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Study on a susceptible–exposed–infected–recovered model with nonlinear incidence rate

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Abstract

A stochastic susceptible–exposed–infected–recovered (SEIR) model with nonlinear incidence rate is investigated. Under suitable conditions, existence and uniqueness of a global solution, stationary distribution with ergodicity, persistence in the mean, and extinction of the disease are obtained. Numerical simulations and conclusions are separately carried out at the end of this paper.

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1 Model formulation

The spread and control of infectious diseases [1–3] was described by governing the mathematical models, aiming to investigate the dynamical properties. Since the pioneering work was established by Kermack and McKendrick [4], the epidemic models have been paid more attention at present. Usually, epidemic models included three states within a total population: the susceptible S , the infected I , and the recovered R . For instance, Cai *et al.* [5] introduced an SIS model incorporating media coverage to investigate the effects of environment fluctuations. However, for some diseases, such as Hepatitis B, Hepatitis C, and AIDS, the exposed hosts E took a vital role when dynamical behaviors were expected to be discussed. As mentioned in recent literature, a class of epidemic models was considered by some scholars; it was called susceptible–exposed–infected–recovered model (short for SEIR model, see [6–18]).

During the development of epidemic models, incidence rates which describe the relationship between the susceptible and the infected/the exposed play an important role, and meanwhile change its form ranging from bilinear case to nonlinear case when investigation of epidemic models is conducted. For example, [19–22] governed bilinear incidence rate βSI to explore epidemic models with fluctuation in epidemic models. If the number of the infected within a population was large, then three types of saturated incidence rates were usually used in epidemiological models: the mixing standard incidence rate $\frac{\beta SI}{N}$ [23–25], the nonlinear incidence rate $\beta S^q I^p$ [26, 27], and the saturation incidence rate $\frac{\beta SI}{1+\alpha S}$

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[28–30]. Later, some generalized and nonlinear incidence rates were studied in recent literature (see [31–33]).

In this paper, we still use four states of epidemic models, that is, the susceptible S , the exposed E , the infected I , and the recovered R , to describe our model with environmental fluctuations. Motivated by the above-mentioned discussions, we assume that individuals within total population are well mixed and settle in the same environment. Our model goes with the idea that the susceptible and the infected contact with constant rate β ; after contacting with the infected, the susceptible turns into the exposed when time exceeds incubation period (also called the latent period, see [8, 11, 18]); that the exposed individuals become the infected and then the recovered; and that part of recovered individuals enter again into the susceptible state. According to this spread cycle, we establish our model by equations, and we start with an equation of the susceptible as follows:

$$\dot{S}(t) = A - \mu S(t) + \delta R(t) - \frac{\beta S(t)I(t)}{\varphi(I(t))},$$

where A and μ respectively denote the new recruitment rate and the disease-free death rate, δ is the rate at which the recovered individuals become susceptible, $\frac{\beta SI}{\varphi(I)}$ is a nonlinear incidence rate with property that $\varphi(I)$ is increasing and $\varphi(0) = 1$, and that, for some constant $l > 0$, the following property is valid:

$$M_l := \sup_{0 < I \leq l} \frac{\varphi(I)}{I} < \infty.$$

For the exposed, we have that

$$\dot{E}(t) = \frac{\beta S(t)I(t)}{\varphi(I(t))} - (\mu + \sigma)E(t),$$

where σ is the rate at which exposed individuals become infected individuals. Further, the changes of infected and recovered individuals at time t are assumed to follow two ordinary differential equations:

$$\dot{I}(t) = \sigma E(t) - (\mu + \rho + \gamma)I(t)$$

and

$$\dot{R}(t) = \gamma I(t) - (\mu + \delta)R(t),$$

where ρ is the death rate caused by diseases and γ is the recovery rate of infected individuals. Now, we derive a system that consists of four ordinary differential equations:

$$\begin{aligned} \dot{S}(t) &= A - \mu S(t) + \delta R(t) - \frac{\beta S(t)I(t)}{\varphi(I(t))}, \\ \dot{E}(t) &= \frac{\beta S(t)I(t)}{\varphi(I(t))} - (\mu + \sigma)E(t), \\ \dot{I}(t) &= \sigma E(t) - (\mu + \rho + \gamma)I(t), \\ \dot{R}(t) &= \gamma I(t) - (\mu + \delta)R(t). \end{aligned} \tag{1}$$

It follows from (1) that

$$\frac{d}{dt}(S + E + I + R) = A - \mu(S + E + I + R) - \rho I \leq A - \mu(S + E + I + R).$$

Then $\limsup_{t \rightarrow +\infty} (S(t) + E(t) + I(t) + R(t)) \leq \frac{A}{\mu}$. Thus the feasible region for system (1) is

$$\Omega = \left\{ (S, E, I, R) \mid S + E + I + R \leq \frac{A}{\mu}, S > 0, E \geq 0, I \geq 0, R \geq 0 \right\}.$$

Let $\text{Int } \Omega$ denote the interior of Ω . It is easy to verify that the region Ω is positively invariant with respect to system (1) (i.e., the solutions with initial conditions in Ω remain in Ω). Hence, system (1) will be considered mathematically and epidemiologically well posed in Ω .

One concern for further investigation is to find out an expression for the basic reproduction number R_0 of model (1) by using of the next generation matrix (see [34]). The basic reproduction number, sometimes called basic reproductive rate or basic reproductive ratio, is one of the most useful threshold parameters that characterize mathematical problems concerning infectious diseases. This metric is useful because it helps determine whether or not an infectious disease will spread through a population. Next, we calculate the basic reproduction number of system (1). Let $x = (S, E, I, R)^T$, where T denotes the transpose of matrix (or vector). Then model (1) can be written as

$$\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$\mathcal{F} = \begin{pmatrix} 0 \\ \frac{\beta SI}{\varphi(I)} \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} -A + \mu S - \delta R + \frac{\beta SI}{\varphi(I)} \\ (\mu + \sigma)E \\ -\sigma E + (\mu + \rho + \gamma)I \\ -\gamma I + (\mu + \delta)R \end{pmatrix}.$$

So the infected classes can be referred to as $m = 2$, that is, the exposed compartment (E) and the infected compartment (I), and the disease-free equilibrium of model (1) is $x_0 = (\frac{A}{\mu}, 0, 0, 0)^T$. Based on the detailed documentations in [34–36], we can easily get

$$F = \begin{pmatrix} 0 & \frac{\beta A}{\mu} \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu + \sigma & 0 \\ -\sigma & \mu + \rho + \gamma \end{pmatrix},$$

and the inverse matrix of V is

$$V^{-1} = \frac{1}{(\mu + \sigma)(\mu + \rho + \gamma)} \begin{pmatrix} \mu + \rho + \gamma & 0 \\ \sigma & \mu + \sigma \end{pmatrix}.$$

