Qiu et al. *Advances in Difference Equations* (2020) 2020:231
https://doi.org/10.1186/s13662-020-02678-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of nonoscillatory solutions tending to zero of third-order neutral dynamic equations on time scales

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Abstract

We study the existence of nonoscillatory solutions tending to zero of a class of third-order nonlinear neutral dynamic equations on time scales by employing Krasnoselskii's fixed point theorem. Two examples are given to illustrate the significance of the conclusions.

MSC: 34K11; 34N05; 39A10; 39A13

Keywords: Nonoscillatory solution; Neutral dynamic equation; Third-order; Time scale

1 Introduction

In this paper, we consider the existence of nonoscillatory solutions tending to zero of a class of third-order nonlinear neutral dynamic equations

$$\Bigl(r_1(t)\Bigl(r_2(t)\bigl(x(t)+p(t)x\bigl(g(t)\bigr)\bigr)^{\Delta}\Bigr)^{\Delta}\Bigr)^{\Delta}+f\bigl(t,x\bigl(h(t)\bigr)\bigr)=0 \tag{1.1}$$

on a time scale $\mathbb{T}$ satisfying $\sup\mathbb{T}=\infty$, where $t\in[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$ with $t_0\in\mathbb{T}$. The following conditions are assumed to hold throughout this paper:

(C1) $r_1,r_2\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$;

(C2) $p\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $\lim_{t\to\infty}p(t)=p_0$, where $|p_0|<1$;

(C3) $g,h\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $g(t)\le t$, and $\lim_{t\to\infty}g(t)=\lim_{t\to\infty}h(t)=\infty$; if $p_0\in(-1,0]$, then there exists a sequence $\{c_k\}_{k\ge0}$ such that $\lim_{k\to\infty}c_k=\infty$ and $g(c_{k+1})=c_k$;

(C4) $f\in C([t_0,\infty)_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$, $f(t,x)$ is nondecreasing in x, and $xf(t,x)>0$ for $x\neq0$.

The details of the theory of time scales can be found in [1–4, 8, 9] and hence they are omitted here. In recent years, the existence of nonoscillatory solutions of neutral dynamic equations on time scales has been studied successively in [6, 7, 11, 13–17]. Zhu and Wang [17] were concerned with a first-order neutral dynamic equation

$$\bigl[x(t)+p(t)x\bigl(g(t)\bigr)\bigr]^{\Delta}+f\bigl(t,x\bigl(h(t)\bigr)\bigr)=0.$$

Afterward, Deng and Wang [6] and Gao and Wang [7] investigated a second-order neutral dynamic equation

$$\left[r(t)\big(x(t)+p(t)x\big(g(t)\big)\big)^{\Delta}\right]^{\Delta}+f\big(t,x\big(h(t)\big)\big)=0$$

under the different assumptions $\int_{t_0}^{\infty} 1/r(t)\Delta t=\infty$ and $\int_{t_0}^{\infty} 1/r(t)\Delta t<\infty$, respectively. Furthermore, Qiu [11] studied (1.1) with $\int_{t_0}^{\infty} 1/r_1(t)\Delta t=\int_{t_0}^{\infty} 1/r_2(t)\Delta t=\infty$, whereas other cases of the convergence and divergence of $\int_{t_0}^{\infty} 1/r_1(t)\Delta t$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t$ were considered in [14–16]. Similar sufficient conditions for the existence of nonoscillatory solutions tending to zero of neutral dynamic equations have been presented. However, it is not easy to find a necessary condition for equations to have a nonoscillatory solution tending to zero asymptotically.

Mojsej and Tartaľová [10] studied the asymptotic behavior of nonoscillatory solutions to a third-order differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)'+q(t)f\big(x(t)\big)=0.$$

They stated some necessary and sufficient conditions ensuring the existence of nonoscillatory solutions tending to zero. Motivated by [10], Qiu [12] studied the existence of nonoscillatory solutions tending to zero of (1.1) under the conditions $0\le p_0<1$ and $g(t)\ge t$. The conclusions extend and improve the results reported in the papers [11, 14–16].

