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Random attractors for non-autonomous stochastic wave equations with nonlinear damping and white noise

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Abstract

This paper is concerned with the asymptotic behavior of solutions to a non-autonomous stochastic wave equation with additive white noise, for which the nonlinear damping has a critical cubic growth rate. By showing the pullback asymptotic compactness of the stochastic dynamic systems, we prove the existence of a random attractor in $H_0^1 \times L^2$.

Keywords: Random attractors; Additive white noise; Nonlinear damping; Non-autonomous stochastic wave equation

1 Introduction

This paper deals with the existence of random attractors for the following non-autonomous stochastic wave equation with white noise in a bounded domain $U \subset \mathbb{R}^3$ with smooth boundary ∂U :

$$u_{tt} + q(u)u_t + \alpha u_t - \Delta u + f(u, x) = g(x, t) + ah(x)\dot{W}(t), \quad (1)$$

$$u(x, t)|_{x \in \partial U} = 0, \quad t \geq \tau, \tau \in R, \quad (2)$$

$$u(x, \tau) = u_{0\tau}(x), \quad u_t(x, \tau) = u_{1\tau}(x), \quad x \in U, \tau \in R, \quad (3)$$

for $(x, t) \in U \times (\tau, +\infty)$ with $\tau \in R$, where $h \in H_0^1(U) \cap H^2(U)$ and $\alpha \geq 0$ is the damping coefficient. Here $u(x, t)$ is a real-valued function on $U \times [\tau, +\infty)$; $g(x, \cdot) \in C_b(R, H_0^1(U))$ is a time-dependent driving force; $C_b(R, H_0^1(U))$ denotes the set of continuous bounded functions from R into $H_0^1(U)$; and $W(t)$ is a two-sided real-valued Wiener processes on the probability space (Ω, \mathcal{F}, P) . In addition, the function $q: R \rightarrow R$ and the nonlinear function f satisfy the following assumptions:

(H₁) The function $q \in C^1$ is not identically equal to zero, and there exist three constants $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_2 \geq |\alpha_1|$ such that

$$-\alpha < \alpha_1 \leq q(s) \leq \alpha_2 < +\infty, \quad |q'(s)| \leq \alpha_3, \quad \forall s \in R. \quad (4)$$

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(H_2) Let $f(u, x) = f_1(u, x) + f_2(u, x)$ and $F_i = \int_0^u f_i(r, x) dr$, where $f_1(\cdot, x) \in \mathcal{C}^2(R, R)$, $f_2(\cdot, x) \in \mathcal{C}^1(R, R)$. Furthermore, f_1, f_2 meet conditions that there exist constants $c_1, c_2, c_3, c_4 > 0$ and functions $\phi_i(x) \in L^1(U)$, $i = 1, 2$, such that

$$f'_{1,u}(0, x) = 0, \quad f_1(u, x)u \geq 0, \quad |f''_{1,u}(u, x)| \leq c_1(1 + |u|), \quad (5)$$

$$f_2(0, x) = 0, \quad |f'_{2,u}(u, x)| \leq c_2(1 + |u|^p), \quad 0 \leq p \leq 2, \quad (6)$$

$$c_3u^4 - \phi_1(x) \leq F_i(u, x) \leq c_4uf_i(u, x) + \phi_2(x), \quad \forall u \in R, x \in U. \quad (7)$$

In the deterministic damped wave equation (i.e., $a = 0$), global attractors have been studied by many authors, such as [1–3] and the reference therein. In addition, uniform attractors and pullback attractors also attracted many experts' attention, cf. [4–8]. If the function g does not depend on time, (1)–(3) is an autonomous stochastic wave equation, and its random attractors have been explored in [9–13]. For many problems, such as wave propagation through the atmosphere or the ocean, the more realistic models must take the random fluctuation into account. So it is important and interesting to study random attractors. For non-autonomous random dynamical systems, Wang established an efficacious theory about the existence of random attractors [14–17]. Particularly, Li [18] studied the asymptotic dynamics for a stochastic damped wave equation with multiplicative noise defined on unbounded domains and proved the existence of random attractors. For the non-autonomous stochastic strongly damped wave equation, the existence of random attractors is proved in [19–21]. Lv and Wang [10] also studied the existence of random attractors for the stochastic wave equation and showed the upper semicontinuous dependence of the random attractor on parameters. The authors in [22] studied the asymptotic behavior of a class of non-autonomous nonlocal fractional stochastic parabolic equations driven by multiplicative white noise on the entire space R^n .

In this paper, (1)–(3) is a non-autonomous stochastic system where the external term g is time-dependent. We shall transform the stochastic wave equation into a deterministic one with random parameter and random initial data through an Ornstein–Uhlenbeck process $z(\theta_t\omega)$, then prove the existence of a random attractor for the random dynamical system generated by (1)–(3). It is well known that the key step in proving the existence of attractors in both deterministic and random systems is to establish the compactness of the system in some sense. Motivated by [23], we will work out this problem.

The paper is arranged as follows. In Sect. 2, we collect some basic concepts and background material about random attractor for the random dynamical system generated by (1)–(3), then the existence and uniqueness of solutions is established in Sect. 3. In Sect. 4, we consider the concrete bounds of the solution and decompose the solution of (12)–(13) into two parts. In Sect. 5, we establish the asymptotic compactness of the random dynamical system and obtain the existence of the random attractor.

2 Random dynamical systems

In this section, we collect some basic definitions and known results about general random dynamical systems (see [17, 24, 25] for details).

Let (Ω, \mathcal{F}, P) be a probability space, where $\Omega = \{\omega \in C(R, R) : \omega(0) = 0\}$ is endowed with compact-open topology. \mathcal{F} is the Borel σ -algebra on Ω and P is the corresponding Wiener

measure on \mathcal{F} . For any t , let $(\theta_t)_{t \in \mathbb{R}}$ on Ω via

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

thus $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system [24]. In the following, X labels as a Banach or Hilbert space with the Borel σ -algebra $B(X)$.

Definition 2.1 Let $\{\theta_t\}_{t \in \mathbb{R}}$ be a family of $(B(\mathbb{R} \times \mathcal{F}), \mathcal{F})$ -measurable mappings, $\theta_t : \mathbb{R} \times \Omega \rightarrow \Omega$ such that $\theta_0(\cdot)$ is the identity on Ω , $\theta_{s+t}(\cdot) = \theta_t(\cdot) \circ \theta_s(\cdot)$ for all $t, s \in \mathbb{R}$ and $P\theta_t = P$ for all $t \in \mathbb{R}$.

Definition 2.2 Let $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ be a parametric dynamical system. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \rightarrow X$ is called a continuous cocycle on X over \mathbb{R} and $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ if, for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $t, s \in \mathbb{R}^+$, the following conditions (i)–(iv) are satisfied:

- (i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \rightarrow X$ is a $(B(\mathbb{R}^+) \times \mathcal{F} \times B(X), B(X))$ -measurable mapping;
- (ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X ;
- (iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;
- (iv) $\Phi(t, \tau, \omega, \cdot) : X \rightarrow X$ is continuous.

Definition 2.3

- (1) Let 2^X be the collection of all subsets of X . A set-valued mapping $(\tau, \omega) \rightarrow D(\tau, \omega) : \mathbb{R} \times \Omega \rightarrow 2^X$ is called measurable with respect to \mathcal{F} in Ω if $D(\tau, \omega)$ is a (usually closed) nonempty subset of X and the mapping $\omega \in \Omega \rightarrow d(x, D(\tau, \omega))$ is $(\mathcal{F}, B(\mathbb{R}))$ -measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$, then $D = D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega$ is called a random set.
- (2) Let D be a collection of random sets in a Polish space X . A continuous cocycle Φ is said to be pullback \mathcal{D} -asymptotically compact (\mathcal{D} -a.c.) in X if, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D \in \mathcal{D}$ and any sequences $t_n \rightarrow +\infty$, $x_n \in D(\tau - t_n, \theta_{-t_n} \omega)$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, x_n)$ has a convergent subsequence in X .
- (3) Let $K = K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \in D$. Then K is called a pullback \mathcal{D} -absorbing set for Φ if, for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $t_0(K, \tau, \omega) > 0$ such that $\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subset K(\tau, \omega)$ for any $t \geq t_0$.
- (4) A family $C = C(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \in \mathcal{D}$ is said to be pullback \mathcal{D} -attracting if $\lim_{t \rightarrow +\infty} d(\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)), C(\tau, \omega)) = 0$ for all $D \in \mathcal{D}$.
- (5) A compact set $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a pullback \mathcal{D} -attractor for Φ if A attracts every $B \in \mathcal{D}$ and A is invariant in the sense that $\Phi(t, \tau, \omega, A(\tau, \omega)) = A(\tau + t, \theta_t \omega)$ for every $t \geq 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$.

