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# Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations

Omar Bazighifan[1*]

*Correspondence:
o.bazighifan@gmail.com
[1]Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout, Yemen

**Abstract**

This work is concerned with the oscillatory behavior of solutions of fourth-order neutral differential equations. By using the Riccati transformation and integral averaging techniques we obtain some new Kamenev-type and Philos-type oscillation criteria. Our results extend and improve some known results in the literature. An example is given to illustrate our main results.

**MSC:** 34C10; 34K11

**Keywords:** Fourth-order neutral differential equations; Oscillatory solutions

## 1 Introduction

In this paper, we establish some oscillation criteria for the fourth-order neutral differential equation of the form

$$L_y' + q(t)y^{\beta}\big(\delta(t)\big) = 0, \quad t \geq t_0, \tag{1}$$

where $L_y = r(t)(z'''(t))^{\gamma}$ and $z(t) := y(t) + p(t)y(\tau(t))$. We suppose that:

$(S_1)$ $\gamma$ and $\beta$ are quotients of odd positive integers,

$(S_2)$ $r, p, q \in C[t_0, \infty)$, $r(t) > 0$, $r'(t) \geq 0$, $q(t) > 0$, $0 \leq p(t) < p_0 < 1$, $\tau, \delta \in C[t_0, \infty)$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$. and

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(s)}\,\mathrm{d}s = \infty. \tag{2}$$

By a solution of (1) we mean a function $y \in C^3[t_y, \infty)$, $t_y \geq t_0$, satisfying (1) on $[t_y, \infty)$ and such that $r(t)(z'''(t))^{\gamma} \in C^1[t_y, \infty)$. We consider only those solutions $y$ of (1) that satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq t_y$.

A solution $y$ of (1) is said to be nonoscillatory if it is ultimately positive or negative; otherwise, it is said to be oscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Delay differential equations play an important role in applications of real-world life. One area of active research in recent years is studying the sufficient conditions for oscillation of delay differential equations, see [1–23] and the references therein.

In particular, the Emden–Fowler delay differential equations have numerous applications in mathematical, theoretical, and chemical physics; see, for instance, [24–27].

Let us briefly comment on a number of related results, which motivated our study. The authors in [28, 29] were concerned with oscillatory behavior of solutions of fourth-order neutral differential equations and established some new oscillation criteria.

In [30, 31] the authors considered the equation

$$\big(y(t)+p(t)y\big(\tau(t)\big)\big)^{(n)}+q(t)f\big(y\big(\delta(t)\big)\big)=0 \tag{3}$$

and established the criteria for the solutions to be oscillatory when $0 \le p(t) < 1$.

Xing et al. [32] proved that the equation

$$\big(r(t)\big(\big(y(t)+p(t)y\big(\tau(t)\big)\big)^{(n-1)}\big)^{\gamma}\big)'+q(t)y^{\gamma}\big(\delta(t)\big)=0 \tag{4}$$

is oscillatory if

$$\big(\delta^{-1}(t)\big)'\ge\delta_0>0, \qquad \tau'(t)\ge\tau_0>0, \qquad \tau^{-1}\big(\delta(t)\big)<t, \tag{5}$$

and

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\delta(t))}^{t}\frac{\widehat{q}(s)}{r(s)}\big(s^{n-1}\big)^{\gamma}\,\mathrm{d}s>\left(\frac{1}{\delta_0}+\frac{p_0^{\gamma}}{\delta_0\tau_0}\right)\frac{((n-1)!)^{\gamma}}{\mathrm{e}}, \tag{6}$$

where $n$ is even, and $\widehat{q}(t):=\min\{q(\delta^{-1}(t)),q(\delta^{-1}(\tau(t)))\}$.