Therefore FV^{-1} is the next generation matrix for model (1). It follows that the spectral radius of matrix FV^{-1} is $\rho(FV^{-1}) = \frac{A\beta\sigma}{\mu(\mu + \sigma)(\mu + \rho + \gamma)}$. According to Theorem 2 in [34], the basic reproduction number is

$$R_0 = \frac{A\beta\sigma}{\mu(\mu + \sigma)(\mu + \rho + \gamma)}.$$

When diseases attack total population in real circumstances, the effects of comprehensive and external fluctuation are inevitable and distinct. We here assume that the effects are proportional to states of models. For instance, the temperature, air humidity, and other factors normally are regarded as comprehensive and external fluctuations. Therefore we consider the following epidemic model with environmental fluctuations and nonlinear incidence rate:

$$\begin{aligned}
 dS(t) &= \left[A - \mu S(t) + \delta R(t) - \frac{\beta S(t)I(t)}{\varphi(I(t))} \right] dt + \sigma_1 S(t) dB_1(t), \\
 dE(t) &= \left[\frac{\beta S(t)I(t)}{\varphi(I(t))} - (\mu + \sigma)E(t) \right] dt + \sigma_2 E(t) dB_2(t), \\
 dI(t) &= [\sigma E(t) - (\mu + \rho + \gamma)I(t)] dt + \sigma_3 I(t) dB_3(t), \\
 dR(t) &= [\gamma I(t) - (\mu + \delta)R(t)] dt + \sigma_4 R(t) dB_4(t),
 \end{aligned} \tag{2}$$

where $B_i(t)$ are standard one-dimensional independent Wiener processes, σ_i are the intensities of white noise for $\sigma_i > 0$ and $i = 1, 2, 3, 4$. Throughout the paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, that is, it is increasing and right continuous, while \mathcal{F}_0 contains all \mathbb{P} -null sets.

The rest of this paper is organized as follows. In Sect. 2, we show that model (2) admits a unique global positive solution with any initial value. In Sect. 3, we establish sufficient conditions for extinction of the disease. In Sect. 4, we verify the persistence in the mean under some conditions. Finally, we prove that there is an ergodic stationary distribution of model (2) by constructing suitable Lyapunov functions.

2 Existence and uniqueness of positive solution

Theorem 2.1 *There is a unique solution $(S(t), E(t), I(t), R(t))$ to model (2) on $t \geq 0$ for any given initial value $(S(0), E(0), I(0), R(0))$ in \mathbb{R}_+^4 with probability one.*

Proof Because the coefficients of system (2) satisfy locally Lipschitz continuity, there exists a unique local solution $(S(t), E(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that the solution is global, we just need to prove that $\tau_e = \infty$. Let $m_0 > 1$ be a sufficiently large integer such that each component of $(S(0), E(0), I(0), R(0))$ lies within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, we define the stopping time

$$\tau_m := \inf \left\{ t \in [0, \tau_e) : \min \{S(t), E(t), I(t), R(t)\} \leq \frac{1}{m} \text{ or } \max \{S(t), E(t), I(t), R(t)\} \geq m \right\}.$$

Throughout this paper, we set $\inf \emptyset = \infty$. It is obvious that τ_m is increasing as $m \rightarrow \infty$ (details can be seen in [37]). We also denote $\lim_{m \rightarrow \infty} \tau_m = \tau_\infty$. Obviously, there is $\tau_\infty \leq \tau_e$. If we confirm that $\tau_\infty = \infty$, then we get $\tau_e = \infty$ for all $t \geq 0$. The proof goes by contradiction. Assuming that $\tau_\infty \neq \infty$, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} \geq \varepsilon$. Hence there exists an integer $m_1 \geq m_0$ such that $\mathbb{P}\{\tau_m \leq T\} \geq \varepsilon$ for each integer $m \geq m_1$. We define a C^2 -function $V : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ as follows:

$$V(S, E, I, R) = S - b - b \ln \frac{S}{b} + E - 1 - \ln E + I - 1 - \ln I + R - 1 - \ln R,$$

where b is a positive constant that will be determined later. Then, making use of Itô’s formula on $V(S, E, I, R)$, we obtain that

$$\begin{aligned}
 & dV(S(t), E(t), I(t), R(t)) \\
 &= \left(1 - \frac{b}{S}\right) dS + \left(1 - \frac{1}{E}\right) dE + \left(1 - \frac{1}{I}\right) dI + \left(1 - \frac{1}{R}\right) dR \\
 &\quad + \frac{b}{2S^2} (dS)^2 + \frac{1}{2E^2} (dE)^2 + \frac{1}{2I^2} (dI)^2 + \frac{1}{2R^2} (dR)^2 \\
 &= \left[A - \mu S + \delta R - \frac{\beta SI}{\varphi(I)} - \frac{Ab}{S} + \mu b - \frac{b\delta R}{S} + \frac{\beta bI}{\varphi(I)} \right] dt + \sigma_1(S - b) dB_1(t) \\
 &\quad + \left[\frac{\beta SI}{\varphi(I)} - (\sigma + \mu)E - \frac{\beta SI}{E\varphi(I)} + \sigma + \mu \right] dt + \sigma_2(E - 1) dB_2(t) \\
 &\quad + \left[\sigma E - (\mu + \rho + \gamma)I - \frac{\sigma E}{I} + \mu + \rho + \gamma \right] dt + \sigma_3(I - 1) dB_3(t) \\
 &\quad + \left[\gamma I - (\mu + \delta)R - \frac{\gamma I}{R} + \mu + \delta \right] dt + \sigma_4(R - 1) dB_4(t) \\
 &\quad + \frac{b\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} dt \\
 &= \mathcal{L}V(S(t), E(t), I(t), R(t)) dt + \sigma_1(S - b) dB_1(t) + \sigma_2(E - 1) dB_2(t) \\
 &\quad + \sigma_3(I - 1) dB_3(t) + \sigma_4(R - 1) dB_4(t),
 \end{aligned}$$

where

$$\begin{aligned}
 & \mathcal{L}V(S(t), E(t), I(t), R(t)) \\
 &= A + \mu b + \frac{\beta bI}{\varphi(I)} + \sigma + 3\mu + \rho + \gamma + \delta + \frac{b\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
 &\quad - \left[\mu S + \frac{Ab}{S} + \frac{b\delta R}{S} + \mu E + \frac{\beta SI}{E\varphi(I)} + (\mu + \rho)I + \frac{\sigma E}{I} + \mu R + \frac{\gamma I}{R} \right] \\
 &\leq \beta bI - (\mu + \rho)I + A + \mu b + \sigma + 3\mu + \rho + \gamma + \delta + \frac{b\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2}.
 \end{aligned}$$

We choose $b = \frac{\mu + \rho}{\beta}$ and denote $K := A + \mu b + \sigma + 3\mu + \rho + \gamma + \delta + \frac{(\mu + \rho)\sigma_1^2 + \beta\sigma_2^2 + \beta\sigma_3^2 + \beta\sigma_4^2}{2\beta}$, so we have

$$\mathcal{L}V(S(t), E(t), I(t), R(t)) \leq K,$$

where K is a positive constant. Then

$$\begin{aligned}
 & dV(S(t), E(t), I(t), R(t)) \\
 &\leq K dt + \sigma_1(S - b) dB_1(t) + \sigma_2(E - 1) dB_2(t) \\
 &\quad + \sigma_3(I - 1) dB_3(t) + \sigma_4(R - 1) dB_4(t).
 \end{aligned}$$

For any $t \in [0, T]$ and $m \geq m_1$, we take integration from 0 to $\tau_m \wedge T$, and then take expectation on both sides, which gives

$$\begin{aligned} &\mathbb{E}V(S(\tau_m \wedge T), E(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)) \\ &\leq V(S(0), E(0), I(0), R(0)) + K\mathbb{E}(\tau_m \wedge T) \\ &\leq V(S(0), E(0), I(0), R(0)) + KT < \infty. \end{aligned}$$