In addition, if there exists $T > 0$ such that $A(\tau + T, \omega) = A(\tau, \omega)$ for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, then A is periodic with period T .

Proposition 2.1 Let D be a neighborhood-closed collection of (τ, ω) -parametrized families of nonempty subsets of X and Φ be a continuous cocycle on X over \mathbb{R} and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. Then Φ has a pullback D -attractor A in D if and only if Φ is pullback \mathcal{D} -asymptotically compact in X and Φ has a closed \mathcal{F} -measurable pullback \mathcal{D} -absorbing set K in D . The unique pullback D -attractor $A = A(\tau, \omega)$ is given by

$$A(\tau, \omega) = \bigcap_{r \geq 0} \bigcup_{t \geq r} \overline{\Phi(t, \tau - t, \theta_{-t} \omega), K(\tau - t, \theta_{-t} \omega)}, \quad \tau \in \mathbb{R}, \omega \in \Omega.$$

Proposition 2.2 *Let D be a neighborhood-closed collection of (τ, ω) -parametrized families of nonempty subsets of X . If Φ is a continuous τ -periodic cocycle with period $T > 0$ on X over R and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in R})$ and if Φ has a pullback \mathcal{D} -attractor $A \in D$, then A is τ -periodic with period T if and only if Φ has a closed \mathcal{F} -measurable pullback \mathcal{D} -absorbing set $K \in D$ with $K = K(\tau, \omega)$ being periodic in τ with period T for each $\omega \in \Omega$.*

Notation Set $E = H_0^1(U) \times L^2(U)$ with its inner product and norm as follows:

$$(z_1, z_2)_E = ((u_1, u_2)) + (v_1, v_2), \quad \|z\|_E = (z, z)_E^{\frac{1}{2}}$$

for all $z_i = (u_i, v_i)^T$, $i = 1, 2$, and $z = (u, v)^T$ in E .

$$(u, v) = \int_U u(x)v(x) dx, \quad \|u\| = \|u\|_{L^2} = (u, u)^{\frac{1}{2}}$$

for all $u, v \in L^2(U)$, and

$$((u, v)) = \int_U \nabla u(x) \nabla v(x) dx, \quad \|u\|_1 = \|u\|_{H_0^1} = ((u, u))^{\frac{1}{2}}$$

for all $u, v \in H_0^1(U)$. More generally, denote $E_s = W^{s,2}(U) \cap H_0^1 \times W^{s-1,2}(U)$ for $s \in R$.

The letters c and c_i ($i = 1, 2, \dots$) are generic positive constants which do not depend on ω, τ, t, a .

3 Existence and uniqueness of solutions

In this section, motivated by [26, 27], we establish the existence and uniqueness of solutions for Eqs. (1)–(3). Let λ be the first eigenvalue of the operator $A := -\Delta$ on U with Dirichlet boundary conditions. Note that $A : H_0^1(U) \cap H^2(U) \rightarrow L^2(U)$, so $D(A) = H_0^1(U) \cap H^2(U)$. In the following, we convert problem (1)–(3) into a random system without noise terms. Identify $\omega(t)$ with $W(t)$, i.e., $\omega(t) = W(t)$, $t \in R$, and let $z(\theta_t \omega) := -\int_{-\infty}^0 e^s(\theta_t \omega)(s) ds$ ($t \in R$) be a Ornstein–Uhlenbeck stationary process which solves the Itô equation $dz + z dx = dW(t)$.

Let $\varepsilon = \frac{(\alpha + \alpha_1)\lambda_1}{2(\alpha + \alpha_2)^2 + 3\lambda_1}$. By the transformation

$$\varphi_1 = u, \quad \varphi_2 = u_t + \varepsilon u - ah(x)z(\theta_t \omega),$$

Equations (1)–(3) are equivalent to the following determined system with random parameters in E :

$$\frac{d\varphi_1}{dt} = \varphi_2 - \varepsilon \varphi_1 + ah(x)z(\theta_t \omega), \quad (8)$$

$$\begin{aligned} \frac{d\varphi_2}{dt} = & \Delta \varphi_1 + \varepsilon(\alpha - \varepsilon)\varphi_1 + (\varepsilon - \alpha)\varphi_2 - q(\varphi_1)(\varphi_2 - \varepsilon \varphi_1) \\ & - (q(\varphi_1) + \alpha - \varepsilon - 1)ah(x)z(\theta_t \omega) - f(\varphi_1, x) + g(x, t), \end{aligned} \quad (9)$$

$$\varphi_1(x, t)|_{\partial U} = 0, \quad (10)$$

$$\varphi_1(\tau, \tau, x) = u_\tau(x), \quad \varphi_2(\tau, \tau, x) = v_\tau(x) = u_{1\tau} + \varepsilon u_\tau(x) - ah(x)z(\theta_\tau \omega). \quad (11)$$

Then (8)–(11) can also be rewritten as a vector form:

$$\dot{\varphi} + L(\varphi) = G(\varphi, \theta_t \omega, t), \quad (12)$$

$$\varphi(\tau, \omega) = \varphi_\tau(\omega) = (u_\tau, u_{1,\tau} + \varepsilon u_\tau - ah(x)z(\theta_\tau \omega))^T, \quad \tau \in R, t \geq \tau, \quad (13)$$

where

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad L = \begin{pmatrix} \varepsilon I & -I \\ A - \varepsilon(\alpha - \varepsilon)I & (\alpha - \varepsilon)I \end{pmatrix} \quad (14)$$

and

$$G(\varphi, \theta_t \omega, t) = \begin{pmatrix} ah(x)z(\theta_t \omega) \\ -q(\varphi_1)(\varphi_2 - \varepsilon \varphi_1) - [q(\varphi_1) + \alpha - \varepsilon - 1]ah(x)z(\theta_t \omega) - f(u, x) + g(x, t) \end{pmatrix}. \quad (15)$$

It is known from [28] that $-L$ is the infinitesimal generator of a C^0 -semigroup e^{-Lt} on E . By assumption (H_2) and the embedding relation $H_0^1(U) \hookrightarrow L^6(U)$, the function $G(\varphi, \theta_t \omega, t) : E \rightarrow E$ is Lipschitz with respect to φ for t in a bounded interval and $\omega \in \Omega$, continuous in (φ, t) , and measurable in ω w.r.t \mathcal{F} . Thus, by the classical semigroup theory on the local existence and the uniqueness of solutions of evolution differential equations in [25], we have the following theorem.

