 10{,}752h^4\beta^4\delta - 10{,}512h^4\beta^3\delta^2 - 5200h^4\beta^2\delta^3 - 1272h^4\beta\delta^4 - 120h^4\delta^5 \\
& + 16{,}384h^3\beta^6 + 48{,}128h^3\beta^5\delta + 59{,}904h^3\beta^4\delta^2 + 41{,}728h^3\beta^3\delta^3 + 16{,}896h^3\beta^2\delta^4 \\
& + 3648h^3\beta\delta^5 + 320h^3\delta^6 - 27{,}520h^2\beta^7 - 92{,}800h^2\beta^6\delta - 141{,}888h^2\beta^5\delta^2 \\
& - 129{,}472h^2\beta^4\delta^3 - 74{,}752h^2\beta^3\delta^4 - 26{,}880h^2\beta^2\delta^5 - 5472h^2\beta\delta^6 - 480h^2\delta^7 \\
& + 7680h\beta^8 + 26{,}880h\beta^7\delta + 58{,}880h\beta^6\delta^2 + 88{,}960h\beta^5\delta^3 + 85{,}632h\beta^4\delta^4 \\
& + 52{,}352h\beta^3\delta^5 + 19{,}840h\beta^2\delta^6 + 4224h\beta\delta^7 + 384h\delta^8 - 2304\beta^7\delta^2 - 6912\beta^6\delta^3 \\
& - 11{,}648\beta^5\delta^4 - 13{,}440\beta^4\delta^5 - 9856\beta^3\delta^6 - 4480\beta^2\delta^7 - 1152\beta\delta^8 - 128\delta^9 - 2h^7\beta \\
& - 2h^7\delta + 120h^6\beta^2 + 92h^6\beta\delta + 21h^6\delta^2 - 1368h^5\beta^3 - 1680h^5\beta^2\delta - 720h^5\beta\delta^2 \\
& - 114h^5\delta^3 + 6976h^4\beta^4 + 11{,}856h^4\beta^3\delta + 8196h^4\beta^2\delta^2 + 2696h^4\beta\delta^3 + 360h^4\delta^4
\end{aligned}
$$

$$
\begin{aligned}
&- 18{,}784h^3\beta^5 - 39{,}840h^3\beta^4\delta - 38{,}080h^3\beta^3\delta^2 - 20{,}048h^3\beta^2\delta^3 - 5664h^3\beta\delta^4 \\
&- 680h^3\delta^5 + 23{,}936h^2\beta^6 + 59{,}712h^2\beta^5\delta + 73{,}648h^2\beta^4\delta^2 + 54{,}976h^2\beta^3\delta^3 \\
&+ 25{,}344h^2\beta^2\delta^4 + 6656h^2\beta\delta^5 + 768h^2\delta^6 - 6272h\beta^7 - 15{,}872h\beta^6\delta - 30{,}208h\beta^5\delta^2 \\
&- 37{,}920h\beta^4\delta^3 - 29{,}952h\beta^3\delta^4 - 14{,}624h\beta^2\delta^5 - 4032h\beta\delta^6 - 480h\delta^7 - 256\beta^7\delta \\
&+ 960\beta^6\delta^2 + 2688\beta^5\delta^3 + 4480\beta^4\delta^4 + 4480\beta^3\delta^5 + 2688\beta^2\delta^6 + 896\beta\delta^7 + 128\delta^8.
