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Global existence of classical solutions for two-dimensional isentropic compressible Navier–Stokes equations with small initial mass

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Abstract

In this paper, we consider the initial-boundary value problem of two-dimensional isentropic compressible Navier–Stokes equations with vacuum on the square domain. Based on the time-weighted uniform estimates, we prove that the classical solution exists globally in time if the initial mass $\|\rho_0\|_{L^1}$ of the fluid is small. Here, we do not require the initial energy or the upper bound of the initial density to be small.

Keywords: Navier–Stokes equations; Small initial mass; Global classical solutions

1 Introduction

In this paper, we consider the following two-dimensional isentropic compressible Navier–Stokes equations in the Eulerian coordinates:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \nabla P(\rho) = 0, \end{cases} \quad (1)$$

where $t \geq 0$ is the time, $x \in \Omega = [0, 1] \times [0, 1]$ is a spatial coordinate. $\rho = \rho(x, t)$, $u = (u_1, u_2)(x, t)$ and

$$P = R\rho^\gamma, \quad R > 0, \gamma > 1$$

are unknown functions denoting fluid density, velocity and pressure, respectively. Without loss of generality, it is assumed that $R = 1$. The constant coefficients μ and λ satisfy the physical restrictions:

$$\mu > 0, \quad \mu + \lambda \geq 0.$$

There is much literature concerning the well-posedness theory of classical and weak solutions for isentropic compressible Navier–Stokes equations. For the three-dimensional

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case, Nash [1] and Itaya [2] established the local existence and uniqueness of classical solutions in the absence of vacuum in 1962 and 1977, respectively. In 1995, Hoff [3, 4] proved the global existence of weak solutions when the initial density would be close to a constant in L^2 and L^∞ norm, and the initial velocity be small in L^2 norm and bounded in L^{2n} norm (n is the space dimension). In 1998, Lions [5] obtained the global existence of weak solutions when the adiabatic exponent γ is suitably large, the main restriction on initial data is that the initial total energy is finite, similar results can be found in [6] given by Feireisl. A few years later, Hoff [7–9] obtained a new type of global weak solutions with small energy, which have more regularity information than the works in [5, 6]. On the other hand, when vacuum is allowed, Cho and Kim [10, 11] proved the existence of unique local strong solutions in bounded and unbounded domains in 2003. In 2012, Huang, Li and Xin [12] established the global classical solutions with small energy but possibly large oscillations. In the same year, Duan [13] generalized the result in [7] and proved the global classical solutions to the half-space problem with the boundary condition proposed by Navier provided the initial energy is small. In 2016, Yu and Zhang [14] studied the nonhomogeneous equations with density-dependent viscosity in a smooth bounded domain and the vacuum is allowed. The global well-posedness of strong solutions is established for the case when the bound of the density is suitably small, or when the total mass is small with large oscillations. Later, in 2017, under the same condition in [12], Yu and Zhao [15] studied the global existence in a cuboid domain, some new ideas being applied to establishing a time-uniform upper bound for the density. Recently, Si, Zhang and Zhao [16] established the global existence of classical solutions with a small initial density but possibly a large energy in the case of $\rho_0 \in L^\gamma$, $\gamma \in (1, 6)$ and $\rho_0 \in L^1$, $\gamma > 1$, respectively, which extends the results in [12].

Compared with the three-dimensional case, there are few results in the two-dimensional space. The pioneering work can be traced back to [17] in 1995, as Vaigant and Kazhikhov first proposed the initial-boundary value problem with the special viscosity coefficients, that is, shear viscosity μ being a positive constant and bulk viscosity

$$\lambda(\rho) = \rho^\beta, \quad \beta > 3. \quad (2)$$

They proved the existence of global strong solution with no restrictions on the size of initial data. In 2012, Luo [18] studied the Cauchy problem and proved local existence and uniqueness of classical solutions with initial density containing vacuum when viscosity coefficients μ and λ are constant. For the case of a viscosity depending on the density, we refer to a later work by Li and Liang [19]. In 2013, under the condition (2), Jiu, Wang and Xin [20, 21] proved the global classical solutions on the torus and in the whole space, respectively, where the initial data may contain vacuum in an open set. In the same year, Ducomet and Necasova [22] studied the initial-boundary value problem with a vorticity-type boundary condition and prove that the results of [17] hold in any smooth bounded domain. In 2014, Zhang, Deng and Zhao [23] established the global classical solutions to the Cauchy problem with smooth initial data under the assumption that the viscosity coefficient μ is large enough. In 2016, Huang and Li [24] relaxed the power index β in (2) to be $\beta > \frac{4}{3}$ and studied the large-time behavior of the solutions, also see a recent work [25] for Cauchy problem. In the same year, Fang and Guo [26] established the global existence and large-time asymptotic behavior of the strong solution to the Cauchy problem in the

case of $\beta \in [0, 1]$ provided that the initial data are of small total energy. In 2018, Ding, Huang and Liu [27] obtained the global classical solutions to the Cauchy problem with $\beta \in [0, 1]$ under the condition of small initial density, which extends the earlier work [26] with small initial energy.

From the well-known results mentioned in the above paragraph, we can see that in the two-dimensional space, the existing work mainly discussed the global existence of system (1) under the condition of density-dependent viscosity, $\beta > \frac{4}{3}$ with general initial data, $\beta \in [0, 1]$ with small initial energy or small initial density. However, whether the unique local classical solution can exist globally for constant viscosity with small initial mass on a bounded domain is still unknown at present. Inspired by the analysis of [12] and [15], in this paper, we consider Dirichlet problem of (1) with the following initial-boundary conditions:

$$(\rho, u)(x, t)|_{t=0} = (\rho_0(x), u_0(x)), \tag{3}$$

$$u(x, t)|_{\partial\Omega} = 0. \tag{4}$$

We hope to establish the global existence of strong solutions for (1), (3)–(4) with constant viscosity on the square domain.

Before stating the main results, we explain the notations and conventions used throughout this paper.

Notations:

- The standard Lebesgue and Sobolev spaces are defined as follows:

$$\begin{cases} L^r = L^r(\Omega), & W^{s,r} = W^{s,r}(\Omega), & H^s = W^{s,2}, \\ W_0^{s,r} = \{f \in W^{s,r} | f = 0 \text{ on } \partial\Omega\}, & H_0^s = W_0^{s,2}. \end{cases}$$

- $\dot{f} = f_t + u \cdot \nabla f$ denotes the material derivative of f .
- $\int f \, dx = \int_{\Omega} f \, dx$ and $\int_0^T \int f \, dx \, dt = \int_0^T \int_{\Omega} f \, dx \, dt$.
- The symbol ∇^l with an integer $l \geq 0$ stands for the usual spatial derivatives of any order l . We define

$$\nabla^k f = \{ \partial_x^\alpha f_i | |\alpha| = k, i = 1, 2 \}, \quad f = (f_1, f_2).$$

- Positive generic constants are denoted by C , which may change in different places.

Now, our main results in this paper can be stated as follows.

Theorem 1.1 *For given numbers $\bar{\rho} > 0, M > 0$ and $q > 2$, suppose that the initial data (ρ_0, u_0) satisfy*

$$\begin{cases} 0 \leq \inf \rho_0 \leq \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, & \|\nabla u_0\|_{L^2} \leq \sqrt{M}, \\ \rho_0 \in H^3, & u_0 \in H_0^1 \cap H^3, \end{cases} \tag{5}$$

and the following compatibility conditions:

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \operatorname{div} u_0 + P(\rho_0) = \rho_0^{\frac{1}{2}} g, \tag{6}$$

for some $g \in L^2$. Then, there exists a positive constant ε_0 depending on $\bar{\rho}, M, \mu, \lambda$, and some other known constants but independent of T , such that, if

$$\|\rho_0\|_{L^1} \leq \varepsilon_0, \tag{7}$$

the initial-boundary value problem (1), (3)–(4) admits a unique global classical solution (ρ, u) in $\Omega \times (0, +\infty)$ satisfying, for any $0 < T < +\infty$,

$$0 \leq \rho(x, t) \leq 2\bar{\rho}, \quad x \in \Omega, t \geq 0,$$

and

$$\begin{cases} 0 \leq \rho \in C([0, T]; H^3), & \rho_t \in L^\infty([0, T]; H^1), \\ u \in C([0, T]; H_0^1 \cap H^3) \cap L^2(0, T; H^4), \\ u_t \in L^\infty(0, T; H_0^1 \cap H^2), & \sqrt{\rho}u_t \in L^\infty(0, T; L^2). \end{cases} \tag{8}$$

Remark 1.1 Cho and Kim [10, 11] proved the existence and uniqueness of local strong solution to (1), (3)–(4) with initial vacuum in the three-dimensional space, where Ω can be bounded domain or the whole space. If Ω is a bounded domain in \mathbb{R}^2 and the initial data (ρ_0, u_0) are smooth enough, and u satisfies the boundary condition (4), it is not difficult to verify that the proofs in [10, 11] are still valid for local existence of classical solutions in two-dimensional space.

Remark 1.2 In Theorem 1.1, we give the global existence of classical solution to the initial-boundary value problem (1), (3)–(4) provided the initial mass $\|\rho_0\|_{L^1}$ is small. In fact, if we take the same vorticity-type boundary condition (Navier-slip boundary condition) in [15] instead of Dirichlet boundary condition, by applying the same method in three-dimensional space, similar results in Theorem 1.1 can also be proved. Thus, our results extended the one due to Yu and Zhao [15], where the global well-posedness of classical solutions with small initial energy was proved. Moreover, under the condition (7), we can prove the global existence of classical solution to the Cauchy problem in three-dimensional space by using effective viscous flux method, which extend the results of [12] for small initial energy and [16] for small initial density.

We now make some comments on the global existence of classical solution to the isentropic compressible Navier–Stokes equations. Compared with the three-dimensional case, it causes some essential difficulties. Similar to the procedure of [12, 15, 16], a key ingredient in our proof is to obtain a uniform priori upper bound for the density function. However, due to the invalidity of the Sobelov embedding inequality $\|u\|_{L^6} \leq C\|\nabla u\|_{L^2}$, and there is no boundary information of effective viscous flux $F \triangleq (2\mu + \lambda) \operatorname{div} u - P$ in the two-dimensional bounded domain, time-weighted estimates are needed the ensure the better integrability of the velocity, which is quite differs from three-dimensional Cauchy problem. In this paper, we use the Poincaré inequality and the following decomposition of the velocity $u = v + w$ to overcome this difficulty, where v solves the elliptic system:

$$\begin{cases} \mu \Delta v + (\mu + \lambda) \nabla \operatorname{div} v = \nabla P, & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases} \tag{9}$$

Then, from the momentum, Eqs. (1) and (9), we can see that w satisfies

$$\begin{cases} \mu \Delta w + (\mu + \lambda) \nabla \operatorname{div} w = \rho \dot{u}, & \text{in } \Omega, \\ w|_{\partial \Omega} = 0. \end{cases} \tag{10}$$

Hence, $\|\nabla u\|_{L^p}$, $p \geq 2$, is controlled by the standard L^p -estimate of elliptic system (9) and (10).