Theorem 1 *Consider the initial value problem (12)–(13), if assumptions (H_1) and (H_2) hold, then for each $\tau \in R$, $\omega \in \Omega$ and any $\varphi_\tau \in E$, there exists $T > 0$ such that Eqs. (12)–(13) have a unique mild function $\varphi(\cdot) = \varphi(\cdot, \tau, \omega, \varphi_\tau) \in C([\tau, \tau + T]; E)$, where $\varphi(\tau, \tau, \omega, \varphi_\tau) = \varphi_\tau$ and $\varphi(t)$ satisfies the integral equation*

$$\begin{aligned} \varphi(t, \tau, \omega, \varphi_\tau) &= e^{-L(t-\tau)} \varphi_\tau(\omega) \\ &+ \int_\tau^t e^{-L(t-r)} G(\varphi(r, \tau, \omega, \varphi_\tau), \theta_r \omega, r) dr, \quad \forall t \geq \tau. \end{aligned} \quad (16)$$

System (12)–(13) generates a continuous random dynamical system over R and $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in R})$

$$\Phi : R^+ \times R \times \Omega \times E \rightarrow E, \quad (t, \tau, \omega, \varphi_\tau) \rightarrow \Phi(t, \tau, \omega, \varphi_\tau), \quad (17)$$

where

$$\begin{aligned} \Phi(t, \tau, \omega, \varphi_\tau(\omega)) &= \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega)) \\ &= \begin{pmatrix} u(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_{-\tau}(\theta_{-\tau} \omega)) \\ u_t(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_{-\tau}(\theta_{-\tau} \omega)) + \varepsilon u(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_{-\tau}(\theta_{-\tau} \omega)) - ah(x)z(\theta_t \omega) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
& \Phi(0, \tau, \omega, \varphi_\tau \omega) \\
&= \varphi_\tau(\theta_{-\tau} \omega) \\
&= \begin{pmatrix} u(\tau, \tau, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega)) \\ u_t(\tau, \tau, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega)) + \varepsilon u(\tau, \tau, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega)) - ah(x)z(\omega) \end{pmatrix},
\end{aligned}$$

and

$$\Phi(\tau, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau-t}(\theta_{-\tau} \omega)) = \varphi(\tau, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau-t}(\theta_{-\tau} \omega)).$$

So we have $\Gamma(t, \tau, \omega, Z_\tau) = R_{\varepsilon, \theta_t \omega}^{-1} \Phi(t, \tau, \omega, \varphi_\tau) R_{\varepsilon, \theta_t \omega} : Z_\tau \rightarrow Z(t + \tau, \tau, \theta_{-\tau} \omega, Z_\tau)$. Next, we use the transformation

$$\psi_1 = u, \quad \psi_2 = u_t + \varepsilon u.$$

By using

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{G}(\psi) = \begin{pmatrix} 0 \\ g(x, t) - f(u, x) + ah(x)z(\theta_t \omega) \end{pmatrix} \quad (18)$$

and

$$H(\psi) = \begin{pmatrix} \varepsilon \psi_1 - \psi_2 \\ A\psi_1 - \varepsilon(\alpha - \varepsilon)\psi_1 + (\alpha - \varepsilon)\psi_2 + q(\psi_1)(\psi_2 - \varepsilon\psi_1) \end{pmatrix}, \quad (19)$$

Eqs. (1)–(3) can be rewritten as

$$\dot{\psi} + H(\psi) = \tilde{G}(\psi), \quad \psi_\tau(\omega) = (u_\tau, u_{1,\tau} + \varepsilon u_\tau)^T. \quad (20)$$

Thus

$$\Psi(t, \tau, \omega, \psi_\tau) = T_\varepsilon \Gamma(t, \tau, \omega, Z_\tau) T_{-\varepsilon} : \psi_\tau \rightarrow \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \psi_\tau), \quad (21)$$

where

$$\begin{aligned}
\Psi(t, \tau, \omega, \psi_\tau) &= \psi(t + \tau, \tau, \theta_{-\tau} \omega, \psi_\tau) \\
&= \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega)) + (0, ah(x)z(\theta_t \omega))^T.
\end{aligned} \quad (22)$$

Since $R_{\varepsilon, \theta_t \omega} : (a, b)^T \rightarrow (a, b + \varepsilon a - ah(x)z(\theta_t \omega))^T$ is an isomorphism of E , then Φ , Γ , Ψ are equivalent to each other in dynamics.

Therefore, the existence of random attractors in any of these stochastic dynamical systems means that random attractors also exist in other dynamical systems. We will consider the existence of a random attractor for RDS Φ in the following.

4 Pullback absorbing set

Let $\varphi = (\varphi_1, \varphi_2)^T$ be a solution of system (12)–(13). Rewriting system (12)–(13) as

$$\dot{\varphi} + Q\varphi = \bar{G}(\varphi, \theta_t\omega), \quad \varphi(-\tau, \omega) = (u_0, u_1 + \varepsilon u_0 - ah(x)z(\theta_{-\tau}\omega))^T, \quad (23)$$

where

$$Q\varphi = \begin{pmatrix} \varepsilon\varphi_1 - \varphi_2 \\ A\varphi_1 - \varepsilon(\alpha - \varepsilon)\varphi_1 + (\alpha - \varepsilon)\varphi_2 + q(\varphi_1)(\varphi_2 - \varepsilon\varphi_1) \end{pmatrix}$$

and

$$\bar{G}(\varphi, \theta_t\omega) = \begin{pmatrix} ah(x)z(\theta_t\omega) \\ -(q(\varphi_1) + \alpha - \varepsilon - 1)ah(x)z(\theta_t\omega) - f(\varphi_1, x) + g(x, t) \end{pmatrix},$$

we have the following lemmas.

Lemma 1 ([26, 27]) *For any $\varphi = (\varphi_1, \varphi_2)^T \in E$, $(Q\varphi, \varphi)_E \geq \frac{\varepsilon}{2}\|\varphi\|_E^2 + \frac{\varepsilon}{4}\|\varphi_1\|_{H_1^0}^2 + \frac{\alpha+\alpha_1}{2}\|\varphi_2\|_{L^2}^2$.*

Lemma 2 *If assumptions (H_1) – (H_2) hold, then for any $\tau \in R$, $\omega \in \Omega$, there exists a tempered variable $M_0(\omega)$ (independent of τ) such that, for any set $B \in D(E)$ and $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B(\tau-t, \theta_{-t}\omega)$, there exists $T = T(\tau, \omega, B) \geq 0$ such that, for $t \geq T$, the solution $\varphi(\tau, \tau-t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) \in E$ of (12)–(13) satisfies*

$$\|\varphi(\tau, \tau-t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \leq M_0^2(\omega), \quad \forall t \geq T(\tau, \omega, B). \quad (24)$$

Proof For any $\tau \in R$, $\omega \in \Omega$, let $\varphi(r) = \varphi(r, \tau-t, \theta_{\tau-t}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) = (\varphi_1, \varphi_2)^T \in E$ ($r > \tau-t$) be a solution of (12)–(13) with

$$\varphi(\tau-t) = \varphi_{\tau-t}(\theta_{-\tau}\omega) = (u_{\tau-t}, u_{1,\tau-t} + \varepsilon u_{\tau-t} - ah(x)z(\theta_{-\tau}\omega))^T \in E.$$

Taking the inner product $(\cdot, \cdot)_E$ of (12) with $\varphi(r)$, according to Lemma 1, we have

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|_E^2 + \frac{\varepsilon}{2} \|\varphi\|_E^2 + \frac{\varepsilon}{4} \|\varphi_1\|_{H_1^0}^2 + \frac{\alpha+\alpha_1}{2} \|\varphi_2\|_{L^2}^2 \leq (\bar{G}(\varphi, \theta_{r-\tau}\omega, t), \varphi)_E \quad (25)$$

and

$$\begin{aligned} & (\bar{G}(\varphi, \theta_{r-\tau}\omega, t), \varphi)_E \\ &= ((ah(x)z(\theta_{r-\tau}\omega), \varphi_1)) - ((\alpha - \varepsilon - 1 + q(\varphi_1))ah(x)z(\theta_{r-\tau}\omega), \varphi_2) \\ &+ (g(x, r), \varphi_2) - (f(u, x), \varphi_2). \end{aligned} \quad (26)$$

By some simple computations, we obtain

$$((ah(x)z(\theta_{r-\tau}\omega), \varphi_1)) \leq \frac{1}{\sigma} |az(\theta_{r-\tau}\omega)|^2 \|h(x)\|_1^2 + \frac{\sigma}{4} \|\varphi_1\|_1^2, \quad (27)$$

$$\begin{aligned} & (-q(\varphi_1)ah(x)z(\theta_{r-\tau}\omega), \varphi_2) \\ &= \delta \|\varphi_2\|_{L^2}^2 + C_\delta (\alpha + \alpha_2)^2 |a|^2 \|h(x)\|_{L^2}^2 |z(\theta_{r-\tau}\omega)|^2, \end{aligned} \quad (28)$$

and

$$\begin{aligned} & (-(\alpha - \varepsilon - 1)ah(x)z(\theta_{r-t}\omega), \varphi_2) \\ & \leq \delta \|\varphi_2\|_{L^2}^2 + C_\delta (\alpha - \varepsilon)^2 a^2 \|h(x)\|^2 |z(\theta_{r-t}\omega)|^2, \end{aligned} \quad (29)$$

$$(g(x, r), \varphi_2) \leq \frac{1}{\alpha} \|g\|^2 + \frac{\alpha}{2} \|\varphi_2\|^2, \quad (30)$$

where $\|g\|^2 = \sup_{r \in \mathbb{R}} \|g(\cdot, r)\|^2 < \infty$, $\varphi_2 = u_t + \varepsilon u - ah(x)z(\theta_t\omega)$.