Moaaz et al. [33] proved that if there exist positive functions $\eta,\zeta\in C^1([t_0,\infty),R)$ such that the equations

$$\psi'(t)+\left(\frac{\mu(\tau^{-1}(\eta(t)))^{n-1}}{(n-1)!r^{1/\gamma}(\tau^{-1}(\eta(t)))}\right)^{\gamma}q(t)P_n^{\gamma}\big(\delta(t)\big)\psi\big(\tau^{-1}\big(\eta(t)\big)\big)=0 \tag{7}$$

and

$$\phi'(t)+\tau^{-1}\big(\zeta(t)\big)R_{n-3}(t)\phi\big(\tau^{-1}\big(\zeta(t)\big)\big)=0 \tag{8}$$

are oscillatory, where

$$P_n(t)=\frac{1}{p(\tau^{-1}(t))}\left(1-\frac{(\tau^{-1}(\tau^{-1}(t)))^{n-1}}{(\tau^{-1}(t))^{n-1}p(\tau^{-1}(\tau^{-1}(t)))}\right),$$
$$R_{n-3}(t)=\int_t^{\infty}R_{n-4}(s)\mathrm{d}s,$$

and

$$R_0(t)=\left(\frac{1}{r(t)}\int_t^{\infty}q(s)P_2^{\gamma}\big(\sigma(s)\big)\,\mathrm{d}s\right)^{1/\gamma}, \tag{9}$$

then (1) is oscillatory.

Our aim in the present paper is employing the Riccati technique to establish some new Kamenev-type and Philos-type conditions for the oscillation of all solutions of equation (1) under condition (2).

The paper is organized as follows. In Sect. 2, we give four lemmas to prove the main results. In Sect. 3, we establish new oscillation results for (1) by using Riccati transformation. In Sect. 4, we establish some new Kamenev-type oscillation criteria for (1). In Sect. 5, we use the integral averaging technique to establish some new Philos-type conditions for the oscillation of all solutions of equation (1). Finally, we present an example and some conclusions to illustrate the main results.

*Remark* 1.1 All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all $t$ large enough.

*Remark* 1.2 Without loss of generality, we can deal only with the positive solutions of (1).

**Notation** For convenience, we use the following notation:

$$A_1(t) = q(t)(1-p_0)^{\beta} M^{\beta-\gamma}\bigl(\delta(t)\bigr),$$
$$A_2(t) = \gamma\varepsilon \frac{\delta^2(t)\zeta\delta'(t)}{r^{1/\gamma}(t)},$$
$$\tilde{A}_1(t) = \int_t^{\infty} A_1(s)\,ds, \qquad B_1(t) = \frac{\pi'(t)}{\pi(t)},$$
$$B_2(t) = \pi(t)q(t)(1-p_0)^{\beta} M^{\beta-\gamma}\bigl(\delta(t)\bigr),$$

and

$$B_3(t) = \gamma\varepsilon \frac{\delta^2(t)\zeta\delta'(t)}{(\pi(t)r(t))^{1/\gamma}}. \tag{10}$$

## 2 Some auxiliary lemmas

We will employ the following lemmas:

**Lemma 2.1** ([34], Lemma 2.1) *Let $\gamma \geq 1$ be the ratio of two odd numbers, and let $V > 0$ and $U$ be constants. Then*

$$Uy - Vy^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{U^{\gamma+1}}{V^{\gamma}}. \tag{11}$$

**Lemma 2.2** ([1, Lemma 2.2.3]) *Let $y \in C^n([t_0,\infty),(0,\infty))$. Assume that $y^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists $t_1 \geq t_0$ such that $y^{(n-1)}(t)y^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} y(t) \neq 0$, then for every $\mu \in (0,1)$, there exists $t_\mu \geq t_1$ such that*

$$y(t) \geq \frac{\mu}{(n-1)!} t^{n-1}\bigl|y^{(n-1)}(t)\bigr| \quad \textit{for } t \geq t_\mu. \tag{12}$$

**Lemma 2.3** ([35]) *Let $y(t)$ be a positive and $n$-times differentiable function on an interval $[T,\infty)$ with its $n$th derivative $y^{(n)}(t)$ nonpositive on $[T,\infty)$, not identically zero on any interval of the form $[T',\infty)$, $T' \geq T$, and such that $y^{(n-1)}(t)y^{(n)}(t) \leq 0$, $t \geq t_y$. Then there exist* *constants* $0 < \theta < 1$*and* $N > 0$ *such that*

$$y'(\theta t) \geq N t^{n-2} y^{(n-1)}(t) \tag{13}$$

*for all sufficient large* $t$.

