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α-ψ-contractions and solutions of a q-fractional differential inclusion with three-point boundary value conditions via computational results

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Abstract

We review the existence of solutions for a three-point nonlinear q-fractional differential equation and also its related inclusion. In this way, we use α-ψ-contractions and multifunctions. Also, we provide two examples to illustrate our main results. Finally by providing some algorithms and tables, we give some numerical computations for the results.

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Keywords: α-ψ-contraction; q-fractional differential equation; Approximate endpoint property; Boundary value inclusion problem

1 Introduction

The subject of q-difference equations was introduced by Jackson in 1908 and 1910 [1, 2]. Later, some researchers reviewed q-difference equations [3–19]. On the other hand, there was published recently much contemporary work on integro-differential equations by using different views and fractional derivatives which young researchers could use as the main idea for their work (see, for example, [20–55]). It is notable that young researchers can consider this idea as a future direction for their work by using the numerical methodologies of [56, 57]. Also, they can use the idea for some applied modeling [58–61].

In 2012, Ahmad et al. investigated the fractional q-difference equation ${}^c\mathcal{D}_q^\alpha y(t) = f(t, y(t))$ with boundary conditions $\alpha_1 y(0) - \beta_1 \mathcal{D}_q y(0) = \gamma_1 y(\eta_1)$ and $\alpha_2 y(1) + \beta_2 \mathcal{D}_q y(1) = \gamma_2 y(\eta_2)$, where $0 \le t \le 1$, $1 < \alpha \le 2$, ${}^c\mathcal{D}_q^\alpha$ is the Caputo fractional q-derivative and $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ [7]. In 2015, Alsaedi et al. studied the fractional q-difference inclusion ${}^c\mathcal{D}_q^\nu y(t) \in \mathcal{F}(t, y(t))$ with nonlocal and sub-strip boundary conditions $y(0) = g(y)$ and $y(w) = b\int_\delta^1 y(s)\,\mathrm{d}_q s$, where $0 \le t \le 1$, $1 \le \nu < 2$, $0 < w < \delta < 1$ and ${}^c\mathcal{D}_q^\nu$ denotes the fractional q-derivative of Caputo type of order ν [8]. In 2019, Ntouyas and Samei investigated the existence of solutions for the multi-term nonlinear fractional q-integro-difference equation

$$ {}^c\mathcal{D}_q^\alpha x(t) = w\big(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), {}^c\mathcal{D}_q^{\beta_1} x(t), {}^c\mathcal{D}_q^{\beta_2} x(t), \dots, {}^c\mathcal{D}_q^{\beta_n} x(t)\big), $$

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

```
function p = powerfunction(a, b, n, q)
%Power Gamma (a-b)^(n)
s=1;
if n==0
    p=1
    else
    for k=1:n-1
        s=s*(a-b*q^k)/(a- b*q^(alpha+k));
    end
    p=a^alpha * s;
end
end
```

with boundary value conditions $x(0)+ax(1)=0$ and $x'(0)+bx'(1)=0$, where $t\in[0,1]$, $0<q<1$, $1<\alpha<2$, $\beta_i\in(0,1)$ with $i=1,2,\dots,n$, $a,b\neq-1$, the maps φ_j are defined by $(\varphi_j u)(t)=\int_0^t\gamma_j(t,s)u(s)\,\mathrm{d}_q s$ for $j=1,2$ and $w:[0,1]\times\mathbb{R}^{n+3}\to\mathbb{R}$ is a continuous mapping with respect to all variables [17].

We investigate the existence of solutions for the q-fractional differential equation

$$ {}^c\mathcal{D}_q^{\vartheta}y(t)=\Phi\bigl(t,y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t)\bigr) \tag{1}$$

with three-point boundary value conditions

$$\begin{cases} y(0)=0,\\ \mathcal{D}_q y(0)+{}^c\mathcal{D}_q^{\sigma}y(\nu)+\mathcal{D}_q^2 y(1)=0,\\ \mathcal{J}_q^{\kappa}y(0)+\mathcal{J}_q^{\kappa}y(\nu)+\mathcal{J}_q^{\kappa}y(1)=0,\end{cases} \tag{2}$$

where $0<q<1$, $0<t<1$, ${}^c\mathcal{D}_q^{\vartheta}$ denotes the fractional q-derivative of the Caputo type of order ϑ, $\vartheta\in(2,3]$, $0<\nu<1$, $1<\sigma<2$ and $\Phi:[0,1]\times\mathbb{R}^3$ is a continuous mapping. Let $\mathcal{J}_q^{\kappa}$ denote the fractional q-integral of the Riemann–Liouville type of order $\kappa>0$. Also, we review the existence of solutions for the q-fractional differential inclusion

$$ {}^c\mathcal{D}_q^{\vartheta}y(t)\in\mathcal{G}\bigl(t,y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t)\bigr), \tag{3}$$

with the three-point boundary value conditions

$$\begin{cases} y(0)=0,\\ \mathcal{D}_q y(0)+{}^c\mathcal{D}_q^{\sigma}y(\nu)+\mathcal{D}_q^2 y(1)=0,\\ \mathcal{J}_q^{\kappa}y(0)+\mathcal{J}_q^{\kappa}y(\nu)+\mathcal{J}_q^{\kappa}y(1)=0,\end{cases} \tag{4}$$

where $0<q<1$, $0<t<1$, $\mathcal{G}:[0,1]\times\mathbb{R}^3\to\mathcal{P}(\mathbb{R})$ is a compact set-valued map.

The paper is organized as follows: In Sect. 2, some basic definitions and applied results are presented. In Sect. 3, we state our main existence results and used techniques in this direction. Finally by using the Algorithms 1–5, Fig. 1 and Tables 1–3, two illustrative examples of the corresponding existence results are given in Sect. 4.

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

```
function g = qGamma(q, x, n)
%q-Gamma Function
p=1;
for k=0:n
    p=p*(1-q^(k+1))/(1- q^(x+k));
end;
g=p/(1-q)^(x-1);
end
```

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

```
function g = Dq(q, x, n, fun)
if x==0
    g=limit ((fun(x)-fun(q*x))/((1-q)*x),x,0);
    else
    g=(fun(x)-fun(q*x))/((1-q)*x);
end;
end
```

2 Preliminaries

Let $0 < q < 1$. The q-analogue of the power function $(a_1 - a_2)^n$ with $n \in \mathbb{N}_0$ defined by $(a_1 - a_2)^{(0)} = 1$ and $(a_1 - a_2)^{(n)} = \prod_{j=0}^{n-1}(a_1 - a_2 q^j)$, where $a_1, a_2 \in \mathbb{R}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ [2, 18]. Now, let ϑ be a real number. Then define

$$(a_1 - a_2)^{(\vartheta)} = a_1^{\vartheta} \prod_{j=0}^{\infty} \frac{a_1 - a_2 q^j}{a_1 - a_2 q^{\vartheta + j}},$$

with $a_1 \neq 0$. It is clear that, if $a_2 = 0$, then $a_1^{(\vartheta)} = a_1^{\vartheta}$ [2, 18]. For each real number ϑ, $[\vartheta]_q$ is defined by $[\vartheta]_q = \frac{1-q^{\vartheta}}{1-q}$ [2]. The q-Gamma function is defined by $\Gamma_q(\vartheta) = \frac{(1-q)^{(\vartheta-1)}}{(1-q)^{\vartheta-1}}$, where $\vartheta \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ [2, 18]. We show in Algorithm 2, a pseudo-code for estimating q-Gamma function. Also by definition of $[\vartheta]_q$, the property $\Gamma_q(\vartheta + 1) = [\vartheta]_q \Gamma_q(\vartheta)$ holds [2]. The definition of the q-derivative of a real-valued function y is given by $(\mathcal{D}_q y)(t) = \frac{y(t)-y(qt)}{(1-q)t}$ and $(\mathcal{D}_q y)(0) = \lim_{t\to 0}(\mathcal{D}_q y)(t)$ [18]. The q-derivative of higher order of a function y is given by $(\mathcal{D}_q^0 y)(t) = y(t)$ and $(\mathcal{D}_q^n y)(t) = \mathcal{D}_q(\mathcal{D}_q^{n-1} y)(t)$ for all $n \geq 1$ [2]. One can find in Algorithm 3 a pseudo-code for calculating q-derivative of a function f. The q-integral of a function y defined in the interval $[0, a_2]$ is given by

$$(\mathcal{J}_q y)(t) = \int_0^t y(s)\,\mathrm{d}_q s = t(1-q)\sum_{j=0}^{\infty} y(tq^j)q^j, \bigl(t \in [0, a_2]\bigr)$$

such that the sum is absolutely convergent which is shown in Algorithm 5 [18]. Now, assume that $a_1 \in [0, a_2]$. In this case the q-integral of y from a_1 to a_2 is given by

$$\begin{aligned}\int_{a_1}^{a_2} y(t)\,\mathrm{d}_q t &= \mathcal{J}_q y(a_2) - \mathcal{J}_q y(a_1) = \int_0^{a_2} y(t)\,\mathrm{d}_q t - \int_0^{a_1} y(t)\,\mathrm{d}_q t \\ &= (1-q)\sum_{j=0}^{\infty}\bigl[a_2 y\bigl(a_2 q^j\bigr) - a_1 y\bigl(a_1 q^j\bigr)\bigr]q^j\end{aligned}$$

Algorithm 4 The proposed method for calculated $\int_a^b f(r) d_q r$

```
function g = Iq(q, x, n, fun)
p=1;
for k=0:n
    p=p+ q^k*fun(x*q^k);
end;
g=x* (1-q) * p;
end
```