By (5)–(7), we have

$$\begin{aligned} (f(u, x), \varphi_2) &= (f(u, x), u_t + \varepsilon u - ah(x)z(\theta_{r-t}\omega)) \\ &= \frac{d}{dt} \int_U F(u(r, x), x) dx + \varepsilon (f(u, x), u) \\ &\quad - (f(u, x), ah(x)z(\theta_{r-t}\omega)). \end{aligned} \quad (31)$$

From assumption (H_2) , it is clear that

$$(f(u, x), u) = \int_U f(u, x)u dx \geq \frac{1}{c_1} \left(\int_U F(u(r, x), x) dx - \int_U \phi_2 dx \right). \quad (32)$$

With $u^4 \leq \frac{1}{c_2^2} F(u, x) + \phi_1$ and $|f(u, x)| \leq c_3(1 + u^4)$, we get

$$\begin{aligned} & (f(u, x), ah(x)z(\theta_{r-t}\omega)) \\ & \leq c_4 |a| \|h(x)\| |z(\theta_{r-t}\omega)| + c_5 |a| \left(\int_U |u|^4 dx \right)^{\frac{3}{4}} \|h(x)\|_{L^4} |z(\theta_{r-t}\omega)| \\ & \leq c_4 |a| \|h\| |z(\theta_{r-t}\omega)| + \frac{\varepsilon}{2c_2} \bar{F}(u, x) \\ & \quad + c_6 \int_U \phi_1 dx + c_7 a^4 \|h(x)\|_1^4 |z(\theta_{r-t}\omega)|^4, \end{aligned} \quad (33)$$

here $\bar{F}(u, x) = \int_U F(u, x) dx$. By taking (26)–(33) into (25), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|\varphi\|^2 + 2\bar{F}(u, x)] + \frac{1}{2} \|\varphi\|_E^2 + \frac{\varepsilon}{4} \|\varphi_1\|_{H_0^1}^2 + \frac{\alpha + \alpha_2}{2} \|\varphi_2\|_{L^2}^2 \\ & \quad + \frac{\varepsilon}{c_1} \left[\bar{F}(u, x) - 2 \int_U \phi_2(x) dx \right] + c_4 |a| \|h(x)\| |z(\theta_{r-t}\omega)| + \frac{\varepsilon}{2c_2} \bar{F}(u, x) \\ & \quad + c_6 \int_U \phi_1(x) dx + c_7 a^4 \|h(x)\|_1^4 |z(\theta_{r-t}\omega)|^4 \\ & \leq \frac{1}{\varepsilon} a^2 |z(\theta_{r-t}\omega)|^2 \|h(x)\|_1^2 + \frac{\varepsilon}{4} \|u\|_1^2 + \frac{1}{\alpha} [\|g\|^2 + a^2 |z(\theta_{r-t}\omega)|^2 \|h(x)\|^2] \\ & \quad + \frac{\alpha}{2} \|\varphi_2\|^2 + \delta \|\varphi_2\|_{L^2}^2 + C_\delta (2\alpha + \alpha_2)^2 a^2 \|h(x)\|_{L^2}^2 |z(\theta_{r-t}\omega)|^2. \end{aligned} \quad (34)$$

Let

$$\begin{aligned}
 \beta(\theta_{r-\tau}\omega) &= -\frac{2\varepsilon}{c_1} \int_U \phi_2(x) dx + c_4 |a| \|h(x)\| |z(\theta_{r-\tau}\omega)| \\
 &\quad + c_6 \int_U \phi_1(x) dx + c_7 a^4 \|h(x)\|_1^4 |z(\theta_{r-\tau}\omega)|^4 \\
 &\quad + \frac{1}{\alpha} [\|g\|^2 + a^2 |z(\theta_{r-\tau}\omega)|^2 \|h(x)\|^2] \\
 &\quad + C_\delta (2\alpha + \alpha_2)^2 a^2 \|h(x)\|_{L^2}^2 |z(\theta_{r-\tau}\omega)|^2 \\
 &= c_8 + c_9 a^4 |z(\theta_{r-\tau}\omega)|^4.
 \end{aligned} \tag{35}$$

By choosing δ small enough, we get

$$\frac{d}{dt}y(r) + \rho y(r) \leq \beta(\theta_{r-\tau}\omega), \quad \forall r \geq \tau - t, \tag{36}$$

where $y(r) = \|\varphi(r)\|_E^2 + 2\bar{F}(u, x)$ and $\rho = \min\{\frac{\varepsilon+2}{4}, \frac{\alpha+\alpha_2+2}{2}, \frac{2\varepsilon}{c_1}\}$. By Gronwall's inequality to Eq. (36), we have

$$\begin{aligned}
 &y(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) \\
 &\leq y(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) e^{-\rho(r+\tau-t)} \\
 &\quad + \int_{\tau-t}^r \beta(\theta_{s-\tau}\omega) e^{-\rho(r-s)} ds,
 \end{aligned} \tag{37}$$

where

$$\begin{aligned}
 &y(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) \\
 &= \|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2 \int_U F(u(\tau - t, x), x) dx \\
 &\leq \|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2c_{10} \left(|U| + \|u_{\tau-t}\|_1^4 + 2 \int_U \phi_2(x) dx \right),
 \end{aligned} \tag{38}$$

$$\int_{\tau-t}^r \beta(\theta_{s-\tau}\omega) e^{-\sigma(r-s)} ds = \frac{c_8}{\sigma} + c_9 a^4 \int_{\tau-t}^r |z(\theta_{s-r}\omega)|^4 e^{-\sigma(r-s)} ds \tag{39}$$

and

$$\begin{aligned}
 &y(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) \\
 &\geq \|\varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 - 2 \int_U \phi_1(x) dx.
 \end{aligned} \tag{40}$$

Thus by (37)–(40), for $r \geq \tau - t$, we have

$$\begin{aligned}
 &\|\varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \\
 &\leq y(r) + 2 \int_U \phi_1(x) dx
 \end{aligned}$$

$$\begin{aligned} &\leq \left(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2c_{10} \left(|U| + \|u_{\tau-t}\|_1^4 + 2 \int_U \phi_2(x) dx \right) \right) e^{-\sigma(r+t-\tau)} \\ &\quad + 2 \int_U \phi_1(x) dx + c_{11} + c_{12}a^4 \int_{\tau-t}^r |z(\theta_{s-r}\omega)|^4 e^{-\sigma(r-s)} ds. \end{aligned} \quad (41)$$

Therefore

$$\begin{aligned} &\|\varphi(r, \tau-t, \theta_{-\tau}\omega), \varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 \\ &\leq \left(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2c_{10} \left(|U| + \|u_{\tau-t}\|_1^4 + 2 \int_U \phi_2(x) dx \right) \right) e^{-\sigma t} \\ &\quad + 2 \int_U \phi_1(x) dx + c_{11} + c_{12}a^4 \int_{\tau-t}^r |z(\theta_{s-r}\omega)|^4 e^{-\sigma t} ds. \end{aligned} \quad (42)$$