**Lemma 2.4** *Assume that* $y$ *is an eventually positive solution of* (1). *Then*

$$\left(r(t)\left(z'''(t)\right)^{\gamma}\right)' \leq -q(t)(1-p_0)^{\beta} z^{\beta}\left(\delta(t)\right). \tag{14}$$

*Proof* Let $y$ be an eventually positive solution of (1). Then there exists $t_1 \geq t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1$. Since $r'(t) > 0$, we have

$$z(t) > 0, \qquad z'(t) > 0, \qquad z'''(t) > 0, \qquad z^{(4)}(t) < 0, \qquad \left(r(t)\left(z'''(t)\right)^{\gamma}\right)' \leq 0 \tag{15}$$

for $t \geq t_1$. From the definition of $z$ we get

$$\begin{aligned} y(t) &\geq z(t) - p_0 y\left(\tau(t)\right) \geq z(t) - p_0 z\left(\tau(t)\right) \\ &\geq (1-p_0) z(t), \end{aligned}$$

which, together with (1), gives

$$\left(r(t)\left(z'''(t)\right)^{\gamma}\right)' + q(t)(1-p_0)^{\beta} z^{\beta}\left(\delta(t)\right) \leq 0. \tag{16}$$

The proof is complete. □

## 3 Oscillation criteria

In this section, we establish new oscillation results for (1) by using the Riccati transformation.

**Lemma 3.1** *Let* $y$ *be an eventually positive solution of* (1). *If there exist constants* $\varepsilon \in (0,1)$ *and* $\zeta > 0$*such that*

$$\varphi(t) := \frac{r(t)(z'''(t))^{\gamma}}{z^{\gamma}(\zeta\delta(t))}, \tag{17}$$

*then*

$$\varphi'(t) + A_1(t) + A_2(t)\varphi^{(\gamma+1)/\gamma}(t) \leq 0. \tag{18}$$

*Proof* Let $y$ be an eventually positive solution of (1). Using Lemma 2.4, we obtain that (14) holds. From (17) we see that $\varphi(t) > 0$ for $t \geq t_1$, and using (14), we obtain

$$\varphi'(t) \leq \frac{-q(t)(1-p_0)^{\beta} z^{\beta}(\delta(t)).}{z^{\gamma}(\zeta\delta(t))} - \gamma \frac{r(t)(z'''(t))^{\gamma} z'(\zeta\delta(t))\zeta\delta'(t)}{z^{\gamma+1}(\zeta\delta(t))}. \tag{19}$$

From Lemma 2.3 we have

$$\varphi'(t) \leq -q(t)(1-p_0)^{\beta} z^{\beta-\gamma}\left(\delta(t)\right) - \gamma \frac{r(t)(z'''(t))^{\gamma}\varepsilon\delta^2(t) z'''(\delta(t))\zeta\delta'(t)}{z^{\gamma+1}(\zeta\delta(t))}, \tag{20}$$

which is

$$\varphi'(t) \leq -q(t)(1-p_0)^{\beta} z^{\beta-\gamma}\big(\delta(t)\big) - \gamma\varepsilon \frac{r(t)\delta^2(t)\zeta\delta'(t)(z'''(t))^{\gamma+1}}{z^{\gamma+1}(\zeta\delta(t))}. \tag{21}$$

Using (17) we have

$$\varphi'(t) \leq -q(t)(1-p_0)^{\beta} z^{\beta-\gamma}\big(\delta(t)\big) - \gamma\varepsilon \frac{\delta^2(t)\zeta\delta'(t)}{r^{1/\gamma}(t)}\varphi^{(\gamma+1)/\gamma}(t). \tag{22}$$