On the one hand, under the condition of (7), we have the following key observation:

$$\sup_{0 \leq t \leq T} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\rho\|_{\gamma}^{\gamma}) + \int_0^T \mu \|\nabla u\|_{L^2}^2 ds \leq m_0^{\frac{1}{4}}, \tag{11}$$

which is derived from (1)₁ and (1)₂. Then, by applying the method in [15], we get the uniform bound for $\|\nabla u\|_{L^2}$ and time-dependent bound for $\|\dot{\nabla} u\|_{L^2(t_1, t_2; L^2)}$, by which, together with Zlotnik inequality, we have the uniform upper bound of density. It is worth mentioning that, these boundness can be obtained by the smallness of the initial mass $\|\rho_0\|_{L^1}$ instead of the smallness of the upper bound of the density in [16] and the initial energy in [12, 15], respectively. At last, higher-order regularity estimates for (ρ, u) can be proved by standard methods after some modifications, see [12] for example. Finally, after all the required a priori estimates obtained, by using the continuity argument, we can extend the local classical solution to a global one.

The rest of the paper is organized as follows: In Sect. 2, we list some elementary inequalities which will be used in later analysis. Section 3 is devoted to deriving the necessary a priori estimates on classical solution which extend the local solution to a global one.

2 Preliminaries

In this section, we recall some well-known inequalities, which will be used frequently throughout this paper. First, we give the Sobolev–Poincaré lemma [28].

Lemma 2.1 *There exists a positive constant C depending only on Ω such that every function $f \in H^1(\Omega)$ satisfies for $2 < p < \infty$,*

$$\|f - \bar{f}\|_{L^p} \leq Cp^{\frac{1}{2}} \|f - \bar{f}\|_{L^2}^{\frac{2}{p}} \|\nabla f\|_{L^2}^{1 - \frac{2}{p}}, \quad \|f\|_{L^p} \leq Cp^{\frac{1}{2}} \|f\|_{L^2}^{\frac{2}{p}} \|f\|_{H^1}^{1 - \frac{2}{p}}, \tag{12}$$

where

$$\bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f dx.$$

Next, we give some regularity results for the following Lamé system with the Dirichlet boundary condition (see [29]):

$$\begin{cases} \mathcal{L}U \triangleq \mu \Delta U + (\mu + \lambda) \nabla \operatorname{div} U = F, & x \in \Omega, \\ U = 0, & x \in \partial \Omega. \end{cases} \tag{13}$$

Suppose $U \in H_0^1$ is a weak solution to the Lamé system, we could denote $U = \mathcal{L}^{-1}F$ due to the uniqueness of solution.

Lemma 2.2 *Let $r \in (1, +\infty)$, then there exists some generic constant $C > 0$ depending only on μ, λ, r and Ω such that*

(1) *If $F \in L^r$, then*

$$\|U\|_{W^{2,r}(\Omega)} \leq C\|F\|_{L^r(\Omega)}. \tag{14}$$

(2) *If $F \in W^{-1,r}$ (i.e., $F = \operatorname{div} f$ with $f = (f_{i,j})_{2 \times 2}, f_{i,j} \in L^r$), then*

$$\|U\|_{W^{1,r}(\Omega)} \leq C\|f\|_{L^r(\Omega)}. \tag{15}$$

(3) *Moreover, for the endpoint case, if $f_{i,j} \in L^2 \cap L^\infty$, then $\nabla U \in \operatorname{BMO}(\Omega)$ and*

$$\|U\|_{\operatorname{BMO}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)}), \tag{16}$$

where $\operatorname{BMO}(\Omega)$ stands for the John–Nirenberg space of mean oscillation whose norm is defined by

$$\|f\|_{\operatorname{BMO}} \triangleq \|f\|_{L^2} + [f]_{\operatorname{BMO}(\Omega)},$$

with

$$[f]_{\operatorname{BMO}(\Omega)} \triangleq \sup_{x \in \Omega, r \in (0,d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| dy,$$

and

$$f_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy.$$

In the following, we give two critical Sobolev inequalities of logarithmic type, which are originally due to Brezis–Gallouet [30] and Brezis–Wainger [31].

Lemma 2.3 *Let $\Omega \in \mathbb{R}^2$ be a bounded Lipschitz domain and $f \in W^{1,q}$ with $q > 2$, then we have*

$$\|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{\operatorname{BMO}(\Omega)} \ln(e + \|f\|_{W^{1,q}})) \tag{17}$$

with a constant C depending only on q .

Lemma 2.4 *Let $\Omega \in \mathbb{R}^2$ be a smooth domain and $f \in L^2(s, t; H_0^1) \cap L^2(s, t; W^{1,q})$, with some $q > 2$ and $0 \leq s < t \leq \infty$. Then we have*

$$\|f\|_{L^2(s,t;L^\infty)}^2 \leq C(1 + \|f\|_{L^2(s,t;H^1)}^2 \ln(e + \|f\|_{L^2(s,t;W^{1,q})})) \tag{18}$$

with a constant C depending only on q .

Finally, we give the following lemma arises from Zlotnik [32], which will be used to prove the uniform upper bound for the density.

Lemma 2.5 *Let $y \in W^{1,1}(0, T)$ satisfy the ODE system:*

$$y'(t) = g(y) + b'(t) \quad \text{on } [0, T], y(0) = y_0,$$

where $b \in W^{1,1}(0, T)$, $g \in C(\mathbb{R})$ and $g(+\infty) = -\infty$. Assume that there are two constants $N_0 \geq 0$ and $N_1 \geq 0$ such that, for all $0 \leq t_1 \leq t_2 \leq T$,

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1). \tag{19}$$

Then

$$y(t) \leq \max\{y_0, \xi^*\} + N_0 < +\infty \quad \text{on } [0, T],$$

where $\xi^* \in \mathbb{R}$ is a constant such that $g(\xi) \leq -N_1$ for $\xi \geq \xi^*$.

3 Global classical solution

In this section, we establish some necessary a priori estimates for the classical solutions of initial-boundary value problem (1), (3)–(4). We assume that, for any $T > 0$, let (ρ, u) be a classical solution of (1), (3)–(4) in the solution space (8) with the initial data satisfying (5) and (6). In Sects. 3.1 and 3.2, we will show the lower-order and the higher-order estimates of the solutions, which guarantee the local classical solution can be extended to a global one.

3.1 Lower-order estimates of the solutions

First, we give the following proposition to prove the uniform upper bounds of $\|\nabla u\|_{L^2}$ and ρ .

Proposition 3.1 *Assume that the initial data satisfy (5)–(6), and the local classical solution satisfies*

$$\sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 \leq 2K, \quad \sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{L^2}^2) \leq 2m_0^{\frac{1}{8}}, \quad 0 \leq \rho \leq 2\bar{\rho} + 1, \tag{20}$$

where $(x, t) \in \Omega \times [0, T]$, $\sigma(t) \triangleq \min\{1, t\}$. Then there exists

$$\varepsilon_2 = \min\{\varepsilon_1, \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*, \varepsilon_5^*\}$$

depending on $\bar{\rho}, M, \mu, \lambda$, and some other known constants but independent of T such that

$$\sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 \leq \frac{3}{2}K, \quad \sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{L^2}^2) \leq \frac{3}{2}m_0^{\frac{1}{8}}, \quad 0 \leq \rho \leq \frac{3}{2}\bar{\rho} + 1, \tag{21}$$

provided that $m_0 \leq \varepsilon_2$ is suitable small.

In order to prove Proposition 3.1, we give the following mass conservation identity and the uniform bound of $\|\nabla u\|_{L^2(0, T; L^2)}$, which are the foundation of our proof in this paper.

Lemma 3.2 *Let (ρ, u) be a classical solution of (1), (3)–(4) on $\Omega \times (0, T]$, then we have*

$$\sup_{0 \leq t \leq T} \|\rho\|_{L^1} = \|\rho_0\|_{L^1} = m_0, \tag{22}$$

$$\sup_{0 \leq t \leq T} (\|\sqrt{\rho}u\|_{L^2}^2 + \|\rho\|_{L^\gamma}^\gamma) + \int_0^T \mu \|\nabla u\|_{L^2}^2 ds \leq m_0^{\frac{1}{4}}, \tag{23}$$

provided there exists a positive constant ε_1 such that $m_0 \leq \varepsilon_1$.

Proof Integrating (1)₁ over Ω , (22) can be easily obtained. In order to prove (23), multiplying (1)₁ by $\rho^{\gamma-1}$, it yields

$$\frac{1}{\gamma - 1} \frac{d}{dt} \int \rho^\gamma dx = \int u \cdot \nabla \rho^\gamma dx. \tag{24}$$

On the other hand, multiplying (1)₂ by u , integrating the result over Ω , and using (1)₁, we have

$$\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \mu \int |\nabla u|^2 dx + (\mu + \lambda) \int |\operatorname{div} u|^2 dx = - \int \nabla P \cdot u dx. \tag{25}$$

Adding (25) and (24) together, and integrating the resulting equality over time interval $(0, t)$, we get

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\frac{1}{2} \|\sqrt{\rho}u\|_{L^2}^2 + \frac{1}{\gamma - 1} \|\rho\|_{L^\gamma}^\gamma \right) + \int_0^T \mu \|\nabla u\|_{L^2}^2 ds \\ & \leq \frac{1}{2} \|\rho_0\|_{L^2} \|u_0\|_{L^4}^2 + \frac{1}{\gamma - 1} \|\rho_0\|_{L^\gamma}^\gamma \\ & \leq C \bar{\rho}^{\frac{1}{2}} \|\nabla u_0\|_{L^2}^2 m_0^{\frac{1}{2}} + C \bar{\rho}^{\gamma-1} m_0 \leq m_0^{\frac{1}{4}}, \end{aligned}$$

provided there exists a positive constant ε_1 such that $m_0 \leq \varepsilon_1$. This completes the proof of Lemma 3.2. □

Next, in Lemma 3.3, we give the uniform upper bound of $\|\nabla u\|_{L^2}$.