Algorithm 5 The proposed method for calculated $I_q^\sigma[x]$

```
function g = Iq_sigma(q, sigma,  x, n, fun)
p=0;
for k=0:n
    s1=1;
    for i=0:k-1
        s1=s1*(1-q^(sigma+i));
    end
    s2=1;
    for i=0:k-1
        s2=s2*(1-q^(i+1));
    end
    p=p + q^k*s1*fun(x*q^k)/s2;
end;
g=round((x^sigma)* ((1-q)^sigma)* p, 6);
end
```

whenever the series exists which is shown in Algorithm 4 [18]. Similar to q-derivatives, we define the operator $\mathcal{J}_q^n$ by $(\mathcal{J}_q^0 y)(t) = y(t)$ and $(\mathcal{J}_q^n y)(t) = \mathcal{J}_q(\mathcal{J}_q^{n-1} y)(t)$ for all $n \geq 1$ [18]. Note that $(\mathcal{D}_q \mathcal{J}_q y)(t) = y(t)$ and if y is continuous at $t = 0$, then $(\mathcal{J}_q \mathcal{D}_q y)(t) = y(t) - y(0)$ [18]. Assume that $\vartheta > 0$ is a real number with $n - 1 \leq \vartheta < n$, that is, $n = [\vartheta] + 1$. The Riemann–Liouville q-integral of a function $y \in C([t_1, t_2], \mathbb{R})$ is given by

$$\mathcal{J}_q^\vartheta y(t) = \frac{1}{\Gamma_q(\vartheta)} \int_0^t (t - q\tau)^{(\vartheta-1)} y(\tau) \, d_q\tau$$

whenever the integral exists [12, 14]. The Caputo q-derivative of y belongs to $C^{(n)}([t_1, t_2], \mathbb{R})$; it is defined by ${}^c\mathcal{D}_q^\vartheta y(t) = \frac{1}{\Gamma_q(n-\vartheta)} \int_0^t (t - q\tau)^{(n-\vartheta-1)} \mathcal{D}_q^{(n)} y(\tau) \, d_q\tau$ [12, 14]. We need the following results.

Lemma 1 ([11]) *Let $\vartheta_1, \vartheta_2 \geq 0$ and y be a function defined on $[0, 1]$. Then $(\mathcal{J}_q^{\vartheta_2} \mathcal{J}_q^{\vartheta_1} y)(t) = (\mathcal{J}_q^{\vartheta_1+\vartheta_2} y)(t)$ and $(\mathcal{D}_q^{\vartheta_1} \mathcal{J}_q^{\vartheta_1} y)(t) = y(t)$.*

Lemma 2 ([11]) *Let $\vartheta > 0$ and n be a positive integer. Then*

$$\left(\mathcal{J}_q^\vartheta \mathcal{D}_q^n y\right)(t) = \left(\mathcal{D}_q^n \mathcal{J}_q^\vartheta y\right)(t) - \sum_{j=0}^{n-1} \frac{t^{\vartheta-n+j}}{\Gamma_q(\vartheta + j - n + 1)} \left(\mathcal{D}_q^j y\right)(0).$$

It is well known that a general solution for the q-fractional differential equation ${}^c\mathcal{D}_q^\vartheta y(t) = 0$ is given by $y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_0, \ldots, c_{n-1}$ are some real numbers and $n = [\vartheta] + 1$ [11]. Also, for every positive real number T^* and every continuous function y on $[0, T^*]$, we have $(\mathcal{J}_q^\vartheta {}^c\mathcal{D}_q^\vartheta) y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_0, \ldots, c_{n-1}$ are constants belonging to $\mathbb{R}$ and $n = [\vartheta] + 1$ [11].

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed space. Denote by $\mathcal{P}(\mathcal{X})$, $\mathcal{P}_{cl}(\mathcal{X})$, $\mathcal{P}_b(\mathcal{X})$, $\mathcal{P}_{cp}(\mathcal{X})$ and $\mathcal{P}_{cp,cv}(\mathcal{X})$, the set of all subsets of $\mathcal{X}$, the set of all closed subsets of $\mathcal{X}$, the set of all bounded subsets of $\mathcal{X}$ and the set of all compact subsets of $\mathcal{X}$ and the set of all convex subsets of $\mathcal{X}$, respectively. An element $y^* \in \mathcal{X}$ is called a fixed point of a multivalued map $\mathcal{G} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ whenever $y^* \in \mathcal{G}(y^*)$. The set of all fixed points of the multifunction $\mathcal{G}$ is denoted by $Fix(\mathcal{G})$ [62]. A multifunction $\mathcal{G}$ is called convex-valued whenever the set $\mathcal{G}(y)$ is convex for all $y \in \mathcal{X}$.

We say that a multifunction $\mathcal{G}$ is an upper semi-continuous (u.s.c.) on the space $\mathcal{X}$ whenever for each element $y^* \in \mathcal{X}$, the set $\mathcal{G}(y^*) \in \mathcal{P}_{cl}(\mathcal{X})$ and there exists an open neighborhood $\mathcal{N}_0^*$ of y^* such that $\mathcal{G}(\mathcal{N}_0^*) \subseteq \mathcal{V}$ for all open set $\mathcal{V}$ of $\mathcal{X}$ containing $\mathcal{H}(y^*)$ [62]. A real-valued function $y : \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} y(\lambda_n) \leq y(\lambda)$ for all sequence $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \to \lambda$ [62]. Assume that $(\mathcal{X}, d)$ is a metric space. The Pompeiu–Hausdorff metric $\mathcal{H}_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathcal{H}_d\big(M^*, N^*\big) = \max\Big\{ \sup_{m^*\in M^*} d\big(m^*, N^*\big), \sup_{n^*\in N^*} d\big(M^*, n^*\big)\Big\},$$

where $d(M^*, n^*) = \inf_{m^*\in M^*} d(m^*, n^*)$ and $d(m^*, N^*) = \inf_{n^*\in N^*} d(m^*, n^*)$. A multivalued map $\mathcal{G} : \mathcal{X} \to \mathcal{P}_{cl}(\mathcal{X})$ is called Lipschitzian with Lipschitz constant $\beta > 0$ whenever

$$\mathcal{H}_d\big(\mathcal{G}(y_1), \mathcal{G}(y_2)\big) \leq \beta d(y_1, y_2),$$

for all $y_1, y_2 \in \mathcal{X}$ [62]. A Lipschitz map $\mathcal{G}$ is called a contraction whenever $\beta \in (0,1)$ [62].

A multifunction $\mathcal{G} : [0,1] \to \mathcal{P}_{cl}(\mathbb{R})$ is called measurable whenever the map $t \to d(\omega, \mathcal{G}(t))$ is measurable for all real constants ω [6, 62]. We say that $\mathcal{G} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Caratheodory whenever $t \mapsto \mathcal{G}(t,y)$ is measurable map for all $y \in \mathbb{R}$ and $y \mapsto \mathcal{G}(a,y)$ is upper semi-continuous map for almost all $t \in [0,1]$ [6, 62]. Also, a Caratheodory multifunction $\mathcal{G} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be $\mathcal{L}^1$-Caratheodory whenever for each constant $\mu > 0$ there exists a function $\phi_\mu \in \mathcal{L}^1([0,1], \mathbb{R}^+)$ such that $\|\mathcal{G}(t,y)\| = \sup_{\tau\in[0,1]}\{|\tau| : \tau \in \mathcal{G}(t,y)\} \leq \phi_\mu(t)$ for all $|y| \leq \mu$ and for almost all $t \in [0,1]$ [6, 62]. The set of selections of the multifunction $\mathcal{G}$ at point $y \in C([0,1], \mathbb{R})$ is defined by $S_{\mathcal{G},y} := \{v \in \mathcal{L}^1([0,1], \mathbb{R}) : v(t) \in \mathcal{G}(t, y(t))\}$ for almost all $t \in [0,1]$. It has been proved that $S_{\mathcal{G},y} \neq \emptyset$ for all $y \in C([0,1], \mathcal{X})$ if $\dim \mathcal{X} < \infty$ [62]. We say that an element $y \in \mathcal{X}$ is an endpoint of a multifunction $\mathcal{G} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ whenever $\mathcal{G}y = \{y\}$ [62]. Also, the multifunction $\mathcal{G}$ has an approximate endpoint property whenever $\inf_{x\in\mathcal{X}} \sup_{y\in\mathcal{G}x} d(x,y) = 0$ [62].

We denote by Ψ, the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ [63]. It is clear that $\psi(t) < t$ for all $t > 0$ [63]. In 2012, Samet et al. introduced the notion of α-ψ-contractive mappings [63]. We say that the selfmap $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is an α-ψ-contraction whenever $\alpha(y_1,y_2)d(\mathcal{T}y_1, \mathcal{T}y_2) \leq \psi(d(y_1,y_2))$ for all $y_1, y_2 \in \mathcal{X}$ [63]. Also, the selfmap $\mathcal{T}$ is called α-admissible whenever $\alpha(y_1,y_2) \geq 1$ implies $\alpha(\mathcal{T}y_1, \mathcal{T}y_2) \geq 1$ [63]. We say that $\mathcal{X}$ has the property (B) whenever for each sequence $\{y_n\}$ in $\mathcal{X}$ with $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \geq 1$ and $y_n \to y$, we have $\alpha(y_n, y) \geq 1$ for all n [63].

In 2013, Mohammadi et al. generalized this notion to multifunctions [64]. A multifunction $\mathcal{G} : \mathcal{X} \to CB(\mathcal{X})$ is called α-ψ-contraction whenever

$$\alpha(y_1,y_2)\mathcal{H}_d(\mathcal{G}y_1, \mathcal{G}y_2) \leq \psi\big(d(y_1,y_2)\big),$$

for all $y_1, y_2 \in \mathcal{X}$ [64]. Similarly, the space $\mathcal{X}$ has the property (C_α) whenever for each sequence $\{y_n\}$ in $\mathcal{X}$ with $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(y_{n_k}, y) \geq 1$ for all $k \in \mathbb{N}$. The multivalued map $\mathcal{G}$ is α-admissible whenever for each $y_1 \in \mathcal{X}$ and $y_2 \in \mathcal{G}y_1$ with $\alpha(y_1, y_2) \geq 1$, we have $\alpha(y_2, y_3) \geq 1$ for all $y_3 \in \mathcal{G}y_2$ [64]. We need the following results.