The purpose of this paper is to further discuss the same problem of (1.1) with $|p_0|<1$ and $g(t)\le t$. The existence of nonoscillatory solutions tending to zero of (1.1) is established by employing Krasnoselskii's fixed point theorem. Finally, two examples are presented to show the versatility of the conclusions.

2 Auxiliary results

Let $\mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ denote the Banach space of all bounded continuous functions mapping $[T_0,\infty)_{\mathbb{T}}$ into $\mathbb{R}$ with the norm $\|x\|=\sup_{t\in[T_0,\infty)_{\mathbb{T}}}|x(t)|$. For the sake of convenience, we define

$$z(t)=x(t)+p(t)x\big(g(t)\big), \tag{2.1}$$

and state the following lemmas which will be used in the sequel.

Lemma 2.1 (see [5, Krasnoselskii's fixed point theorem]) *Let X be a Banach space and Ω be a bounded, convex, and closed subset of X. If there exist two operators $U,V:\Omega\to X$ such that $Ux+Vy\in\Omega$ for all $x,y\in\Omega$, where U is a contraction mapping and V is completely continuous, then $U+V$ has a fixed point in Ω.*

Lemma 2.2 *Suppose that x is an eventually positive solution of (1.1) and there exists a constant $a\ge 0$ such that $\lim_{t\to\infty} z(t)=a$. Then*

$$\lim_{t\to\infty} x(t)=\frac{a}{1+p_0}.$$

The proof is similar to those of [6, Lemma 2.3], [7, Theorem 1], and [17, Theorem 7], and thus is omitted.

3 Main results

In this section, our existence criteria for eventually positive solutions tending to zero as $t \to \infty$ of (1.1) are established by employing Krasnoselskii's fixed point theorem.

Theorem 3.1 *Assume that*

$$H_1(t_0) < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \int_{t_0}^{s} \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s < \infty, \tag{3.1}$$

where

$$H_1(t) = \int_t^{\infty} \frac{\Delta v}{r_2(v)},$$

which satisfies $\lim_{t\to\infty} H_1(g(t))/H_1(t) = 1$. *Then* (1.1) *has an eventually positive solution x with* $\lim_{t\to\infty} x(t) = 0$, *where* $r_2 z^{\Delta}$ *and* $r_1(r_2 z^{\Delta})^{\Delta}$ *are both eventually negative.*

Proof Suppose that (3.1) holds. There will be two cases to be considered.

Case (i). $0 \le p_0 < 1$. Take p_1 such that $p_0 < p_1 < (1 + 4p_0)/5 < 1$. When $p_0 > 0$, choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\begin{aligned} &p(t) > 0, \qquad \frac{5p_1 - 1}{4} \le p(t) \le p_1 < 1, \qquad p(t)\frac{H_1(g(t))}{H_1(t)} \ge \frac{5p_1 - 1}{4}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \\ &\int_{T_0}^{\infty} \int_{T_0}^{s} \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s \le \frac{1 - p_1}{4}. \end{aligned} \tag{3.2}$$

When $p_0 = 0$, choose p_1 such that $|p(t)| \le p_1 \le 1/13$ for $t \in [T_0, \infty)_{\mathbb{T}}$. In view of (C3), there exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define

$$\Omega_1 = \bigl\{ x \in \mathrm{BC}[T_0, \infty)_{\mathbb{T}} : H_1(t) \le x(t) \le 2H_1(t) \bigr\}.$$

It is easy to prove that Ω_1 is a bounded, convex, and closed subset of $\mathrm{BC}[T_0, \infty)_{\mathbb{T}}$. Define U_1 and V_1: $\Omega_1 \to \mathrm{BC}[T_0, \infty)_{\mathbb{T}}$ as follows:

$$\begin{aligned} (U_1x)(t) &= \begin{cases} (U_1x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3p_1 H_1(t)/2 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases} \\ (V_1x)(t) &= \begin{cases} (V_1x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3H_1(t)/2 \\ \quad + \int_t^{\infty} \int_{T_1}^{v} \int_{T_1}^{s} f(u, x(h(u)))/(r_1(s) r_2(v)) \Delta u \Delta s \Delta v, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \end{aligned} \tag{3.3}$$