For any set $B(\tau, \omega) \in B \in D(E)$,

$$\begin{aligned} &\varphi_{\tau-t}(\theta_{-\tau}\omega) \\ &= (u_{\tau-t}, u_{1,\tau-t} + \varepsilon u_{\tau-t} - ah(x)z(\theta_{-t}\omega))^T \\ &\in B(\tau-t, \theta_{-t}\omega) \in D(E). \end{aligned} \quad (43)$$

We have

$$\begin{aligned} &\lim_{t \rightarrow +\infty} \sup \left(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 \right. \\ &\quad \left. + 2c_{10} \left(|U| + \|u_{\tau-t}\|_1^4 + 2 \int_U \phi_2(x) dx \right) \right) e^{-\rho t} = 0. \end{aligned} \quad (44)$$

Taking

$$M_0^2(\omega) = 2c_{11} + 2c_{12}a^4 \int_{-\infty}^0 |z(\theta_s\omega)|^4 e^{\rho s} ds < \infty,$$

which is a tempered random variable, $B_0(\omega) = \{\varphi \in E : \|\varphi\|_E \leq M_0(\omega)\}$ is a close measurable absorbing ball in $D(E)$, and there exists $T(\tau, \omega, B) \geq 0$ for all $t \geq T(\tau, \omega, B)$ such that

$$\varphi(\tau, \tau-t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) \in B_0(\omega). \quad (45)$$

$B_0(\omega)$ is the random absorbing set for Φ . The proof is completed. \square

5 Decomposition of the equations

In this section, for proving asymptotic compactness of the random dynamical system Φ , we decompose the solution of Eq. (12)–(13) with different initial data into a sum of two parts, one part decays exponentially and another one is bounded in a higher regular space by using the method in [4, 13].

For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, assume that

$$B_1(\tau, \omega) = \bigcup_{t \geq T(\tau, \omega, B_0)} \varphi(\tau, \tau-t, \theta_{-t}\omega, B_0(\theta_{-t}\omega)) \subseteq B_0(\omega).$$

Let $\varphi(r) = \varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ be a solution of system (12)–(13), with $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau - t, \theta_{-\tau}\omega) \subseteq B_0(\theta_{-\tau}\omega)$. Thus $\varphi(r) \in B_0(\theta_{\tau-t}\omega)$ for all $r \geq \tau - t$. We decompose $\varphi(r)$ into

$$\varphi(r) = \varphi_L(r) + \varphi_N(r), \quad \varphi_L(r) = (u_L, v_L)^T, \varphi_N(r) = (u_N, \varphi_{2N})^T,$$

where $\varphi_L(r)$ and $\varphi_N(r)$ satisfy

$$\begin{cases} \dot{\varphi}_L + Q\varphi_L + \begin{pmatrix} 0 \\ f_1(u_L, x) \end{pmatrix} = 0, \\ \varphi_L(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) = \varphi_{L, \tau-t} = (u_{\tau-t}, u_{1, \tau-t} + \varepsilon u_{\tau-t})^T \end{cases} \quad (46)$$

and

$$\begin{cases} \dot{\varphi}_N + Q\varphi_N + \begin{pmatrix} 0 \\ f_1(u, x) - f_1(u_L, x) \end{pmatrix} = \begin{pmatrix} ah(x)z(\theta_t\omega) \\ -(q(u) + \alpha - \varepsilon)ah(x)z(\theta_t\omega) + g(x, r) \end{pmatrix}, \\ \varphi_N(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) = (0, -ah(x)z(\theta_{-\tau}\omega))^T. \end{cases} \quad (47)$$

First, let us estimate the component φ_L which decays exponentially.

Lemma 3 *Under assumptions (H_1) – (H_2) , for any $\tau \in R$, $\omega \in \Omega$, $t \geq 0$, $r \geq \tau - t$, and $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-\tau}\omega)$, the solution $\varphi_L(r) = \varphi_L(r, \tau - t, \varphi_{L, \tau-t})$ of (46) satisfies that*

$$\|\varphi_L(r, \tau - t, \varphi_{L, \tau-t})\|_E^2 \leq M_L^2 e^{-2\sigma_1(t+r-\tau)} \quad (48)$$

holds.

Proof Let $\varphi_L = (\varphi_{L,1}, \varphi_{L,2}) = (u_L, v_L) = (u_L, u_{L,t} + \varepsilon u)$. Taking the inner product $(\cdot, \cdot)_E$ of Eq. (46) with $\varphi_L(r)$, we have

$$\frac{1}{2} \frac{d}{dt} \|\varphi_L(r)\|_E^2 + 2(Q\varphi_L, \varphi_L)_E = \left(\begin{pmatrix} 0, \\ -f_1(u_L, x) \end{pmatrix}, \varphi_L \right), \quad (49)$$

where $(\begin{pmatrix} 0, \\ -f_1(u_L, x) \end{pmatrix}, \varphi_L) = -\frac{d}{dt} \int_U F(\varphi_{L,1}, x) - \varepsilon \int_U f(\varphi_{L,1}, x) \varphi_{L,1}$. By Lemma 1, we see that

$$2(Q(\varphi_L), \varphi_L) \geq \varepsilon \|\varphi_L\|_E^2 + \frac{\varepsilon}{2} \|\varphi_{L,1}\|_{H_0^2}^2 + (\alpha + \alpha_1) \|\varphi_{L,2}\|_{L^2}^2. \quad (50)$$

By assumption (H_2) , we obtain

$$f(\varphi_{L,1}, x, \varphi_{L,1}) = (f(u, x), u) \geq \frac{1}{C_4} \bar{F}_1(u_L, x) - \varepsilon \int_U \phi_2(x) dx. \quad (51)$$

Thus, by Eqs. (49)–(51), we have

$$\begin{aligned} & \frac{d}{dt} [\|\varphi_L\|_E^2 + 2\bar{F}_1(u_L, x)] + \varepsilon \|\varphi_L\|_E^2 + \frac{\varepsilon}{2} \|\varphi_{L,1}\|_{H_0^2}^2 \\ & \quad + (\alpha + \alpha_1) \|\varphi_{L,2}\|_{L^2}^2 + \frac{2\varepsilon}{C_4} \bar{F}_1(u_L, x) \\ & \leq 2\varepsilon \int_U \phi_2(x) dx. \end{aligned} \quad (52)$$

Hence we can conclude that there exists $\sigma_L = \min\{\frac{2\varepsilon}{C_4}, \frac{\varepsilon}{2}, \frac{\alpha+\alpha_2}{2}\}$ such that

$$\frac{d}{dt}y_L + \sigma_L y_L(r) \leq \beta_L, \quad (53)$$

where $y_L(r) = \|\varphi_L(r)\|_E^2 + 2\bar{F}_L(u_L, x) \geq 0$, $\beta_L = 2\varepsilon \int_U \phi_2(x) dx$. Since $\varphi_{\tau-t}(\theta_{-\tau}\omega) + (0, ah(x)z(\theta_{-t}\omega))^T \in B_0(\theta_{-t}\omega)$, we have

$$\|\varphi_{L,\tau-t}\|_E \leq M_0(\theta_{-t}\omega) + |a| \|h(x)\| \cdot |z(\theta_{-t}\omega)|.$$

Notice that $\varphi_{L,\tau-t}$ is independent of ω , so replacing ω by $\theta_{-t}\omega$, then

$$\|\varphi_{L,\tau-t}\|_E \leq M_0(\omega) + |a| \|h(x)\| \cdot |z(\omega)|.$$

Applying Gronwall's inequality to Eq. (53), we have

$$\begin{aligned} \|\varphi_L(r, \tau-t, \varphi_{L,\tau-t})\|_E^2 &\leq y_L(r, \tau-t, \varphi_{L,\tau-t}) \\ &\leq y(\tau-t, \tau-t, \varphi_{L,\tau-t}) e^{-\sigma_L(t+r-\tau)} + \frac{\beta_L}{\sigma_L} \\ &\leq \left(\|\varphi_{L,\tau-t}\|_E^2 + c_{10} \left(|U| + \|u_{\tau-t}\|_1^4 + \int_U \phi_2 dx \right) \right) e^{-\sigma_L(t+r-\tau)} + \frac{\beta_L}{\sigma_L} \\ &= M_1^2(\omega). \end{aligned} \quad (54)$$