Since $z'(t) > 0$, there exist $t_2 \geq t_1$ and a constant $M > 0$ such that

$$z(t) > M. \tag{23}$$

Then (22) turns into

$$\varphi'(t) \leq -q(t)(1-p_0)^{\beta} M^{\beta-\gamma}\big(\delta(t)\big) - \gamma\varepsilon \frac{\delta^2(t)\zeta\delta'(t)}{r^{1/\gamma}(t)}\varphi^{(\gamma+1)/\gamma}(t), \tag{24}$$

that is,

$$\varphi'(t) + A_1(t) + A_2(t)\varphi^{(\gamma+1)/\gamma}(t) \leq 0. \tag{25}$$

The proof is complete. □

**Theorem 3.1** *Assume that* (2) *holds. If*

$$\liminf_{t\to\infty} \frac{1}{\tilde{A}_1(t)} \int_t^{\infty} A_2(s)\tilde{A}_1^{\frac{\gamma+1}{\gamma}}(s)\,\mathrm{d}s > \frac{\gamma}{(\gamma+1)^{\frac{\gamma+1}{\gamma}}}, \tag{26}$$

*then* (1) *is oscillatory.*

*Proof* Let $y$ be an eventually positive solution of (1). Then there exists $t_1 \geq t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geq t_1$. By Lemma 3.1 we get that (18) holds.

Integrating (18) from $t$ to $l$, we get

$$\varphi(l) - \varphi(t) + \int_t^l A_1(s)\,\mathrm{d}s + \int_t^l A_2(s)\varphi^{\frac{\gamma+1}{\gamma}}(s)\,\mathrm{d}s \leq 0. \tag{27}$$

Letting $l \to \infty$ and using $\varphi > 0$ and $\varphi' < 0$, we have

$$\varphi(t) \geq \tilde{A}_1(t) + \int_t^{\infty} A_2(s)\varphi^{\frac{\gamma+1}{\gamma}}(s)\,\mathrm{d}s. \tag{28}$$

This implies

$$\frac{\varphi(t)}{\tilde{A}_1(t)} \geq 1 + \frac{1}{\tilde{A}_1(t)} \int_t^{\infty} A_2(s)\tilde{A}_1^{\frac{\gamma+1}{\gamma}}(s)\left(\frac{\varphi(s)}{\tilde{A}_1(s)}\right)^{\frac{\gamma+1}{\gamma}}\mathrm{d}s. \tag{29}$$

Let $\lambda = \inf_{t\geq T}\varphi(t)/\tilde{A}_1(t)$. Then obviously $\lambda \geq 1$. Thus from (26) and (29) we see that

$$\lambda \geq 1 + \gamma\left(\frac{\lambda}{\gamma+1}\right)^{(\gamma+1)/\gamma} \tag{30}$$

or

$$\frac{\lambda}{\gamma+1} \geq \frac{1}{\gamma+1} + \frac{\gamma}{\gamma+1}\left(\frac{\lambda}{\gamma+1}\right)^{(\gamma+1)/\gamma}, \tag{31}$$

which contradicts the admissible values of $\lambda \geq 1$ and $\gamma > 0$. Therefore the proof is complete. □

## 4 Kamenev-type criteria

In this section, we establish new Kamenev-type oscillation criteria for (1).

**Lemma 4.1** *Let $y$ be an eventually positive solution of* (1), *and suppose that* (15) *holds. If there exist a function $\pi \in C^1([t_0,\infty), R^+)$ and constants $\varepsilon \in (0,1)$ and $\zeta > 0$ such that*

$$\varpi(t) := \pi(t)\frac{r(t)(z'''(t))^\gamma}{z^\gamma(\zeta\delta(t))}, \tag{32}$$

*then*

$$\varpi'(t) - B_1(t)\varpi(t) + B_2(t) + B_3(t)\varpi^{(\gamma+1)/\gamma}(t) \leq 0. \tag{33}$$

*Proof* Let $y$ be an eventually positive solution of (1). Using Lemma 2.4, we obtain that (14) holds. From (32) we see that $\varpi(t) > 0$ for $t \geq t_1$, and using (14), we obtain

$$\begin{aligned}\varpi'(t) \leq{}& \pi'(t)\frac{r(t)(z'''(t))^\gamma}{z^\gamma(\zeta\delta(t))} + \pi(t)\frac{-q(t)(1-p_0)^\beta z^\beta(\delta(t))}{z^\gamma(\zeta\delta(t))}\\ &- \gamma\pi(t)\frac{r(t)(z'''(t))^\gamma z'(\zeta\delta(t))\zeta\delta'(t)}{z^{\gamma+1}(\zeta\delta(t))}.\end{aligned}$$

