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Sharing values of q-difference-differential polynomials

Jian Li[1] and Kai Liu[1*]

*Correspondence:
liukai418@126.com;
liukai@ncu.edu.cn
[1]Department of Mathematics, Nanchang University, Nanchang, Jiangxi, China

Abstract

This paper is devoted to the uniqueness of q-difference-differential polynomials of different types. Using the idea of common zeros and common poles (Chin. Ann. Math., Ser. A 35:675–684, 2014), we improve the conditions of the former theorems and obtain some new results on the uniqueness of q-difference-differential polynomials of meromorphic functions.

MSC: 30D35

Keywords: Meromorphic functions; q-Difference-differential polynomials; Sharing values; Common zeros; Common poles

1 Introduction and main results

In this paper, a meromorphic function is assumed meromorphic in the whole complex plane. We assume that the reader is familiar with the basic symbols and fundamental results of Nevanlinna theory; see, for example, [2, 3, 10]. We say that two meromorphic functions f and g share a point a CM (IM) if $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities (ignoring multiplicities). The logarithmic density of the set E is defined by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap E} \frac{1}{t}\, dt.$$

Denote by $S(r,f)$ a quantity of $o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set E of logarithmic density 0.

Yang and Hua [9] obtained an important result on the uniqueness when the differential polynomials $f^n f'$ and $g^n g'$ share one value CM. Recently, many studies are devoted to the uniqueness of difference and q-difference polynomials; see [4–6, 11–14]. Zhang [12] obtained the following result.

Theorem A ([12]) *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order, and let n, m, d be positive integers. If $n \geq m + 5d$ and $f(z)^n(f(z)^m - 1)\prod_{i=1}^{d} f(q_i z)$ and $g(z)^n \times (g(z)^m - 1)\prod_{i=1}^{d} g(q_i z)$ share 1 CM, then $f(z) \equiv tg(z)$, $t^{n+d} = t^m = 1$.*

Liu, Liu, and Cao [4] and Zhang and Korhonen [11] obtained the following two theorems.

Theorem B ([4, Theorem 1.5]) *Let $f(z)$ and $g(z)$ be transcendental zero-order entire functions, and let m be a positive integer. If $n \geq m + 5$ and $f(z)^n(f(z)^m - a)f(qz + c)$ and $g(z)^n(g(z)^m - a)g(qz + c)$ share a nonzero polynomial $p(z)$ CM, then $f(z) \equiv g(z)$.*

Theorem C ([11, Theorem 5.1]) *Let $f(z)$ and $g(z)$ be transcendental zero-order meromorphic functions. If $n \geq 8$ and $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share 1 and ∞ CM, then $f(z) \equiv tg(z)$, $t^{n+1} = 1$.*

Zhao and Zhang [13] proved the following theorem.

Theorem D ([13, Theorem 1.4]) *Let $f(z)$ and $g(z)$ be transcendental zero-order entire functions, and let k be a positive integer. If $n \geq 2k + 6$ and $(f(z)^n f(qz + c))^{(k)}$ and $(g(z)^n g(qz + c))^{(k)}$ share 1 CM, then $f(z) \equiv tg(z)$, where $t^{n+1} = 1$.*

Wang and Ye [8] improved the conditions of Theorems B and C to $n \geq m + 4$ and $n \geq 6$, respectively, by using the idea of common zeros and common poles. Here we give the main idea of common zeros and common poles. Let f, g be two nonconstant meromorphic functions. Denote by $\overline{n}_0(r)$ or $\overline{n}_1(r)$ the numbers of common zeros or poles of fg and g, ignoring multiplicities. Let p, q be positive integers. We assume that the Laurent series of f and g at z_0 are as follows:

$$f(z) = \frac{1}{(z - z_0)^p} f_1(z), \qquad g(z) = (z - z_0)^q g_1(z),$$

where $f_1(z)$ and $g_1(z)$ are analytic functions at z_0, and $f_1(z_0) \neq 0$, $g_1(z_0) \neq 0$; the other cases can be discussed in a similar way. So z_0 is a zero of $g(z)$ with multiplicities q. If $q > p$, then z_0 is a zero of $f(z)g(z)$ with multiplicity $q - p$, and thus the contribution to $\overline{n}_0(r)$ is 1 at z_0. If $q \leq p$, then z_0 is a pole of $f(z)g(z)$ with multiplicity $p - q$ or an analytic point of $f(z)g(z)$, and thus the contribution to $\overline{n}_0(r)$ is 0 at z_0. A similar method can be discussed for $\overline{n}_1(r)$. As usual, denote by $\overline{N}_0(r)$ or $\overline{N}_1(r)$ the counting functions of the common zeros or poles of fg and g, ignoring multiplicities. Thus we have $\overline{N}(r, \frac{1}{fg}) \leq \overline{N}(r, \frac{1}{f}) + \overline{N}_0(r)$ and $\overline{N}(r, fg) \leq \overline{N}(r, f) + \overline{N}_1(r)$. In this paper, we continue to consider the uniqueness of q-difference-differential polynomials. Firstly, we improve the condition $n \geq m + 5d$ in Theorem A to $n \geq m + d + 3$ in Theorem 1.1. Set

$$L(z, f) = \prod_{i=1}^{d} f(q_i z + c_i),$$

where c_i and $q_i \neq 0$ $(i = 1, \ldots, d)$ are constants, and d is a positive integer.

Theorem 1.1 *Let $f(z)$ and $g(z)$ be transcendental zero-order entire functions, and let m be a positive integer. If $n \geq m + d + 3$ and $f(z)^n(f(z)^m - 1)L(z, f)$ and $g(z)^n(g(z)^m - 1)L(z, g)$ share 1 CM, then $f(z) \equiv c_1 g(z)$, $c_1^{n+d} = c_1^m = 1$.*

In the following theorem, we improve the condition $n \geq 2k + 6$ in Theorem D to $n \geq 6$.

Theorem 1.2 *Let $f(z)$ and $g(z)$ be transcendental zero-order meromorphic functions, and let k be a positive integer. If $n \geq 6$ and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share 1 and ∞ CM, then $f(z) \equiv c_2 g(z)$, $c_2^{n+1} = 1$.*

We also consider the following theorems for q-difference polynomials of different types. The following theorem is also an improvement of Theorem C.

Theorem 1.3 *Let $f(z)$ and $g(z)$ be transcendental zero-order meromorphic functions, and let s be a positive integer. If $n \geq (d+1)s + 4$ and $f(z)^n L(z,f)^s$ and $g(z)^n L(z,g)^s$ share 1 and ∞ CM, then $f(z) \equiv c_3 g(z)$, $c_3^{n+sd} = 1$.*

Theorem 1.4 *Let $f(z)$ and $g(z)$ be transcendental zero-order meromorphic functions, $q, c \in \mathbb{C}$, and $q \neq 0$. If $n \geq 7$, and $f(z)^n(f(qz+c) - f(z))$ and $g(z)^n(g(qz+c) - g(z))$ share 1 and ∞ CM, then*

$$f(z)^n\big(f(qz+c) - f(z)\big) = g(z)^n\big(g(qz+c) - g(z)\big).$$

If $\frac{g(z)}{g(qz+c)}$ is transcendental with only finitely many zeros, then $f(z) \equiv c_4 g(z)$, where $c_4^{n+1} = 1$.

2 Lemmas

Combining [11, Theorem 1.1] and [1, Theorem 2.1], we easily get the following lemma.

Lemma 2.1 *Let $f(z)$ be a transcendental zero-order meromorphic function, $q, c \in \mathbb{C}$, and $q \neq 0$. Then*

$$T\big(r, f(qz+c)\big) = T(r,f) + S(r,f)$$

on a set of logarithmic density 1.

Lemma 2.2 ([7]) *Let $f(z)$ be a zero-order meromorphic function $q, c \in \mathbb{C}$, and $q \neq 0$. Then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r,f)$$

on a set of logarithmic density 1.

Lemma 2.3 *If f is a transcendental zero-order entire function, then*

$$T\big(r, f(z)^n\big(f(z)^m - 1\big)L(z,f)\big) = (m+n+d)T(r,f) + S(r,f)$$

on a set of logarithmic density 1.