We set $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_1$, then $\mathbb{P}(\Omega_m) \geq \varepsilon$. Each component of $(S(\tau_m \wedge T), E(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T))$ equals either m or $\frac{1}{m}$ for all $w \in \Omega_m$. Therefore, we get

$$\begin{aligned} \infty &> V(S(0), E(0), I(0), R(0)) + KT \\ &\geq \mathbb{P}\{\tau_m \leq T\} \min\left\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\right\} \\ &\geq \varepsilon \min\left\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\right\}. \end{aligned}$$

Let $m \rightarrow \infty$, which implies the contradiction

$$\infty > V(S(0), E(0), I(0), R(0)) + KT \geq \infty,$$

as a consequence, we have $\tau_\infty = \infty$. The proof is complete. □

3 Extinction of diseases

Extinction and persistence are two most important issues in the study of epidemic models. For the sake of simplicity, we denote

$$\langle x(t) \rangle = \frac{1}{t} \int_0^t x(s) \, ds.$$

Lemma 3.1 *For any initial value $(S(0), E(0), I(0), R(0)) \in \mathbb{R}_+^4$, the solution $(S(t), E(t), I(t), R(t))$ has the following properties:*

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{E(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{I(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{R(t)}{t} = 0, \quad a.s.$$

and

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln S(t)}{t} &\leq 0, & \limsup_{t \rightarrow \infty} \frac{\ln E(t)}{t} &\leq 0, \\ \limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} &\leq 0, & \limsup_{t \rightarrow \infty} \frac{\ln R(t)}{t} &\leq 0, \quad a.s. \end{aligned}$$

Moreover, if $\mu > \frac{\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2}{2}$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(s) \, dB_1(s) &= 0, & \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t E(s) \, dB_2(s) &= 0, \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) \, dB_3(s) &= 0, & \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(s) \, dB_4(s) &= 0. \end{aligned}$$

The proof of Lemma 3.1 is similar to the approach used in [38, 39], and we omit the proof here.

Lemma 3.2 (Strong law of large numbers [37]) *Let $M = \{M_t\}_{t \geq 0}$ be a real-value continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \infty, \quad a.s. \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{\langle M, M \rangle_t} = 0, \quad a.s.$$

and also

$$\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < \infty, \quad a.s. \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{t} = 0, \quad a.s.$$

Theorem 3.1 *Let $(S(t), E(t), I(t), R(t))$ be the solution of model (2) with any initial value in \mathbb{R}_+^4 . If the basic reproduction number satisfies*

$$R_0 = \frac{A\beta\sigma}{\mu(\sigma + \mu)(\mu + \rho + \gamma)} < 1$$

and

$$\mu > \frac{\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2}{2},$$

then

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad a.s.,$$

then the density of the infected individuals reaches extinction.

Proof Let

$$N(t) = S(t) + E(t) + I(t) + R(t),$$

we obtain that

$$\begin{aligned} dN(t) = & (A - \mu N(t) - \rho I(t)) dt + \sigma_1 S(t) dB_1(t) + \sigma_2 E(t) dB_2(t) \\ & + \sigma_3 I(t) dB_3(t) + \sigma_4 R(t) dB_4(t). \end{aligned} \tag{3}$$

Taking integration on both sides of (3) from 0 to t and according to Lemma 3.1, we see that

$$\limsup_{t \rightarrow \infty} \langle N(t) \rangle \leq \frac{A}{\mu}. \tag{4}$$

Now we define a C^2 -function $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ as follows:

$$W(E, I) = m_1 E(t) + m_2 I(t),$$

where

$$m_1 = \frac{\sigma}{(\sigma + \mu)(\mu + \rho + \gamma)}, \quad m_2 = \frac{\sqrt{R_0}}{\mu + \rho + \gamma}. \tag{5}$$

Making use of Itô’s formula, we have

$$\begin{aligned} d \ln W(E(t), I(t)) &= \frac{1}{W} (m_1 dE + m_2 dI) - \frac{1}{2W^2} [m_1^2 (dE)^2 + m_2^2 (dI)^2] \\ &= \mathcal{L}(\ln W(E(t), I(t))) dt + \frac{m_1 \sigma_2 E dB_2(t) + m_2 \sigma_3 I dB_3(t)}{W}. \end{aligned}$$

Based on the fundamental inequality $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$ for positive $a, b, c,$ and $d,$ we obtain that

$$(m_1 E + m_2 I)^2 = \left(m_1 \sigma_2 E \frac{1}{\sigma_2} + m_2 \sigma_3 I \frac{1}{\sigma_3} \right)^2 \leq (m_1^2 \sigma_2^2 E^2 + m_2^2 \sigma_3^2 I^2) \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2} \right). \tag{6}$$

Therefore, from (5) and (6)

$$\begin{aligned} &\mathcal{L} \ln W(E(t), I(t)) \\ &= \frac{1}{W} \left[\frac{m_1 \beta S I}{\varphi(I)} - m_1 (\sigma + \mu) E + m_2 \sigma E - m_2 (\mu + \rho + \gamma) I \right] - \frac{m_1^2 \sigma_2^2 E^2 + m_2^2 \sigma_3^2 I^2}{2(m_1 E + m_2 I)^2} \\ &\leq \frac{1}{W} \left[\frac{m_1 \beta S I}{\varphi(I)} + m_1 \frac{\beta A I}{\mu} - m_1 (\sigma + \mu) E + m_2 \sigma E - m_2 (\mu + \rho + \gamma) I \right] \\ &\quad - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} \\ &\leq \frac{m_1 \beta S}{m_2} + \frac{1}{W} \left\{ \frac{\sigma}{(\sigma + \mu)(\mu + \rho + \gamma)} \left[\frac{\beta A I}{\mu} - (\sigma + \mu) E \right] \right. \\ &\quad \left. + \frac{\sqrt{R_0}}{\mu + \rho + \gamma} [\sigma E - (\mu + \rho + \gamma) I] \right\} - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} \\ &= \frac{m_1 \beta S}{m_2} + \frac{1}{W} \left(R_0 I - \frac{\sigma}{\mu + \rho + \gamma} E + \frac{\sqrt{R_0} \sigma E}{\mu + \rho + \gamma} - \sqrt{R_0} I \right) - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} \\ &= \frac{m_1 \beta S}{m_2} + \frac{1}{W} (\sqrt{R_0} - 1) \left(\frac{\sigma}{\mu + \rho + \gamma} E + \sqrt{R_0} I \right) - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} \\ &= \frac{m_1 \beta S}{m_2} + \frac{1}{W} (\sqrt{R_0} - 1) [(\sigma + \mu) m_1 E + (\mu + \rho + \gamma) m_2 I] - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} \\ &\leq \frac{m_1 \beta S}{m_2} + \min\{\sigma + \mu, \mu + \rho + \gamma\} (\sqrt{R_0} - 1) - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1}, \end{aligned}$$

therefore

$$\begin{aligned} &d \ln W(E(t), I(t)) \\ &\leq \left\{ \frac{m_1 \beta S}{m_2} + \min\{\sigma + \mu, \mu + \rho + \gamma\} (\sqrt{R_0} - 1) - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} \right\} dt \\ &\quad + \frac{m_1 \sigma_2 E dB_2(t)}{W} + \frac{m_2 \sigma_3 I dB_3(t)}{W}. \end{aligned} \tag{7}$$