Lemma 3.3 *Let (ρ, u) be a classical solution of (1), (3)–(4) on $\Omega \times (0, T]$, if the assumption of Proposition 3.1 holds, then*

$$\sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 ds \leq \frac{3}{2}K, \tag{26}$$

$$\sup_{i-1 \leq t \leq i+1} (\sigma \|\nabla u\|_{L^2}^2) + \int_{i-1}^{i+1} \sigma_i \|\sqrt{\rho}\dot{u}\|_{L^2}^2 ds \leq \frac{3}{2}m_0^{\frac{1}{8}}, \tag{27}$$

provided there exist constant ε_1^ and ε_2^* such that $m_0 \leq \min\{\varepsilon_1, \varepsilon_1^*, \varepsilon_2^*\}$, where $\sigma_i(t) \triangleq \sigma(t + 1 - i)$, $1 \leq i \leq [T] - 1$, $t \in (i - 1, i + 1]$.*

Proof From (1)₂, we get

$$\rho \dot{u} + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u. \tag{28}$$

Multiplying (28) by $\eta \dot{u}$, $\eta = \eta(t) \geq 0$ is a piecewise smooth function, integrating the resulting equation over Ω , it yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\int \mu \eta |\nabla u|^2 dx + \int (\mu + \lambda) \eta |\operatorname{div} u|^2 dx \right) + \int \eta \rho |\dot{u}|^2 dx \\ &= \frac{d}{dt} \int \eta P \operatorname{div} u dx - \int \eta' P \operatorname{div} u dx - \int \eta (P_t + \operatorname{div}(Pu)) \operatorname{div} u dx \\ & \quad + \int \eta P \nabla u : \nabla u^\top dx - \mu \int \eta \nabla u : (\nabla u \nabla u) dx + \frac{1}{2} \mu \int \eta \operatorname{div} u |\nabla u|^2 dx \\ & \quad + \frac{1}{2} \mu \int \eta' |\nabla u|^2 dx + \frac{1}{2} (\mu + \lambda) \int \eta' |\operatorname{div} u|^2 dx \\ & \quad - (\mu + \lambda) \int \eta (\nabla u : \nabla u^\top) \operatorname{div} u dx + \frac{1}{2} (\mu + \lambda) \int \eta |\operatorname{div} u|^3 dx \\ & := \frac{d}{dt} \int \eta P \operatorname{div} u dx + \sum_{i=1}^9 I_i. \end{aligned} \tag{29}$$

Now, we estimate I_i , $i = 1, 2, 3, \dots, 9$, one by one:

$$\begin{aligned} I_1 &\leq |\eta'| \|P\|_{L^2} \|\nabla u\|_{L^2} \leq |\eta'| \|P\|_{L^2}^2 + |\eta'| \|\nabla u\|_{L^2}^2 \\ &\leq |\eta'| \bar{\rho}^{2\gamma-1} m_0 + |\eta'| \|\nabla u\|_{L^2}^2, \end{aligned} \tag{30}$$

$$I_2 + I_3 \leq C \eta \|\nabla u\|_{L^2}^2, \tag{31}$$

where we have used the identity $P_t + \operatorname{div}(Pu) = (1 - \gamma) \rho^\gamma \operatorname{div} u$.

The terms I_4, I_5, I_9 can be estimated as

$$\begin{aligned} I_4 + I_5 + I_8 + I_9 &\leq C \eta \|\nabla u\|_{L^3}^3 \leq C \eta (\|\nabla v\|_{L^3}^2 + \|\nabla w\|_{L^3}^3) \\ &\leq C \eta (\|P\|_{L^3}^3 + \|\nabla w\|_{L^2}^2 \|\nabla^2 w\|_{L^2}) \\ &\leq C \eta (\bar{\rho}^{3\gamma-1} m_0 + (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) \|\nabla^2 w\|_{L^2}) \\ &\leq C \eta (\bar{\rho}^{3\gamma-1} m_0 + (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2) \|\nabla^2 w\|_{L^2}) \\ &\leq C \eta (\bar{\rho}^{3\gamma-1} m_0 + (\|\nabla u\|_{L^2}^2 + \bar{\rho}^{2\gamma-1} m_0) \|\sqrt{\rho} \dot{u}\|_{L^2}) \\ &\leq C \eta (m_0 + m_0^2) + \frac{1}{4} \eta \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \eta \|\nabla u\|_{L^2}^4. \end{aligned} \tag{32}$$

It remains to estimate I_6 and I_7 , we get

$$I_6 + I_7 \leq C |\eta'| \|\nabla u\|_{L^2}^2. \tag{33}$$

Then, inserting (30)–(33) into (29), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\int \mu \eta |\nabla u|^2 dx + \int (\mu + \lambda) \eta |\operatorname{div} u|^2 dx \right) + \frac{1}{2} \int \eta \rho |\dot{u}|^2 dx \\ & \leq \frac{d}{dt} \int \eta P \operatorname{div} u dx + C |\eta'| m_0 + C \eta (m_0 + m_0^2) + C (|\eta'| + \eta) \|\nabla u\|_{L^2}^2 \\ & \quad + C \eta \|\nabla u\|_{L^2}^4. \end{aligned} \tag{34}$$

In order to prove (26), taking $\eta = 1$ and integrating (34) over $(0, t)$, for $0 < t \leq \sigma(T)$, we get

$$\begin{aligned} & \int \mu |\nabla u|^2 dx + \int_0^{\sigma(T)} \int \rho |\dot{u}|^2 dx ds \\ & \leq \mu \|\nabla u_0\|_{L^2}^2 + 2(\mu + \lambda) \|\operatorname{div} u_0\|_{L^2}^2 + \|P\|_{L^2} \|\nabla u\|_{L^2} + \|P_0\|_{L^2} \|\nabla u_0\|_{L^2} \\ & \quad + C(m_0 + m_0^2) + C m_0^{\frac{1}{4}} + CK m_0^{\frac{1}{4}} \\ & \leq \mu M + 2(\mu + \lambda)M + C(\sqrt{K} m_0^{\frac{1}{2}} + \sqrt{M} m_0^{\frac{1}{2}} + m_0 + m_0^2 + m_0^{\frac{1}{4}} + K m_0^{\frac{1}{4}}), \end{aligned} \tag{35}$$

where we have used (23) and (20). Then we have

$$\sup_{0 \leq t \leq \sigma(T)} \int \mu |\nabla u|^2 dx + \int_0^{\sigma(T)} \int \rho |\dot{u}|^2 dx ds \leq K + \frac{1}{2}K = \frac{3}{2}K, \tag{36}$$

provided there exists a constant ε_1^* , $m_0 \leq \varepsilon_1^*$ such that

$$\begin{aligned} & \mu M + 2(\mu + \lambda)M \leq K, \\ & C(\sqrt{K} m_0^{\frac{1}{2}} + \sqrt{M} m_0^{\frac{1}{2}} + m_0 + m_0^2 + m_0^{\frac{1}{4}} + K m_0^{\frac{1}{4}}) \leq \frac{1}{2}K. \end{aligned}$$

In order to prove (27), taking $\eta = \sigma_i$ in (34), integrating (34) over $(i - 1, t)$, we get

$$\begin{aligned} & \sigma_i \int \mu |\nabla u|^2 dx + \int_{i-1}^t \int \sigma_i \rho |\dot{u}|^2 dx ds \\ & \leq \sigma_i \|P\|_{L^2} \|\nabla u\|_{L^2} + C(m_0 + m_0^2 + m_0^{\frac{1}{4}}) + C \int_{i-1}^t \sigma_i \|\nabla u\|_{L^2}^4 ds \\ & \leq \frac{1}{2} \sigma_i \mu \|\nabla u\|_{L^2}^2 + C(m_0 + m_0^2 + m_0^{\frac{1}{4}}) + C \sup_{i-1 \leq t \leq i+1} \sigma_i \|\nabla u\|_{L^2}^2 \int_{i-1}^t \|\nabla u\|_{L^2}^2 ds \\ & \leq \frac{1}{2} \sigma_i \mu \|\nabla u\|_{L^2}^2 + C(m_0 + m_0^2 + m_0^{\frac{1}{4}}) + C \sup_{i-1 \leq t \leq i+1} \sigma_i \|\nabla u\|_{L^2}^2 m_0^{\frac{1}{4}} \\ & \leq \frac{1}{2} \sigma_i \mu \|\nabla u\|_{L^2}^2 + C(m_0 + m_0^2 + m_0^{\frac{1}{4}} + m_0^{\frac{3}{8}}). \end{aligned} \tag{37}$$

Then, we have

$$\sup_{i-1 \leq t \leq i+1} \sigma_i \int \mu |\nabla u|^2 dx + \int_{i-1}^{i+1} \int \sigma_i \rho |\dot{u}|^2 dx ds \leq \frac{3}{2} m_0^{\frac{1}{8}}, \tag{38}$$

provided there exists a constant ε_2^* , $m_0 \leq \varepsilon_2^*$ such that

$$C(m_0 + m_0^2 + m_0^{\frac{1}{4}} + m_0^{\frac{3}{8}}) \leq \frac{3}{2} m_0^{\frac{1}{8}}.$$

Hence, if we take $m_0 \leq \min\{\varepsilon_1, \varepsilon_1^*, \varepsilon_2^*\}$, this completes the proof of Lemma 3.3. □

In Lemma 3.4, we will give the bound for $\int_{t_1}^{t_2} \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 ds$ which will be used to prove the uniform upper bound of ρ . It should be noted that the constant C on the right-hand side of (39) and (40) is independent of time.

Lemma 3.4 *Let (ρ, u) be a classical solution of (1), (3)–(4) on $\Omega \times (0, T]$, if the assumption of Proposition 3.1 holds, then*

$$\sup_{0 \leq t \leq T} (\sigma^2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2) \leq C m_0^{\frac{1}{8}}, \tag{39}$$

$$\int_{t_1}^{t_2} \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 ds \leq C m_0 (t_2 - t_1) + C m_0^{\frac{1}{8}}, \tag{40}$$

for any $t_1, t_2 \in (0, T]$, provided $m_0 \leq \min\{\varepsilon_1, \varepsilon_1^*, \varepsilon_2^*\}$.