We can prove that U_1 and V_1 satisfy all conditions in Lemma 2.1. The proof is expatiatory but similar to those of [6, Theorem 2.5], [7, Theorem 2], [12, Theorem 3.1], and [17, Theorem 8]; so we omit it here. By virtue of Lemma 2.1, there exists an $x \in \Omega_1$ such that $(U_1 + V_1)x = x$. Hence, for $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1+p_1)}{2} H_1(t) - p(t)x\bigl(g(t)\bigr) + \int_t^\infty \int_{T_1}^v \int_{T_1}^s \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

Since

$$\int_t^\infty \int_{T_1}^v \int_{T_1}^s \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le H_1(t) \int_{T_1}^\infty \int_{T_1}^s \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s$$

and

$$\lim_{t\to\infty} H_1(t) \int_{T_1}^\infty \int_{T_1}^s \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s = 0,$$

we arrive at $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$ with the help of Lemma 2.2. For $t \in [T_1, \infty)_{\mathbb{T}}$, we obtain

$$r_2(t)z^\Delta(t) = -\frac{3(1+p_1)}{2} - \int_{T_1}^t \int_{T_1}^s \frac{f(u, x(h(u)))}{r_1(s)} \Delta u \Delta s < 0$$

and

$$r_1(t)\bigl(r_2(t)z^\Delta(t)\bigr)^\Delta = -\int_{T_1}^t f\bigl(u, x\bigl(h(u)\bigr)\bigr) \Delta u < 0.$$

Case (ii). $-1 < p_0 < 0$. Take p_1 satisfying $-p_0 < p_1 < (1-4p_0)/5 < 1$. Choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that (3.2) holds and

$$p(t) < 0, \qquad \frac{5p_1 - 1}{4} \le -p(t) \le p_1 < 1, \qquad -p(t)\frac{H_1(g(t))}{H_1(t)} \ge \frac{5p_1 - 1}{4}, \quad t \in [T_0, \infty)_{\mathbb{T}}.$$

Proceeding as in the proof of Case (i), define V_1 as in (3.3) and U_1' on Ω_1 as follows:

$$\bigl(U_1'x\bigr)(t) = \begin{cases} (U_1'x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ -3p_1 H_1(t)/2 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly, there exists an $x \in \Omega_1$ such that $(U_1' + V_1)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1-p_1)}{2} H_1(t) - p(t)x\bigl(g(t)\bigr) + \int_t^\infty \int_{T_1}^v \int_{T_1}^s \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v$$

and we arrive at the same conclusions as in Case (i). This completes the proof. □

Theorem 3.2 *Assume that*

$$H_1(t_0) = \infty \quad \textit{or} \quad \int_{t_0}^\infty \int_{t_0}^v \frac{1}{r_1(s)r_2(v)} \Delta s \Delta v = \infty.$$

Then (1.1) *has no eventually positive solutions* x *satisfying that* $r_2 z^\Delta$ *and* $r_1(r_2 z^\Delta)^\Delta$ *are both eventually negative.*

Proof Suppose that x is an eventually positive solution of (1.1), and there exists a $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$r_2(t)z^{\Delta}(t) < 0, \qquad r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta} < 0, \quad t \in [T_0,\infty)_{\mathbb{T}}.$$

From (C3), there exists a $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $h(t) \geq T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Integrating (1.1) from T_1 to s, $s \in [\sigma(T_1),\infty)_{\mathbb{T}}$, by (C4) we obtain

$$r_1(s)\bigl(r_2(s)z^{\Delta}(s)\bigr)^{\Delta} - r_1(T_1)\bigl(r_2(T_1)z^{\Delta}(T_1)\bigr)^{\Delta} = -\int_{T_1}^{s} f\bigl(u,x\bigl(h(u)\bigr)\bigr)\Delta u < 0,$$

which yields

$$\bigl(r_2(s)z^{\Delta}(s)\bigr)^{\Delta} < \frac{r_1(T_1)(r_2(T_1)z^{\Delta}(T_1))^{\Delta}}{r_1(s)}. \tag{3.4}$$