\end{aligned}
$$

$\widetilde{r}_{13}$ and $\widetilde{r}_{14}$ in (3.1) are given by

$$
\begin{aligned}
\widetilde{r}_{13} = {} & 16\beta^5x_1^3 + 24\beta^4\delta x_1^3 - 28\beta^4x_1^4 + 12\beta^3\delta^2x_1^3 - 30\beta^3\delta x_1^4 + 28\beta^3x_1^5 + 2\beta^2\delta^3x_1^3 \\
& - 10\beta^2\delta^2x_1^4 + 19\beta^2\delta x_1^5 - 28\beta^2x_1^6 - \beta\delta^3x_1^4 + 3\beta\delta^2x_1^5 - 16\beta\delta x_1^6 + 9\beta x_1^7 - 2\delta^2x_1^6 \\
& + 2\delta x_1^7 - 48\beta^5x_1^2 - 48\beta^4\delta x_1^2 + 76\beta^4x_1^3 - 12\beta^3\delta^2x_1^2 + 52\beta^3\delta {x_1}^3 - 60\beta^3x_1^4 \\
& + 8\beta^2\delta^2x_1^3 - 20\beta^2\delta x_1^4 + 48\beta^2x_1^5 + 12\beta\delta x_1^5 - 10\beta x_1^6 + 48\beta^5x_1 + 24\beta^4\delta x_1 - 68\beta^4x_1^2 \\
& - 22\beta^3\delta x_1^2 + 36\beta^3x_1^3 + \beta^2\delta x_1^3 - 20\beta^2x_1^4 + \beta x_1^5 - 16\beta^5 + 20\beta^4x_1 - 4\beta^3x_1^2, \\
\widetilde{r}_{14} = {} & 16{,}384\beta^{12}x_1^8 + 65{,}536\beta^{11}\delta x_1^8 - 1024\beta^{11}x_1^9 + 114{,}688\beta^{10}\delta^2x_1^8 + 22{,}784\beta^{10}\delta x_1^9 \\
& - 3584\beta^{10}x_1^{10} + 114{,}688\beta^9\delta^3x_1^8 + 82{,}176\beta^9\delta^2x_1^9 - 56{,}320\beta^9\delta x_1^{10} - 12{,}032\beta^9x_1^{11} \\
& + 71{,}680\beta^8\delta^4x_1^8 + 119{,}744\beta^8\delta^3x_1^9 - 143{,}488\beta^8\delta^2x_1^{10} + 38{,}256\beta^8\delta x_1^{11} + 5392\beta^8x_1^{12} \\
& + 28{,}672\beta^7\delta^5x_1^8 + 95{,}360\beta^7\delta^4x_1^9 - 165{,}152\beta^7\delta^3x_1^{10} + 142{,}176\beta^7\delta^2x_1^{11} \\
& - 123{,}024\beta^7\delta x_1^{12} - 77{,}392\beta^7x_1^{13} + 7168\beta^6\delta^6x_1^8 + 45{,}168\beta^6\delta^5x_1^9 - 105{,}120\beta^6\delta^4x_1^{10} \\
& + 164{,}428\beta^6\delta^3{x_1}^{11} - 303{,}016\beta^6\delta^2x_1^{12} - 73{,}196\beta^6\delta x_1^{13} + 99{,}076\beta^6x_1^{14} \\
& + 1024\beta^5\delta^7x_1^8 + 12{,}752\beta^5\delta^6{x_1}^9 - 39{,}152\beta^5\delta^5x_1^{10} + 94{,}516\beta^5\delta^4x_1^{11} \\
& - 293{,}222\beta^5\delta^3x_1^{12} + 26{,}914\beta^5\delta^2x_1^{13} + 92{,}054\beta^5\delta x_1^{14} - 88{,}632\beta^5x_1^{15} \\
& + 64\beta^4\delta^8x_1^8 + 1988\beta^4\delta^7x_1^9 - 8360\beta^4\delta^6x_1^{10} + 28{,}928\beta^4\delta^5x_1^{11} - 145{,}936\beta^4\delta^4x_1^{12} \\