Now we take integration on both sides of (7) and divide it by t , which then implies the following expression:

$$\begin{aligned} & \frac{\ln W(E(t), I(t))}{t} \\ & \leq \frac{\ln W(E(0), I(0))}{t} + \frac{1}{t} \int_0^t \frac{m_1 \beta S(s)}{m_2} ds + \min\{\sigma + \mu, \mu + \rho + \gamma\}(\sqrt{R_0} - 1) \\ & \quad - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned} \tag{8}$$

where

$$M_1(t) = \int_0^t \frac{m_1 \sigma_2 E(s)}{W(s)} dB_2(s), \quad M_2(t) = \int_0^t \frac{m_2 \sigma_3 I(s)}{W(s)} dB_3(s),$$

are local martingales, whose quadratic variations are $\langle M_1(t), M_1(t) \rangle \leq \sigma_2^2 t$, $\langle M_2(t), M_2(t) \rangle \leq \sigma_3^2 t$ respectively. Applying Lemma 3.2, we conclude that

$$\limsup_{t \rightarrow \infty} \frac{M_i(t)}{t} = 0, \quad i = 1, 2, \text{ a.s.} \tag{9}$$

Then taking the upper limit on both sides, from (7), (8), and (9), we can get

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{\ln W(E(t), I(t))}{t} \\ & \leq \frac{m_1 \beta A}{m_2 \mu} + \min\{\sigma + \mu, \mu + \rho + \gamma\}(\sqrt{R_0} - 1) - [2(\sigma_2^{-2} + \sigma_3^{-2})]^{-1} := \nu, \quad \text{a.s.} \end{aligned}$$

If $\nu < 0$, we obtain that

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} < 0, \quad \text{a.s.,}$$

which suggests that $\lim_{t \rightarrow \infty} I(t) = 0$. This indicates that the disease would tend to extinction. The proof is complete. \square

4 Persistence in the mean

In this section, we will demonstrate some useful results about the persistence of the diseases.

Theorem 4.1 *Let $(S(t), E(t), I(t), R(t))$ be a solution of system (2) with any initial value in \mathbb{R}_+^4 . If*

$$\tilde{R}_0 = \frac{2A\beta\sigma}{(\mu + \frac{\sigma_2^2}{2})(\mu + \sigma + \frac{\sigma_2^2}{2})(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2})} > 1,$$

then system (2) has the following property:

$$\liminf_{t \rightarrow \infty} \langle I \rangle_t \geq \frac{1}{\gamma} \left(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \right) (\tilde{R}_0 - 1) > 0,$$

where

$$\bar{\gamma} = \frac{A\beta^2\sigma + A\beta\sigma(\mu + \rho + \gamma)(\mu + \frac{\sigma_1^2}{2})}{(\mu + \frac{\sigma_1^2}{2})^2(\mu + \sigma + \frac{\sigma_2^2}{2})}.$$

That is to say, the disease will be prevalent.

Proof In order to testify the persistence, we establish a C^2 -function $V_1 : \mathbb{R}_+^4 \rightarrow \mathbb{R}$ as follows:

$$V_1(S, E, I, R) = -c_1 \ln S - c_2 \ln E - c_3 I - \ln R - \ln I, \tag{10}$$

where c_1, c_2, c_3 are positive constants to be determined later. Next we apply Itô's formula to (10). Then we get the following result:

$$\begin{aligned} dV_1(S(t), E(t), I(t), R(t)) &= -\frac{c_1}{S} dS - \frac{c_2}{E} dE - c_3 dI - \frac{1}{R} dR + \frac{c_1}{2S^2} (dS)^2 \\ &\quad + \frac{c_2}{2E^2} (dE)^2 + \frac{1}{2R^2} (dR)^2 - \frac{1}{I} dI + \frac{1}{2I^2} (dI)^2 \\ &= -\frac{c_1}{S} \left[A - \mu S + \delta R - \frac{\beta SI}{\varphi(I)} \right] dt - c_1 \sigma_1 dB_1(t) - \frac{c_2}{E} \left[\frac{\beta SI}{\varphi(I)} - (\mu + \sigma)E \right] dt \\ &\quad - c_2 \sigma_2 dB_2(t) - c_3 [\sigma E - (\mu + \rho + \gamma)I] dt - c_3 \sigma_3 I dB_3(t) - \frac{1}{R} [\gamma I - (\mu + \delta)R] dt \\ &\quad - \sigma_4 dB_4(t) + \left(\frac{c_1 \sigma_1^2}{2} + \frac{c_2 \sigma_2^2}{2} + \frac{\sigma_4^2}{2} \right) dt + \left(-\frac{\sigma E}{I} + \mu + \rho + \gamma + \frac{\sigma_3^2}{2} \right) dt - \sigma_3 dB_3(t) \\ &= \left[-\frac{c_1 A}{S} + c_1 \mu - \frac{c_1 \delta R}{S} + \frac{c_1 \beta I}{\varphi(I)} - \frac{c_2 \beta SI}{E \varphi(I)} + c_2 (\mu + \sigma) - c_3 \sigma E + c_3 (\mu + \rho + \gamma) I \right. \\ &\quad \left. - \frac{\gamma I}{R} + \mu + \delta + \frac{c_1 \sigma_1^2}{2} + \frac{c_2 \sigma_2^2}{2} + \frac{\sigma_4^2}{2} - \frac{\sigma E}{I} + \mu + \rho + \gamma + \frac{\sigma_3^2}{2} \right] dt \\ &\quad - c_1 \sigma_1 dB_1(t) - c_2 \sigma_2 dB_2(t) - c_3 \sigma_3 I dB_3(t) - \sigma_4 dB_4(t) - \sigma_3 dB_3(t), \tag{11} \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}V_1(S(t), E(t), I(t), R(t)) &= -\frac{c_1 A}{S} + c_1 \mu - \frac{c_1 \delta R}{S} + \frac{c_1 \beta I}{\varphi(I)} - \frac{c_2 \beta SI}{E \varphi(I)} + c_2 (\mu + \sigma) - c_3 \sigma E + c_3 (\mu + \rho + \gamma) I \\ &\quad - \frac{\gamma I}{R} + \mu + \delta + \frac{c_1 \sigma_1^2}{2} + \frac{c_2 \sigma_2^2}{2} + \frac{\sigma_4^2}{2} - \frac{\sigma E}{I} + \mu + \rho + \gamma + \frac{\sigma_3^2}{2} \\ &< -\frac{c_1 A}{S} + c_1 \mu + [c_1 \beta + c_3 (\mu + \rho + \gamma)] I - \frac{c_2 \beta SI}{E \varphi(I)} + c_2 (\mu + \sigma) - c_3 \sigma E + \mu + \delta \\ &\quad + \frac{c_1 \sigma_1^2 + c_2 \sigma_2^2 + \sigma_4^2}{2} - \frac{\varphi(I)}{I} + \frac{\varphi(I)}{I} + \mu + \rho + \gamma + \frac{\sigma_3^2}{2} \\ &\leq -\frac{c_1 A}{S} - \frac{c_2 \beta SI}{E \varphi(I)} - c_3 \sigma E - \frac{\varphi(I)}{I} + c_1 \left(\mu + \frac{\sigma_1^2}{2} \right) + [c_1 \beta + c_3 (\mu + \rho + \gamma)] I \end{aligned}$$

$$\begin{aligned}
 &+ c_2 \left(\mu + \sigma + \frac{\sigma_2^2}{2} \right) + \mu + \delta + \frac{\sigma_4^2}{2} + M_l + \mu + \rho + \gamma + \frac{\sigma_3^2}{2} \\
 &\leq -4(c_1 c_2 c_3 A \beta \sigma)^{\frac{1}{4}} + c_1 \left(\mu + \frac{\sigma_1^2}{2} \right) + [c_1 \beta + c_3(\mu + \rho + \gamma)]I + c_2 \left(\mu + \sigma + \frac{\sigma_2^2}{2} \right) \\
 &\quad + 2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2}.
 \end{aligned}$$