Theorem 3 ([63]) *Let $(\mathcal{X}, d)$ be a complete metric space, $\psi \in \Psi$, $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ a map and $\mathcal{T}$ an α-admissible and α-ψ-contractive selfmap on $\mathcal{X}$ such that $\alpha(y_0, \mathcal{T}y_0) \geq 1$, for some $y_0 \in \mathcal{X}$. If $\mathcal{X}$ has the property (B), then $\mathcal{T}$ has a fixed point.*

Theorem 4 ([65], Krasnoselskii) *Let M be a closed, bounded, convex and nonempty subset of a Banach space $\mathcal{X}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two operators such that*

(i) *$\mathcal{A}x + \mathcal{B}y \in M$ whenever $x, y \in M$,*
(ii) *$\mathcal{A}$ is compact and continuous,*
(iii) *$\mathcal{B}$ is a contraction mapping.*

Then there exists $z \in M$ such that $z = \mathcal{A}z + \mathcal{B}z$.

Theorem 5 ([64]) *Let $(\mathcal{X}, d)$ be a complete metric space, $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ a map, $\psi \in \Psi$ a strictly increasing map, $\mathcal{G} : \mathcal{X} \to CB(\mathcal{X})$ an α-admissible and α-ψ-contractive multifunction and $\alpha(y_0, y_1) \geq 1$ for some $y_0 \in \mathcal{X}$ and $y_1 \in \mathcal{G}y_0$. If the space $\mathcal{X}$ has the property (C_α), then $\mathcal{G}$ has a fixed point.*

Theorem 6 ([62]) *Let $(\mathcal{X}, d)$ be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be an upper semi-continuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ for all $t > 0$. Suppose that $\mathcal{T} : \mathcal{X} \to CB(\mathcal{X})$ is a multifunction such that $\mathcal{H}_d(\mathcal{T}y_1, \mathcal{T}y_2) \leq \psi(d(y_1, y_2))$ for all $y_1, y_2 \in \mathcal{X}$. Then $\mathcal{T}$ has a unique endpoint if and only if $\mathcal{T}$ has approximate endpoint property.*

3 Main results

In this work, we consider the Banach space

$$\mathcal{X} = \{y(t) : y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t) \in C([0,1], \mathbb{R})\},$$

via the norm

$$\|y\| = \sup_{t\in[0,1]} |y(t)| + \sup_{t\in[0,1]} |\mathcal{D}_q y(t)| + \sup_{t\in[0,1]} |\mathcal{D}_q^2 y(t)|.$$

Lemma 7 *Let $\varphi \in C([0,1], \mathcal{X})$. Then solution of the q-fractional boundary value problem*

$$\begin{cases} {}^c\mathcal{D}_q^{\vartheta} y(t) = \varphi(t), \\ y(0) = 0, \\ \mathcal{D}_q y(0) + {}^c\mathcal{D}_q^{\sigma} y(\nu) + \mathcal{D}_q^2 y(1) = 0, \\ \mathcal{J}_q^{\kappa} y(0) + \mathcal{J}_q^{\kappa} y(\nu) + \mathcal{J}_q^{\kappa} y(1) = 0, \end{cases}$$

is given by

$$y(t) = \int_0^t \frac{(t - q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \varphi(\tau)\, d_q\tau$$

$$+\frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\varphi(\tau)\,\mathrm{d}_q\tau$$
$$+\frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\varphi(\tau)\,\mathrm{d}_q\tau$$
$$+\frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\varphi(\tau)\,\mathrm{d}_q\tau$$
$$+\frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\varphi(\tau)\,\mathrm{d}_q\tau, \tag{5}$$

where $\Delta_1 = 2\nu^{2-\sigma} + (1+q)\Gamma_q(3-\sigma)$, $\Delta_2 = (1+\nu^{\kappa+1})\Gamma_q(3-\sigma)$,

$$\Delta_3 = \left|\frac{-\Gamma_q(\kappa+3)(\nu^{\kappa+1}+1)\Delta_1 + \Gamma_q(3-\sigma)\Gamma_q(\kappa+2)(1+q)[\nu^{\kappa+2}+1]}{\Gamma_q(\kappa+2)\Gamma_q(\kappa+3)\Gamma_q(3-\sigma)}\right|, \tag{6}$$

$\Delta_4 = (1+\nu^{\kappa+1})\Delta_1 + \Delta_3\Gamma_q(\kappa+2)\Gamma_q(3-\sigma)$ *and* $\Delta_5 = \Delta_3\Gamma_q(3-\sigma)\Gamma_q(\kappa+2) \neq 0$.

Proof Let $\hat{y}_0$ be a solution for the q-problem. Choose the constants a_0, a_1 and $a_2 \in \mathbb{R}$ such that

$$\hat{y}_0(t) = \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\varphi(\tau)\,\mathrm{d}_q\tau + a_0 + a_1 t + a_2 t^2. \tag{7}$$

Thus, we have

$$\mathcal{D}_q\hat{y}_0(t) = \int_0^t \frac{(t-q\tau)^{(\vartheta-2)}}{\Gamma_q(\vartheta-1)}\varphi(\tau)\,\mathrm{d}_q\tau + a_1 + a_2(1+q)t,$$
$${}^c\mathcal{D}_q^\sigma\hat{y}_0(t) = \int_0^t \frac{(t-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\varphi(\tau)\,\mathrm{d}_q\tau + a_2\frac{2t^{2-\sigma}}{\Gamma_q(3-\sigma)},$$
$$\mathcal{D}_q^2\hat{y}_0(t) = \int_0^t \frac{(t-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\varphi(\tau)\,\mathrm{d}_q\tau + a_2(1+q),$$

and

$$\mathcal{J}_q^\kappa\hat{y}_0(t) = \int_0^t \frac{(t-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\varphi(\tau)\,\mathrm{d}_q\tau + a_0\frac{t^\kappa}{\Gamma_q(\kappa+1)}$$
$$+ a_1\frac{t^{\kappa+1}}{\Gamma_q(\kappa+2)} + a_2\frac{(1+q)t^{\kappa+2}}{\Gamma_q(\kappa+3)}.$$

By using the boundary value conditions, we obtain $a_0 = 0$,

$$a_1 = \frac{\Delta_1}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\varphi(\tau)\,\mathrm{d}_q\tau$$
$$-\frac{\Delta_4}{\Delta_5}\int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\varphi(\tau)\,\mathrm{d}_q\tau$$
$$-\frac{\Delta_4}{\Delta_5}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\varphi(\tau)\,\mathrm{d}_q\tau$$

$$+\frac{\Delta_1}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\varphi(\tau)\,\mathrm{d}_q\tau$$

and

$$\begin{aligned} a_2 = {} & \frac{1+\nu^{\kappa+1}}{\Delta_3\Gamma_q(\kappa+2)}\int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\varphi(\tau)\,\mathrm{d}_q\tau \\ & - \frac{1}{\Delta_4}\int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\varphi(\tau)\,\mathrm{d}_q\tau \\ & - \frac{1}{\Delta_3}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\varphi(\tau)\,\mathrm{d}_q\tau \\ & + \frac{1+\vartheta^{\kappa+1}}{\Delta_3\Gamma_q(\kappa+2)}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\varphi(\tau)\,\mathrm{d}_q\tau. \end{aligned}$$

By substituting the values of the a_i in (7), we obtain the q-integral equation (5). This completes the proof. □

Now, consider the operator $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ defined by

$$\begin{aligned} (\mathcal{T}y)(t) = {} & \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\Phi\bigl(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\bigr)\,\mathrm{d}_q\tau \\ & + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \\ & \times \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Phi\bigl(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\bigr)\,\mathrm{d}_q\tau \\ & + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \\ & \times \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\Phi\bigl(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\bigr)\,\mathrm{d}_q\tau \\ & + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \\ & \times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\Phi\bigl(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\bigr)\,\mathrm{d}_q\tau \\ & + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \\ & \times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Phi\bigl(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\bigr)\,\mathrm{d}_q\tau. \end{aligned}$$