Integrating (3.4) from T_1 to v, $v \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we get

$$r_2(v)z^{\Delta}(v) - r_2(T_1)z^{\Delta}(T_1) < r_1(T_1)\bigl(r_2(T_1)z^{\Delta}(T_1)\bigr)^{\Delta}\int_{T_1}^{v}\frac{1}{r_1(s)}\Delta s$$

or

$$z^{\Delta}(v) < \frac{r_2(T_1)z^{\Delta}(T_1)}{r_2(v)} + \frac{r_1(T_1)(r_2(T_1)z^{\Delta}(T_1))^{\Delta}}{r_2(v)}\int_{T_1}^{v}\frac{1}{r_1(s)}\Delta s. \tag{3.5}$$

Integrating (3.5) from T_1 to t, $t \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} z(t) &< z(T_1) + r_2(T_1)z^{\Delta}(T_1)\int_{T_1}^{t}\frac{1}{r_2(v)}\Delta v \\ &\quad + r_1(T_1)\bigl(r_2(T_1)z^{\Delta}(T_1)\bigr)^{\Delta}\int_{T_1}^{t}\int_{T_1}^{v}\frac{1}{r_1(s)r_2(v)}\Delta s\Delta v. \end{aligned}$$

Letting $t \to \infty$, we have $z(t) \to -\infty$. From (2.1), it follows that $p_0 \in (-1,0]$, and then there exist a $T_2 \in [T_1,\infty)_{\mathbb{T}}$ and a p_1 with $-p_0 < p_1 < 1$ such that $z(t) < 0$ or

$$x(t) < -p(t)x\bigl(g(t)\bigr) \leq p_1 x\bigl(g(t)\bigr), \quad t \in [T_2,\infty)_{\mathbb{T}}.$$

By (C3), choose some positive integer k_0 such that $c_k \in [T_2,\infty)_{\mathbb{T}}$ for all $k \geq k_0$. Then, for any $k \geq k_0 + 1$, we have

$$x(c_k) < p_1 x(c_{k-1}) < p_1^2 x(c_{k-2}) < \cdots < p_1^{k-k_0}x(c_{k_0}).$$

This inequality implies that $\lim_{k\to\infty} x(c_k) = 0$. It follows from (2.1) that $\lim_{k\to\infty} z(c_k) = 0$ which contradicts $z(t) \to -\infty$ as $t \to \infty$. The proof is complete. □

Theorem 3.3 *Assume that*

$$H_2(t_0) < \infty \quad \text{and} \quad \int_{t_0}^{\infty} f\bigl(t, 2H_2\bigl(h(t)\bigr)\bigr)\Delta t < \infty, \tag{3.6}$$

where

$$H_2(t) = \int_t^\infty \int_v^\infty \frac{1}{r_1(s)r_2(v)} \Delta s \Delta v,$$

which satisfies $\lim_{t\to\infty} H_2(g(t))/H_2(t) = 1$. *Then* (1.1) *has an eventually positive solution* x *with* $\lim_{t\to\infty} x(t) = 0$, *where* $r_2 z^\Delta$ *is eventually negative and* $r_1(r_2 z^\Delta)^\Delta$ *is eventually positive.*

Proof Suppose that (3.6) holds. There are two cases to be considered.