Let

$$\begin{aligned}
 c_1 &= \frac{A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right)^2 \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right)}, & c_2 &= \frac{A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right) \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right)^2}, \\
 c_3 &= \frac{A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right) \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right)}.
 \end{aligned}$$

Then

$$\begin{aligned}
 &\mathcal{L}V_1(S(t), E(t), I(t), R(t)) \\
 &\leq \frac{-4A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right) \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right)} + \frac{A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right) \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right)} + [c_1 \beta + c_3(\mu + \rho + \gamma)]I \\
 &\quad + \frac{A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right) \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right)} + 2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \\
 &= \frac{-2A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right) \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right)} + 2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} + [c_1 \beta + c_3(\mu + \rho + \gamma)]I \\
 &= -\left(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \right) \\
 &\quad \times \left[\frac{2A\beta\sigma}{\left(\mu + \frac{\sigma_1^2}{2}\right) \left(\mu + \sigma + \frac{\sigma_2^2}{2}\right) \left(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \right)} - 1 \right] \\
 &\quad + [c_1 \beta + c_3(\mu + \rho + \gamma)]I \\
 &= -\left(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \right) (\tilde{R}_0 - 1) + [c_1 \beta + c_3(\mu + \rho + \gamma)]I \\
 &\leq -\lambda + \bar{\gamma}I, \tag{12}
 \end{aligned}$$

where

$$\lambda = \left(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \right) (\tilde{R}_0 - 1), \quad \bar{\gamma} = c_1 \beta + c_3(\mu + \rho + \gamma).$$

Hence, from (11) and (12) we get

$$\begin{aligned}
 &dV_1(S(t), E(t), I(t), R(t)) \\
 &\leq (-\lambda + \bar{\gamma}I) dt - c_1 \sigma_1 dB_1(t) - c_2 \sigma_2 dB_2(t) - c_3 \sigma_3 I dB_3(t) - \sigma_4 dB_4(t) - \sigma_3 dB_3(t),
 \end{aligned}$$

where

$$M_3(t) = \int_0^t c_1 \sigma_1 dB_1(s) + \int_0^t c_2 \sigma_2 dB_2(s) + \int_0^t c_3 \sigma_3 I(s) dB_3(s) + \int_0^t \sigma_4 dB_4(s) + \int_0^t \sigma_3 dB_3(s)$$

is a local martingale. Using Lemma 3.2 yields

$$\lim_{t \rightarrow \infty} \frac{M_3(t)}{t} = 0.$$

Therefore

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \bar{\gamma}(I)_t \\ & \geq \left(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \right) (\tilde{R}_0 - 1) + \liminf_{t \rightarrow \infty} \frac{V_1(S(t), E(t), I(t), R(t))}{t} \\ & \quad - \limsup_{t \rightarrow \infty} \frac{V_1(S(0), E(0), I(0), R(0))}{t} + \liminf_{t \rightarrow \infty} \frac{M_3(t)}{t} \\ & \geq \left(2\mu + \delta + \frac{\sigma_4^2}{2} + M_l + \rho + \gamma + \frac{\sigma_3^2}{2} \right) (\tilde{R}_0 - 1) > 0. \end{aligned}$$

The proof is complete. □

5 Stationary distribution

In this section, we will establish sufficient conditions for the existence of a unique ergodic stationary distribution. First of all, we present a lemma which will be used later.

Let $x(t)$ be a homogeneous Markov process in E_l (E_l denotes an l -dimensional Euclidean space) and be described by the following stochastic differential equation:

$$dx(t) = b(x) dt + \sum_{r=1}^k g_r(x) dB_r(t).$$

The diffusion matrix is defined as follows:

$$\tilde{A}(x) = (a_{ij}(x)) = \sum_{r=1}^k g_r^i(x) g_r^j(x).$$

Lemma 5.1 ([40]) *The Markov process $x(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$ if there exists a bounded domain $U \subset E_l$ with regular boundary Γ and*

(A1) *There is a positive number M such that $\sum_{i,j=1}^l a_{ij}(x) \xi_i \xi_j \geq M |\xi|^2$, $x \in U$, $\xi \in \mathbb{R}^l$.*

(A2) *There exists a nonnegative C^2 -function V such that $\mathcal{L}V$ is negative for any $E_l \setminus U$.*

Then

$$P_x \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x(t)) dt = \int_{E_l} f(y) \mu(dy) \right\} = 1$$

for all $x \in E_l$, where $f(\cdot)$ is a function integrable with respect to the measure μ .

Theorem 5.1 *Assume that $\tilde{R}_0 > 1$. Then, for any initial value $(S(0), E(0), I(0), R(0)) \in \mathbb{R}_+^4$, there is a stationary distribution $\mu(\cdot)$ for system (2) and the ergodicity holds.*

Proof The diffusion matrix of system (2) is given by $\tilde{A} = \text{diag}(\sigma_1^2 S^2, \sigma_2^2 E^2, \sigma_3^2 I^2, \sigma_4^2 R^2)$. We choose $M = \min_{(S,E,I,R) \in D_k \subset \mathbb{R}_+^4} \{\sigma_1^2 S^2, \sigma_2^2 E^2, \sigma_3^2 I^2, \sigma_4^2 R^2\}$. We get

$$\begin{aligned} \sum_{i,j=1}^4 a_{ij}(S, E, I, R) \xi_i \xi_j &= (\xi_1, \xi_2, \xi_3, \xi_4) \tilde{A} (\xi_1, \xi_2, \xi_3, \xi_4)^T \\ &= \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 E^2 \xi_2^2 + \sigma_3^2 I^2 \xi_3^2 + \sigma_4^2 R^2 \xi_4^2 \\ &\geq M \|\xi\|^2 \end{aligned}$$

for any $(S, E, I, R) \in D_k$, $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}_+^4$, where $D_k = [\frac{1}{k}, k] \times [\frac{1}{k}, k] \times [\frac{1}{k}, k] \times [\frac{1}{k}, k]$ and $k > 1$ is a sufficiently large integer. Then condition (A1) holds, where $E_l = \mathbb{R}_+^4$, $U = D_k$.

Next we construct a nonnegative C^2 -function $\tilde{V} : \mathbb{R}_+^4 \rightarrow \mathbb{R}$ in the following form:

$$\begin{aligned} \tilde{V}(S, E, I, R) &= M(-c_1 \ln S - c_2 \ln E - c_3 I - \ln R - \ln I) \\ &\quad - \ln S - \ln R - \ln(S + E + I + R) + \frac{(S + E + I + R)^{n+1}}{n + 1}. \end{aligned}$$

It is easy to check that

$$\liminf_{k \rightarrow \infty, (S,E,I,R) \in \mathbb{R}_+^4 \setminus U_k} \tilde{V}(S, E, I, R) = +\infty,$$

where $U_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k)$. Besides, $\tilde{V}(S, E, I, R)$ is a continuous function. Hence, $\tilde{V}(S, E, I, R)$ must admit a minimum point (S^*, E^*, I^*, R^*) in the interior of \mathbb{R}_+^4 . Then we define a nonnegative C^2 -function \tilde{V} as follows:

$$\begin{aligned} \tilde{V}(S, E, I, R) &= \tilde{V}(S, E, I, R) - \tilde{V}(S^*, E^*, I^*, R^*) \\ &= MV_1 + V_2 + V_3 + V_4 + V_5 - \tilde{V}(S^*, E^*, I^*, R^*), \end{aligned}$$

where V_1 is presented in (10), and

$$V_2 = -\ln S, \quad V_3 = -\ln R, \quad V_4 = -\ln(S + E + I + R), \quad V_5 = \frac{(S + E + I + R)^{n+1}}{n + 1},$$

where n is a sufficiently small constant and $M > 0$ satisfying the following condition:

$$\begin{aligned} \eta &= \mu - \frac{n(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)}{2} > 0, \\ -M\lambda + 5\mu + \delta + \rho + \frac{2\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_4^2}{2} + B &\leq -2, \end{aligned}$$

where

$$B = \sup_{(S,E,I,R) \in \mathbb{R}_+^4} \left\{ A(S + E + I + R)^n - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) \right\}.$$