It is clear that y_0 is a solution for the problem (1) if and only if y_0 is a fixed point of the operator $\mathcal{T}$. Put

$$\begin{aligned} \Xi_1 = {} & \frac{1}{\Gamma_q(\vartheta+1)} + \frac{(\Delta_1+\Gamma_q(3-\sigma))(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)} \\ & + \frac{\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)} + \frac{(\Delta_2+\Delta_4)\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}, \end{aligned}$$

$$
\begin{aligned}
\varXi_2 &= \frac{1}{\varGamma_q(\vartheta)} + \frac{[\Delta_1 + (1+q)\varGamma_q(3-\sigma)](\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\varGamma_q(3-\sigma)\varGamma_q(\vartheta+\kappa+1)} \\
&\quad + \frac{(1+q)\Delta_2 + \Delta_4}{\Delta_5\varGamma_q(\vartheta-1)} + \frac{[(1+q)\Delta_2+\Delta_4]\nu^{(\vartheta-\sigma)}}{\Delta_5\varGamma_q(\vartheta-\sigma+1)}, \\
\varXi_3 &= \frac{1}{\varGamma_q(\vartheta-1)} + \frac{(1+q)(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\varGamma_q(\vartheta+\kappa+1)} \\
&\quad + \frac{(1+q)\Delta_2}{\Delta_5\varGamma_q(\vartheta-1)} + \frac{(1+q)\Delta_2\nu^{(\vartheta-\sigma)}}{\Delta_5\varGamma_q(\vartheta-\sigma+1)},
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\Delta^{(1)} &= \frac{(\Delta_1 + \varGamma_q(3-\sigma))(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\varGamma_q(3-\sigma)\varGamma_q(\vartheta+\kappa+1)} + \frac{\Delta_2+\Delta_4}{\Delta_5\varGamma_q(\vartheta-1)} \\
&\quad + \frac{(\Delta_2+\Delta_4)\nu^{(\vartheta-\sigma)}}{\Delta_5\varGamma_q(\vartheta-\sigma+1)}, \\
\Delta^{(2)} &= \frac{[\Delta_1 + (1+q)\varGamma_q(3-\sigma)](\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\varGamma_q(3-\sigma)\varGamma_q(\vartheta+\kappa+1)} + \frac{(1+q)\Delta_2+\Delta_4}{\Delta_5\varGamma_q(\vartheta-1)} \\
&\quad + \frac{[(1+q)\Delta_2+\Delta_4]\nu^{(\vartheta-\sigma)}}{\Delta_5\varGamma_q(\vartheta-\sigma+1)}, \\
\Delta^{(3)} &= \frac{(1+q)(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\varGamma_q(\vartheta+\kappa+1)} + \frac{(1+q)\Delta_2}{\Delta_5\varGamma_q(\vartheta-1)} + \frac{(1+q)\Delta_2\nu^{(\vartheta-\sigma)}}{\Delta_5\varGamma_q(\vartheta-\sigma+1)},
\end{aligned}
\tag{9}
$$

and

$$
\varSigma_1 = \|m\|\varXi_1, \qquad \varSigma_2 = \|m\|\varXi_2, \qquad \varSigma_3 = \|m\|\varXi_3. \tag{10}
$$

Now, we are ready to prove our main results.

Theorem 8 *Let $\psi \in \varPsi$, $\chi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a map and $\varPhi : [0,1] \times \mathcal{X}^3 \to \mathcal{X}$ a continuous function. Suppose that*

($H1$)

$$
\begin{aligned}
&\bigl|\varPhi\bigl(t, x_1(t), y_1(t), z_1(t)\bigr) - \varPhi\bigl(t, x_2(t), y_2(t), z_2(t)\bigr)\bigr| \\
&\quad \le \lambda\psi\bigl(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\bigr),
\end{aligned}
$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathcal{X}$ with

$$
\chi\bigl(\bigl(x_1(t), y_1(t), z_1(t)\bigr), \bigl(x_2(t), y_2(t), z_2(t)\bigr)\bigr) \ge 0,
$$

for all $t \in [0,1]$, where $\lambda = \frac{1}{\varXi_1+\varXi_2+\varXi_3}$,

($H2$) *There exists $y_0 \in \mathcal{X}$ such that*

$$
\chi\bigl(\bigl(y_0(t), \mathcal{D}_q y_0(t), \mathcal{D}_q^2 y_0(t)\bigr), \bigl(\mathcal{T}y_0(t), \mathcal{D}_q\bigl(\mathcal{T}y_0(t)\bigr), \mathcal{D}_q^2\bigl(\mathcal{T}y_0(t)\bigr)\bigr)\bigr) \ge 0,
$$

for all $t \in [0,1]$ and

$$
\chi\bigl(\bigl(y_1(t), \mathcal{D}_q y_1(t), \mathcal{D}_q^2 y_1(t)\bigr), \bigl(y_2(t), \mathcal{D}_q y_2(t), \mathcal{D}_q^2 y_2(t)\bigr)\bigr) \ge 0,
$$

which implies

$$\chi\big(\big(\mathcal{T}y_1(t),\mathcal{D}_q\big(\mathcal{T}y_1(t)\big),\mathcal{D}_q^2\big(\mathcal{T}y_1(t)\big)\big),\big(\mathcal{T}y_2(t),\mathcal{D}_q\big(\mathcal{T}y_2(t)\big),\mathcal{D}_q^2\big(\mathcal{T}y_2(t)\big)\big)\big)\geq 0$$

for all $t\in[0,1]$ *and* $y_1,y_2\in\mathcal{X}$,

$(H3)$ *For each convergent sequence* $\{y_n\}_{n\geq 1}$ *in* $\mathcal{X}$ *with* $y_n\to y$ *and*

$$\chi\big(\big(y_n(t),\mathcal{D}_q y_n(t),\mathcal{D}_q^2 y_n(t)\big),\big(y_{n+1}(t),\mathcal{D}_q y_{n+1}(t),\mathcal{D}_q^2 y_{n+1}(t)\big)\big)\geq 0,$$

for all n *and* $t\in[0,1]$, *we have*

$$\chi\big(\big(y_n(t),\mathcal{D}_q y_n(t),\mathcal{D}_q^2 y_n(t)\big),\big(y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t)\big)\big)\geq 0.$$

Then the q-fractional boundary value problem (1)–(2) *has at least one solution.*

Proof Let $y_1,y_2\in\mathcal{X}$ be such that

$$\chi\big(\big(y_1(t),\mathcal{D}_q y_1(t),\mathcal{D}_q^2 y_1(t)\big),\big(y_2(t),\mathcal{D}_q y_2(t),\mathcal{D}_q^2 y_2(t)\big)\big)\geq 0,$$

for all $t\in[0,1]$. Then we have

$$\begin{aligned}
&\big|\mathcal{T}y_1(t)-\mathcal{T}y_2(t)\big|\\
&\quad\leq\int_0^t\frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\\
&\qquad\times\big|\Phi\big(\tau,y_1(\tau),\mathcal{D}_q y_1(\tau),\mathcal{D}_q^2 y_1(\tau)\big)-\Phi\big(\tau,y_2(\tau),\mathcal{D}_q y_2(\tau),\mathcal{D}_q^2 y_2(\tau)\big)\big|\,\mathrm{d}_q\tau\\
&\qquad+\frac{|t\Delta_1-t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1\frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\\
&\qquad\times\big|\Phi\big(\tau,y_1(\tau),\mathcal{D}_q y_1(\tau),\mathcal{D}_q^2 y_1(\tau)\big)-\Phi\big(\tau,y_2(\tau),\mathcal{D}_q y_2(\tau),\mathcal{D}_q^2 y_2(\tau)\big)\big|\,\mathrm{d}_q\tau\\
&\qquad+\frac{|t^2\Delta_2-t\Delta_4|}{\Delta_5}\int_0^1\frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\\
&\qquad\times\big|\Phi\big(\tau,y_1(\tau),\mathcal{D}_q y_1(\tau),\mathcal{D}_q^2 y_1(\tau)\big)-\Phi\big(\tau,y_2(\tau),\mathcal{D}_q y_2(\tau),\mathcal{D}_q^2 y_2(\tau)\big)\big|\,\mathrm{d}_q\tau\\
&\qquad+\frac{|t^2\Delta_2-t\Delta_4|}{\Delta_5}\int_0^\nu\frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\\
&\qquad\times\big|\Phi\big(\tau,y_1(\tau),\mathcal{D}_q y_1(\tau),\mathcal{D}_q^2 y_1(\tau)\big)-\Phi\big(\tau,y_2(\tau),\mathcal{D}_q y_2(\tau),\mathcal{D}_q^2 y_2(\tau)\big)\big|\,\mathrm{d}_q\tau\\
&\qquad+\frac{|t\Delta_1-t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu\frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\\
&\qquad\times\big|\Phi\big(\tau,y_1(\tau),\mathcal{D}_q y_1(\tau),\mathcal{D}_q^2 y_1(\tau)\big)-\Phi\big(\tau,y_2(\tau),\mathcal{D}_q y_2(\tau),\mathcal{D}_q^2 y_2(\tau)\big)\big|\,\mathrm{d}_q\tau\\
&\quad\leq\frac{1}{\Gamma_q(\vartheta+1)}\\
&\qquad\times\lambda\psi\big(\big|y_1(\tau)-y_2(\tau)\big|+\big|D_q y_1(\tau)-D_q y_2(\tau)\big|+\big|D_q^2 y_1(\tau)-D_q^2 y_2(\tau)\big|\big)\\
&\qquad+\frac{\Delta_1+\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}
\end{aligned}$$

$$
\begin{aligned}
&\times \lambda\psi\bigl(\bigl|y_1(\tau)-y_2(\tau)\bigr|+\bigl|D_q y_1(\tau)-D_q y_2(\tau)\bigr|+\bigl|D_q^2 y_1(\tau)-D_q^2 y_2(\tau)\bigr|\bigr)\\
&+\frac{\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)}\\
&\times \lambda\psi\bigl(\bigl|y_1(\tau)-y_2(\tau)\bigr|+\bigl|D_q y_1(\tau)-D_q y_2(\tau)\bigr|+\bigl|D_q^2 y_1(\tau)-D_q^2 y_2(\tau)\bigr|\bigr)\\
&+\frac{(\Delta_2+\Delta_4)\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\\
&\times \lambda\psi\bigl(\bigl|y_1(\tau)-y_2(\tau)\bigr|+\bigl|D_q y_1(\tau)-D_q y_2(\tau)\bigr|+\bigl|D_q^2 y_1(\tau)-D_q^2 y_2(\tau)\bigr|\bigr)\\
&+\frac{(\Delta_1+\Gamma_q(3-\sigma))\nu^{(\vartheta+\kappa)}}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\\
&\times \lambda\psi\bigl(\bigl|y_1(\tau)-y_2(\tau)\bigr|+\bigl|D_q y_1(\tau)-D_q y_2(\tau)\bigr|+\bigl|D_q^2 y_1(\tau)-D_q^2 y_2(\tau)\bigr|\bigr)\\
\le{}& \frac{1}{\Gamma_q(\vartheta+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&+\frac{\Delta_1+\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&+\frac{\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&+\frac{(\Delta_2+\Delta_4)\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&+\frac{(\Delta_1+\Gamma_q(3-\sigma))\nu^{(\vartheta+\kappa)}}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
={}& \lambda\Xi_1\psi\bigl(\|y_1-y_2\|\bigr).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\bigl|\mathcal{D}_q\mathcal{T}y_1(t)-\mathcal{D}_q\mathcal{T}y_2(t)\bigr|\\
&\quad\le \frac{1}{\Gamma_q(\vartheta)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\qquad+\frac{\Delta_1+(1+q)\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\qquad+\frac{(1+q)\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\qquad+\frac{[(1+q)\Delta_2+\Delta_4]\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\qquad+\frac{[\Delta_1+(1+q)\Gamma_q(3-\sigma)]\nu^{(\vartheta+\kappa)}}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\quad= \lambda\Xi_2\psi\bigl(\|y_1-y_2\|\bigr)
\end{aligned}
$$