From Lemma 2.3 we have

$$\begin{aligned}\varpi'(t) \leq{}& \pi'(t)\frac{r(t)(z'''(t))^\gamma}{z^\gamma(\zeta\delta(t))} - \pi(t)q(t)(1-p_0)^\beta z^{\beta-\gamma}\big(\delta(t)\big)\\ &- \gamma\pi(t)\frac{r(t)(z'''(t))^\gamma\varepsilon\delta^2(t)z'''(\delta(t))\zeta\delta'(t)}{z^{\gamma+1}(\zeta\delta(t))},\end{aligned}$$

which is

$$\begin{aligned}\varpi'(t) \leq{}& \pi'(t)\frac{r(t)(z'''(t))^\gamma}{z^\gamma(\zeta\delta(t))} - \pi(t)q(t)(1-p_0)^\beta z^{\beta-\gamma}\big(\delta(t)\big)\\ &- \gamma\varepsilon\pi(t)\frac{r(t)\delta^2(t)\zeta\delta'(t)(z'''(t))^{\gamma+1}}{z^{\gamma+1}(\zeta\delta(t))}.\end{aligned}$$

By (32) we have

$$\varpi'(t) \le \frac{\pi'(t)}{\pi(t)}\varpi(t) - \pi(t)q(t)(1-p_0)^{\beta} z^{\beta-\gamma}\big(\delta(t)\big) - \gamma\varepsilon \frac{\delta^2(t)\zeta\delta'(t)}{(\pi(t)r(t))^{1/\gamma}}\varpi^{(\gamma+1)/\gamma}(t).$$

Since $z'(t) > 0$, there exist $t_2 \ge t_1$ and $M > 0$ such that

$$z(t) > M. \tag{34}$$

Hence we obtain

$$\varpi'(t) \le \frac{\pi'(t)}{\pi(t)}\varpi(t) - \pi(t)q(t)(1-p_0)^{\beta} M^{\beta-\gamma}\big(\delta(t)\big) - \gamma\varepsilon \frac{\delta^2(t)\zeta\delta'(t)}{(\pi(t)r(t))^{1/\gamma}}\varpi^{(\gamma+1)/\gamma}(t),$$

that is,

$$\varpi'(t) - B_1(t)\varpi(t) + B_2(t) + B_3(t)\varpi^{(\gamma+1)/\gamma}(t) \le 0. \tag{35}$$

The proof is complete. □

**Theorem 4.1** *Assume that* (2) *holds. If there exist a function* $\pi \in C^1([t_0,\infty), R^+)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_0}^{t} (t-s)^n \left( B_2(t) - \frac{r(s)}{(\gamma+1)^{\gamma+1}} \frac{(\pi'(s))^{\gamma+1}}{(\varepsilon\pi(s)\delta^2(t)\zeta\delta'(s))^{\gamma}} \right) \mathrm{d}s = \infty, \tag{36}$$

*then* (1) *is oscillatory.*

*Proof* Let $y$ be a nonoscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality, we can assume that $u$ is eventually positive. Using Lemma 4.1, we get that (33) holds. From Lemma 2.1 we set

$$U = \pi'/\pi, \qquad V = \gamma\varepsilon\delta^2(t)\zeta\delta'(t)/\big(\pi(t)r(t)\big)^{1/\gamma} \quad \text{and} \quad y = \varpi(t). \tag{37}$$

Thus we have

$$\varpi'(t) \le -B_2(t) + \frac{r(t)}{(\gamma+1)^{\gamma+1}} \frac{(\pi'(t))^{\gamma+1}}{(\varepsilon\pi(t)\delta^2(t)\zeta\delta'(t))^{\gamma}} \tag{38}$$

and

$$-\int_{t_0}^{t} (t-s)^n \varpi'(s)\,\mathrm{d}s \ge \int_{t_0}^{t} (t-s)^n \left( B_2(t) - \frac{r(s)}{(\gamma+1)^{\gamma+1}} \frac{\big(\pi'(s)\big)^{\gamma+1}}{\big(\varepsilon\pi(s)\delta^2(t)\zeta\delta'(s)\big)^{\gamma}} \right) \mathrm{d}s. \tag{39}$$