Proof Set $F(z) = f(z)^n(f(z)^m - 1)L(z,f)$. By Lemma 2.2 and the standard Valiron–Mohon'ko theorem, if f is a transcendental zero-order entire function, then

$$\begin{aligned}(n+m+d)T(r,f) &= T\big(r, f^{n+d}\big(f^m - 1\big)\big) + S(r,f) \\ &= m\big(r, f^{n+d}\big(f^m - 1\big)\big) + S(r,f)\end{aligned}$$

$$\begin{aligned}&\le m\left(r,\frac{f^{n+d}(f^m-1)}{f^n(f^m-1)L(z,f)}\right)+m(r,F)+S(r,f)\\&\le m\left(r,\frac{f^d}{L(z,f)}\right)+m(r,F)+S(r,f)\\&\le T(r,F)+S(r,f)\end{aligned}$$

on a set of logarithmic density 1. On the other hand, combining Lemma 2.1 with the fact that f is a transcendental zero-order function, we have

$$\begin{aligned}T(r,F)&\le T\big(r,f^n\big(f^m-1\big)\big)+T\big(r,L(z,f)\big)\\&\le (n+m+d)T(r,f)+S(r,f)\end{aligned}$$

on a set of logarithmic density 1. □

3 Proofs of theorems

Proof of Theorem 1.1 Let $F(z)=f^n(f^m-1)L(z,f)$ and $G(z)=g^n(g^m-1)L(z,g)$. Since $F(z)$ and $G(z)$ share 1 and ∞ CM, we have that $\frac{F-1}{G-1}=B$, that is,

$$F=BG+1-B, \tag{1}$$

where B is a nonzero constant.

If $B\neq 1$, then from the second main theorem of Nevanlinna theory, Lemma 2.1, and Lemma 2.3 we obtain

$$\begin{aligned}(n+m+d)T(r,f)&=T(r,F)+S(r,f)\\&\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1+B}\right)+S(r,f)\\&\le \overline{N}\left(r,\frac{1}{f^n}\right)+\overline{N}\left(r,\frac{1}{f^m-1}\right)+\overline{N}\left(r,\frac{1}{L(z,f)}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\&\le (m+d+1)\big(T(r,f)+T(r,g)\big)+S(r,f)+S(r,g).\end{aligned}\tag{2}$$

Using the same method, we have

$$(n+m+d)T(r,g)\le (m+d+1)\big(T(r,f)+T(r,g)\big)+S(r,f)+S(r,g). \tag{3}$$

Combining (2) with (3), we have

$$(n-m-d-2)\big(T(r,f)+T(r,g)\big)\le S(r,f)+S(r,g),$$

which contradicts to $n\ge m+d+3$. Thus $B=1$, and from (1) we have

$$f^n\big(f^m-1\big)L(z,f)=g^n\big(g^m-1\big)L(z,g). \tag{4}$$

Let $h(z)=\frac{f(z)}{g(z)}$. So $\frac{L(z,f)}{L(z,g)}=\prod_{i=1}^{d}\frac{f(q_iz+c_i)}{g(q_iz+c_i)}=\prod_{i=1}^{d}h(q_iz+c_i)=L(z,h)$, and then (4) can be written as

$$g^m\bigl(h^{n+m}L(z,h)-1\bigr)=h^nL(z,h)-1. \tag{5}$$

Next, we prove that $h(z)\equiv c_1$ and $c_1^{n+d}=c_1^m=1$, where c_1 is a constant. Assume on the contrary that $h(z)$ is not a constant. From Lemma 2.1 we have

$$T\bigl(r,h^{n+m}L(z,h)\bigr)\le(n+m+d)T(r,h)+S(r,h).$$

We also have

$$\begin{aligned}(n+m)T(r,h)&=T\bigl(r,h^{n+m}\bigr)\\&\le T\left(r,\frac{1}{h^{n+m}L(z,h)}\right)+T\bigl(r,L(z,h)\bigr)+O(1)\\&\le T\bigl(r,h^{n+m}L(z,h)\bigr)+dT(r,h)+S(r,h).\end{aligned}$$

Since $n\ge m+d+3$, from the last two inequalities it follows that $S(r,h^{n+m}L(z,h))=S(r,h)$. Denote by $\overline{N}_1(r)$ the counting function of the common poles of $h^{n+m}L(z,h)$ and $L(z,h)$ ignoring multiplicities. Then

$$\overline{N}\bigl(r,h^{n+m}L(z,h)\bigr)\le\overline{N}(r,h)+\overline{N}_1(r).$$

Here we should remark that the poles of $L(z,h)$ may be the zeros of h and the zeros of $L(z,h)$ may be the poles of h. Similarly, denote by $\overline{N}_0(r)$ the counting function of the common zeros of $h^{n+m}L(z,h)$ and $L(z,h)$ ignoring multiplicities, and then

$$\overline{N}\left(r,\frac{1}{h^{n+m}L(z,h)}\right)\le\overline{N}\left(r,\frac{1}{h}\right)+\overline{N}_0(r).$$