and

$$
\bigl|\mathcal{D}_q^2\mathcal{T}y_1(t)-\mathcal{D}_q^2\mathcal{T}y_2(t)\bigr|
$$

$$\begin{aligned}
&\le \frac{1}{\Gamma_q(\vartheta-1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\quad+\frac{(1+q)}{\Delta_3\Gamma_q(\vartheta+\kappa+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\quad+\frac{(1+q)\Delta_2}{\Delta_5\Gamma_q(\vartheta-1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\quad+\frac{(1+q)\Delta_2\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&\quad+\frac{(1+q)\nu^{(\vartheta+\kappa)}}{\Delta_3\Gamma_q(\vartheta+\kappa+1)}\lambda\psi\bigl(\|y_1-y_2\|\bigr)\\
&=\lambda\Xi_3\psi\bigl(\|y_1-y_2\|\bigr).
\end{aligned}$$

Hence $\|\mathcal{T}y_1(t)-\mathcal{T}y_2(t)\|\le(\Xi_1+\Xi_2+\Xi_3)\lambda\psi(\|y_1-y_2\|)=\psi(\|y_1-y_2\|)$. Now, we define the non-negative function α on $\mathcal{X}\times\mathcal{X}$ as follows:

$$\alpha(y_1,y_2)=\begin{cases}1 & \text{if } \chi((y_1(t),\mathcal{D}_q y_1(t),\mathcal{D}_q^2 y_1(t)),(y_2(t),\mathcal{D}_q y_2(t),\mathcal{D}_q^2 y_2(t)))\ge 0,\\ 0 & \text{otherwise,}\end{cases}$$

for all $y_1,y_2\in\mathcal{X}$. Then we have $\alpha(y_1,y_2)d(\mathcal{T}y_1,\mathcal{T}y_2)\le\psi(d(y_1,y_2))$ for all $y_1,y_2\in\mathcal{X}$. This means that $\mathcal{T}$ is an α-ψ-contractive operator. Furthermore, it is easy to check that $\mathcal{T}$ is α-admissible and $\alpha(y_0,\mathcal{T}y_0)\ge 1$. Besides, we suppose that $\{y_n\}_{n\ge1}$ is a sequence that belongs to $\mathcal{X}$ with $y_n\to y$ and $\alpha(y_n,y_{n+1})\ge 1$ for all n. The definition of the non-negative function α implies that

$$\chi\bigl((y_n(t),\mathcal{D}_q y_n(t),\mathcal{D}_q^2 y_n(t)),(y_{n+1}(t),\mathcal{D}_q y_{n+1}(t),\mathcal{D}_q^2 y_{n+1}(t))\bigr)\ge 0.$$

Thus, by the hypothesis, we get

$$\chi\bigl((y_n(t),\mathcal{D}_q y_n(t),\mathcal{D}_q^2 y_n(t)),(y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t))\bigr)\ge 0.$$

This shows that, for all n, $\alpha(y_n,y)\ge 1$. Hence the Banach space $\mathcal{X}$ has the property (B). Now, Theorem 3 implies that the operator $\mathcal{T}$ has fixed point $y^*\in\mathcal{X}$ which is a solution for the q-fractional BVP (1)–(2). This completes the proof. □

Theorem 9 *Let $\Phi:[0,1]\times\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ be a continuous function. Suppose that*:

$(H4)$ *there exists a continuous real-valued function L on the closed interval $[0,1]$ such that*

$$\bigl|\Phi(t,x_1,y_1,z_1)-\Phi(t,x_2,y_2,z_2)\bigr|\le L(t)\bigl(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\bigr).$$

for all $t\in[0,1]$ and $x_1,x_2,y_1,y_2,z_1,z_2\in\mathcal{X}$,

$(H5)$ *there exist a continuous function $\zeta:[0,1]\to\mathbb{R}^+$ and a nondecreasing continuous function $\psi:[0,1]\to\mathbb{R}^+$ such that*

$$\bigl|\Phi(t,y_1,y_2,y_3)\bigr|\le\zeta(t)\psi\bigl(|y_1|+|y_2|+|y_3|\bigr),$$

for all $t\in[0,1]$ and $y_1,y_2,y_3\in\mathcal{X}$.

Then the fractional q-difference equation (1) *with the boundary value conditions* (2) *has at least one solution whenever*

$$K := \|L\|\big(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}\big) < 1, \tag{11}$$

where $\|L\| = \sup_{t\in[0,1]} |L(t)|$ *and* $\Delta^{(1)}$, $\Delta^{(2)}$ *and* $\Delta^{(3)}$ *are given by* (10).

Proof Put $\|\zeta\| = \sup_{t\in[0,1]} |\zeta(t)|$ and choose a suitable positive constant ϵ such that

$$\epsilon \geq \psi\big(\|y\|\big)\|\zeta\|\{\varXi_1 + \varXi_2 + \varXi_3\}, \tag{12}$$

where the $\Delta^{(i)}$ are given by (10). Consider the set $V_\epsilon = \{y \in \mathcal{X} : \|y\| \leq \epsilon\}$, where ϵ is given in (12). One can check that the set V_ϵ is a closed, convex, bounded and nonempty subset of the Banach space $\mathcal{X}$. Now, consider the fractional operators $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$ on the set V_ϵ defined by

$$\big(\mathcal{T}^{(1)}y\big)(t) = \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \varPhi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\,\mathrm{d}_q\tau$$

and

$$\begin{aligned}
\big(\mathcal{T}^{(2)}y\big)(t) = {}& \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \\
&\times \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \varPhi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\,\mathrm{d}_q\tau \\
&+ \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \\
&\times \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \varPhi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\,\mathrm{d}_q\tau \\
&+ \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \\
&\times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \varPhi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\,\mathrm{d}_q\tau \\
&+ \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \\
&\times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \varPhi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\,\mathrm{d}_q\tau
\end{aligned}$$

for all $t \in [0,1]$. Put $\hat{m} = \sup_{y\in\mathbb{R}} \psi(\|y\|)$. For $y_1, y_2 \in V_\epsilon$, we have

$$\begin{aligned}
&\big|\big(\mathcal{T}^{(1)}y_1 + \mathcal{T}^{(2)}y_2\big)(t)\big| \\
&\quad\leq \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \big|\varPhi\big(\tau, y_1(\tau), \mathcal{D}_q y_1(\tau), \mathcal{D}_q^2 y_1(\tau)\big)\big|\,\mathrm{d}_q\tau \\
&\qquad+ \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \big|\Phi\big(\tau, y_2(\tau), \mathcal{D}_q y_2(\tau), \mathcal{D}_q^2 y_2(\tau)\big)\big| \,\mathrm{d}_q\tau \\
&+ \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5} \\
&\times \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \big|\Phi\big(\tau, y_2(\tau), \mathcal{D}_q y_2(\tau), \mathcal{D}_q^2 y_2(\tau)\big)\big| \,\mathrm{d}_q\tau \\
&+ \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5} \\
&\times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \big|\Phi\big(\tau, y_2(\tau), \mathcal{D}_q y_2(\tau), \mathcal{D}_q^2 y_2(\tau)\big)\big| \,\mathrm{d}_q\tau \\
&+ \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)} \\
&\times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \big|\Phi\big(\tau, y_2(\tau), \mathcal{D}_q y_2(\tau), \mathcal{D}_q^2 y_2(\tau)\big)\big| \,\mathrm{d}_q\tau \\
\le{}& \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \psi\big(\big|y_1(\tau)\big| + \big|\mathcal{D}_q y_1(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau)\big|\big) \,\mathrm{d}_q\tau \\
&+ \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)} \\
&\times \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \psi\big(\big|y_2(\tau)\big| + \big|\mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_2(\tau)\big|\big) \,\mathrm{d}_q\tau \\
&+ \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5} \\
&\times \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \psi\big(\big|y_2(\tau)\big| + \big|\mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_2(\tau)\big|\big) \,\mathrm{d}_q\tau \\
&+ \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5} \\
&\times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \psi\big(\big|y_2(\tau)\big| + \big|\mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_2(\tau)\big|\big) \,\mathrm{d}_q\tau \\
&+ \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)} \\
&\times \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \psi\big(\big|y_2(\tau)\big| + \big|\mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_2(\tau)\big|\big) \,\mathrm{d}_q\tau \\
\le{}& \hat{m}\|\zeta\| \biggl[\frac{1}{\Gamma_q(\vartheta+1)} + \frac{(\Delta_1 + \Gamma_q(3-\sigma))(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)} \\
&+ \frac{\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)} + \frac{(\Delta_2+\Delta_4)\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\biggr] = \hat{m}\|\zeta\|\Xi_1.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\big|\big(\mathcal{D}_q\mathcal{T}^{(1)}y_1 + \mathcal{D}_q\mathcal{T}^{(2)}y_2\big)(t)\big| \\
&\quad\le \int_0^t \frac{(t-q\tau)^{(\vartheta-2)}}{\Gamma_q(\vartheta-1)} \psi\big(\big|y_1(\tau)\big| + \big|\mathcal{D}_q y_1(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau)\big|\big) \,\mathrm{d}_q\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\Delta_1-(1+q)t\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)}\\
&\times\int_0^1\frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\psi\left(\left|y_2(\tau)\right|+\left|\mathcal{D}_q y_2(\tau)\right|+\left|\mathcal{D}_q^2 y_2(\tau)\right|\right)\mathrm{d}_q\tau\\
&+\frac{|(1+q)t\Delta_2-\Delta_4|}{\Delta_5}\\
&\times\int_0^1\frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\psi\left(\left|y_2(\tau)\right|+\left|\mathcal{D}_q y_2(\tau)\right|+\left|\mathcal{D}_q^2 y_2(\tau)\right|\right)\mathrm{d}_q\tau\\
&+\frac{|(1+q)t\Delta_2-\Delta_4|}{\Delta_5}\\
&\times\int_0^\nu\frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\psi\left(\left|y_2(\tau)\right|+\left|\mathcal{D}_q y_2(\tau)\right|+\left|\mathcal{D}_q^2 y_2(\tau)\right|\right)\mathrm{d}_q\tau\\
&+\frac{|\Delta_1-(1+q)t\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)}\\
&\times\int_0^\nu\frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\psi\left(\left|y_2(\tau)\right|+\left|\mathcal{D}_q y_2(\tau)\right|+\left|\mathcal{D}_q^2 y_2(\tau)\right|\right)\mathrm{d}_q\tau\\
\leq{}&\hat{m}\|\zeta\|\left[\frac{1}{\Gamma_q(\vartheta)}+\frac{[\Delta_1+(1+q)\Gamma_q(3-\sigma)](\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\right.\\
&\left.+\frac{(1+q)\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)}+\frac{[(1+q)\Delta_2+\Delta_4]\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\right]=\hat{m}\|\zeta\|\Xi_2,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\left|\left(\mathcal{D}_q^2\mathcal{T}^{(1)}y_1+\mathcal{D}_q^2\mathcal{T}^{(2)}y_2\right)(t)\right|\\
&\quad\leq\hat{m}\|\zeta\|\left[\frac{1}{\Gamma_q(\vartheta-1)}+\frac{(1+q)(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\Gamma_q(\vartheta+\kappa+1)}\right.\\
&\qquad\left.+\frac{(1+q)\Delta_2}{\Delta_5\Gamma_q(\vartheta-1)}+\frac{(1+q)\Delta_2\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\right]=\hat{m}\|\zeta\|\Xi_3.
\end{aligned}
$$