Proof Operating $\eta \dot{u}^j (\partial_t + \operatorname{div}(u \cdot))$ to (1)₂^j, summing with respect to j , and integrating the resulting equation over Ω , we obtain

$$\begin{aligned} & \frac{d}{dt} \int \eta \rho |\dot{u}|^2 dx - \eta' \int \rho |\dot{u}|^2 dx \\ &= -2\eta \int \dot{u}^j (\partial_j P_t + \operatorname{div}(u \partial_j P)) dx + 2\mu \eta \int \dot{u}^j (\partial_t \Delta u^j + \operatorname{div}(u \Delta u^j)) dx \\ & \quad + 2(\mu + \lambda) \eta \int \dot{u}^j (\partial_j \partial_t \operatorname{div} u + \operatorname{div}(u \partial_j \operatorname{div} u)) dx \\ & := \sum_{i=1}^3 J_i. \end{aligned} \tag{41}$$

It follows from integration by parts and using Eq. (1)₁ that

$$\begin{aligned} J_1 &= -2\eta \int \dot{u}^j (\partial_j P_t + \partial_j \operatorname{div}(uP) - \operatorname{div}(P \partial_j u)) dx \\ &= 2\eta \int \operatorname{div} \dot{u} (P_t + \operatorname{div}(uP)) dx - 2\eta \int (P \partial_j u) \cdot \nabla \dot{u}^j dx \\ &\leq C \eta \int \rho^\nu |\operatorname{div} \dot{u}| |\operatorname{div} u|^2 dx + C \eta \int \rho^\nu |\nabla u| |\nabla \dot{u}| dx \\ &\leq \frac{\mu \eta}{4} \|\nabla \dot{u}\|_{L^2}^2 + C \eta \|\nabla u\|_{L^2}^2, \end{aligned} \tag{42}$$

$$\begin{aligned} J_2 &= 2\mu \eta \int \dot{u}^j [\Delta \dot{u}^j + \partial_i (\operatorname{div} u \partial_i u^j - \partial_i u \cdot \nabla u^j) - \operatorname{div}(\partial_i u \partial_i u^j)] dx \\ &= -2\mu \eta \int |\nabla \dot{u}^j|^2 dx - 2\mu \eta \int \partial_i \dot{u}^j (\operatorname{div} u \partial_i u^j - \partial_i u \cdot \nabla u^j) dx \\ & \quad + 2\mu \eta \int \nabla \dot{u}^j \partial_i u \partial_i u^j dx \\ &\leq -\mu \eta \|\nabla \dot{u}^j\|_{L^2}^2 + C \eta \|\nabla u\|_{L^4}^4. \end{aligned} \tag{43}$$

Similarly, we get

$$J_3 \leq -(\mu + \lambda) \eta \|\operatorname{div} \dot{u}\|_{L^2}^2 + C \eta \|\nabla u\|_{L^4}^4, \tag{44}$$

where $\|\nabla u\|_{L^4}^4$ can be estimated as

$$\begin{aligned} \|\nabla u\|_{L^4}^4 &\leq \|\nabla v\|_{L^4}^4 + \|\nabla w\|_{L^4}^4 \\ &\leq \|P\|_{L^4}^4 + \|\nabla w\|_{L^2}^2 \|\nabla^2 w\|_{L^2}^2 \\ &\leq C(\bar{\rho}^{4\gamma-1} m_0 + (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) \|\nabla^2 w\|_{L^2}^2) \\ &\leq C(\bar{\rho}^{4\gamma-1} m_0 + (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2) \|\nabla^2 w\|_{L^2}^2) \\ &\leq C(\bar{\rho}^{4\gamma-1} m_0 + (\|\nabla u\|_{L^2}^2 + \bar{\rho}^{2\gamma-1} m_0) \|\sqrt{\rho} \dot{u}\|_{L^2}^2). \end{aligned} \tag{45}$$

Substituting the estimates J_1, J_2, J_3 and (45) into (41), we arrive at

$$\begin{aligned} &\frac{d}{dt} \int \eta \rho |\dot{u}|^2 dx + \mu \eta \|\nabla \dot{u}\|_{L^2}^2 \\ &\leq |\eta'| \int \rho |\dot{u}|^2 dx + C\eta \|\nabla u\|_{L^2}^2 + C\eta (\|\nabla u\|_{L^2}^2 + \bar{\rho}^{2\gamma-1} m_0) \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\ &\quad + C\eta \bar{\rho}^{4\gamma-1} m_0. \end{aligned} \tag{46}$$

In order to prove (39), taking $\eta = \sigma_i^2$ in (46), integrating (46) over $(i-1, t)$ and taking (27) into consideration, we get

$$\begin{aligned} &\sigma_i^2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \int_{i-1}^t \mu \sigma_i^2 \|\nabla \dot{u}\|_{L^2}^2 ds \\ &\leq \int_{i-1}^t \sigma_i \sigma_i' \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds + C \int_{i-1}^t \sigma_i^2 (\|\nabla u\|_{L^2}^2 + \bar{\rho}^{2\gamma-1} m_0) \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds \\ &\quad + C(m_0^{\frac{1}{4}} + m_0) \\ &\leq C m_0^{\frac{1}{8}} + C \left(\sup_{i-1 \leq t \leq i+1} \sigma_i \|\nabla u\|_{L^2}^2 + m_0 \right) \int_{i-1}^t \sigma_i \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds \\ &\quad + C(m_0^{\frac{1}{4}} + m_0) \\ &\leq C(m_0^{\frac{1}{8}} + m_0^{\frac{1}{4}} + m_0^{\frac{5}{8}} + m_0^{\frac{9}{8}} + m_0) \\ &\leq C m_0^{\frac{1}{8}}, \end{aligned} \tag{47}$$

which proves (39).

Furthermore, from (47), we can see that, if we take $\eta = \sigma^2$, then integrating (46) over $(t_1, t_2) \in [0, T]$, we have

$$\begin{aligned} &\sigma^2 \|\sqrt{\rho} \dot{u}(t_2)\|_{L^2}^2 + \int_{t_1}^{t_2} \mu \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 ds \\ &\leq \sigma^2 \|\sqrt{\rho} \dot{u}(t_1)\|_{L^2}^2 + \int_{t_1}^{t_2} \sigma \sigma' \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds \\ &\quad + C \int_{t_1}^{t_2} \sigma^2 (\|\nabla u\|_{L^2}^2 + \bar{\rho}^{2\gamma-1} m_0) \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds \\ &\quad + C m_0^{\frac{1}{4}} + C m_0 (t_2 - t_1) \end{aligned}$$

$$\begin{aligned}
 &\leq C(m_0^{\frac{1}{8}} + m_0^{\frac{1}{4}}) + C\left(1 + \sup_{0 \leq t \leq T} \sigma \|\nabla u\|_{L^2}^2 + m_0\right) \int_{t_1}^{t_2} \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds \\
 &\quad + Cm_0(t_2 - t_1) \\
 &\leq Cm_0^{\frac{1}{8}} + C(1 + m_0^{\frac{1}{8}} + m_0) \int_{t_1}^{t_2} \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds + Cm_0(t_2 - t_1), \tag{48}
 \end{aligned}$$

where we have used (23), (20) and (39).

In order to estimate the second term on the right-hand side of the above inequality, taking $\eta = \sigma$ in (29), integrating (29) over $(t_1, t_2) \in [0, T]$, we get

$$\begin{aligned}
 &\sigma \int \mu |\nabla u|^2 dx + \int_{t_1}^{t_2} \int \sigma \rho |\dot{u}|^2 dx ds \\
 &\leq \mu \sigma \|\nabla u(t_1)\|_{L^2}^2 + 2(\mu + \lambda) \sigma \|\operatorname{div} u(t_1)\|_{L^2}^2 + \sigma \|P\|_{L^2} \|\nabla u\|_{L^2} \\
 &\quad + C(m_0 + m_0^2)(t_2 - t_1) + Cm_0^{\frac{1}{4}} + C \int_{i-1}^t \sigma_i \|\nabla u\|_{L^2}^4 ds \\
 &\leq \frac{1}{2} \sigma \mu \|\nabla u\|_{L^2}^2 + C(m_0 + m_0^2)(t_2 - t_1) + C(m_0^{\frac{1}{4}} + m_0^{\frac{1}{8}}) \\
 &\quad + C \sup_{i-1 \leq t \leq i+1} \sigma_i \|\nabla u\|_{L^2}^2 \int_{i-1}^t \|\nabla u\|_{L^2}^2 ds \\
 &\leq \frac{1}{2} \sigma \mu \|\nabla u\|_{L^2}^2 + C(m_0 + m_0^2)(t_2 - t_1) + C(m_0^{\frac{1}{4}} + m_0^{\frac{1}{8}}) \\
 &\quad + C \sup_{i-1 \leq t \leq i+1} \sigma_i \|\nabla u\|_{L^2}^2 m_0^{\frac{1}{4}} \\
 &\leq \frac{1}{2} \sigma \mu \|\nabla u\|_{L^2}^2 + C(m_0 + m_0^2)(t_2 - t_1) + C(m_0^{\frac{1}{4}} + m_0^{\frac{1}{8}} + m_0^{\frac{3}{8}}), \tag{49}
 \end{aligned}$$

from which one deduces

$$\sigma \int \mu |\nabla u|^2 dx + \int_{t_1}^{t_2} \int \sigma \rho |\dot{u}|^2 dx ds \leq Cm_0(t_2 - t_1) + Cm_0^{\frac{1}{8}}. \tag{50}$$

Then, inserting (50) into (48), (40) can be obtained. This completes the proof of Lemma 3.4. \square

Inspired by the methods in Refs. [12, 15], in the following lemma, we use the Zlotnik inequality to prove the uniform upper bound of the density ρ .

Lemma 3.5 *Under the condition of Proposition 3.1, we have*

$$\rho \leq \frac{3}{2} \bar{\rho} + 1, \tag{51}$$

provided there exist constants ε_3^* , ε_4^* and ε_5^* such that $m_0 \leq \min\{\varepsilon_1, \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*, \varepsilon_5^*\}$.

Proof For any given $(x, t) \in \Omega \times [0, T]$, denoting $X(s; x, t)$ the solution to the initial value problem

$$\begin{cases} \frac{d}{ds} X(s; x, t) = u(X(s; x, t), s), & 0 \leq s < t, \\ X(t; x, t) = x. \end{cases}$$

It is easy to verify that

$$\frac{d}{ds} \rho(X(s; x, t), s) + \rho(X(s; x, t), s) \operatorname{div} u(X(s; x, t), s) = 0,$$

due to (1)₁. This gives

$$Y'(s) = g(s) + b'(s), \tag{52}$$

where

$$Y(s) = \rho(X(s; x, t), s), \quad g(s) = -\frac{\rho^{\gamma+1}(X(s; x, t), s)}{2\mu + \lambda},$$

$$b(s) = -\int_0^s \rho(X(s; x, t), s) \left(\frac{C(t)}{2\mu + \lambda} + \operatorname{div} w(X(s; x, t), s) \right) ds,$$

and $C(t) = (2\mu + \lambda)\operatorname{div} v - P$.