Next, we consider Eq. (49). Due to $(f_1(u_L, x), u_L) \geq 0$, $|f_1(u_L(r), x)| \leq c_{13}(|u_L(r)|^3 + |u_L(r)|)$ and assumption (H_2) , according to Sobolev embedding $H_0^1(U) \subset L^4(U) \subset L^2(U)$, there exists $M_2(\omega) > 0$ such that

$$0 \leq \bar{F}_1(u_L(r), x) \leq c_{14}(\|u_L(r)\|_{L^4}^4 + \|u_L(r)\|^2) \leq M_2(\omega) \|u_L(r)\|_1^2. \quad (55)$$

That is, $\|u_L(r)\|_1^2 \geq \frac{1}{M_2(\omega)} \bar{F}_1(u_L(r), x)$. From (49)–(50) and (55), for any $r \geq \tau-t$, we have

$$\frac{d}{dt} [\|\varphi_L\|_E^2 + 2\bar{F}_1(u_L, x)] + \frac{\varepsilon}{2} \|\varphi_L\|_E^2 + \frac{\varepsilon}{2M_2(\omega)} \bar{F}_1(u_L, x) \leq 0. \quad (56)$$

So the inequality

$$\frac{d}{dt} [\|\varphi_L\|_E^2 + 2\bar{F}_1(u_L, x)] + 2\sigma_1(\omega) [\|\varphi_L\|_E^2 + 2\bar{F}_1(u_L, x)] \leq 0$$

holds, where $\sigma_1 = \min\{\frac{\varepsilon}{4}, \frac{\varepsilon}{8M_2(\omega)}\} > 0$. Thus, we get

$$\begin{aligned} \|\varphi_L(r, \tau-t, \varphi_{L,\tau-t})\|_E^2 &\leq [\|\varphi_{L,\tau-t}\|_E^2 + 2\bar{F}_1(u_{L,\tau-t}, x)] e^{-2\sigma_1(r+t-\tau)} \\ &\leq \left(\|\varphi_{L,\tau-t}\|_E^2 + c_{10} \left(|U| + \|u_{\tau-t}\|_1^4 + \int_U \phi_2 dx \right) \right) e^{-2\sigma_1(t+r-\tau)} \\ &= M_L^2 e^{-2\sigma_1(r+t-\tau)}. \end{aligned} \quad (57)$$

The proof is completed. \square

For the component φ_N , which is ultimately pullback bounded in a higher regular space, we have the following estimate.

Lemma 4 *If assumptions (H_1) – (H_2) hold, then for any $\tau \in R$, $\omega \in \Omega$, and $t \geq 0$, there exist a positive constant $\nu \in (0, \min\{\frac{1}{4}, \frac{3-p}{4}\})$ and a positive-value random variable $M_N(\omega) > 0$ such that the solution $\varphi_N(r) = (\varphi_{1N}(r), \varphi_{2N}(r))^T$ of Eq. (47) satisfies the following:*

$$\|A^\nu \varphi_N(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \leq M_N^2(\omega) \quad (58)$$

for $t \geq 0$ and $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-\tau}\omega)$.

Proof Taking the inner product of Eq. (47) in E with

$$A^{2\nu} \varphi_N = (A^{2\nu} \varphi_{1N}, A^{2\nu} \varphi_{2N}) = (A^{2\nu} u_N, A^{2\nu} \varphi_{2N}),$$

we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[A^\nu \|\varphi_N\|^2 + 2 \int_U [f(u, x) - f_1(u_L, x)] A^{2\nu} u_N dx \right] + (Q(\varphi_N, A^{2\nu})) \\ & + \varepsilon \int_U [f(u, x) - f_1(u_L, x)] A^{2\nu} u_N dx \\ & - \int_U ([f'_{1,u}(u, x) - f'_{1,u}(u_L, x)] u_t + f'_{1,u}(u_L, x) u_{Nt} + f'_{2,u}(u, x) u_t) A^{2\nu} u_N dx \\ & = (ah(x)z(\theta_{r-\tau}\omega), A^{2\nu} u_N)_1 - (f(u, x) - f_1(u_L, x), A^{2\nu} ah(x)z(\theta_{r-\tau}\omega)) \\ & + (g(x, r) - (\alpha - \varepsilon - 1)ah(x)z(\theta_{r-\tau}\omega), A^{2\nu} \varphi_{2N}) \\ & + (-q(\varphi_{1N})ah(x)z(\theta_{r-\tau}\omega), A^{2\nu} \varphi_{2N}). \end{aligned} \quad (59)$$

Similar to (50), we get

$$\frac{\varepsilon}{2} \|A^\nu \varphi_N\|_E^2 + \frac{\varepsilon}{4} \|A^\nu u_N\|_{H_0^1}^2 + \frac{\alpha + \alpha_1}{2} \|A^\nu \varphi_{2N}\|_{L^2}^2 \leq (Q\varphi_N, A^{2\nu} \varphi_N)_E. \quad (60)$$

By some computations, we have

$$(ah(x)z(\theta_{r-\tau}\omega), A^{2\nu} u_N) \leq \frac{2a^2}{\varepsilon} z^2(\theta_{r-\tau}\omega) \|h(x)\|_1^2 + \frac{\varepsilon}{8} \|A^{\nu+\frac{1}{2}} u_N\|^2, \quad (61)$$

$$\begin{aligned} & (f(u, x) - f_1(u_L, x), A^{2\nu} ah(x)z(\theta_{r-\tau}\omega)) \\ & \leq K_1(r, \tau - t, \theta_{-t}\omega) + c_{15} a^2 \|h(x)\|_1^2 |z(\theta_{r-\tau}\omega)|^2, \end{aligned} \quad (62)$$

and

$$\begin{aligned} & (g(x, r) - (\alpha - \varepsilon - 1)ah(x)z(\theta_{r-\tau}\omega), A^{2\nu} \varphi_{2N}) \\ & \leq \frac{2}{\alpha} [\|g\|_1^2 + (\alpha - \varepsilon - 1)^2 a^2 \|h(x)\|_2^2 z^2(\theta_{r-\tau}\omega)] + \frac{\alpha}{4} \|A^\nu \varphi_{2N}\|^2, \end{aligned} \quad (63)$$

where $\|g\|_1^2 = \sup_{r \in \mathbb{R}} \|g(\cdot, r)\|_1^2 < \infty$. Thus, by taking (60)–(63) into (59), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|A^v \varphi_N\|_E^2 + 2 \int_U [f(u, x) - f_1(u_L, x)] A^{2v} u_N dx \right) + \sigma_2 \|A^v \varphi_N\|_E^2 \\ & + \varepsilon \int_U [f(u, x) - f_1(u_L, x) \cdot A^{2v} u_N dx] - \int_U ([f'_{1,u}(u, x) \\ & - f'_{1,u}(u_L, x)] u_t + f'_{1,u}(u_L, x) u_{N,t} + f'_{2,u}(u, x) u_t) \cdot A^{2v} u_N dx \\ & \leq K_2(r, r - t, \theta_{-t} \omega) + c_{16} a^2 z^2 (\theta_{r-\tau} \omega). \end{aligned} \quad (64)$$