According to similar discussions as shown in Theorem 4.1, we have

$$\mathcal{L}V_2 = -\frac{A}{S} + \mu - \frac{\delta R}{S} + \frac{\beta I}{\varphi(I)} + \frac{\sigma_1^2}{2} < -\frac{A}{S} + \mu + \beta I + \frac{\sigma_1^2}{2},$$

$$\begin{aligned} \mathcal{L}V_3 &= -\frac{\gamma I}{R} + \mu + \delta + \frac{\sigma_4^2}{2}, \\ \mathcal{L}V_4 &= -\frac{1}{S+E+I+R} [A - \mu S - \mu E - (\mu + \rho)I - \mu R] + \frac{\sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2 + \sigma_4^2 R^2}{2(S+E+I+R)^2} \\ &< \frac{\mu S + (\mu + \rho)I + \mu R}{S+E+I+R} + \frac{\mu E}{S+E+I+R} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\ &\leq 3\mu + \rho + \frac{\mu E}{S+E+I+R} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}V_5 &= (S+E+I+R)^n [A - \mu(S+E+I+R) - \rho I] \\ &\quad + \frac{n(S+E+I+R)^{n-1}}{2} [\sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2 + \sigma_4^2 R^2] \\ &< A(S+E+I+R)^n - \left[\mu - \frac{n(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)}{2} \right] (S+E+I+R)^{n+1} \\ &\leq B - \frac{\eta}{2} (S+E+I+R)^{n+1} \\ &< B - \frac{\eta}{2} (S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}). \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}\bar{V} &\leq -M\lambda + M\bar{\gamma}I - \frac{A}{S} + \mu + \beta I + \frac{\sigma_1^2}{2} - \frac{\gamma I}{R} + \mu + \delta + \frac{\sigma_4^2}{2} + 3\mu + \rho + \frac{\mu E}{S+E+I+R} \\ &\quad + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} + B - \frac{\eta}{2} (S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) \\ &= -M\lambda + (M\bar{\gamma} + \beta)I - \frac{A}{S} - \frac{\gamma I}{R} + \frac{\mu E}{S+E+I+R} - \frac{\eta}{2} (S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) \\ &\quad + 5\mu + \delta + \rho + \frac{2\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_4^2}{2} + B \\ &\leq -M\lambda + (M\bar{\gamma} + \beta)I - \frac{A}{S} - \frac{\gamma I}{R} + \frac{\mu E}{S+E+I+R} \\ &\quad - \frac{\eta}{2} (S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N, \end{aligned}$$

where $N = 5\mu + \delta + \rho + \frac{2\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_4^2}{2} + B$.

Now we are in a position to construct a compact subset D such that condition (A2) in Lemma 5.1 holds. Define the bounded closed set

$$D = \left\{ (S, E, I, R) \in \mathbb{R}_+^4 : \varepsilon_1 \leq S \leq \frac{1}{\varepsilon_1}, \varepsilon_2 \leq E \leq \frac{1}{\varepsilon_2}, \varepsilon_3 \leq I \leq \frac{1}{\varepsilon_3}, \varepsilon_4 \leq R \leq \frac{1}{\varepsilon_4} \right\},$$

where $\varepsilon_i > 0$ ($i = 1, 2, 3, 4$) are sufficiently small constants satisfying the following conditions:

$$-\frac{A}{\varepsilon_1} + P \leq -1, \tag{13}$$

$$-M\lambda + (M\bar{\gamma} + \beta)\varepsilon_2 + Q \leq -1, \tag{14}$$

$$-M\lambda + \frac{\mu\varepsilon_3}{\varepsilon_1 + \varepsilon_2} + T \leq -1, \tag{15}$$

$$-\frac{\gamma\varepsilon_2}{\varepsilon_4} + P \leq -1, \tag{16}$$

$$-\frac{\eta}{4\varepsilon_1^{n+1}} + F \leq -1, \tag{17}$$

$$-\frac{\eta}{4\varepsilon_2^{n+1}} + G \leq -1, \tag{18}$$

$$-\frac{\eta}{4\varepsilon_3^{n+1}} + H \leq -1, \tag{19}$$

$$-\frac{\eta}{4\varepsilon_4^{n+1}} + L \leq -1, \tag{20}$$

where P, Q, T, F, G, H, L are presented in (21), (22), (23), (24), (25), (26), (27), respectively. For convenience, we divide $\mathbb{R}_+^4 \setminus D$ into eight domains:

$$\begin{aligned} D_1 &= \{(S, E, I, R) \in \mathbb{R}_+^4 : 0 < S < \varepsilon_1\}, & D_2 &= \{(S, E, I, R) \in \mathbb{R}_+^4 : 0 < I < \varepsilon_2, S \geq \varepsilon_1\}, \\ D_3 &= \{(S, E, I, R) \in \mathbb{R}_+^4 : S \geq \varepsilon_1, I \geq \varepsilon_2, 0 < E < \varepsilon_3\}, \\ D_4 &= \{(S, E, I, R) \in \mathbb{R}_+^4 : 0 < R < \varepsilon_4, I \geq \varepsilon_2\}, \\ D_5 &= \left\{ (S, E, I, R) \in \mathbb{R}_+^4 : S > \frac{1}{\varepsilon_1} \right\}, & D_6 &= \left\{ (S, E, I, R) \in \mathbb{R}_+^4 : E > \frac{1}{\varepsilon_2} \right\}, \\ D_7 &= \left\{ (S, E, I, R) \in \mathbb{R}_+^4 : I > \frac{1}{\varepsilon_3} \right\}, & D_8 &= \left\{ (S, E, I, R) \in \mathbb{R}_+^4 : R > \frac{1}{\varepsilon_4} \right\}. \end{aligned}$$

Obviously, $D^C = D_1 \cup D_2 \cup \dots \cup D_8$. Next we only need to show that $\mathcal{L}\bar{V}(S, E, I, R) \leq -1$ on D^C .