& + 54{,}948\beta^4\delta^3x_1^{13} - 26{,}074\beta^4\delta^2x_1^{14} - 79{,}829\beta^4\delta x_1^{15} + 75{,}624\beta^4x_1^{16} + 132\beta^3\delta^8x_1^9 \\
& - 914\beta^3\delta^7x_1^{10} + 4222\beta^3\delta^6x_1^{11} - 39{,}134\beta^3\delta^5x_1^{12} + 21{,}313\beta^3\delta^4x_1^{13} - 60{,}308\beta^3\delta^3x_1^{14} \\
& + 6274\beta^3\delta^2x_1^{15} + 89{,}784\beta^3\delta x_1^{16} - 34{,}413\beta^3x_1^{17} - 36\beta^2\delta^8x_1^{10} + 122\beta^2\delta^7x_1^{11} \\
& - 5252\beta^2\delta^6x_1^{12} + 1582\beta^2\delta^5x_1^{13} - 26{,}593\beta^2\delta^4{x_1}^{14} + 25{,}832\beta^2\delta^3x_1^{15} \\
& + 35{,}571\beta^2\delta^2x_1^{16} - 36{,}585\beta^2\delta x_1^{17} + 5616\beta^2x_1^{18} - 23\beta\delta^8x_1^{11} - 263\beta\delta^7x_1^{12} \\
& - 617\beta\delta^6{x_1}^{13} - 4692\beta\delta^5x_1^{14} + 8576\beta\delta^4x_1^{15} + 5121\beta\delta^3x_1^{16} - 12{,}174\beta\delta^2x_1^{17} \\
& + 4056\beta\delta {x_1}^{18} - 94\delta^7x_1^{13} - 278\delta^6x_1^{14} + 834\delta^5x_1^{15} + 162\delta^4x_1^{16} - 1248\delta^3x_1^{17} \\
& + 624\delta^2x_1^{18} - 131{,}072\beta^{12}x_1^7 - 458{,}752\beta^{11}\delta x_1^7 - 26{,}112\beta^{11}x_1^8 - 688{,}128\beta^{10}\delta^2x_1^7 \\
& - 279{,}552\beta^{10}\delta x_1^8 + 137{,}728\beta^{10}x_1^9 - 573{,}440\beta^9\delta^3x_1^7 - 673{,}152\beta^9\delta^2x_1^8 \\
& + 713{,}344\beta^9\delta x_1^9 - 112{,}928\beta^9x_1^{10} - 286{,}720\beta^8\delta^4x_1^7 - 748{,}800\beta^8\delta^3x_1^8 \\
& + 1{,}250{,}816\beta^8\delta^2x_1^9 - 753{,}728\beta^8\delta {x_1}^{10} + 250{,}592\beta^8x_1^{11} - 86{,}016\beta^7\delta^5x_1^7 \\
& - 456{,}480\beta^7\delta^4x_1^8 + 1{,}080{,}960\beta^7\delta^3x_1^9 - 1{,}279{,}928\beta^7\delta^2x_1^{10} + 1{,}360{,}552\beta^7\delta x_1^{11}
\end{aligned}
$$

$$
\begin{aligned}
&+ 167{,}264\beta^7x_1^{12} - 14{,}336\beta^6\delta^6x_1^7 - 158{,}016\beta^6\delta^5x_1^8 + 515{,}360\beta^6\delta^4x_1^9 \\
&- 975{,}888\beta^6\delta^3x_1^{10} + 2{,}034{,}944\beta^6\delta^2x_1^{11} - 219{,}388\beta^6\delta x_1^{12} - 275{,}684\beta^6x_1^{13} \\
&- 1024\beta^5\delta^7x_1^7 - 29{,}256\beta^5\delta^6x_1^8 + 136{,}744\beta^5\delta^5x_1^9 - 375{,}816\beta^5\delta^4x_1^{10} \\
&+ 1{,}374{,}780\beta^5\delta^3x_1^{11} - 539{,}720\beta^5\delta^2x_1^{12} + 57{,}612\beta^5\delta x_1^{13} + 257{,}536\beta^5x_1^{14} \\