Hence, $\|(\mathcal{T}^{(1)}y_1+\mathcal{T}^{(2)}y_2)(t)\|\leq\epsilon$ and so $(\mathcal{T}^{(1)}y_1+\mathcal{T}^{(2)}y_2)(t)\in V_\epsilon$. Clearly, the continuity of the function Φ implies the continuity of the fractional operator $\mathcal{T}^{(1)}$. Also,

$$
\begin{aligned}
\left|\left(\mathcal{T}^{(1)}y\right)(t)\right|&\leq\int_0^t\frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\left|\Phi\left(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\right)\right|\mathrm{d}_q\tau\\
&\leq\frac{1}{\Gamma_q(\vartheta+1)}\|\zeta\|\psi\left(\|y\|\right),\\
\left|\left(\mathcal{D}_q\mathcal{T}^{(1)}y\right)(t)\right|&\leq\int_0^t\frac{(t-q\tau)^{(\vartheta-2)}}{\Gamma_q(\vartheta-1)}\left|\Phi\left(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\right)\right|\mathrm{d}_q\tau\\
&\leq\frac{1}{\Gamma_q(\vartheta)}\|\zeta\|\psi\left(\|y\|\right)
\end{aligned}
$$

and

$$
\left|\left(\mathcal{D}_q^2\mathcal{T}^{(1)}y\right)(t)\right|\leq\int_0^t\frac{(t-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\left|\Phi\left(\tau,y(\tau),\mathcal{D}_q y(\tau),\mathcal{D}_q^2 y(\tau)\right)\right|\mathrm{d}_q\tau
$$

$$\leq \frac{1}{\Gamma_q(\vartheta - 1)} \|\zeta\| \psi\big(\|y\|\big),$$

for all $y \in V_\epsilon$. Hence,

$$\big\|\mathcal{T}^{(1)} y\big\| \leq \left\{ \frac{1}{\Gamma_q(\vartheta + 1)} + \frac{1}{\Gamma_q(\vartheta)} + \frac{1}{\Gamma_q(\vartheta - 1)} \right\} \|\zeta\| \psi\big(\|y\|\big).$$

This proves that the operator $\mathcal{T}^{(1)}$ is uniformly bounded on V_ϵ. Now for checking the compactness of the fractional operator $\mathcal{T}^{(1)}$ on V_ϵ, assume that $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\big|\big(\mathcal{T}^{(1)} y\big)(t_2) - \big(\mathcal{T}^{(1)} y\big)(t_1)\big| \\
&= \bigg| \int_0^{t_2} \frac{(t_2 - q\tau)^{(\vartheta - 1)}}{\Gamma_q(\vartheta)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \\
&\quad - \int_0^{t_1} \frac{(t_1 - q\tau)^{(\vartheta - 1)}}{\Gamma_q(\vartheta)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \bigg| \\
&\leq \bigg| \int_0^{t_1} \frac{(t_2 - q\tau)^{(\vartheta - 1)} - (t_1 - q\tau)^{(\vartheta - 1)}}{\Gamma_q(\vartheta)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \bigg| \\
&\quad + \bigg| \int_{t_1}^{t_2} \frac{(t_2 - q\tau)^{(\vartheta - 1)}}{\Gamma_q(\vartheta)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \bigg| \\
&\leq \int_0^{t_1} \frac{(t_2 - q\tau)^{(\vartheta - 1)} - (t_1 - q\tau)^{(\vartheta - 1)}}{\Gamma_q(\vartheta)} \big|\Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\big| \, \mathrm{d}\tau \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - q\tau)^{(\vartheta - 1)}}{\Gamma_q(\vartheta)} \big|\Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\big| \, \mathrm{d}\tau \\
&\leq \left\{ \frac{t_2^\vartheta - t_1^\vartheta - (t_2 - t_1)^\vartheta}{\Gamma_q(\vartheta + 1)} + \frac{(t_2 - t_1)^\vartheta}{\Gamma_q(\vartheta + 1)} \right\} \|\zeta\| \psi\big(\|y\|\big).
\end{aligned}$$

Thus, $|(\mathcal{T}^{(1)} y)(t_2) - (\mathcal{T}^{(1)} y)(t_1)| \to 0$ as $t_2 \to t_1$. Also, we have

$$\begin{aligned}
&\big|\big(\mathcal{D}_q \mathcal{T}^{(1)} y\big)(t_2) - \big(\mathcal{D}_q \mathcal{T}^{(1)} y\big)(t_1)\big| \\
&= \bigg| \int_0^{t_2} \frac{(t_2 - q\tau)^{(\vartheta - 2)}}{\Gamma_q(\vartheta - 1)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \\
&\quad - \int_0^{t_1} \frac{(t_1 - q\tau)^{(\vartheta - 2)}}{\Gamma_q(\vartheta - 1)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \bigg| \\
&\leq \bigg| \int_0^{t_1} \frac{(t_2 - q\tau)^{(\vartheta - 2)} - (t_1 - q\tau)^{(\vartheta - 2)}}{\Gamma_q(\vartheta - 1)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \bigg| \\
&\quad + \bigg| \int_{t_1}^{t_2} \frac{(t_2 - q\tau)^{(\vartheta - 2)}}{\Gamma_q(\vartheta - 1)} \Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big) \, \mathrm{d}\tau \bigg| \\
&\leq \int_0^{t_1} \frac{(t_2 - q\tau)^{(\vartheta - 2)} - (t_1 - q\tau)^{(\vartheta - 2)}}{\Gamma_q(\vartheta - 1)} \big|\Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\big| \, \mathrm{d}\tau \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - q\tau)^{(\vartheta - 2)}}{\Gamma_q(\vartheta - 1)} \big|\Phi\big(\tau, y(\tau), \mathcal{D}_q y(\tau), \mathcal{D}_q^2 y(\tau)\big)\big| \, \mathrm{d}\tau
\end{aligned}$$

$$\leq \left\{ \frac{t_2^{(\vartheta-1)} - t_1^{(\vartheta-1)} - (t_2 - t_1)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} + \frac{(t_2 - t_1)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \right\} \|\zeta\| \psi\big(\|y\|\big)$$

and so $|(\mathcal{D}_q\mathcal{T}^{(1)}y)(t_2) - (\mathcal{D}_q\mathcal{T}^{(1)}y)(t_1)| \to 0$ as $t_2 \to t_1$. Similarly, we can show that

$$\big|\big(\mathcal{D}_q^2\mathcal{T}^{(1)}y\big)(t_2) - \big(\mathcal{D}_q^2\mathcal{T}^{(1)}y\big)(t_1)\big| \to 0,$$

as $t_2 \to t_1$ Hence, $\|(\mathcal{T}^{(1)}y)(t_2) - (\mathcal{T}^{(1)}y)(t_1)\|$ tends to zero as $t_2 \to t_1$. Thus, $\mathcal{T}^{(1)}$ is equicontinuous and so $\mathcal{T}^{(1)}$ is relatively compact on V_ϵ. Now by using the Arzela–Ascoli theorem, the fractional operator $\mathcal{T}^{(1)}$ is compact on V_ϵ. Finally, we prove that $\mathcal{T}^{(2)}$ is a contraction map. Let $y_1, y_2 \in V_\epsilon$. Then we have

$$\begin{aligned}
&\big|\big(\mathcal{T}^{(2)}y_1\big)(t) - \big(\mathcal{T}^{(2)}y_2\big)(t)\big| \\
&\quad\leq \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)} \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau \\
&\qquad+ \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5} \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau \\
&\qquad+ \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau \\
&\qquad+ \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau.
\end{aligned}$$