Case 1. If $(S, E, I, R) \in D_1$, by (13) we get that

$$\begin{aligned} \mathcal{L}\bar{V} &\leq -M\lambda + (M\bar{\gamma} + \beta)I - \frac{A}{S} - \frac{\gamma I}{R} + \frac{\mu E}{S + E + I + R} \\ &\quad - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &< (M\bar{\gamma} + \beta)I - \frac{A}{S} + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &\leq -\frac{A}{\varepsilon_1} + P \leq -1, \end{aligned}$$

where

$$P = \sup_{(S, E, I, R) \in \mathbb{R}_+^4} \left\{ (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \right\}. \tag{21}$$

Case 2. If $(S, E, I, R) \in D_2$, by (14) we have that

$$\begin{aligned} \mathcal{L}\bar{V} &< -M\lambda + (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &\leq -M\lambda + (M\bar{\gamma} + \beta)I + Q \\ &\leq -M\lambda + (M\bar{\gamma} + \beta)\varepsilon_2 + Q \leq -1, \end{aligned}$$

where

$$Q = \sup_{(S,E,I,R) \in \mathbb{R}_+^4} \left\{ \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \right\}. \tag{22}$$

Case 3. If $(S, E, I, R) \in D_3$, by (15) we have that

$$\begin{aligned} \mathcal{L}\bar{V} &< -M\lambda + \frac{\mu E}{S+I} + (M\bar{\gamma} + \beta)I - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &\leq -M\lambda + \frac{\mu E}{S+I} + T \\ &\leq -M\lambda + \frac{\mu \varepsilon_3}{\varepsilon_1 + \varepsilon_2} + T \leq -1, \end{aligned}$$

where

$$T = \sup_{(S,E,I,R) \in \mathbb{R}_+^4} \left\{ (M\bar{\gamma} + \beta)I - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \right\}. \tag{23}$$

Case 4. If $(S, E, I, R) \in D_4$, by (16) we get that

$$\begin{aligned} \mathcal{L}\bar{V} &< (M\bar{\gamma} + \beta)I - \frac{\gamma I}{R} + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &\leq -\frac{\gamma I}{R} + P \\ &\leq -\frac{\gamma \varepsilon_2}{\varepsilon_4} + P \leq -1. \end{aligned}$$

Case 5. If $(S, E, I, R) \in D_5$, by (17) we get that

$$\begin{aligned} \mathcal{L}\bar{V} &< (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &= -\frac{\eta}{4}S^{n+1} + (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}S^{n+1} - \frac{\eta}{2}(E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &\leq -\frac{\eta}{4\varepsilon_1^{n+1}} + F \leq -1, \end{aligned}$$

where

$$F = \sup_{(S,E,I,R) \in \mathbb{R}_+^4} \left\{ (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}S^{n+1} - \frac{\eta}{2}(E^{n+1} + I^{n+1} + R^{n+1}) + N \right\}. \tag{24}$$

Case 6. If $(S, E, I, R) \in D_6$, by (18) we get that

$$\begin{aligned} \mathcal{L}\bar{V} &< (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &= -\frac{\eta}{4}E^{n+1} + (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}E^{n+1} - \frac{\eta}{2}(S^{n+1} + I^{n+1} + R^{n+1}) + N \\ &\leq -\frac{\eta}{4\varepsilon_2^{n+1}} + G \leq -1, \end{aligned}$$

where

$$G = \sup_{(S,E,I,R) \in \mathbb{R}_+^4} \left\{ (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}E^{n+1} - \frac{\eta}{2}(S^{n+1} + I^{n+1} + R^{n+1}) + N \right\}. \tag{25}$$

Case 7. If $(S, E, I, R) \in D_7$, by (19) we get that

$$\begin{aligned} \mathcal{L}\bar{V} &< (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &= -\frac{\eta}{4}I^{n+1} + (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}I^{n+1} - \frac{\eta}{2}(S^{n+1} + E^{n+1} + R^{n+1}) + N \\ &\leq -\frac{\eta}{4\epsilon_3^{n+1}} + H \leq -1, \end{aligned}$$

where

$$H = \sup_{(S,E,I,R) \in \mathbb{R}_+^4} \left\{ (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}I^{n+1} - \frac{\eta}{2}(S^{n+1} + E^{n+1} + R^{n+1}) + N \right\}. \tag{26}$$

Case 8. If $(S, E, I, R) \in D_8$, by (20) we get that

$$\begin{aligned} \mathcal{L}\bar{V} &< (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1} + R^{n+1}) + N \\ &= -\frac{\eta}{4}R^{n+1} + (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}R^{n+1} - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1}) + N \\ &\leq -\frac{\eta}{4\epsilon_4^{n+1}} + L \leq -1, \end{aligned}$$

where

$$L = \sup_{(S,E,I,R) \in \mathbb{R}_+^4} \left\{ (M\bar{\gamma} + \beta)I + \mu - \frac{\eta}{4}R^{n+1} - \frac{\eta}{2}(S^{n+1} + E^{n+1} + I^{n+1}) + N \right\}. \tag{27}$$

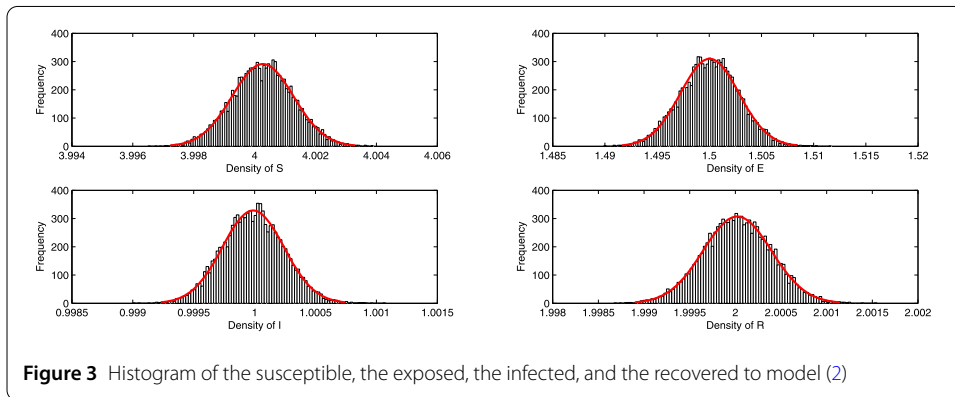
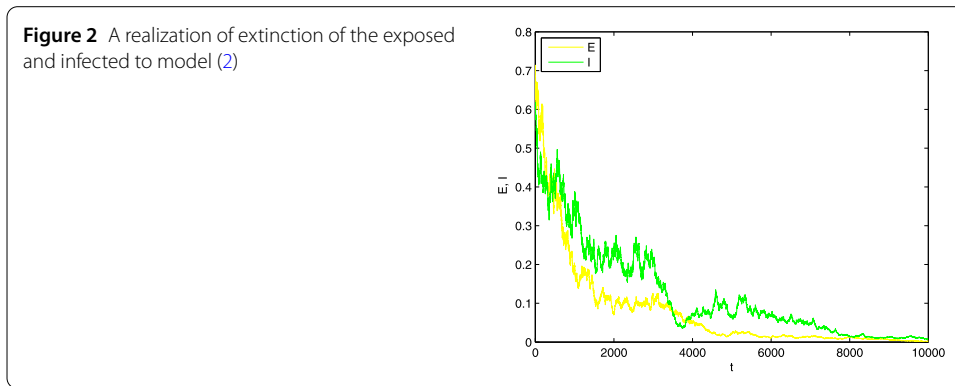
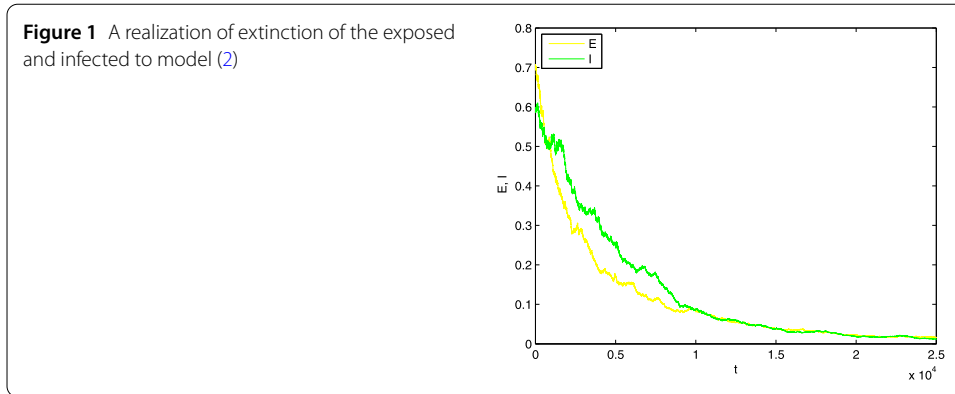
The proof is complete. □