&- 2256\beta^4\delta^7x_1^8 + 18{,}512\beta^4\delta^6x_1^9 - 68{,}092\beta^4\delta^5x_1^{10} + 468{,}512\beta^4\delta^4x_1^{11} \\
&- 299{,}474\beta^4\delta^3x_1^{12} + 406{,}892\beta^4\delta^2x_1^{13} + 49{,}154\beta^4\delta x_1^{14} - 235{,}944\beta^4x_1^{15} \\
&+ 948\beta^3\delta^7x_1^9 - 3148\beta^3\delta^6x_1^{10} + 77{,}056\beta^3\delta^5x_1^{11} - 48{,}904\beta^3\delta^4x_1^{12} + 251{,}810\beta^3\delta^3x_1^{13} \\
&- 158{,}498\beta^3\delta^2x_1^{14} - 188{,}268\beta^3\delta x_1^{15} + 94{,}632\beta^3x_1^{16} + 362\beta^2\delta^7x_1^{10} + 4660\beta^2\delta^6x_1^{11} \\
&+ 4654\beta^2\delta^5x_1^{12} + 56{,}508\beta^2\delta^4x_1^{13} - 81{,}166\beta^2\delta^3x_1^{14} - 39{,}246\beta^2\delta^2x_1^{15} \\
&+ 65{,}808\beta^2\delta x_1^{16} - 11{,}856\beta^2x_1^{17} + 1400\beta\delta^6x_1^{12} + 4050\beta\delta^5x_1^{13} - 10{,}576\beta\delta^4x_1^{14} \\
&- 1134\beta\delta^3x_1^{15} + 10{,}620\beta\delta^2x_1^{16} - 4368\beta\delta {x_1}^{17} + 458{,}752\beta^{12}x_1^6 + 1{,}376{,}256\beta^{11}\delta x_1^6 \\
&+ 211{,}456\beta^{11}x_1^7 + 1{,}720{,}320\beta^{10}\delta^2x_1^6 + 1{,}198{,}848\beta^{10}\delta {x_1}^7 - 840{,}704\beta^{10}x_1^8 \\
&+ 1{,}146{,}880\beta^9\delta^3x_1^6 + 2{,}133{,}120\beta^9\delta^2x_1^7 - 3{,}028{,}224\beta^9\delta x_1^8 + 970{,}208\beta^9{x_1}^9 \\
&+ 430{,}080\beta^8\delta^4x_1^6 + 1{,}797{,}760\beta^8\delta^3x_1^7 - 4{,}015{,}360\beta^8\delta^2x_1^8 + 3{,}289{,}488\beta^8\delta x_1^9 \\
&- 1{,}539{,}232\beta^8x_1^{10} + 86{,}016\beta^7\delta^5x_1^6 + 797{,}280\beta^7\delta^4x_1^7 - 2{,}614{,}720\beta^7\delta^3x_1^8 \\
&+ 3{,}781{,}240\beta^7\delta^2x_1^9 - 4{,}760{,}864\beta^7\delta x_1^{10} + 365{,}600\beta^7x_1^{11} + 7168\beta^6\delta^6x_1^6 \\
&+ 180{,}528\beta^6\delta^5{x_1}^7 - 893{,}760\beta^6\delta^4x_1^8 + 1{,}968{,}552\beta^6\delta^3x_1^9 - 4{,}910{,}872\beta^6\delta^2x_1^{10} \\
&+ 1{,}668{,}756\beta^6\delta x_1^{11} - 6204\beta^6{x_1}^{12} + 16{,}504\beta^5\delta^6x_1^7 - 151{,}712\beta^5\delta^5x_1^8 \\
&+ 463{,}392\beta^5\delta^4x_1^9 - 2{,}264{,}124\beta^5\delta^3x_1^{10} + 1{,}469{,}674\beta^5\delta^2x_1^{11} - 956{,}142\beta^5\delta x_1^{12} \\
&- 139{,}128\beta^5x_1^{13} - 9792\beta^4\delta^6{x_1}^8 + 32{,}348\beta^4\delta^5x_1^9 - 473{,}008\beta^4\delta^4x_1^{10} \\