Case (i). $0 \le p_0 < 1$. Take p_1 as in Case (i) of Theorem 3.1. When $p_0 > 0$, choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\begin{aligned} &p(t) > 0, \qquad \frac{5p_1 - 1}{4} \le p(t) \le p_1 < 1, \qquad p(t)\frac{H_2(g(t))}{H_2(t)} \ge \frac{5p_1 - 1}{4}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \\ &\int_{T_0}^\infty f\big(t, 2H_2\big(h(t)\big)\big) \Delta t \le \frac{1 - p_1}{4}. \end{aligned}$$

When $p_0 = 0$, choose p_1 such that $|p(t)| \le p_1 \le 1/13$ for $t \in [T_0, \infty)_{\mathbb{T}}$. By virtue of (C3), there exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define

$$\Omega_2 = \big\{x \in \mathrm{BC}[T_0, \infty)_{\mathbb{T}} : H_2(t) \le x(t) \le 2H_2(t)\big\} \tag{3.7}$$

and U_2, V_2: $\Omega_2 \to \mathrm{BC}[T_0, \infty)_{\mathbb{T}}$ as follows:

$$\begin{aligned} &(U_2 x)(t) = \begin{cases} (U_2 x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3p_1 H_2(t)/2 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases} \\ &(V_2 x)(t) = \begin{cases} (V_2 x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3H_2(t)/2 \\ \quad + \int_t^\infty \int_v^\infty \int_s^\infty f(u, x(h(u)))/(r_1(s)r_2(v)) \Delta u \Delta s \Delta v, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \end{aligned} \tag{3.8}$$

The remainder of the proof is similar to that of Theorem 3.1 and so is omitted. By Lemma 2.1, there exists an $x \in \Omega_2$ such that $(U_2 + V_2)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1 + p_1)}{2} H_2(t) - p(t)x\big(g(t)\big) + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

Since

$$\int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le H_2(t) \int_{T_1}^\infty f\big(u, 2H_2\big(h(u)\big)\big) \Delta u$$

and

$$\lim_{t\to\infty} H_2(t) \int_{T_1}^\infty f\big(u, 2H_2\big(h(u)\big)\big) \Delta u = 0,$$

we get $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$ due to Lemma 2.2. For $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$r_2(t)z^{\Delta}(t) = -\frac{3(1+p_1)}{2}\int_t^{\infty}\frac{1}{r_1(s)}\Delta s - \int_t^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)}\Delta u\Delta s < 0$$

and

$$r_1(t)\big(r_2(t)z^{\Delta}(t)\big)^{\Delta} = \frac{3(1+p_1)}{2} + \int_t^{\infty} f\big(u,x\big(h(u)\big)\big)\Delta u > 0.$$

Case (ii). $-1 < p_0 < 0$. Introduce $\mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ and its subset Ω_2 as in (3.7). Define V_2 as in (3.8) and U_2' on Ω_2 as follows:

$$\big(U_2'x\big)(t) = \begin{cases} (U_2'x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ -3p_1H_2(t)/2 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

The following proof is similar to Case (i) and we omit it here. There exists an $x \in \Omega_2$ such that $(U_2' + V_2)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1-p_1)}{2}H_2(t) - p(t)x\big(g(t)\big) + \int_t^{\infty}\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v$$

and obtain the similar results as in Case (i). This completes the proof. □

4 Examples

In this section, two examples are presented to show the applications of our results. The first example is given to illustrate Theorem 3.1.

Example 4.1 Let $\mathbb{T} = \bigcup_{n=1}^{\infty}[3n-2,3n]$. For $t \in [4,\infty)_{\mathbb{T}}$, consider

$$\left(t^4\left(t^2\left(x(t) - \frac{t-1}{2t}x(t-3)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + tx^3(t) + \frac{x(t)}{t^2} = 0. \tag{4.1}$$

Here, $r_1(t) = t^4$, $r_2(t) = t^2$, $p(t) = -(t-1)/(2t)$, $g(t) = t-3$, $h(t) = t$, and $f(t,x) = tx^3 + x/t^2$. It is obvious that the coefficients of (4.1) satisfy (C1)–(C4). Since

$$\int_{t_0}^{\infty}\frac{\Delta t}{r_2(t)} = \int_4^{\infty}\frac{\Delta t}{t^2} < 1$$

and

$$H_1(t) = \int_t^{\infty}\frac{\Delta v}{v^2} < 1,$$

we obtain

$$f\big(u,2H_1\big(h(u)\big)\big) = u\cdot\big(2H_1(u)\big)^3 + \frac{2H_1(u)}{u^2} < 8u + \frac{2}{u^2} < 9u$$

and

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{f(u,2H_1(h(u)))}{r_1(s)}\Delta u\Delta s < 9\int_4^{\infty}\int_4^{s}\frac{u}{s^4}\Delta u\Delta s$$
$$< 9\int_4^{\infty}\int_4^{s}\frac{1}{s^3}\Delta u\Delta s < 9\int_4^{\infty}\frac{1}{s^2}\Delta s < \infty.$$