From the second main theorem of Nevanlinna theory and the last two inequalities we have

$$\begin{aligned}T\bigl(r,h^{n+m}L(z,h)\bigr)&\le\overline{N}\bigl(r,h^{n+m}L(z,h)\bigr)+\overline{N}\left(r,\frac{1}{h^{n+m}L(z,h)}\right)\\&\quad+\overline{N}\left(r,\frac{1}{h^{n+m}L(z,h)-1}\right)+S\bigl(r,h^{n+m}L(z,h)\bigr)\\&\le\overline{N}(r,h)+\overline{N}_1(r)+\overline{N}\left(r,\frac{1}{h}\right)+\overline{N}_0(r)\\&\quad+\overline{N}\left(r,\frac{1}{h^{n+m}L(z,h)-1}\right)+S(r,h).\end{aligned} \tag{6}$$

On the other hand,

$$(n+m)m(r,h)=m\bigl(r,h^{n+m}\bigr)\le m\bigl(r,h^{n+m}L(z,h)\bigr)+m\left(r,\frac{1}{L(z,h)}\right)+O(1), \tag{7}$$

$$
\begin{aligned}
(n+m)N(r,h) &= N\left(r,h^{n+m}\right) = N\left(\frac{h^{n+m}L(z,h)}{L(z,h)}\right) \\
&\leq N\left(r,h^{n+m}L(z,h)\right) + N\left(r,\frac{1}{L(z,h)}\right) - \overline{N}_1(r) - \overline{N}_0(r).
\end{aligned}
\tag{8}
$$

From (7) and (8) we get

$$
\begin{aligned}
(n+m)T(r,h) \leq{} & T\left(r,h^{n+m}L(z,h)\right) + T\left(r,\frac{1}{L(z,h)}\right) \\
& - \overline{N}_1(r) - \overline{N}_0(r) + O(1).
\end{aligned}
\tag{9}
$$

From (6) and (9) we get

$$
\begin{aligned}
(n+m)T(r,h) &\leq \overline{N}(r,h) + \overline{N}\left(r,\frac{1}{h}\right) + \overline{N}\left(r,\frac{1}{h^{n+m}L(z,h)-1}\right) \\
&\quad + T\left(r,\frac{1}{L(z,h)}\right) + S(r,h) \\
&\leq (d+2)T(r,h) + \overline{N}\left(r,\frac{1}{h^{n+m}L(z,h)-1}\right) + S(r,h).
\end{aligned}
\tag{10}
$$

Since $n \geq m+d+3$, the value 1 is not the Picard exceptional value of $h^{n+m}L(z,h)$ from (10). Furthermore, we prove $h^{n+m}L(z,h) \equiv 1$ and $h(z) \equiv c_1$ is a nonzero constant. If $h^{n+m}L(z,h) \not\equiv 1$,then since 1 is not the Picard exceptional value of $h^{n+m}L(z,h)$, there exists a point z_0 satisfying $h(z_0)^{n+m}L(z,h(z_0)) = 1$. From the condition that $g(z)$ is an entire function and (5) we have $h(z_0)^m = 1$ and

$$
\overline{N}\left(r,\frac{1}{h^{n+m}L(z,h)-1}\right) \leq \overline{N}\left(r,\frac{1}{h^m-1}\right) \leq mT(r,h) + S(r,h).
\tag{11}
$$

Substituting (11) into (10), we get a contradiction to $n \geq m+d+3$, so $h^{n+m}L(z,h) \equiv 1$, that is, $h^{n+m} = \frac{1}{L(z,h)}$. From Lemma 2.1 we have

$$
(n+m)T(r,h) = T\left(r,L(z,h)\right) \leq dT(r,h) + S(r,h),
$$

which also contradicts to $n \geq m+d+3$, so $h(z) \equiv c_1$, where c_1 is a nonzero constant, that is, $f(z) \equiv c_1 g(z)$, and $L(z,h) = c_1^d$. From (5) we can get $c_1^m = c_1^{n+d} = 1$. Thus the theorem is proved. □