Next, we use Lemma 2.5 to prove the uniform upper bound of the density. We have

$$b(t_2) - b(t_1) = \int_{t_1}^{t_2} \left\| \frac{\rho C(t)}{2\mu + \lambda} \right\|_{\infty} ds + \int_{t_1}^{t_2} \|\rho \operatorname{div} w\|_{\infty} ds$$

$$= K_1 + K_2. \tag{53}$$

In the following, we estimate the terms on the right-hand side of Eq. (53) one by one. In order to estimate $C(t)$, from Eq. (9), we have

$$\nabla((2\mu + \lambda)\operatorname{div} v - P) - \mu \nabla \times (\nabla \times v) = 0. \tag{54}$$

We have $(\nabla \times (\nabla \times v)) = (\partial_2(\partial_1 v^2 - \partial_2 v^1), -\partial_1(\partial_1 v^2 - \partial_2 v^1))$ and the boundary condition (4) implies

$$\begin{cases} v^1 = \partial_2 v^2 = 0, & x_1 = 0, 1, \\ v^2 = \partial_1 v^1 = 0, & x_2 = 0, 1. \end{cases} \tag{55}$$

Then, we have $(\nabla \times (\nabla \times v)) \cdot n = 0$ a.e. on $\partial\Omega$ and $\operatorname{div}(\nabla \times (\nabla \times v)) = 0$.

Multiplying (54) by $\nabla((2\mu + \lambda)\operatorname{div} v - P)$ and integrating the resulting equation over Ω , we arrive at

$$\|\nabla((2\mu + \lambda)\operatorname{div} v - P)\|_{L^2} = 0,$$

which implies that there exists $C(t)$ such that

$$C(t) = (2\mu + \lambda)\operatorname{div} v - P. \tag{56}$$

Using (9), we have $\|\nabla v\|_{L^2} \leq C\|P\|_{L^2}$. Integrating (56), we get

$$C(t) \leq C(\|\nabla v\|_{L^2} + \|P\|_{L^2}) \leq C\|P\|_{L^2} \leq C\bar{\rho}^{-\frac{2\gamma-1}{2}} m_0^{\frac{1}{2}}. \tag{57}$$

Then, we have

$$K_1 = \int_{t_1}^{t_2} \left\| \frac{\rho C(t)}{2\mu + \lambda} \right\|_{\infty} ds \leq \frac{C \bar{\rho}^{-\frac{2\gamma+1}{2}} m_0^{\frac{1}{2}}}{2\mu + \lambda} (t_2 - t_1) \leq \frac{1}{4(2\mu + \lambda)} (t_2 - t_1), \tag{58}$$

provided there exists a constant ε_3^* , $m_0 \leq \varepsilon_3^*$ such that

$$C \bar{\rho}^{-\frac{2\gamma+1}{2}} m_0^{\frac{1}{2}} \leq \frac{1}{4}.$$

In order to estimate K_2 , we consider the following three cases:

(1) $0 \leq t_1 \leq t_2 \leq \sigma(T)$.

$$\begin{aligned} K_2 &= \int_{t_1}^{t_2} \|\rho \operatorname{div} w\|_{L^\infty} ds \leq C \int_0^{\sigma(T)} \|\nabla w\|_{L^2}^{\frac{1}{3}} \|\nabla w\|_{W^{1,4}}^{\frac{2}{3}} ds \\ &\leq C \int_0^{\sigma(T)} (\|\nabla u\|_{L^2} + \|P\|_{L^2})^{\frac{1}{3}} \|\nabla \dot{u}\|_{L^2}^{\frac{2}{3}} ds \\ &\leq C \int_0^{\sigma(T)} \sigma^{-\frac{1}{2}} (\sigma^{\frac{1}{2}} \|\nabla u\|_{L^2} + \sigma^{\frac{1}{2}} \|P\|_{L^2})^{\frac{1}{3}} (\sigma \|\nabla \dot{u}\|_{L^2}^2)^{\frac{1}{3}} ds \\ &\leq C \int_0^{\sigma(T)} \sigma^{-\frac{1}{2}} (m_0^{\frac{1}{16}} + \sigma^{\frac{1}{2}} \bar{\rho}^{-\frac{2\gamma-1}{2}} m_0^{\frac{1}{2}})^{\frac{1}{3}} (\sigma \|\nabla \dot{u}\|_{L^2}^2)^{\frac{1}{3}} ds \\ &\leq C (m_0^{\frac{1}{16}} + \bar{\rho}^{-\frac{2\gamma-1}{2}} m_0^{\frac{1}{2}})^{\frac{1}{3}} \left(\int_0^{\sigma(T)} \sigma^{-\frac{3}{4}} ds \right)^{\frac{2}{3}} \left(\int_0^{\sigma(T)} \sigma \|\nabla \dot{u}\|_{L^2}^2 ds \right)^{\frac{1}{3}} \\ &\leq C (m_0^{\frac{1}{16}} + \bar{\rho}^{-\frac{2\gamma-1}{2}} m_0^{\frac{1}{2}})^{\frac{1}{3}} \left(\int_0^{\sigma(T)} \sigma \|\nabla \dot{u}\|_{L^2}^2 ds \right)^{\frac{1}{3}}, \end{aligned} \tag{59}$$

where we have used Lemma 3.3. It remains to estimate the term on the right-hand side of inequality (59). To do this, we taking $\eta = \sigma$ in (46) and integrating the resulting inequality over $(0, \sigma(T))$, we have

$$\begin{aligned} &\int \eta \rho |\dot{u}|^2 dx + \int_0^{\sigma(T)} \mu \sigma \|\nabla \dot{u}\|_{L^2}^2 ds \\ &\leq C + \int_0^{\sigma(T)} |\sigma'| \int \rho |\dot{u}|^2 dx ds + C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^2 ds \\ &\quad + C \int_0^{\sigma(T)} \sigma (\|\nabla u\|_{L^2}^2 ds + \bar{\rho}^{-2\gamma-1} m_0) \|\sqrt{\rho} \dot{u}\|_{L^2}^2 ds + C \sigma \bar{\rho}^{4\gamma-1} m_0 \\ &\leq C(1 + K + m_0^{\frac{1}{4}}) + C(m_0^{\frac{1}{8}} + \bar{\rho}^{-2\gamma-1} m_0)K + C \sigma \bar{\rho}^{4\gamma-1} m_0 \\ &\leq C. \end{aligned} \tag{60}$$

From (59) and (60), we can see that

$$K_2 \leq C (m_0^{\frac{1}{16}} + \bar{\rho}^{-\frac{2\gamma-1}{2}} m_0^{\frac{1}{2}})^{\frac{1}{3}} \leq \frac{1}{8} \bar{\rho}, \tag{61}$$

provided there exists a constant ε_4^* such that $m_0 \leq \varepsilon_4^*$.

(2) $\sigma(T) \leq t_1 \leq t_2 \leq T$.

$$\begin{aligned}
 K_2 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) + 4(2\mu + \lambda) \int_{t_1}^{t_2} \|\rho \operatorname{div} w\|_{L^\infty}^2 ds \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) + 4\bar{\rho}^2(2\mu + \lambda) \int_{t_1}^{t_2} \|\operatorname{div} w\|_{L^\infty}^2 ds \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) + C\bar{\rho}^2 \int_{t_1}^{t_2} \|\nabla w\|_{L^2}^{\frac{2}{3}} \|\nabla w\|_{W^{1,4}}^{\frac{4}{3}} ds \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) + C\bar{\rho}^2 \int_{t_1}^{t_2} (\|\nabla u\|_{L^2} + \|P\|_{L^2})^{\frac{2}{3}} \|\nabla \dot{u}\|_{L^2}^{\frac{4}{3}} ds \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) \\
 &\quad + C\bar{\rho}^2 \left(\int_{t_1}^{t_2} (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2) ds \right)^{\frac{1}{3}} \left(\int_{t_1}^{t_2} \|\nabla \dot{u}\|_{L^2}^2 ds \right)^{\frac{2}{3}} \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) \\
 &\quad + C\bar{\rho}^2 (m_0^{\frac{1}{4}} + \bar{\rho}^{2\gamma-1} m_0(t_2 - t_1))^{\frac{1}{3}} \left(\int_{t_1}^{t_2} \|\nabla \dot{u}\|_{L^2}^2 ds \right)^{\frac{2}{3}} \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) + C\bar{\rho}^2 m_0^{\frac{1}{12}} \left(\int_{t_1}^{t_2} \|\nabla \dot{u}\|_{L^2}^2 ds \right)^{\frac{2}{3}} \\
 &\quad + C\bar{\rho}^{\frac{2\gamma+5}{3}} m_0^{\frac{1}{3}} (t_2 - t_1)^{\frac{1}{3}} \left(\int_{t_1}^{t_2} \|\nabla \dot{u}\|_{L^2}^2 ds \right)^{\frac{2}{3}} \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) + C \int_{t_1}^{t_2} \|\nabla \dot{u}\|_{L^2}^2 ds + C m_0^{\frac{1}{4}} + C m_0^{\frac{1}{3}} (t_2 - t_1) \\
 &\leq \frac{1}{4(2\mu + \lambda)}(t_2 - t_1) + C(m_0 + m_0^{\frac{1}{3}})(t_2 - t_1) + C(m_0^{\frac{1}{8}} + m_0^{\frac{1}{4}}), \tag{62}
 \end{aligned}$$

where in the last inequality we have used (40). Then, we get

$$K_2 \leq \frac{1}{2(2\mu + \lambda)}(t_2 - t_1) + \frac{1}{8}\bar{\rho}, \tag{63}$$

provided there exists a constant ε_5^* , $m_0 \leq \varepsilon_5^*$, such that

$$C(m_0 + m_0^{\frac{1}{3}}) \leq \frac{1}{4(2\mu + \lambda)}, \quad C(m_0^{\frac{1}{8}} + m_0^{\frac{1}{4}}) \leq \frac{1}{8}\bar{\rho}.$$

(3) $0 \leq t_1 \leq \sigma(T) \leq t_2 \leq T$.