6 Examples and simulations

For the numerical simulation, we use Milstein’s higher order method mentioned in [41] to obtain the following discretization transformation of system (2):

$$\begin{aligned} S(j+1) &= S(j) + \left[A - \mu S(j) + \delta R(j) - \frac{\beta S(j)I(j)}{\varphi(I(j))} \right] \Delta t + \sigma_1 S(j) \xi_1 \sqrt{\Delta t} + \frac{\sigma_1^2}{2} S(j) (\xi_1^2 - 1) \Delta t, \\ E(j+1) &= E(j) + \left[\frac{\beta S(j)I(j)}{\varphi(I(j))} - (\mu + \sigma) E(j) \right] \Delta t + \sigma_2 E(j) \xi_2 \sqrt{\Delta t} + \frac{\sigma_2^2}{2} E(j) (\xi_2^2 - 1) \Delta t, \\ I(j+1) &= I(j) + [\sigma E(j) - (\mu + \rho + \gamma) I(j)] \Delta t + \sigma_3 I(j) \xi_3 \sqrt{\Delta t} + \frac{\sigma_3^2}{2} I(j) (\xi_3^2 - 1) \Delta t, \\ R(j+1) &= R(j) + [\gamma I(j) - (\mu + \delta) R(j)] \Delta t + \sigma_4 R(j) \xi_4 \sqrt{\Delta t} + \frac{\sigma_4^2}{2} R(j) (\xi_4^2 - 1) \Delta t, \end{aligned}$$

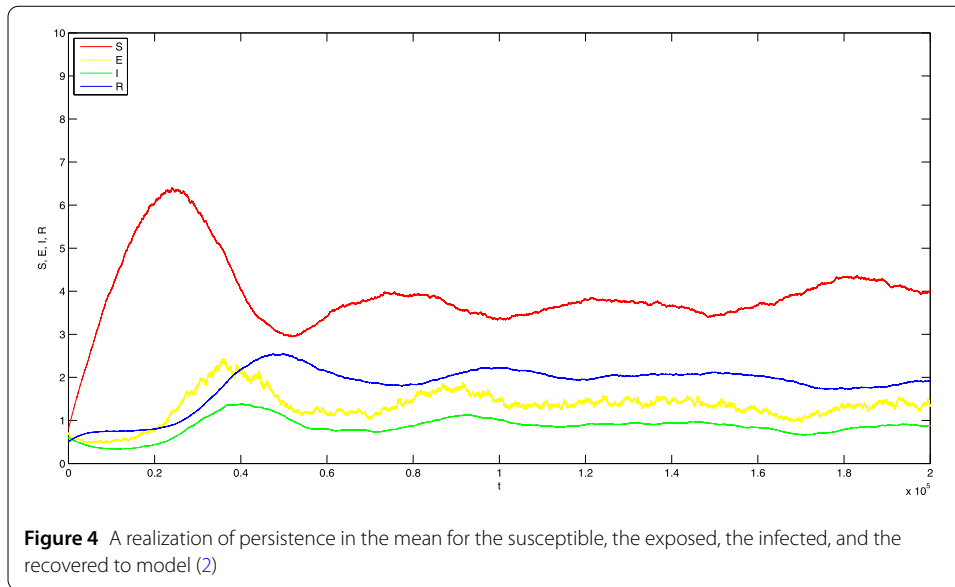
where the time increment $\Delta t > 0$, $\sigma_i^2 > 0$ ($i = 1, 2, 3, 4$) are the intensities of the white noise, ξ_i ($i = 1, 2, 3, 4$) are independent Gaussian random variables which follow the distribution $N(0, 1)$ for $j = 0, 1, 2, \dots, n$.



We take the parameters of model (2) as follows: $A = 0.7$, $\beta = 0.055$, $\mu = 0.1$, $\gamma = 0.2$, $\rho = 0.15$, $\sigma = 0.3$, $\Delta t = 0.001$, $\varphi(I) = 1 + 0.6I$, $\delta = 0.04$, $\sigma_1 = 0.009$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$, $\sigma_4 = 0.1$, $S(0) = 0.8$, $E(0) = 0.7$, $I(0) = 0.6$, $R(0) = 0.5$. By condition of Theorem 3.1, we have

$$R_0 = 0.6417 < 1, \quad \mu = 0.1 > \frac{\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2}{2} = 0.005,$$

which indicates the extinction of infected individuals. The corresponding realizations of model (2) demonstrate their properties in Fig. 1, when $n = 25,000$. At the same time, we get that the disease will reach extinction faster as the environmental disturbance increases. For example, when $\sigma_1 = 0.054$, $\sigma_2 = 0.6$, $\sigma_3 = 0.6$, $\sigma_4 = 0.6$, for the corresponding dynamics see Fig. 2, when $n = 5000$.



We further assume that the parameters of model (2) are $A = 0.5$, $\beta = 0.1$, $\mu = 0.01$, $\gamma = 0.2$, $\rho = 0.1$, $\sigma = 0.2$, $\sigma_1 = 0.008$, $\sigma_2 = 0.06$, $\sigma_3 = 0.008$, $\sigma_4 = 0.006$, $\delta = 0.08$, $\varphi(I) = 1 + 0.15I$. By the condition of Theorem 4.1 and Theorem 5.1, we have $\tilde{R}_0 = 6.0725 > 1$, and $\liminf_{t \rightarrow \infty} \langle I \rangle_t \geq 0.1625$, model (2) admits a unique stationary distribution as shown in Fig. 3, and the solution of model (2) is persistent in the mean as shown in Fig. 4, when $n = 200,000$.

7 Conclusions

In this paper, we intend to investigate an epidemic model of having four stages: the susceptible, the exposed, the infected, and the recovered. And we focus on extinction, persistence, and stationary distribution of a positive solution to epidemic model with nonlinear incidence rate and independent environmental fluctuations.

We firstly, by constructing an appropriate function, show that model (2) admits a unique global positive solution with any initial value. Moreover, we also find that the extinction of disease depends on the basic reproduction number R_0 (a threshold for its corresponding deterministic model). When $R_0 < 1$ and $\nu < 0$, the disease under independent environmental fluctuations dies out as demonstrated in Theorem 3.1, and its corresponding dynamics could be found in Fig. 1. While, by constructing several C^2 -functions, under the condition $\tilde{R}_0 > 1$, we derive sufficient conditions for persistence and existence of a unique ergodic stationary distribution to model (2), the corresponding realizations could be found in Fig. 2 and Fig. 3, respectively.

We further present numerical simulations on ergodicity of model (2) at the end of this paper and point out that extinction time of infected individuals decreases when intensities of environmental fluctuations σ_i ($i = 1, 2, 3, 4$) increase. These results provide readers a biological perspective when understanding an epidemic model with fluctuated environments.

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Availability of data and materials

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Competing interests

The authors state that they have no competing interests in the manuscript.

Authors' contributions

We declare that all the authors conceived of the study and carried out the proof. All authors of this paper read and approved the final manuscript.

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