Let $\sigma_2 = \min\{\frac{\varepsilon}{4}, \frac{2\varepsilon + \alpha}{4}\}$, then by Hölder's inequality, we have

$$\begin{aligned} & |(f'_{1,u}(u_L, x) u_{N,t}, A^{2v} u_N)| \\ & \leq c_{17} \left(\int_U (1 + u_L^2)^3 dx \right)^{\frac{1}{3}} \left(\int_U |A^{2v} u_N|^{\frac{6}{1+4v}} dx \right)^{\frac{1+4v}{6}} \left(\int_U |u_{N,t}|^{\frac{6}{3-4v}} dx \right)^{\frac{3-4v}{6}} \\ & \leq c_{18} \|A^{v+\frac{1}{2}} u_N\| \cdot \|A^v u_{N,t}\| \end{aligned} \quad (65)$$

for $r > \tau - t$. We have

$$\begin{aligned} & |(f'_{2,u}(u, x) u_t, A^{2v} u_N)| \\ & \leq c_{19} \int_U |u_t| \cdot (1 + |u|^p) \cdot |A^{2v} u_N| dx \\ & \leq \left(\int_U |u_t|^2 dx \right)^{\frac{1}{2}} \left(\int_U (1 + |u|^p)^{\frac{6}{2-4v}} dx \right)^{\frac{2-4v}{6}} \left(\int_U |A^{2v} u_N|^{\frac{6}{1+4v}} dx \right)^{\frac{1+4v}{6}} \\ & \leq \|u_t\|_0 (1 + \|u\|_1^p) \|A^{v+\frac{1}{2}} u_N\| \leq K_3(r, \tau - t, \theta_{-t} \omega) \|A^{v+\frac{1}{2}} u_N\| \end{aligned} \quad (66)$$

and

$$\begin{aligned} & |([f'_{1,u}(u, x) - f'_{1,u}(u_L, x)] u_t, A^{2v} u_N)| \\ & \leq \int_U |u_t| (1 + |u_N| + |u_L|) |u_N| |A^{2v} u_N| dx \\ & \leq \left(\int_U |u_t|^2 dx \right)^{\frac{1}{2}} \left(\int_U (1 + |u_N| + |u_L|)^6 dx \right)^{\frac{1}{6}} \\ & \quad \times \left(\int_U |u_N|^{\frac{6}{3-4v}} dx \right)^{\frac{3-4v}{6}} \left(\int_U |A^{2v} u_N|^{\frac{6}{1+4v}} dx \right)^{\frac{1+4v}{6}} \\ & \leq K_4(r, \tau - t, \theta_{-t} \omega) \|A^{v+\frac{1}{2}} u_N\|. \end{aligned} \quad (67)$$

By putting the above inequalities into (64), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|A^v \varphi_N\|_E^2 + 2 \int_U [f(u, x) - f_1(u_L, x)] A^{2v} u_N dx \right) \\ & + \sigma_1 \left[\|A^v \varphi_N\|_E^2 + 2 \int_U [f(u, x) - f_1(u_L, x)] A^{2v} u_N dx \right] \end{aligned}$$

$$\begin{aligned} &\leq c_{20} \|A^{v+\frac{1}{2}} u_N\| \|A^v u_{N,t}\| + K_2(r, \tau - t, \theta_{-t}\omega) + c_{16} a^2 z^2(\theta_{r-\tau}\omega) \\ &\quad + \|u_t\|_0 (1 + \|u\|_1^p \|A^{v+\frac{1}{2}} u_N\|) + K_4(r, \tau - t, \theta_{-t}\omega) \|A^{v+\frac{1}{2}} u_N\|. \end{aligned} \quad (68)$$

That is,

$$\begin{aligned} &\frac{d}{dt} y_1(r, \tau - t, \theta_{-t}\omega, \varphi_{\tau-t}(\theta_{-t}\omega)) + \sigma_1 y_1(r, \tau - t, \theta_{-t}\omega, \varphi_{\tau-t}(\theta_{-t}\omega)) \\ &\leq K_5(r, \tau - t, \theta_{-t}\omega) + c_{21} a^2 z^2(\theta_{r-\tau}\omega) \end{aligned} \quad (69)$$

and

$$y_1(r) = \|A^v \varphi_N\|_E^2 + 2 \int_U [f(u, x) - f_1(u_L, x)] A^{2v} u_N, \quad (70)$$

$$\beta_2(\theta_{r-\tau}\omega) = K_5(r, \tau - t, \theta_{-t}\omega) + c_{21} a^2 z^2(\theta_{r-\tau}\omega). \quad (71)$$

It follows from (69) that

$$\frac{d}{dt} y_1(r) + \sigma_1 y_1(r) \leq \beta_2(\theta_{r-\tau}\omega), \quad \forall r \geq \tau - t. \quad (72)$$

Note that $y_1(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}\omega) \leq a^2 \|A^v h(x)\|^2 z^2(\theta_{-t}\omega)$, then by applying Gronwall's inequality to (72) on $[\tau - t, r]$ ($r \geq \tau - t$), we have

$$\begin{aligned} &y_1(r, \tau - t, \theta_{-t}\omega, \varphi_{\tau-t}\omega) \\ &\leq y_1(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) + \int_{\tau-t}^r \beta_2(\theta_{\tau-t}\omega) e^{r-\tau+t} \\ &\leq c_{21} \|h(x)\|_1^2 a^2 z^2(\theta_{-t}\omega) e^{r-\tau+t} + \int_{\tau-t}^r \beta_2(\theta_{\tau-t}\omega) e^{r-\tau+t} \\ &\leq a^2 M_8(\omega) + M_9(\tau, \omega) \end{aligned} \quad (73)$$

for

$$\begin{aligned} &\left| \left[\int_U f(u, x) - f_1(u_L, x) \right] A^{2v} u_N dx \right| \\ &= \int_U [f_2(u, x) + f_1(u, x) - f_1(u_L, x)] A^{2v} u_N dx, \end{aligned} \quad (74)$$

where

$$\begin{aligned} \left| \int_U f_2(u, x) A^{2v} u_N dx \right| &\leq c_{31} \int_U (1 + |u|^p) A^{2v} u_N dx \\ &\leq c_{31} \left(\int_U (1 + |u|^p)^{\frac{6}{5-4v}} \right)^{\frac{5-4v}{6}} \left(\int_U |A^{2v} u_N|^{\frac{6}{1+4v}} \right)^{\frac{1+4v}{6}} \\ &\leq K_8(r, \tau - t, \theta_{-t}\omega) \|A^{v+\frac{1}{2}} u_N\| \\ &\leq K_8^2(r, \tau - t, \theta_{-t}\omega) + \frac{1}{2} \|A^{v+\frac{1}{2}} u_N\|^2 \end{aligned} \quad (75)$$

and

$$\begin{aligned}
 & \int_U |f_1(u, x) - f_1(u_L, x)| A^{2v} u_N |dx \\
 & \leq c_{32} \int_U (1 + |u_N|^2 + |u_L|^2) |u_N| |A^{2v} u_N| dx \\
 & \leq c_{33} \left(\int_U (1 + |u_N|^2 + |u_L|^2)^3 \right)^{\frac{1}{3}} \left(\int_U |u_N|^{\frac{6}{3-4v}} \right)^{\frac{3-4v}{6}} \\
 & \quad \times \left(\int_U |A^{2v} u_N|^{\frac{6}{1+4v}} \right)^{\frac{1+4v}{6}} \\
 & \leq K_9^2(r, \tau - t, \theta_{-t}\omega) + \frac{1}{8} \|A^{v+\frac{1}{2}} u_N\|^2.
 \end{aligned} \tag{76}$$

It follows from above that

$$y_1(r) = \|A^v \varphi_N\|^2 + 2 \int_U [f(u, x) - f_1(u_L, x)] A^{2v} u_N dx,$$

where

$$\begin{aligned}
 & \int_U [f(u, x) - f_1(u_L, x)] A^{2v} u_N dx \\
 & \leq K_8^2(r, \tau - t, \theta_{-t}\omega) + \frac{1}{8} \|A^{v+\frac{1}{2}} u_N\|^2 \\
 & \quad + K_9^2(r, \tau - t, \theta_{-t}\omega) + \frac{1}{8} \|A^{v+\frac{1}{2}} u_N\|^2 \\
 & \leq K_{10}^2(r, \tau - t, \theta_{-t}\omega) + \frac{1}{4} \|A^{v+\frac{1}{2}} u_N\|^2.
 \end{aligned} \tag{77}$$

So we obtain

$$\begin{aligned}
 \|A^v \varphi_N\| &= \|A^{v+\frac{1}{2}} u_N\|^2 + \|A^v u_{N,t}\|^2 \\
 &\leq y_1(r, \tau - t, \theta_{-t}\omega, \varphi_{\tau-t}(\theta_{-t}\omega)) + K_{10}^2(r, \tau - t, \theta_{-t}\omega) \\
 &\leq \alpha^2 M_8(\omega) + M_9 + K_{10}^2(r, \tau - t, \theta_{-t}\omega) \leq M_N^2(\omega).
 \end{aligned} \tag{78}$$