Also

$$\begin{aligned}
&\big|\big(\mathcal{D}_q\mathcal{T}^{(2)}y_1\big)(t) - \big(\mathcal{D}_q\mathcal{T}^{(2)}y_2\big)(t)\big| \\
&\quad\leq \frac{|\Delta_1 - (1+q)t\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)} \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau \\
&\qquad+ \frac{|(1+q)t\Delta_2 - \Delta_4|}{\Delta_5} \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau \\
&\qquad+ \frac{|(1+q)t\Delta_2 - \Delta_4|}{\Delta_5} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau \\
&\qquad+ \frac{|\Delta_1 - (1+q)t\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \\
&\qquad\times L(\tau)\big(\big|y_1(\tau) - y_2(\tau)\big| + \big|\mathcal{D}_q y_1(\tau) - \mathcal{D}_q y_2(\tau)\big| + \big|\mathcal{D}_q^2 y_1(\tau) - \mathcal{D}_q^2 y_2(\tau)\big|\big)\,\mathrm{d}_q\tau
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(\mathcal{D}_q^2\mathcal{T}^{(2)}y_1\right)(t)-\left(\mathcal{D}_q^2\mathcal{T}^{(2)}y_2\right)(t)\right| \\
&\quad\leq \frac{(1+q)\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1\frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \\
&\qquad\times L(\tau)\left(\left|y_1(\tau)-y_2(\tau)\right|+\left|\mathcal{D}_qy_1(\tau)-\mathcal{D}_qy_2(\tau)\right|+\left|\mathcal{D}_q^2y_1(\tau)-\mathcal{D}_q^2y_2(\tau)\right|\right)\,\mathrm{d}_q\tau \\
&\qquad+\frac{(1+q)\Delta_2}{\Delta_5}\int_0^1\frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \\
&\qquad\times L(\tau)\left(\left|y_1(\tau)-y_2(\tau)\right|+\left|\mathcal{D}_qy_1(\tau)-\mathcal{D}_qy_2(\tau)\right|+\left|\mathcal{D}_q^2y_1(\tau)-\mathcal{D}_q^2y_2(\tau)\right|\right)\,\mathrm{d}_q\tau \\
&\qquad+\frac{(1+q)\Delta_2}{\Delta_5}\int_0^\nu\frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \\
&\qquad\times L(\tau)\left(\left|y_1(\tau)-y_2(\tau)\right|+\left|\mathcal{D}_qy_1(\tau)-\mathcal{D}_qy_2(\tau)\right|+\left|\mathcal{D}_q^2y_1(\tau)-\mathcal{D}_q^2y_2(\tau)\right|\right)\,\mathrm{d}_q\tau \\
&\qquad+\frac{(1+q)\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu\frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \\
&\qquad\times L(\tau)\left(\left|y_1(\tau)-y_2(\tau)\right|+\left|\mathcal{D}_qy_1(\tau)-\mathcal{D}_qy_2(\tau)\right|+\left|\mathcal{D}_q^2y_1(\tau)-\mathcal{D}_q^2y_2(\tau)\right|\right)\,\mathrm{d}_q\tau.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\sup_{t\in[0,1]}\left|\left(\mathcal{T}^{(2)}y_1\right)(t)-\left(\mathcal{T}^{(2)}y_2\right)(t)\right|\leq\|L\|\Delta^{(1)}\|y_1-y_2\|, \\
&\sup_{t\in[0,1]}\left|\left(\mathcal{D}_q\mathcal{T}^{(2)}y_1\right)(t)-\left(\mathcal{D}_q\mathcal{T}^{(2)}y_2\right)(t)\right|\leq\|L\|\Delta^{(2)}\|y_1-y_2\|, \\
&\sup_{t\in[0,1]}\left|\left(\mathcal{D}_q^2\mathcal{T}^{(2)}y_1\right)(t)-\left(\mathcal{D}_q^2\mathcal{T}^{(2)}y_2\right)(t)\right|\leq\|L\|\Delta^{(3)}\|y_1-y_2\|.
\end{aligned}$$

Thus, $\|\mathcal{T}^{(2)}y_1-\mathcal{T}^{(2)}y_2\|\leq\|L\|(\Delta^{(1)}+\Delta^{(2)}+\Delta^{(3)})\|y_1-y_2\|$ or $\|\mathcal{T}^{(2)}y_1-\mathcal{T}^{(2)}y_2\|\leq K\|y_1-y_2\|$. Hence, $\mathcal{T}^{(2)}$ is a contraction on V_ϵ with constant $K<1$. Now by using Theorem 4, the q-fractional boundary value problem (1)–(2) has at least one solution. □

Now we investigate the existence of solutions for the fractional q-difference inclusion problem (3)–(4). A function $y\in C_{\mathcal{X}}([0,1],\mathcal{X})$ is called a solution for the fractional q-difference inclusion (3) whenever it satisfies the boundary conditions and there exists a function $\Theta\in L^1([0,1])$ such that $\Theta(t)\in\mathcal{G}(t,y(t),\mathcal{D}_qy(t),\mathcal{D}_q^2y(t))$ for almost all $t\in[0,1]$ and

$$\begin{aligned}
y(t)={}&\int_0^t\frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\Theta(\tau)\,\mathrm{d}_q\tau \\
&+\frac{t\Delta_1-t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1\frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Theta(\tau)\,\mathrm{d}_q\tau \\
&+\frac{t^2\Delta_2-t\Delta_4}{\Delta_5}\int_0^1\frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\Theta(\tau)\,\mathrm{d}_q\tau \\
&+\frac{t^2\Delta_2-t\Delta_4}{\Delta_5}\int_0^\nu\frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\Theta(\tau)\,\mathrm{d}_q\tau
\end{aligned}$$

$$+ \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^\nu \frac{(\nu - q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta(\tau)\,\mathrm{d}_q\tau$$

for all $t \in [0,1]$. For each $y \in \mathcal{X}$, define the set of selections of the operator $\mathcal{G}$ by

$$\mathcal{S}_{\mathcal{G},y} = \{\Theta \in L^1([0,1]) : \Theta(t) \in \mathcal{G}(t, y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t)) \text{ for all } t \in [0,1]\}.$$

Also, consider the operator $\mathcal{N} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined by

$$\mathcal{N}(y) = \{p \in \mathcal{X} : \text{ there exists } \Theta \in \mathcal{S}_{\mathcal{G},y} : p(t) = w(t) \text{ for all } t \in [0,1]\}, \tag{13}$$

where

$$\begin{aligned} w(t) = {} & \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \Theta(\tau)\,\mathrm{d}_q\tau \\ & + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta(\tau)\,\mathrm{d}_q\tau \\ & + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \Theta(\tau)\,\mathrm{d}_q\tau \\ & + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \Theta(\tau)\,\mathrm{d}_q\tau \\ & + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta(\tau)\,\mathrm{d}_q\tau. \end{aligned}$$

Theorem 10 *Let $\mathcal{G} : [0,1] \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{P}_{cp}(\mathcal{X})$ be a set-valued map. Suppose that*:

($H6$) *The operator $\mathcal{G}$ is integrable bounded and $\mathcal{G}(\cdot, y_1, y_2, y_3) : [0,1] \to \mathcal{P}_{cp}(\mathcal{X})$ is measurable for all $y_1, y_2, y_3 \in \mathcal{X}$.*

($H7$) *There exist $m \in C([0,1],[0,\infty))$ and $\psi \in \Psi$ such that*

$$\begin{aligned} & \mathcal{H}_d(\mathcal{G}(t, y_1, y_2, y_3), \mathcal{G}(t, y_1', y_2', y_3')) \\ & \quad \le \frac{m(t)\lambda}{\|m\|} \psi(|y_1 - y_1'| + |y_2 - y_2'| + |y_3 - y_3'|), \end{aligned} \tag{14}$$

for all $t \in [0,1]$ and $y_1, y_2, y_3, y_1', y_2', y_3' \in \mathcal{X}$, where $\lambda = \frac{1}{\Xi_1 + \Xi_2 + \Xi_3}$ and the constants Ξ_1, Ξ_2 and Ξ_3 are given by (8).

($H8$) *There exists a function $\chi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ such that $\chi((y_1, y_2, y_3), (y_1, y_2', y_3')) \ge 0$ for all $y_1, y_2, y_3, y_1', y_2', y_3' \in \mathcal{X}$.*

($H9$) *If $\{y_n\}_{n\ge1}$ is a sequence in $\mathcal{X}$ with $y_n \to y$ and*

$$\chi((y_n(t), \mathcal{D}_q y_n(t), \mathcal{D}_q^2 y_n(t)), (y_{n+1}(t), \mathcal{D}_q y_{n+1}(t), \mathcal{D}_q^2 y_{n+1}(t))) \ge 0,$$

for all $t \in [0,1]$ and $n \ge 1$, then there exists a subsequence $\{y_{n_j}\}_{j\ge1}$ of $\{y_n\}$ such that

$$\chi((y_{n_j}(t), \mathcal{D}_q y_{n_j}(t), \mathcal{D}_q^2 y_{n_j}(t)), (y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t))) \ge 0,$$

for all $t \in [0,1]$ and $j \ge 1$.

(*H*10) *There exist* $y_0 \in \mathcal{X}$ *and* $p \in \mathcal{N}(y_0)$ *such that*

$$\chi\big(\big(y_0(t), \mathcal{D}_q y_0(t), \mathcal{D}_q^2 y_0(t)\big), \big(p(t), \mathcal{D}_q p(t), \mathcal{D}_q^2 p(t)\big)\big) \geq 0,$$

for all $t \in [0,1]$, *where the operator* $\mathcal{N} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ *is given by* (13).