Combining case (1) and case (2), we can easily obtain

$$\begin{aligned}
 K_2 &= \int_{t_1}^{\sigma(T)} \|\rho \operatorname{div} w\|_{L^\infty} ds + \int_{\sigma(T)}^{t_2} \|\rho \operatorname{div} w\|_{L^\infty} ds \\
 &\leq \frac{1}{2(2\mu + \lambda)}(t_2 - t_1) + \frac{1}{4}\bar{\rho}. \tag{64}
 \end{aligned}$$

Taking (53), (58), (61), (63) and (64) into consideration, we have

$$|b(t_2) - b(t_1)| \leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + \frac{1}{2}\bar{\rho}, \tag{65}$$

provided there exist constants $\varepsilon_3^*, \varepsilon_4^*$ and ε_5^* as mentioned above such that

$$m_0 \leq \min\{\varepsilon_1, \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*, \varepsilon_5^*\}.$$

Then we can choose N_0, N_1 as follows:

$$N_0 = \frac{1}{2}\bar{\rho}, \quad N_1 = \frac{1}{2\mu + \lambda}.$$

Choosing $\xi^* = \bar{\rho} + 1$, we can see that

$$g(\xi) = -\frac{\xi^{\nu+1}}{2\mu + \lambda} \leq -\frac{1}{2\mu + \lambda} = -N_1, \quad \text{for } \xi \geq \xi^*.$$

Using Lemma 2.5, we obtain

$$\sup_{t \in [0, T]} \|\rho\|_{L^\infty} \leq \max\{\rho_0, \xi^*\} + N_0 \leq \max\{\bar{\rho}, \bar{\rho} + 1\} + N_0 \leq \bar{\rho} + 1 + \frac{1}{2}\bar{\rho} = \frac{3}{2}\bar{\rho} + 1.$$

This completes the proof of Lemma 3.5. □

Combining Lemmas 3.2–3.5, if we take $m_0 \leq \min\{\varepsilon_1, \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*, \varepsilon_5^*\}$, then Proposition 3.1 is proved. At the end of this subsection, we give the following second-order a priori estimates, where the constants C on the right-hand side of (66) and (67) may depend on time T .

Lemma 3.6 *Let (ρ, u) be a strong solution of (1), (3)–(4) on $\Omega \times (0, T]$, under the condition of Theorem 1.1, we have*

$$\sup_{0 \leq t \leq T} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 ds \leq C(T), \tag{66}$$

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^p} + \|u\|_{H^2}) \leq C(T). \tag{67}$$

Proof Taking $\eta = 1$ in (46), integrating (46) over $(0, \sigma(T)]$, we get

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \int_0^{\sigma(T)} \mu \|\nabla \dot{u}\|_{L^2}^2 ds \\ & \leq C(1 + m_0^{\frac{1}{4}} + m_0) + C \int_0^{\sigma(T)} (\|\nabla u\|_{L^2}^2 + m_0) \|\sqrt{\rho}\dot{u}\|_{L^2}^2 ds \leq C, \end{aligned} \tag{68}$$

which combines (39) and (40), and we obtain

$$\sup_{0 \leq t \leq T} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 dt \leq C(T). \tag{69}$$

Next, applying the operator ∇ to (1)₁, and multiplying the resulting equation by $p|\nabla\rho|^{p-2}\nabla\rho$, $p > 2$, we obtain

$$\begin{aligned} \frac{d}{dt} \int |\nabla\rho|^p dx &= (1-p) \int |\nabla\rho|^p \operatorname{div} u dx - p \int |\nabla\rho|^{p-2} \nabla\rho (\nabla u \cdot \nabla\rho) dx \\ &\quad - p \int \rho |\nabla\rho|^{p-2} \nabla\rho \cdot \nabla(\operatorname{div} u) dx \\ &\leq C \int |\nabla\rho|^p |\nabla u| dx + C \int |\nabla\rho|^{p-1} |\nabla^2 u| dx \\ &\leq C \|\nabla u\|_{L^\infty} \|\nabla\rho\|_{L^p}^p + C \|\nabla\rho\|_{L^p}^{p-1} \|\nabla^2 u\|_{L^p}, \end{aligned} \tag{70}$$

then we have

$$\frac{d}{dt} \|\nabla\rho\|_p \leq C(\|\nabla u\|_{L^\infty} \|\nabla\rho\|_{L^p} + \|\nabla^2 u\|_{L^p}), \tag{71}$$

where the terms on the right-hand side of the above inequality can be estimated as

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq C(\|\nabla w\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \\ &\leq C(\|\nabla w\|_{L^p} + \|\nabla^2 w\|_{L^p} + \|\nabla v\|_{L^\infty}) \\ &\leq C(\|\nabla \dot{u}\|_{L^2} + \|\nabla v\|_{\text{BMO}} \ln(e + \|\nabla^2 v\|_{L^p}) + 1) \\ &\leq C(\|\nabla \dot{u}\|_{L^2} + (\|P\|_{L^\infty} + \|P\|_{L^2}) \ln(e + \|\nabla^2 v\|_{L^p}) + 1) \\ &\leq C(\|\nabla \dot{u}\|_{L^2} + \ln(e + \|\nabla\rho\|_{L^p}) + 1) \end{aligned} \tag{72}$$

and

$$\begin{aligned} \|\nabla^2 u\|_{L^p} &\leq C(\|\nabla^2 v\|_{L^p} + \|\nabla^2 w\|_{L^p}) \\ &\leq C(\|\nabla P\|_{L^p} + \|\rho \dot{u}\|_{L^p}) \\ &\leq C(\|\nabla\rho\|_{L^p} + \|\nabla \dot{u}\|_{L^2}). \end{aligned} \tag{73}$$

Inserting (72) and (73) into (71), we have

$$\begin{aligned} \frac{d}{dt} (\|\nabla\rho\|_{L^p} + e) &\leq C(\|\nabla \dot{u}\|_{L^2} + e) \ln(e + \|\nabla\rho\|_{L^p}) \|\nabla\rho\|_{L^p} + C(\|\nabla \dot{u}\|_{L^2} + e) (\|\nabla\rho\|_{L^p} + e). \end{aligned} \tag{74}$$

Both sides of (74) divided by $\|\nabla\rho\|_p + e$ lead to

$$\frac{d}{dt} \ln(\|\nabla\rho\|_{L^p} + e) \leq C(\|\nabla \dot{u}\|_{L^2} + e) \ln(e + \|\nabla\rho\|_{L^p}) + C(\|\nabla \dot{u}\|_{L^2} + e). \tag{75}$$

Then, by using the Gronwall inequality and (69), we have

$$\sup_{0 \leq t \leq T} \|\nabla\rho\|_{L^p} \leq C(T). \tag{76}$$

Moreover, from (69), we have

$$\|u\|_{H^2} \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla P\|_{L^2}) \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla \rho\|_{L^p}) \leq C(T). \tag{77}$$

This completes the proof of Lemma 3.6. □

3.2 Higher-order estimates of the solutions

For completeness of our proof, we list the higher-order estimates of the solution (ρ, u) below, which can be derived in a similar manner to those obtained in [12] after some modifications.

Lemma 3.7 *Let (ρ, u) be a classical solution of (1), (3)–(4) on $\Omega \times (0, T]$, under the condition of Theorem 1.1, the following estimates hold:*

$$\sup_{0 \leq t \leq T} \|\sqrt{\rho} u_t\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 ds \leq C(T), \tag{78}$$

$$\sup_{0 \leq t \leq T} (\|\rho\|_{H^2}^2 + \|P(\rho)\|_{H^2}^2) \leq C(T). \tag{79}$$

Proof Estimate (78) follows directly from the following simple facts that

$$\begin{aligned} \int \rho |u_t|^2 dx &\leq \int \rho |\dot{u}|^2 dx + \int \rho |u \cdot \nabla u|^2 dx \leq C + \|\sqrt{\rho} u\|_{L^2} \|u\|_{L^6} \|\nabla u\|_{L^6}^2 \\ &\leq C \end{aligned} \tag{80}$$

and

$$\begin{aligned} \int_0^T \|\nabla u_t\|_{L^2}^2 ds &\leq \int_0^T \|\nabla \dot{u}\|_{L^2}^2 ds + \int_0^T \|\nabla(u \cdot \nabla u)\|_{L^2}^2 ds \\ &\leq C + \int_0^T \|u\|_{L^\infty}^2 \|\nabla^2 u\|_{L^2}^2 ds + \int_0^T \|\nabla u\|_{L^4}^4 ds \\ &\leq C, \end{aligned} \tag{81}$$

where in the last inequality we have used the Sobolev embedding inequalities and Lemma 3.6.

Next, we prove (79). P satisfies

$$P_t + u \cdot \nabla P + \gamma P \operatorname{div} u = 0, \tag{82}$$

which together with (1)₁ yields

$$\begin{aligned} &\frac{d}{dt} (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla P\|_{L^2}^2) \\ &\leq C(\|\nabla^2 u\|_{L^4} \|\nabla \rho\|_{L^4} \|\nabla^2 \rho\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2 \\ &\quad + \|\nabla^2 u\|_{L^4} \|\nabla P\|_{L^4} \|\nabla^2 P\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla^2 P\|_{L^2}^2 + \|\nabla^3 u\|_{L^2} \|\nabla^2 P\|_{L^2}) \\ &\leq C(\|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla^3 u\|_{L^2}^{\frac{1}{2}} \|\nabla \rho\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \rho\|_{L^2}^{\frac{3}{2}} + \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2) \end{aligned}$$

$$\begin{aligned}
 & + \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla^3 u\|_{L^2}^{\frac{1}{2}} \|\nabla P\|_{L^2}^{\frac{1}{2}} \|\nabla^2 P\|_{L^2}^{\frac{3}{2}} + \|\nabla u\|_{L^\infty} \|\nabla^2 P\|_{L^2}^2 \\
 & + \|\nabla^3 u\|_{L^2} \|\nabla^2 P\|_{L^2}) \\
 \leq & C(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2 + \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2 \\
 & + \|\nabla u\|_{L^\infty} \|\nabla^2 P\|_{L^2}^2) \\
 \leq & C\|\nabla^3 u\|_{L^2}^2 + C(\|\nabla u\|_{L^\infty} + 1)(\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) \\
 \leq & C\|\nabla \dot{u}\|_{L^2} + C(\|\nabla \dot{u}\|_{L^2} + \ln(e + \|\nabla \rho\|_{L^p}) + 1)(\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2), \tag{83}
 \end{aligned}$$

where in the last inequality we have used (72) and the following standard L^2 -estimate for elliptic system (9) and (10)

$$\begin{aligned}
 \|\nabla^3 u\|_{L^2} & \leq C(\|\nabla^3 v\|_{L^2} + \|\nabla^3 w\|_{L^2}) \leq C(\|\nabla^2 P\|_{L^2} + \|\nabla(\rho \dot{u})\|_{L^2}) \\
 & \leq C(\|\nabla^2 \rho\|_{L^2} + \|\nabla \dot{u}\|_{L^2}).
 \end{aligned}$$

Then, combining (83), Lemma 3.6 and the Gronwall inequality, we have (79). □

Lemma 3.8 *Let (ρ, u) be a classical solution of (1), (3)–(4) on $\Omega \times (0, T]$, under the condition of Theorem 1.1, the following estimates hold:*

$$\sup_{0 \leq t \leq T} (\|\rho_t\|_{H^1} + \|P_t\|_{H^1}) + \int_0^T (\|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2) ds \leq C(T), \tag{84}$$

$$\sup_{0 \leq t \leq T} \|\nabla u_t\|_{L^2}^2 + \int_0^T \|\sqrt{\rho} u_{tt}\|_{L^2}^2 ds \leq C(T). \tag{85}$$