Since

$$\int_{t_0}^{t}(t-s)^n\varpi'(s)\,ds = n\int_{t_0}^{t}(t-s)^{n-1}\varphi(s)\,ds-(t-t_0)^n\varpi(t_0), \tag{40}$$

we get

$$\begin{aligned}&\left(\frac{t-t_0}{t}\right)^n\varpi'(t_0)-\frac{n}{t^n}\int_{t_0}^{t}(t-s)^{n-1}\varpi(s)\,ds\\&\quad\geq\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(B_2(t)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\frac{(\pi'(s))^{\gamma+1}}{(\varepsilon\pi(s)\delta^2(t)\zeta\delta'(s))^{\gamma}}\right)ds.\end{aligned}$$

Hence

$$\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(B_2(t)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\frac{(\pi'(s))^{\gamma+1}}{(\varepsilon\pi(s)\delta^2(t)\zeta\delta'(s))^{\gamma}}\right)ds\leq\left(\frac{t-t_0}{t}\right)^n\varpi(t_0), \tag{41}$$

and so

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(B_2(t)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\frac{(\pi'(s))^{\gamma+1}}{(\varepsilon\pi(s)\delta^2(t)\zeta\delta'(s))^{\gamma}}\right)ds\to\varpi(t_0), \tag{42}$$

which contradicts (36), and this completes the proof. □

## 5 Philos-type oscillation result

In the section, we employ the integral averaging technique to establish a Philos-type oscillation criterion for (1).

**Definition** Let

$$D=\{(t,s)\in R^2:t\geq s\geq t_0\}\quad\text{and}\quad D_0=\{(t,s)\in R^2:t>s\geq t_0\}. \tag{43}$$

A kernel function $H\in C(D,R)$ is said to belong to the function class $\Im$, written as $H\in\Im$, if

(i) $H(t,s)=0$ for $t\geq t_0$, $H(t,s)>0$, $(t,s)\in D_0$;

(ii) $H(t,s)$ has a continuous and nonpositive partial derivative $\partial H/\partial s$ on $D_0$, and there exist functions $\pi\in C^1([t_0,\infty),(0,\infty))$ and $h\in C(D_0,R)$ such that

$$\frac{\partial}{\partial s}H(t,s)+\frac{\pi'(s)}{\pi(s)}H(t,s)=h(t,s)H^{\gamma/(\gamma+1)}(t,s). \tag{44}$$

**Theorem 5.1** *Assume that* (2) *holds. If there exist a positive function* $\pi\in C^1([t_0,\infty),R)$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(H(t,s)B_2(s)-\frac{h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{\pi(s)r(t)}{(\gamma\varepsilon\delta^2(s)\zeta\delta'(s))^{\gamma}}\right)ds=\infty, \tag{45}$$

*then* (1) *is oscillatory.*

*Proof* Let $y$ is a nonoscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality, we can assume that $u$ is eventually positive. From Lemma 4.1 we get that (33) holds. Multiplying (33) by $H(t,s)$ and integrating the resulting inequality from $t_1$ to $t$, we find that

$$\begin{aligned}\int_{t_1}^{t} H(t,s)B_2(s)\,\mathrm{d}s \le \varpi(t_1)H(t,t_1) + \int_{t_1}^{t}\left(\frac{\partial}{\partial s}H(t,s) + B_1(s)H(t,s)\right)\varpi(s)\,\mathrm{d}s \\ - \int_{t_1}^{t} B_3(s)H(t,s)\varpi^{\frac{\gamma+1}{\gamma}}(s)\,\mathrm{d}s.\end{aligned}$$

From (44) we get

$$\begin{aligned}\int_{t_1}^{t} H(t,s)B_2(s)\,\mathrm{d}s \le \varpi(t_1)H(t,t_1) + \int_{t_1}^{t} h(t,s)H^{\gamma/(\gamma+1)}(t,s)\varpi(s)\,\mathrm{d}s \\ - \int_{t_1}^{t} B_3(s)H(t,s)\varpi^{\frac{\gamma+1}{\gamma}}(s)\,\mathrm{d}s.\end{aligned}$$