Proof of Theorem 1.2 Let $F(z) = f(z)^n f(qz+c)$ and $G(z) = g(z)^n g(qz+c)$. From the condition in Theorem 1.2 we know that $F^{(k)}$ and $G^{(k)}$ share 1 and ∞ CM, so

$$
\frac{F^{(k)}-1}{G^{(k)}-1} = C,
$$

where C is a nonzero constant, that is,

$$
F^{(k)} = CG^{(k)} - C + 1.
\tag{12}
$$

Integrating both sides of (12), we have

$$F = CG + \frac{1-C}{k!}z^k + p_1(z), \tag{13}$$

where $p_1(z)$ is a polynomial of degree at most $k-1$. Denote $\frac{1-C}{k!}z^k + p_1(z)$ by $p(z)$. If $p(z) \not\equiv 0$, then by the second main theorem of Nevanlinna theory, Lemma 2.1, and (13) we obtain

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-p}\right) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}_1(r) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}_0(r) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f) \\ &\le 2T(r,f) + 2T(r,g) + \overline{N}_1(r) + \overline{N}_0(r) + S(r,f) + S(r,g), \end{aligned} \tag{14}$$

where $\overline{N}_0(r)$ denotes the counting function ignoring multiplicities of the common zeros of $F(z)$ and $f(qz+c)$, and $\overline{N}_1(r)$ denotes the counting function ignoring multiplicities of the common poles of $F(z)$ and $f(qz+c)$. On the other hand,

$$nm(r,f) = m(r,f^n) \le m(r,F) + m\left(r,\frac{1}{f(qz+c)}\right) + O(1). \tag{15}$$

$$\begin{aligned} nN(r,f) &= N(r,f^n) = N\left(\frac{F(z)}{f(qz+c)}\right) \\ &\le N(r,F) + N\left(r,\frac{1}{f(qz+c)}\right) - \overline{N}_1(r) - \overline{N}_0(r). \end{aligned} \tag{16}$$

From (15), (16), and Lemma 2.1 we have

$$(n-1)T(r,f) \le T(r,F) - \overline{N}_1(r) - \overline{N}_0(r) + O(1). \tag{17}$$

Substituting (14) into (17), we obtain

$$(n-3)T(r,f) \le 2T(r,g) + S(r,f) + S(r,g). \tag{18}$$

Using the same method, we also get

$$(n-3)T(r,g) \le 2T(r,f) + S(r,f) + S(r,g). \tag{19}$$

Combining (18) with (19), we have

$$(n-5)\bigl(T(r,g) + T(r,f)\bigr) \le S(r,f) + S(r,g),$$

which contradicts to $n \ge 6$, and thus $p(z) \equiv 0$. Since the degree of $p_1(z)$ is at most $k-1$, we have $C = 1$ and $p_1(z) \equiv 0$. From (13) we get

$$f^n f(qz+c) = g^n g(qz+c).$$

Assume that $h(z) = \frac{f(z)}{g(z)}$. Then $h(qz+c)h(z)^n = 1$, that is, $h(z)^n = \frac{1}{h(qz+c)}$, and from Lemma 2.1 we have

$$nT(r,h) = T\bigl(r, h(qz+c)\bigr) \le T(r,h) + S(r,h),$$

which also contradicts to $n \ge 6$, so $h(z)$ is a nonzero constant, say c_2. So $f(z) \equiv c_2 g(z)$, and $c_2^{n+1} = 1$. Thus the theorem is proved. □

Proof of Theorem 1.3 Since $f(z)$ and $g(z)$ are transcendental zero-order meromorphic functions and $f(z)^n L(z,f)^s$ and $g(z)^n L(z,g)^s$ share 1 and ∞ CM, we have

$$\frac{f(z)^n L(z,f)^s - 1}{g(z)^n L(z,g)^s - 1} = E, \tag{20}$$

where E is a nonzero constant. Then (20) can be rewritten as

$$Eg(z)^n L(z,g)^s = f(z)^n L(z,f)^s - 1 + E. \tag{21}$$

Let $F(z) = f(z)^n L(z,f)^s$ and $G(z) = g(z)^n L(z,g)^s$. We affirm that $E = 1$. On the contrary, assume that $E \not\equiv 1$. Using the second main theorem of Nevanlinna theory and Lemma 2.1 for (21), we get

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1+E}\right) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}_1(r) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_0(r) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f) \\ &\le 2T(r,f) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{L(z,g)}\right) + \overline{N}_1(r) + \overline{N}_0(r) + S(r,f) \\ &\le 2T(r,f) + (d+1)T(r,g) + \overline{N}_1(r) + \overline{N}_0(r) + S(r,f) + S(r,g), \end{aligned} \tag{22}$$

where $\overline{N}_0(r)$ denotes the counting function ignoring multiplicities of the common zeros of $F(z)$ and $L(z,f)$, and $\overline{N}_1(r)$ denotes the counting function ignoring multiplicities of the common poles of $F(z)$ and $L(z,f)$.