Proof First, from (82) and Lemma 3.6, we obtain

$$\|P_t\|_{L^2} \leq C(\|u\|_{L^\infty} \|\nabla P\|_{L^2} + \|\nabla u\|_{L^2}) \leq C. \tag{86}$$

Furthermore, differentiating (82) yields

$$\nabla P_t + u \cdot \nabla \nabla P + \nabla u \cdot \nabla P + \gamma \nabla P \operatorname{div} u + \gamma P \nabla \operatorname{div} u = 0, \tag{87}$$

which together with Lemma 3.6 and Lemma 3.7, one gets

$$\|\nabla P_t\|_{L^2} \leq C(\|u\|_{L^\infty} \|\nabla^2 P\|_{L^2} + \|\nabla u\|_{L^4} \|\nabla P\|_{L^4} + \|\nabla^2 u\|_{L^2}) \leq C. \tag{88}$$

The combination of (86) and (88) implies

$$\sup_{0 \leq t \leq T} \|P_t\|_{H^1} \leq C. \tag{89}$$

Note that P_{tt} satisfies

$$P_{tt} + u_t \cdot \nabla P + u \cdot \nabla P_t + \gamma P \operatorname{div} u_t + \gamma P_t \operatorname{div} u = 0, \tag{90}$$

from which, together with (89) and Lemma 3.7, we have

$$\int_0^T \|P_{tt}\|_{L^2}^2 ds \leq \int_0^T C(\|u_t\|_{L^4}\|\nabla P\|_{L^4} + \|\nabla u\|_{L^4}\|P_t\|_{L^4} + \|\nabla u_t\|_{L^2})^2 ds \leq C. \tag{91}$$

Next, we differentiate (1)₂ with respect to t , then multiplying the resulting equation by u_{tt} , one gets after integration by parts

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu|\nabla u_t|^2 + (\lambda + \mu)(\operatorname{div} u_t)^2) dx + \int \rho u_{tt}^2 dx \\ &= \frac{d}{dt} \left(-\frac{1}{2} \int \rho_t |u_t|^2 dx - \int \rho_t u \cdot \nabla u \cdot u_t dx + \int P_t \operatorname{div} u_t dx \right) \\ & \quad + \frac{1}{2} \int \rho_{tt} |u_t|^2 dx + \int (\rho_t u \cdot \nabla u)_t \cdot u_t dx - \int \rho u_t \cdot \nabla u \cdot u_{tt} dx \\ & \quad - \int \rho u \cdot \nabla u_t \cdot u_{tt} dx - \int P_{tt} \operatorname{div} u_t dx \\ & := \frac{d}{dt} L_0 + \sum_{i=1}^5 L_i. \end{aligned} \tag{92}$$

The terms on the right-hand side of Eq. (92) can be estimated as follows:

$$\begin{aligned} L_0 &= -\frac{1}{2} \int \rho_t |u_t|^2 dx - \int \rho_t u \cdot \nabla u \cdot u_t dx + \int P_t \operatorname{div} u_t dx \\ &\leq C(\|\rho_t\|_{L^4}\|u_t\|_{L^4}\|u_t\|_{L^2} + \|\rho_t\|_{L^4}\|u\|_{L^\infty}\|\nabla u\|_{L^4}\|u_t\|_{L^2} + \|P_t\|_{L^2}\|\nabla u_t\|_{L^2}) \\ &\leq C(\|\rho_t\|_{H^1}\|u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}} + \|\rho_t\|_{H^1}\|u\|_{H^2}^2\|\nabla u_t\|_{L^2} + \|P_t\|_{L^2}\|\nabla u_t\|_{L^2}) \\ &\leq \delta\|\nabla u_t\|_{L^2}^2 + C(\|\rho_t\|_{H^1}^4\|u_t\|_{L^2}^2 + \|\rho_t\|_{H^1}^2\|u\|_{H^2}^4 + \|P_t\|_{L^2}^2) \\ &\leq \delta\|\nabla u_t\|_{L^2}^2 + C, \end{aligned} \tag{93}$$

where we have used Lemma 3.6, (84) and the Poincaré inequality. We have

$$\begin{aligned} L_1 &= \frac{1}{2} \int \rho_{tt} |u_t|^2 dx = -\frac{1}{2} \int (\rho_t u + \rho u_t) \cdot \nabla |u_t|^2 dx \\ &\leq C(\|\rho_t\|_{L^4}\|u\|_{L^\infty}\|u_t\|_{L^4}\|\nabla u_t\|_{L^2} + \|u_t\|_{L^4}^2\|\nabla u_t\|_{L^2}) \\ &\leq C(1 + \|\nabla u_t\|_{L^2}^2)\|\nabla u_t\|_{L^2}^2, \\ L_2 &= \int (\rho_t u \cdot \nabla u)_t \cdot u_t dx \\ &= \int (\rho_{tt} u \cdot \nabla u + \rho_t u_t \cdot \nabla u + \rho_t u \cdot \nabla u_t) \cdot u_t dx \\ &\leq C(\|\rho_{tt}\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^4}\|u_t\|_{L^4} + \|\rho_t\|_{L^4}\|\nabla u\|_{L^4}\|u_t\|_{L^4}^2 \\ & \quad + \|\rho_t\|_{L^4}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_t\|_{L^4}) \\ &\leq C(\|\rho_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2), \end{aligned} \tag{94}$$

$$\tag{95}$$

$$\begin{aligned}
 L_3 + L_4 &= - \int \rho u_t \cdot \nabla u \cdot u_{tt} \, dx - \int \rho u \cdot \nabla u_t \cdot u_{tt} \, dx \\
 &\leq \| \sqrt{\rho} u_{tt} \|_{L^2} \| u_t \|_{L^4} \| \nabla u \|_{L^4} + \| \sqrt{\rho} u_{tt} \|_{L^2} \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} \\
 &\leq \delta \| \sqrt{\rho} u_{tt} \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2,
 \end{aligned} \tag{96}$$

$$L_5 = - \int P_{tt} \operatorname{div} u_t \, dx \leq \| P_{tt} \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2. \tag{97}$$

At last, integrating (92) over time $(0, T)$, and inserting estimates (93)–(97), we have

$$\begin{aligned}
 &\int (\mu | \nabla u_t |^2 + (\lambda + \mu) | \operatorname{div} u_t |^2) \, dx + \int_0^T \int \rho u_{tt}^2 \, dx \, ds \\
 &\leq C + \int_0^T C (1 + \| \nabla u_t \|_{L^2}^2) \| \nabla u_t \|_{L^2}^2 \, ds,
 \end{aligned} \tag{98}$$

from which, together with the Gronwall inequality, one obtains (85) immediately. This completes the proof of Lemma 3.8. \square

Lemma 3.9 *Let (ρ, u) be a classical solution of (1), (3)–(4) on $\Omega \times (0, T]$, under the condition of Theorem 1.1, the following estimates hold:*

$$\sup_{0 \leq t \leq T} (\| \rho \|_{H^3} + \| P \|_{H^3}) \leq C(T), \tag{99}$$

$$\sup_{0 \leq t \leq T} (\| \nabla u_t \|_{L^2}^2 + \| \nabla u \|_{H^2}^2) + \int_0^T (\| \nabla u_t \|_{H^1}^2 + \| \nabla u \|_{H^3}^2) \, ds \leq C(T). \tag{100}$$

Proof It follows from Lemma 3.8 that

$$\begin{aligned}
 \| \nabla(\rho \dot{u}) \|_{L^2} &\leq \| \nabla \rho u_t \|_{L^2} + \| \rho \nabla u_t \|_{L^2} + \| \nabla \rho u \cdot \nabla u \|_{L^2} + \| \rho \nabla u \cdot \nabla u \|_{L^2} \\
 &\quad + \| \rho u \cdot \nabla^2 u \|_{L^2} \\
 &\leq \| \nabla \rho \|_{L^4} \| u_t \|_{L^4} + \| \nabla u_t \|_{L^2} + \| \nabla \rho \|_{L^4} \| u \|_{L^\infty} \| \nabla u \|_{L^4} + \| \nabla u \|_{L^4}^2 \\
 &\quad + \| u \|_{L^\infty} \| \nabla^2 u \|_{L^2}^2 \\
 &\leq C,
 \end{aligned} \tag{101}$$

which together with Lemma 3.6 gives

$$\sup_{0 \leq t \leq T} \| \rho \dot{u} \|_{H^1} \leq C. \tag{102}$$

The standard H^1 estimate for elliptic system (28) yields

$$\| \nabla^2 u \|_{H^1} \leq C (\| \rho \dot{u} \|_{H^1} + \| \nabla P \|_{H^1}) \leq C. \tag{103}$$

Then, as a consequence of (67) and (103), we have

$$\sup_{0 \leq t \leq T} \| \nabla u \|_{H^2} \leq C. \tag{104}$$

Moreover, the standard L^2 -estimate for elliptic system (28) and Lemma 3.8 yield

$$\begin{aligned} \|\nabla^2 u_t\|_{L^2} &\leq \|\rho u_{tt}\|_{L^2} + \|\rho_t u_t\|_{L^2} + \|\rho_t u \cdot \nabla u\|_{L^2} + \|\rho u_t \cdot \nabla u\|_{L^2} \\ &\quad + \|\rho u \cdot \nabla u_t\|_{L^2} + \|\nabla P_t\|_{L^2} \\ &\leq \|\rho u_{tt}\|_{L^2} + \|\rho_t\|_{L^4} \|u_t\|_{L^4} + \|\rho_t\|_{L^4} \|u\|_{L^\infty} \|\nabla u\|_{L^4} \\ &\quad + \|u_t\|_{L^4} \|\nabla u\|_{L^4} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} + \|\nabla P_t\|_{L^2} \\ &\leq \|\rho u_{tt}\|_{L^2} + C, \end{aligned} \tag{105}$$

which together with (85) implies

$$\int_0^T \|\nabla u_t\|_{H^1}^2 ds \leq C. \tag{106}$$

On the other hand, applying the standard H^2 -estimate for the elliptic system (28) again leads to

$$\begin{aligned} \|\nabla^2 u\|_{H^2} &\leq C(\|\rho \dot{u}\|_{H^2} + \|\nabla P\|_{H^2}) \\ &\leq C(\|\nabla^2(\rho u_t)\|_{L^2} + \|\nabla^2(\rho u \cdot \nabla u)\|_{L^2} + \|\nabla^3 P\|_{L^2}), \end{aligned} \tag{107}$$