Using Lemma 2.1 with $V = B_3(s)H(t,s)$, $U = h(t,s)H^{\gamma/(\gamma+1)}(t,s)$, and $y = \varpi(s)$, we get

$$\begin{aligned}&h(t,s)H^{\gamma/(\gamma+1)}(t,s)\varpi(s) - B_3(s)H(t,s)\varpi^{\frac{\gamma+1}{\gamma}}(s) \\ &\quad\le \frac{h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{\pi(s)r(t)}{(\gamma\varepsilon\delta^2(s)\zeta\delta'(s))^{\gamma}},\end{aligned}$$

which implies that

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(H(t,s)B_2(s) - \frac{h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{\pi(s)r(t)}{(\gamma\varepsilon\delta^2(s)\zeta\delta'(s))^{\gamma}}\right)\mathrm{d}s \le \varpi(t_1), \tag{46}$$

a contradiction to (45).

Theorem 5.1 is proved. □

**Corollary 5.1** *If condition* (45) *in Theorem* 5.1 *is replaced by the conditions*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t} H(t,s)B_2(s)\,\mathrm{d}s = \infty \tag{47}$$

*and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{\pi(s)r(t)}{(\gamma\varepsilon\delta^2(s)\zeta\delta'(s))^{\gamma}}\,\mathrm{d}s < \infty, \tag{48}$$

*then* (1) *is oscillatory.*

*Example* Consider the differential equation

$$\left(t\left(y(t) + \frac{1}{2}y\left(\frac{t}{3}\right)\right)'''\right)' + \frac{q_0}{t^4}y\left(\frac{t}{2}\right) = 0, \tag{49}$$

where $q_0 > 0$ is a constant. Note that $\gamma = \beta = 1$, $r(t) = t$, $p_0(t) = 1/2$, $q(t) = q_0/t^4$, $\delta(t) = t/2$, and $\tau(t) = t/3$. If we set $\pi(t) = t^2$, then

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\,\mathrm{d}s = \int_{t_0}^{\infty} \frac{1}{s}\,\mathrm{d}s = \infty \tag{50}$$

and

$$B_2(t) = \pi(t)q(t)(1-p_0)^{\beta} M^{\beta-\gamma}\delta(t) = \frac{q_0}{4t}. \tag{51}$$

Thus we get

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \left( B_2(t) - \frac{r(s)}{(\gamma+1)^{\gamma+1}} \frac{(\pi'(s))^{\gamma+1}}{(\varepsilon\pi(s)\delta^2(t)\zeta\delta'(s))^{\gamma}} \right) \mathrm{d}s \\
&\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} (t-s)^2 \frac{1}{s} \left( \frac{q_0}{4} - 8 \right) \mathrm{d}s = \infty.
\end{aligned}$$

Therefore by Theorem 4.1 all solutions of (49) are oscillatory if $q_0 > 32$.

*Remark* 5.1 We can easily see that the results obtained in [32, 33] cannot be applied to (36), so our results are new.

*Remark* 5.2 We can generalize our results by studying the equation

$$\bigl(r(t)\bigl(z'''(t)\bigr)^{\gamma}\bigr)' + \sum_{i=1}^{j} q_i(t) y^{\beta}\bigl(\delta_i(t)\bigr) = 0, \quad t \geq t_0, j \geq 1. \tag{52}$$

For this, we leave the results to interested researchers.

*Remark* 5.3 For interested researchers, there is a good problem of finding new results for (1) where

$$z(t) := y(t) - p(t)y\bigl(\tau(t)\bigr). \tag{53}$$

## 6 Conclusions

The aim of this paper was to provide a study of asymptotic nature for a class of fourth-order neutral delay differential equations. We used a Riccati substitution and the integral averaging technique to ensure that every solution of the studied equation is oscillatory. The results presented complement some of the known results reported in the literature.

A further extension of this paper is using our results to study a class of systems of higher-order neutral differential equations, including those of fractional order. Some research in this area is in progress.

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## Publisher's Note



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