Since $F(z) = f(z)^n L(z,f)^s$, we have

$$\begin{aligned} nm(r,f) &= m\bigl(r,f^n\bigr) \le m(r,F) + sm\left(r, \frac{1}{L(z,f)}\right) + O(1), \\ nN(r,f) &= N\bigl(r,f^n\bigr) = N\left(r, F(z)\left[\frac{1}{L(z,f)}\right]^s\right) \\ &\le N(r,F) + sN\left(r, \frac{1}{L(z,f)}\right) - \overline{N}_1(r) - \overline{N}_0(r). \end{aligned}$$

So

$$\begin{aligned} nT(r,f) &\le T(r,F) + sT\left(r, \frac{1}{L(z,f)}\right) - \overline{N}_1(r) - \overline{N}_0(r) + O(1) \\ &\le T(r,F) + dsT(r,f) - \overline{N}_1(r) - \overline{N}_0(r) + S(r,f). \end{aligned} \tag{23}$$

Substituting (22) into (23), we obtain

$$(n - ds - 2)T(r,f) \leq (d + 1)T(r,g) + S(r,f) + S(r,g). \tag{24}$$

Using the same method, we can get

$$(n - ds - 2)T(r,g) \leq (d + 1)T(r,f) + S(r,f) + S(r,g). \tag{25}$$

Combining (24) with (25), it follows

$$(n - d - ds - 3)\bigl(T(r,g) + T(r,f)\bigr) \leq S(r,f) + S(r,g),$$

which contradicts to $n \geq (d + 1)s + 4$, and thus $E = 1$. From (21) we get

$$f(z)^n L(z,f)^s = g(z)^n L(z,g)^s.$$

Let $h(z) = \frac{f(z)}{g(z)}$. So $\frac{L(z,f)}{L(z,g)} = L(z,h)$. Then

$$h(z)^n \bigl[L(z,h)\bigr]^s = 1. \tag{26}$$

So $nT(r,h) = sT(r,L(z,h)) \leq sdT(r,h) + S(r,h)$. Since $n \geq (d + 1)s + 4$, $h(z)$ must be a constant, say c_3, that is, $f(z) \equiv c_3 g(z)$. Then from (26) it follows that $c_3^{n+sd} = 1$. □

Proof of Theorem 1.4 Letting $L(z,f) = f(qz + c) - f(z)$ and $L(z,g) = g(qz + c) - g(z)$, $s = 1$ in Theorem 1.3, we obtain that if $n \geq 7$, then

$$f(z)^n \bigl(f(qz + c) - f(z)\bigr) = g(z)^n \bigl(g(qz + c) - g(z)\bigr).$$

Let $h(z) = \frac{f(z)}{g(z)}$ and $H(z) = h(qz + c)h(z)^n$. The last equation implies that

$$g(qz + c)\bigl(H(z) - 1\bigr) = g(z)\bigl(h(z)^{n+1} - 1\bigr). \tag{27}$$

We know that $T(r,H) \leq (n+1)T(r,h)+S(r,h)$ from the expression of $H(z)$ and Lemma 2.1. Thus $S(r,H) = S(r,h)$. Next, we prove that $h(z) \equiv c_4$, where $c_4^{n+1} = 1$, when $\frac{g(z)}{g(qz+c)}$ is transcendental with only finitely many zeros. Obviously, $h(z)$ is neither a constant except c_4 nor a rational function from (27) since $\frac{g(z)}{g(qz+c)}$ is transcendental. Thus we assume that $h(z)$ is a transcendental meromorphic function.