where

$$\begin{aligned} \|\nabla^2(\rho u_t)\|_{L^2} &\leq C(\|\nabla^2 \rho u_t\|_{L^2} + \|\nabla \rho \nabla u_t\|_{L^2} + \|\nabla^2 u_t\|_{L^2}) \\ &\leq C(\|\nabla^2 \rho\|_{L^2} \|u_t\|_{L^\infty} + \|\nabla \rho\|_{L^4} \|\nabla u_t\|_{L^4} + \|\nabla^2 u_t\|_{L^2}) \\ &\leq C(\|\nabla^2 \rho\|_{L^2} \|\nabla u_t\|_{H^1} + \|\nabla \rho\|_{H^1} \|\nabla u_t\|_{H^1} + \|\nabla^2 u_t\|_{L^2}) \\ &\leq C\|\nabla u_t\|_{H^1} \end{aligned} \tag{108}$$

and

$$\begin{aligned} \|\nabla^2(\rho u \cdot \nabla u)\|_{L^2} &\leq C(\|\nabla^2 \rho u \cdot \nabla u\|_{L^2} + \|\nabla^2 u \cdot \nabla u\|_{L^2} + \|u \cdot \nabla^3 u\|_{L^2} \\ &\quad + \|\nabla \rho \nabla u \cdot \nabla u\|_{L^2} + \|\nabla \rho u \cdot \nabla^2 u\|_{L^2}) \\ &\leq C(\|\nabla^2 \rho u \cdot \nabla u\|_{L^2} + \|\nabla^2 u \cdot \nabla u\|_{L^2} + \|u \cdot \nabla^3 u\|_{L^2} \\ &\quad + \|\nabla \rho \nabla u \cdot \nabla u\|_{L^2} + \|\nabla \rho u \cdot \nabla^2 u\|_{L^2}) \\ &\leq C(\|\nabla^2 \rho\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^\infty} + \|\nabla^2\|_{L^4} \|\nabla u\|_{L^4} \\ &\quad + \|u\|_{L^\infty} \|\nabla^3 u\|_{L^2} + \|\nabla \rho\|_{L^6} \|\nabla u\|_{L^6}^2 \\ &\quad + \|\nabla \rho\|_{L^4} \|u\|_{L^\infty} \|\nabla^2 u\|_{L^4}) \\ &\leq C. \end{aligned} \tag{109}$$

In order to estimate the third term on the right-hand side of (107), applying ∇^3 to (82) and integrating the resulting equation over Ω , we obtain

$$\begin{aligned}
 \frac{d}{dt} \|\nabla^3 P\|_{L^2}^2 &\leq C(\|\nabla^3 u \cdot \nabla P\|_{L^2} + \|\nabla u \cdot \nabla^3 P\|_{L^2} + \|\nabla^2 u \cdot \nabla^2 P\|_{L^2} \\
 &\quad + \|\nabla^4 u P\|_{L^2}) \\
 &\leq C(\|\nabla^3 u\|_{L^2} \|\nabla P\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\nabla^3 P\|_{L^2} + \|\nabla^2 u\|_{L^4} \|\nabla^2 P\|_{L^4} \\
 &\quad + \|\nabla^4 u\|_{L^2}) \\
 &\leq C(\|\nabla^3 P\|_{L^2} + \|\nabla^4 u\|_{L^2}) \\
 &\leq C(\|\nabla^3 P\|_{L^2} + \|\nabla^2(\rho \dot{u})\|_{L^2}) \\
 &\leq C(\|\nabla^3 P\|_{L^2} + \|\nabla^2(\rho u_t)\|_{L^2} + \|\nabla^2(u \cdot \nabla u)\|_{L^2}) \\
 &\leq C(\|\nabla^3 P\|_{L^2} + \|\nabla^2 \rho\|_{L^2} \|u_t\|_{L^\infty} + \|\nabla^2 u_t\|_{L^2} + \|\nabla \rho\|_{L^4} \|\nabla u_t\|_{L^4} \\
 &\quad + \|\nabla^2 u \cdot \nabla u\|_{L^2} + \|u \cdot \nabla^3 u\|_{L^2} + \|\nabla u \cdot \nabla^2 u\|_{L^2}) \\
 &\leq C(\|\nabla^3 P\|_{L^2} + \|\nabla u_t\|_{H^1} + \|\nabla \rho\|_{H^1} \|\nabla u_t\|_{H^1} + \|\nabla^2 u\|_{L^4} \|\nabla u\|_{L^4} \\
 &\quad + \|u\|_{L^\infty} \|\nabla^3 u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla^2 u\|_{L^2}) \\
 &\leq C(\|\nabla^3 P\|_{L^2} + \|\nabla u_t\|_{H^1} + 1),
 \end{aligned} \tag{110}$$

which, together with the Gronwall inequality (106), implies that

$$\sup_{0 \leq t \leq T} \|\nabla^3 P\|_{L^2} \leq C. \tag{111}$$

Taking (106)–(111) into consideration, we have

$$\int_0^T \|\nabla u\|_{H^3}^2 ds \leq C. \tag{112}$$

It is easy to check that similar arguments work for ρ by using (112). Hence the proof of Lemma 3.9 is completed. \square

Lemma 3.10 *Let (ρ, u) be a classical solution of (1), (3)–(4) on $\Omega \times (0, T]$, under the condition of Theorem 1.1, the following estimates hold:*

$$\sup_{0 \leq t \leq T} (\|\nabla u_t\|_{H^1}^2 + \|\nabla^4 u\|_{L^2}^2) + \int_0^T \|\nabla u_{tt}\|_{H^1}^2 ds \leq C(T). \tag{113}$$

Proof Differentiating (1)₂ with respect to t twice, one can get

$$\begin{aligned}
 &\rho u_{ttt} + \rho u \cdot \nabla u_{tt} - \mu \Delta u_{tt} - (\mu + \lambda) \nabla \operatorname{div} u_{tt} \\
 &= 2 \operatorname{div}(\rho u) u_{tt} + \operatorname{div}(\rho u)_t u_t - 2(\rho u)_t \cdot \nabla u_t - (\rho_{tt} u + 2\rho_t u_t) \cdot \nabla u \\
 &\quad - \rho u_{tt} \cdot \nabla u - \nabla P_{tt}.
 \end{aligned} \tag{114}$$

Multiplying (114) by u_{tt} and then integrating the resulting equation over Ω , after integration by parts, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u_{tt}|^2 dx + \int (\mu |\nabla u_{tt}|^2 + (\mu + \lambda) |\operatorname{div} u_{tt}|^2) dx \\ &= -4 \int \rho u u_{tt} \cdot \nabla u_{tt} dx - \int (\rho u)_t \cdot (\nabla(u_t \cdot u_{tt}) + 2 \nabla u_t \cdot u_{tt}) dx \\ & \quad - \int (\rho_{tt} u + 2 \rho_t u_t) \cdot \nabla u \cdot u_{tt} dx - \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx + \int P_{tt} \operatorname{div} u_{tt} dx \\ & := \sum_{i=1}^5 M_i. \end{aligned} \tag{115}$$

Next, we estimate each term $M_i, i=1,2,3,4,5$, as follows:

$$\begin{aligned} M_1 &= -4 \int \rho u u_{tt} \cdot \nabla u_{tt} dx \leq C \|u\|_{L^\infty} \|\sqrt{\rho} u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \\ & \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C \|\sqrt{\rho} u_{tt}\|_{L^2}^2. \end{aligned} \tag{116}$$

It follows from Lemma 3.7, Lemma 3.8, and Lemma 3.9 that

$$\begin{aligned} M_2 &= - \int (\rho u)_t \cdot (\nabla(u_t \cdot u_{tt}) + 2 \nabla u_t \cdot u_{tt}) dx \\ & \leq C (\|\rho_t u\|_{L^4} + \|\rho u_t\|_{L^4}) (\|\nabla u_t\|_{L^2} \|u_{tt}\|_{L^4} + \|u_t\|_{L^4} \|\nabla u_{tt}\|_{L^2}) \\ & \leq C (\|\rho_t\|_{H^1} + \|u_t\|_{H^1}) (\|u_{tt}\|_{L^2}^{\frac{1}{2}} \|\nabla u_{tt}\|_{L^2}^{\frac{1}{2}} + \|u_t\|_{H^1} \|\nabla u_{tt}\|_{L^2}) \\ & \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C, \end{aligned} \tag{117}$$

$$\begin{aligned} M_3 &= - \int (\rho_{tt} u + 2 \rho_t u_t) \cdot \nabla u \cdot u_{tt} dx \\ & \leq C (\|\rho_{tt}\|_{L^2} \|u\|_{L^\infty} + \|\rho_t\|_{L^4} \|u_t\|_{L^4}) \|\nabla u\|_{L^4} \|u_{tt}\|_{L^4} \\ & \leq C (\|\rho_{tt}\|_{L^2} \|u\|_{H^2} + \|\rho_t\|_{H^1} \|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}}) \|\nabla u\|_{H^1} \|u_{tt}\|_{L^2}^{\frac{1}{2}} \|\nabla u_{tt}\|_{L^2}^{\frac{1}{2}} \\ & \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C \|\rho_{tt}\|_{L^2}^2, \end{aligned} \tag{118}$$

$$\begin{aligned} M_4 + M_5 &= - \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx + \int P_{tt} \operatorname{div} u_{tt} dx \\ & \leq C (\|\sqrt{\rho} u_{tt}\|_{L^2} \|\nabla u\|_{L^4} \|u_{tt}\|_{L^4} + \|P_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2}) \\ & \leq C (\|\sqrt{\rho} u_{tt}\|_{L^2} \|\nabla u\|_{H^1} \|u_{tt}\|_{L^2}^{\frac{1}{2}} \|\nabla u_{tt}\|_{L^2}^{\frac{1}{2}} + \|P_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2}) \\ & \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C (\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2). \end{aligned} \tag{119}$$

Substituting (116)–(119) into (114) and choosing δ suitably small, we get

$$\begin{aligned} & \frac{d}{dt} \int \rho |u_{tt}|^2 dx + \int (\mu |\nabla u_{tt}|^2 + (\mu + \lambda) |\operatorname{div} u_{tt}|^2) dx \\ & \leq C (\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2 + 1), \end{aligned} \tag{120}$$

then, integrating the inequality (120) over $(0, T)$, together with (84) and the Gronwall inequality, yields

$$\sup_{0 \leq t \leq T} \int \rho |u_{tt}|^2 dx + \int_0^T \int |\nabla u_{tt}|^2 dx ds \leq C(T), \quad (121)$$

then (113) follows from (85) and (105). We have finished the proof of Lemma 3.10. \square

Finally, by using the continuity argument, we can extend the local classical solution to a global one, and thus Theorem 1.1 is proved.

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