&+ 407{,}540\beta^4\delta^3x_1^{11} - 923{,}824\beta^4\delta^2x_1^{12} + 335{,}165\beta^4\delta x_1^{13} + 232{,}392\beta^4x_1^{14} \\
&- 2306\beta^3\delta^6x_1^9 - 34{,}892\beta^3\delta^5x_1^{10} + 4698\beta^3\delta^4x_1^{11} - 276{,}618\beta^3\delta^3x_1^{12} \\
&+ 300{,}364\beta^3\delta^2x_1^{13} + 82{,}116\beta^3\delta x_1^{14} - 81{,}813\beta^3x_1^{15} + 82\beta^2\delta^6x_1^{10} - 8150\beta^2\delta^5x_1^{11} \\
&- 23{,}493\beta^2\delta^4x_1^{12} + 56{,}769\beta^2\delta^3x_1^{13} - 5820\beta^2\delta^2x_1^{14} - 26{,}586\beta^2\delta x_1^{15} + 6864\beta^2x_1^{16} \\
&+ 237\beta\delta^5x_1^{12} + 332\beta\delta^4x_1^{13} - 1179\beta\delta^3x_1^{14} + 306\beta\delta^2x_1^{15} + 312\beta\delta {x_1}^{16} \\
&- 917{,}504\beta^{12}x_1^5 - 2{,}293{,}760\beta^{11}\delta x_1^5 - 663{,}040\beta^{11}x_1^6 - 2{,}293{,}760\beta^{10}\delta^2x_1^5 \\
&- 2{,}598{,}400\beta^{10}\delta x_1^6 + 2{,}352{,}640\beta^{10}x_1^7 - 1{,}146{,}880\beta^9\delta^3{x_1}^5 - 3{,}444{,}480\beta^9\delta^2x_1^6 \\
&+ 6{,}412{,}160\beta^9\delta x_1^7 - 2{,}780{,}960\beta^9x_1^8 - 286{,}720\beta^8\delta^4x_1^5 - 2{,}097{,}920\beta^8\delta^3x_1^6 \\
&+ 6{,}423{,}040\beta^8\delta^2x_1^7 - 6{,}492{,}960\beta^8\delta x_1^8 + 3{,}752{,}032\beta^8{x_1}^9 - 28{,}672\beta^7\delta^5x_1^5 \\
&- 606{,}560\beta^7\delta^4x_1^6 + 3{,}009{,}920\beta^7\delta^3x_1^7 - 5{,}181{,}680\beta^7\delta^2x_1^8 + 7{,}895{,}704\beta^7\delta x_1^9 \\
&- 1{,}641{,}376\beta^7x_1^{10} - 67{,}680\beta^6\delta^5x_1^6 + 661{,}920\beta^6\delta^4{x_1}^7 - 1{,}700{,}848\beta^6\delta^3x_1^8 \\
&+ 5{,}539{,}456\beta^6\delta^2x_1^9 - 3{,}043{,}100\beta^6\delta x_1^{10} + 796{,}252\beta^6x_1^{11} + 54{,}120\beta^5\delta^5{x_1}^7 \\
&- 178{,}960\beta^5\delta^4x_1^8 + 1{,}564{,}088\beta^5\delta^3x_1^9 - 1{,}459{,}312\beta^5\delta^2{x_1}^{10} + 1{,}584{,}272\beta^5\delta x_1^{11}
\end{aligned}
$$

$$
\begin{aligned}
&- 259{,}008\beta^5 x_1{}^{12} + 6660\beta^4\delta^5 x_1^8 + 145{,}176\beta^4\delta^4 x_1^9 - 143{,}040\beta^4\delta^3 x_1{}^{10} \\
&+ 709{,}624\beta^4\delta^2 x_1^{11} - 521{,}700\beta^4\delta x_1^{12} - 37{,}752\beta^4 x_1^{13} - 1044\beta^3\delta^5 x_1^9 \\
&+ 22{,}052\beta^3\delta^4 x_1{}^{10} + 73{,}958\beta^3\delta^3 x_1^{11} - 160{,}300\beta^3\delta^2 x_1^{12} + 41{,}544\beta^3\delta x_1{}^{13} \\