First, we affirm that $H(z) - 1$ has infinitely many zeros. Otherwise, by the second main theorem of Nevanlinna theory and Lemma 2.1

$$\begin{aligned} T\bigl(r,H(z)\bigr) &\leq \overline{N}\bigl(r,H(z)\bigr) + \overline{N}\left(r,\frac{1}{H(z)}\right) + \overline{N}\left(r,\frac{1}{H(z) - 1}\right) + S(r,H) \\ &\leq 2T\bigl(r,h(z)\bigr) + 2T\bigl(r,h(qz + c)\bigr) + S(r,h) \\ &\leq 4T\bigl(r,h(z)\bigr) + S(r,h). \end{aligned} \tag{28}$$

From the Valiron–Mohon'ko theorem, Lemma 2.1, and (28) we obtain

$$\begin{aligned} nT(r,h) = T\bigl(r,h(z)^n\bigr) &\le T\bigl(r,H(z)\bigr) + T\biggl(r,\frac{1}{h(qz+c)}\biggr) + S(r,h) \\ &\le 5T\bigl(r,h(z)\bigr) + S(r,h), \end{aligned}$$

which contradicts to $n \ge 7$. Thus $H(z)-1$ has infinitely many zeros.

Then we prove that $H(z)\equiv 1$ and $h(z)\equiv c_4$ is a nonzero constant, that is, $c_4^{n+1}=1$. If $H(z)\not\equiv 1$, then since $H(z)-1$ has infinitely many zeros, we can choose a point z_0 satisfying $H(z_0)=1$, and z_0 is not the zero of $\frac{g(z)}{g(qz+c)}$. From (27) we have $h(qz_0+c)=h(z_0)$. By Lemma 2.1

$$\overline{N}\biggl(r,\frac{1}{H(z)-1}\biggr) \le \overline{N}\biggl(r,\frac{1}{h(qz+c)-h(z)}\biggr) \le 2T\bigl(r,h(z)\bigr) + S(r,h). \tag{29}$$

Using the second main theorem of Nevanlinna theory, Lemma 2.1, and (29), we obtain

$$\begin{aligned} T\bigl(r,H(z)\bigr) &\le \overline{N}\bigl(r,H(z)\bigr) + \overline{N}\biggl(r,\frac{1}{H(z)}\biggr) + \overline{N}\biggl(r,\frac{1}{H(z)-1}\biggr) + S(r,h) \\ &\le \overline{N}(r,h) + \overline{N}_1(r) + \overline{N}\biggl(r,\frac{1}{h}\biggr) + \overline{N}_0(r) + \overline{N}\biggl(r,\frac{1}{H(z)-1}\biggr) + S(r,h) \\ &\le 4T(r,h) + \overline{N}_1(r) + \overline{N}_0(r) + S(r,h), \end{aligned} \tag{30}$$

where $\overline{N}_0(r)$ denotes the counting function ignoring multiplicities of the common zeros of $H(z)$ and $h(qz+c)$, and $\overline{N}_1(r)$ denotes the counting function ignoring multiplicities of the common poles of $H(z)$ and $h(qz+c)$.

On the other hand, from $H(z)=h(qz+c)h(z)^n$ we have

$$nm(r,h) = m\bigl(r,h^n\bigr) \le m\bigl(r,H(z)\bigr) + m\biggl(r,\frac{1}{h(qz+c)}\biggr) + O(1), \tag{31}$$

$$nN(r,h) = N\bigl(r,h^n\bigr) \le N\bigl(r,H(z)\bigr) + N\biggl(r,\frac{1}{h(qz+c)}\biggr) - \overline{N}_1(r) - \overline{N}_0(r). \tag{32}$$

From (31), (32), and Lemma 2.1 we have

$$\begin{aligned} nT(r,h) &\le T\bigl(r,H(z)\bigr) + T\biggl(r,\frac{1}{h(qz+c)}\biggr) - \overline{N}_1(r) - \overline{N}_0(r) + O(1) \\ &\le T\bigl(r,H(z)\bigr) + T(r,h) - \overline{N}_1(r) - \overline{N}_0(r) + S(r,h). \end{aligned} \tag{33}$$

Substituting (30) into (33), we obtain

$$nT(r,h) \le 5T(r,h) + S(r,h),$$

which contradicts to $n \ge 7$, so $H(z)=h(qz+c)h(z)^n \equiv 1$, that is, $h(z)^n = \frac{1}{h(qz+c)}$. From Lemma 2.1 we have

$$nT(r,h) = T\bigl(r,h(qz+c)\bigr) \le T(r,h) + S(r,h),$$

which also contradicts to $n \geq 7$, so $h(z)$ is a nonzero constant, say c_4, and from (27) we get $c_4^{n+1} = 1$. Thus the theorem is proved. $\square$

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