By Theorem 3.1, we see that (4.1) has an eventually positive solution x satisfying $\lim_{t\to\infty} x(t) = 0$, where $r_2 z^{\Delta}$ and $r_1(r_2 z^{\Delta})^{\Delta}$ are both eventually negative.

Now, we give the second example to demonstrate Theorems 3.2 and 3.3.

Example 4.2 Let $\mathbb{T} = \bigcup_{n=1}^{\infty}[2^n - 1, 2^n]$. For $t \in [3,\infty)_{\mathbb{T}}$, consider

$$\left(t^3\left(t\left(\left(\frac{3}{2}+\frac{1}{t}\right)x(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + \frac{1}{t^2}x\left(\frac{t}{2}\right) = 0. \tag{4.2}$$

Here, $r_1(t) = t^3$, $r_2(t) = t$, $p(t) = 1/2 + 1/t$, $g(t) = t$, $h(t) = t/2$, and $f(t,x) = x/t^2$. It is obvious that the coefficients of (4.2) satisfy (C1)–(C4). Since

$$\int_{t_0}^{\infty}\frac{\Delta t}{r_2(t)} = \int_3^{\infty}\frac{\Delta t}{t} = \infty,$$

in terms of Theorem 3.2, we deduce that (4.2) has no eventually positive solutions x satisfying $\lim_{t\to\infty} x(t) = 0$, where $r_2 z^{\Delta}$ and $r_1(r_2 z^{\Delta})^{\Delta}$ are both eventually negative. However, we have

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\frac{1}{r_1(s)r_2(v)}\Delta s\Delta v = \int_3^{\infty}\int_v^{\infty}\frac{1}{s^3 v}\Delta s\Delta v < \infty$$

and

$$H_2(t) = \int_t^{\infty}\int_v^{\infty}\frac{1}{s^3 v}\Delta s\Delta v.$$

Furthermore, there exists a constant $M > 0$ such that

$$f\left(t,2H_2\left(h(t)\right)\right) = \frac{2}{t^2}H_2\left(\frac{t}{2}\right) \le \frac{2M}{t^2}$$

and

$$\int_{t_0}^{\infty} f\left(t,2H_2\left(h(t)\right)\right)\Delta t \le 2M\int_3^{\infty}\frac{\Delta t}{t^2} < \infty.$$

By virtue of Theorem 3.3, we conclude that (4.2) has an eventually positive solution x satisfying $\lim_{t\to\infty} x(t) = 0$, where $r_2 z^{\Delta}$ is eventually negative and $r_1(r_2 z^{\Delta})^{\Delta}$ is eventually positive.

Acknowledgements
The authors express their sincere gratitude to the editors for the careful reading of the original manuscript and useful comments that helped to improve the presentation of the results and accentuate important details.

Funding
The research of the first author was supported by the National Natural Science Foundation of P. R. China (Grant No. 11671406) and Natural Science Program for Young Creative Talents of Innovation Enhancing College Project of Department of Education of Guangdong Province (Grant Nos. 2017GKQNCX111 and 2018-KJZX039). The research of the second author was supported by the Slovak Research and Development Agency (Grant No. APVV-18-0373). The research of the third author was supported by FGI 10-18 DIUMCE and PGI 03-2020 DIUMCE. The research of the fourth author was supported by the National Natural Science Foundation of P. R. China (Grant No. 61503171), China Postdoctoral Science Foundation (Grant No. 2015M582091), and Natural Science Foundation of Shandong Province (Grant No. ZR2016JL021).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

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Received: 27 March 2020 Accepted: 1 May 2020 Published online: 24 May 2020

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