&+ 16{,}776\beta^3 x_1^{14} - 2642\beta^2\delta^4 x_1^{11} - 3128\beta^2\delta^3 x_1^{12} + 9558\beta^2\delta^2 x_1^{13} - 2904\beta^2\delta x_1{}^{14} \\
&- 624\beta^2 x_1^{15} + 1{,}146{,}880\beta^{12} x_1^4 + 2{,}293{,}760\beta^{11}\delta x_1^4 + 1{,}128{,}960\beta^{11} x_1{}^5 \\
&+ 1{,}720{,}320\beta^{10}\delta^2 x_1^4 + 3{,}198{,}720\beta^{10}\delta x_1^5 - 3{,}722{,}240\beta^{10} x_1^6 + 573{,}440\beta^9\delta^3 x_1^4 \\
&+ 3{,}033{,}600\beta^9\delta^2 x_1^5 - 7{,}662{,}080\beta^9\delta x_1^6 + 4{,}146{,}400\beta^9 x_1^7 + 71{,}680\beta^8\delta^4 x_1^4 \\
&+ 1{,}199{,}040\beta^8\delta^3 x_1^5 - 5{,}532{,}800\beta^8\delta^2 x_1^6 + 6{,}812{,}560\beta^8\delta x_1^7 - 4{,}819{,}840\beta^8 x_1^8 \\
&+ 170{,}400\beta^7\delta^4 x_1{}^5 - 1{,}673{,}760\beta^7\delta^3 x_1^6 + 3{,}524{,}240\beta^7\delta^2 x_1^7 - 6{,}879{,}376\beta^7\delta x_1^8 \\
&+ 2{,}271{,}104\beta^7 x_1^9 - 178{,}400\beta^6\delta^4 x_1^6 + 577{,}452\beta^6\delta^3 x_1^7 - 2{,}957{,}336\beta^6\delta^2 x_1^8 \\
&+ 2{,}404{,}508\beta^6\delta x_1{}^9 - 1{,}076{,}660\beta^6 x_1^{10} - 3132\beta^5\delta^4 x_1^7 - 368{,}814\beta^5\delta^3 x_1^8 \\
&+ 530{,}918\beta^5\delta^2 x_1^9 - 972{,}678\beta^5\delta x_1{}^{10} + 363{,}736\beta^5 x_1^{11} + 5256\beta^4\delta^4 x_1^8 \\
&- 20{,}872\beta^4\delta^3 x_1^9 - 144{,}938\beta^4\delta^2 x_1^{10} + 237{,}541\beta^4\delta x_1^{11} - 55{,}944\beta^4 x_1^{12} \\
&- 95\beta^3\delta^4 x_1^9 + 11{,}158\beta^3\delta^3 x_1^{10} + 10{,}898\beta^3\delta^2 x_1^{11} - 25{,}248\beta^3\delta x_1^{12} + 5433\beta^3 x_1^{13} \\
&- 179\beta^2\delta^3 x_1^{11} - 63\beta^2\delta^2 x_1^{12} + 267\beta^2\delta x_1^{13} - 917{,}504\beta^{12} x_1^3 - 1{,}376{,}256\beta^{11}\delta x_1^3 \\
&- 1{,}143{,}296\beta^{11} x_1^4 - 688{,}128\beta^{10}\delta^2 x_1^3 - 2{,}279{,}424\beta^{10}\delta x_1^4 + 3{,}556{,}864\beta^{10} x_1^5 \\
&- 114{,}688\beta^9\delta^3 x_1^3 - 1{,}393{,}536\beta^9\delta^2 x_1^4 + 5{,}278{,}080\beta^9\delta x_1^5 - 3{,}553{,}888\beta^9 x_1^6 \\
&- 269{,}824\beta^8\delta^3 x_1^4 + 2{,}464{,}768\beta^8\delta^2 x_1^5 - 3{,}872{,}256\beta^8\delta x_1^6 + 3{,}471{,}008\beta^8 x_1^7 \\
&+ 362{,}752\beta^7\delta^3 x_1^5 - 1{,}081{,}816\beta^7\delta^2 x_1^6 + 3{,}047{,}768\beta^7\delta x_1^7 - 1{,}502{,}944\beta^7 x_1^8 \\