The proof is completed. \square

Lemma 5 For any $\tau \in R$, $\omega \in \Omega$, and $t > 0$, assume that $B_v(\tau, \omega) \subseteq B_1(\tau, \omega) \subseteq B_0(\omega)$ and $B_v(\tau, \omega) \in D_{E^v}$, where v is as in Lemma 4, then if assumptions (H_1) – (H_2) hold, then there exist a random variable $t_v(\omega) > 0$ and a tempered random variable $M_v(\omega) > 0$ such that, for any $t \geq t_v(\omega)$, $\varphi_{\tau-t}(\theta_{-t}\omega) \subset B_v(\tau - t, \theta_{-t}\omega) \subseteq B_0(\theta_{-t}\omega) \cap D(E^v)$, the solution φ of Eqs. (12)–(13) satisfies

$$\begin{aligned}
 & \|\varphi(\tau, \tau - t, \theta_{-t}\omega, \varphi_{\tau-t}(\theta_{-t}\omega))\|_{E^v}^2 \\
 &= \|A^v \varphi(\tau, \tau - t, \theta_{-t}\omega, \varphi_{\tau-t}(\theta_{-t}\omega))\|_E^2 \leq M_v^2(\omega).
 \end{aligned} \tag{79}$$

Proof Taking the inner product of Eqs. (12)–(13) in E with $A^{2\nu}\varphi = (A^{2\nu}u, A^{2\nu}v)^T$, then for any $r \geq \tau - t$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|A^{2\nu}\varphi\|_E^2 + 2 \int_U f(u, x) A^{2\nu}u \, dx \right) + (Q\varphi, A^{2\nu}\varphi)_E \\ & \quad + \varepsilon (f(u, x), A^{2\nu}\varphi) + (f(u, x), aA^{2\nu}h(x)z(\theta_{r-\tau}\omega)) \\ & = (ah(x)z(\theta_{r-\tau}\omega), A^{2\nu}u) + (g(x, r) \\ & \quad + (\varepsilon - \alpha + 1)ah(x)z(\theta_{r-\tau}\omega), A^{2\nu}v). \end{aligned} \quad (80)$$

The same to (75), the following inequality holds:

$$\begin{aligned} \left| \int_U f(u, x) A^{2\nu}u \, dx \right| & \leq c_3 \int_U |(1 + u^4)| |A^{2\nu}u| \, dx \\ & \leq c_3 \left(\int_U (1 + u^4)^{\frac{6}{5-4\nu}} \, dx \right)^{\frac{5-4\nu}{6}} \left(\int_U |A^{2\nu}u|^{\frac{6}{1+4\nu}} \, dx \right)^{\frac{1+4\nu}{6}} \\ & \leq c_3 M_6(\theta_{r-\tau}\omega) + \frac{\mu}{4} \|A^{\nu+\frac{1}{2}}u\|^2. \end{aligned} \quad (81)$$

Similar to (72), by (80) and (81), we get

$$\frac{d}{dt} y_2 + \sigma_1 y_2 \leq \beta_3(\theta_{r-\tau}\omega), \quad (82)$$

where

$$y_2 = \|A^\nu\varphi(r)\|_E^2 + 2 \int_U f(u, x) A^{2\nu}u \, dx \geq \frac{1}{2} \|A^\nu\varphi(r)\|_E^2 - c_3 M_6(\theta_{r-\tau}\omega), \quad (83)$$

$$\beta_3 = K_{11}(r, \tau - t, \theta_t\omega) + c_{22}a^2z^2(\theta_{r-\tau}\omega), \quad (84)$$

$$y_2(r) \leq y_2(\tau - t)e^{-\sigma_1(r+t-\tau)} + \int_{\tau-t}^r \beta_3(\theta_{\xi-\tau}\omega)e^{-\sigma_1(r-\xi)} \, d\xi. \quad (85)$$

By applying Gronwall's inequality to (82) on $[\tau - t, r]$, one has

$$\begin{aligned} & \|A^\nu\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\| \\ & \leq 2y_2(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) + c_3 M_6(\omega) \\ & \leq 2y_2(\tau - t)e^{-\sigma_1 t} + 2 \int_{\tau-t}^r \beta_3(\theta_{\xi-t}\omega)e^{-\sigma_1(\tau-\xi)} \, d\xi + 4c_3 M_6(\omega) \\ & \leq 2y_2(\tau - t)e^{-\sigma_1 t} + 2 \left[\int_{-\tau}^0 (K_{11}(r, \tau - t, \theta_t\omega) + c_{22}a^2z^2(\theta_{r-\tau}\omega)) \, d\xi \right] e^{\sigma_1 \xi} \\ & \quad + 4c_3 M_6(\omega). \end{aligned} \quad (86)$$

From (81), (83), (85), and $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-t}\omega) \cap D(E)$, it is clear that as $t \rightarrow +\infty$,

$$y_2(\tau - t)e^{-\sigma_1 t} \leq \left(\frac{3}{2} \|A^\nu\varphi_{\tau-t}\| \right)_E^2 + 2c_3 M_6(\theta_{-t}\omega)e^{-\sigma_1 t} \rightarrow 0.$$

Taking

$$M_v^2 = 4c \int_{-\infty}^0 (K_{11}(r, \tau - t, \theta_t \omega) + c_{22} a^2 z^2(\theta_{\tau-t} \omega)) e^{\sigma_1 \xi} d\xi + 8c_3 M_6(\omega),$$

then the proof is completed. \square

6 Existence of random attractor

Lemma 6 *If assumptions (H_1) – (H_2) hold, then for any $\tau \in R$, $\omega \in \Omega$, there exist $T_v(\omega) \geq 0$, a random bounded ball \hat{B}_1 of E , a positive number $\hat{\rho}$, and a tempered random variable $\hat{Q}(\omega)$ such that, for any $t \geq T_v(\omega)$ and $\varphi_{\tau-t}(\theta_{-\tau} \omega) \in B_1(\tau - t, \theta_{-t} \omega)$, the solution φ of (12)–(13) satisfies*

$$d_E(\varphi(\tau, \tau - t, \theta_{-\tau} \omega, B_1(\tau - t, \theta_{-t} \omega)), \hat{B}_1(\omega)) \leq \hat{Q}(\theta_{-t} \omega) e^{-\hat{\rho} t}. \quad (87)$$

By Lemma 7.6 in [2], Lemma 3, and Lemma 4, one can prove Lemma 6. Since the proof of Lemma 6 is similar to that of Lemma 3.8 in [29], we omit it here. From Lemmas 5 and 6, it is easy to see the existence of a random attractor for the cocycle Φ .

Theorem 2 *If assumptions (H_1) and (H_2) hold, then the cocycle Φ associated with (12)–(13) possesses a $D(E)$ -pullback random attractor $A \in D(E)$ such that, for any $\tau \in R$, $\omega \in \Omega$, $A(\tau, \omega) \subseteq \hat{B}_1(\omega) \cap B_0(\omega)$, where $B_0(\omega)$ and $\hat{B}_1(\omega)$ are as in (45) and Lemma 6, respectively.*

Proof For any $\tau \in R$ and $\omega \in \Omega$, by Lemma 6 and the compactness of embedding $E^v \rightarrow E$, $\hat{B}_1(\omega)$ is a compact measurable $D(E)$ -pullback attracting ball in E . By Proposition 2.1, the cocycle Φ has a $D(E)$ -pullback random attractor $\mathcal{A} \in D(E)$ such that, for any $\tau \in R$, $\omega \in \Omega$, $\mathcal{A}(\tau, \omega) \subseteq \hat{B}_1(\omega) \cap B_0(\omega)$. The proof is completed. \square

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Authors' contributions

HY completed the proofs of the main theorems, the rest of this paper was accomplished by JZ. All authors read and approved the final manuscript.

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