&- 33{,}696\beta^6\delta^3 x_1^6 + 583{,}168\beta^6\delta^2 x_1^7 - 814{,}548\beta^6\delta x_1^8 + 568{,}884\beta^6 x_1^9 \\
&- 12{,}708\beta^5\delta^3 x_1^7 - 25{,}480\beta^5\delta^2 x_1^8 + 176{,}508\beta^5\delta x_1^9 - 147{,}648\beta^5 x_1^{10} + 898\beta^4\delta^3 x_1^8 \\
&- 21{,}716\beta^4\delta^2 x_1^9 - 17{,}438\beta^4\delta x_1^{10} + 21{,}480\beta^4 x_1^{11} + 1262\beta^3\delta^2 x_1^{10} + 36\beta^3\delta x_1^{11} \\
&- 624\beta^3 x_1^{12} + 458{,}752\beta^{12} x_1^2 + 458{,}752\beta^{11}\delta x_1^2 + 691{,}712\beta^{11} x_1^3 \\
&+ 114{,}688\beta^{10}\delta^2 x_1^2 + 879{,}872\beta^{10}\delta x_1^3 - 2{,}044{,}928\beta^{10} x_1^4 + 262{,}272\beta^9\delta^2 x_1^3 \\
&- 1{,}963{,}264\beta^9\delta x_1^4 + 1{,}748{,}128\beta^9 x_1^5 - 446{,}976\beta^8\delta^2 x_1^4 + 1{,}085{,}808\beta^8\delta x_1{}^5 \\
&- 1{,}339{,}232\beta^8 x_1^6 + 95{,}768\beta^7\delta^2 x_1^5 - 538{,}048\beta^7\delta x_1^6 + 466{,}400\beta^7 x_1^7 + 13{,}656\beta^6\delta^2 x_1^6 \\
&+ 72{,}892\beta^6\delta x_1^7 - 100{,}916\beta^6 x_1^8 - 2994\beta^5\delta^2 x_1^7 + 18{,}590\beta^5\delta x_1^8 + 10{,}968\beta^5 x_1^9 \\
&+ 36\beta^4\delta^2 x_1^8 - 2893\beta^4\delta x_1^9 + 216\beta^4 x_1^{10} + 36\beta^3\delta x_1^{10} + 9\beta^3 x_1^{11} - 131{,}072\beta^{12} x_1 \\
&- 65{,}536\beta^{11}\delta x_1 - 231{,}936\beta^{11} x_1^2 - 142{,}848\beta^{10}\delta x_1^2 + 653{,}824\beta^{10} x_1^3 \\
&+ 306{,}304\beta^9\delta x_1^3 - 449{,}888\beta^9 x_1^4 - 107{,}168\beta^8\delta x_1^4 + 222{,}752\beta^8 x_1^5 - 2712\beta^7\delta x_1^5 \\
&- 46{,}816\beta^7 x_1^6 + 4076\beta^6\delta x_1^6 - 5036\beta^6 x_1^7 - 216\beta^5\delta x_1^7 + 2176\beta^5 x_1^8 - 72\beta^4 x_1^9 \\
&+ 16{,}384\beta^{12} + 33{,}280\beta^{11} x_1 - 89{,}600\beta^{10} x_1^2 + 44{,}960\beta^9 x_1^3 - 3472\beta^8 x_1^4 \\
&- 1840\beta^7 x_1^5 + 288\beta^6 x_1^6.
\end{aligned}
$$

Acknowledgements
The author would like to thank the editors and reviewers sincerely.

Funding
The author is supported financially by the Program of Chengdu Normal University (2018JY37).

Availability of data and materials
Not applicable.

Competing interests
The author declares that there are no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Received: 20 February 2020 Accepted: 20 April 2020 Published online: 01 May 2020

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