(*H*11) *For each* $y \in \mathcal{X}$ *and* $p \in \mathcal{N}(y)$ *with*

$$\chi\big(\big(y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t)\big), \big(p(t), \mathcal{D}_q p(t), \mathcal{D}_q^2 p(t)\big)\big) \geq 0,$$

there exists $w \in \mathcal{N}(y)$ *such that*

$$\chi\big(\big(p(t), \mathcal{D}_q p(t), \mathcal{D}_q^2 p(t)\big), \big(w(t), \mathcal{D}_q w(t), \mathcal{D}_q^2 w(t)\big)\big) \geq 0,$$

for all $t \in [0,1]$. *Then the q-fractional inclusion problem* (3)–(4) *has a solution.*

Proof It is clear that each fixed point of the operator $\mathcal{N} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ is a solution for the q-fractional inclusion problem (3). Since the multivalued map $t \mapsto \mathcal{G}(t, y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t))$ is measurable and it is closed-valued for all $y \in \mathcal{X}$, $\mathcal{G}$ has a measurable selection and the set $\mathcal{S}_{\mathcal{G},y}$ is not empty. We show that the subset $\mathcal{N}(y)$ of $\mathcal{X}$ is closed for all $y \in \mathcal{X}$. Let $\{y_n\}_{n\geq 1}$ be a sequence in $\mathcal{N}(y)$ converging to y. For each n, there exists $\Theta_n \in \mathcal{S}_{\mathcal{G},y}$ such that

$$\begin{aligned}
y_n(t) = {} & \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \Theta_n(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta_n(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \Theta_n(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \Theta_n(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta_n(\tau)\,\mathrm{d}_q\tau,
\end{aligned}$$

for almost all $t \in [0,1]$. Since $\mathcal{G}$ has compact values, we pass into a subsequence (if necessary) to find that a subsequence $\{\Theta_n\}_{n\geq 1}$ that converges to some $\Theta \in L^1([0,1])$. Hence, $\Theta \in \mathcal{S}_{\mathcal{G},y}$ and

$$\begin{aligned}
y_n(t) \to {} & y(t) \\
= {} & \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \Theta(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \Theta(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \Theta(\tau)\,\mathrm{d}_q\tau
\end{aligned}$$

$$+ \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^\nu \frac{(\nu - q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta(\tau)\, \mathrm{d}_q\tau$$

for all $t \in [0,1]$. This shows that $y \in \mathcal{N}(y)$ and so the operator $\mathcal{N}$ is closed-valued. Since $\mathcal{G}$ is a multifunction with compact values, it is easy to prove that $\mathcal{N}(y)$ is bounded for all $y \in \mathcal{X}$. Now, we show that the operator $\mathcal{N}$ is an α-ψ-contractive set-valued map. For this purpose, define the non-negative function α on $\mathcal{X} \times \mathcal{X}$ by

$$\alpha(y,y') = \begin{cases} 1 & \text{if } \chi((y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t)), (y'(t), \mathcal{D}_q y'(t), \mathcal{D}_q^2 y'(t))) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y, y' \in \mathcal{X}$. Let $y, y' \in \mathcal{X}$ and $p_1 \in \mathcal{N}(y')$. Choose $\Theta_1 \in \mathcal{S}_{\mathcal{G},y'}$ such that

$$\begin{aligned} p_1(t) = {} & \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} \Theta_1(\tau)\, \mathrm{d}_q\tau \\ & + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta_1(\tau)\, \mathrm{d}_q\tau \\ & + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)} \Theta_1(\tau)\, \mathrm{d}_q\tau \\ & + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)} \Theta_1(\tau)\, \mathrm{d}_q\tau \\ & + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)} \int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)} \Theta_1(\tau)\, \mathrm{d}_q\tau, \end{aligned}$$

for all $t \in [0,1]$. By using (14), we have

$$\begin{aligned} & \mathcal{H}_d\big(\mathcal{G}(t, y, \mathcal{D}_q y, \mathcal{D}_q^2 y), \mathcal{G}(t, y', \mathcal{D}_q y', \mathcal{D}_q^2 y')\big) \\ & \quad \leq \frac{m(t)\lambda}{\|m\|} \psi\big(|y - y'| + |\mathcal{D}_q y - \mathcal{D}_q y'| + |\mathcal{D}_q^2 y - \mathcal{D}_q^2 y'|\big), \end{aligned}$$

for all $y, y' \in \mathcal{X}$ with

$$\chi\big((y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t)), (y'(t), \mathcal{D}_q y'(t), \mathcal{D}_q^2 y'(t))\big) \geq 0,$$

for almost all $t \in [0,1]$. Thus, there exists $w \in \mathcal{G}(t, y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t))$ such that

$$|\Theta_1(t) - w| \leq \frac{m(t)\lambda}{\|m\|} \psi\big(|y(t) - y'(t)| + |\mathcal{D}_q y(t) - \mathcal{D}_q y'(t)| + |\mathcal{D}_q^2 y(t) - \mathcal{D}_q^2 y'(t)|\big).$$

Now, consider the multivalued map $B : [0,1] \to \mathcal{P}(\mathcal{X})$ defined by

$$\begin{aligned} B(t) = \bigg\{ & w \in \mathcal{X} : |\Theta_1(t) - w| \leq \frac{m(t)\lambda}{\|m\|} \\ & \times \psi\big(|y(t) - y'(t)| + |\mathcal{D}_q y(t) - \mathcal{D}_q y'(t)| + |\mathcal{D}_q^2 y(t) - \mathcal{D}_q^2 y'(t)|\big) \bigg\} \end{aligned}$$

for all $t \in [0,1]$. Since Θ_1 and

$$\varphi = \frac{m\lambda}{\|m\|}\psi\left(\left|y - y'\right| + \left|\mathcal{D}_q y - \mathcal{D}_q y'\right| + \left|\mathcal{D}_q^2 y - \mathcal{D}_q^2 y'\right|\right)$$

are measurable, the multifunction $B(\cdot) \cap \mathcal{G}(\cdot, y(\cdot), \mathcal{D}_q y(\cdot), \mathcal{D}_q^2 y(\cdot))$ is measurable. Now, select Θ_2 in $\mathcal{G}(t, y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t))$ such that

$$\left|\Theta_1(t) - \Theta_2(t)\right| \le \frac{m(t)\lambda}{\|m\|}\psi\left(\left|y(t) - y'(t)\right| + \left|\mathcal{D}_q y(t) - \mathcal{D}_q y'(t)\right| + \left|\mathcal{D}_q^2 y(t) - \mathcal{D}_q^2 y'(t)\right|\right)$$

for all $t \in [0,1]$. Define $p_2 \in \mathcal{N}(u)$ by

$$\begin{aligned}
p_2(t) = {} & \int_0^t \frac{(t - q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\Theta_2(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1 \frac{(1 - q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Theta_2(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^1 \frac{(1 - q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\Theta_2(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^\nu \frac{(\nu - q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\Theta_2(\tau)\,\mathrm{d}_q\tau \\
& + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu \frac{(\nu - q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Theta_2(\tau)\,\mathrm{d}_q\tau
\end{aligned}$$

for all $t \in [0,1]$. Let $\sup_{t\in[0,1]} |m(t)| = \|m\|$. Then we have

$$\begin{aligned}
|p_1 - p_2| \le {} & \int_0^t \frac{(t - q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\left|\Theta_1(\tau) - \Theta_2(\tau)\right|\mathrm{d}_q\tau \\
& + \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1 \frac{(1 - q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\left|\Theta_1(\tau) - \Theta_2(\tau)\right|\mathrm{d}_q\tau \\
& + \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5}\int_0^1 \frac{(1 - q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\left|\Theta_1(\tau) - \Theta_2(\tau)\right|\mathrm{d}_q\tau \\
& + \frac{|t^2\Delta_2 - t\Delta_4|}{\Delta_5}\int_0^\nu \frac{(\nu - q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\left|\Theta_1(\tau) - \Theta_2(\tau)\right|\mathrm{d}_q\tau \\
& + \frac{|t\Delta_1 - t^2\Gamma_q(3-\sigma)|}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu \frac{(\nu - q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\left|\Theta_1(\tau) - \Theta_2(\tau)\right|\mathrm{d}_q\tau \\
\le {} & \frac{1}{\Gamma_q(\vartheta+1)}\|m\|\psi\left(\left\|y - y'\right\|\right)\left(\frac{p}{\|m\|}\right) \\
& + \frac{\Delta_1 + \Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\|m\|\psi\left(\left\|y - y'\right\|\right)\left(\frac{\lambda}{\|m\|}\right) \\
& + \frac{\Delta_2 + \Delta_4}{\Delta_5\Gamma_q(\vartheta-1)}\|m\|\psi\left(\left\|y - y'\right\|\right)\left(\frac{\lambda}{\|m\|}\right) \\
& + \frac{[\Delta_2 + \Delta_4]\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-1)}\|m\|\psi\left(\left\|y - y'\right\|\right)\left(\frac{\lambda}{\|m\|}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{[\Delta_1+\Gamma_q(3-\sigma)]\nu^{(\vartheta+\kappa)}}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\|m\|\psi\big(\|y-y'\|\big)\left(\frac{\lambda}{\|m\|}\right)\\
&=\left[\frac{1}{\Gamma_q(\vartheta+1)}+\frac{(\Delta_1+\Gamma_q(3-\sigma))(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}+\frac{\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)}\right.\\
&\quad\left.+\frac{(\Delta_2+\Delta_4)\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\right]\|m\|\psi\big(\|y-y'\|\big)\left(\frac{\lambda}{\|m\|}\right)\\
&=\Xi_1\|m\|\psi\big(\|y-y'\|\big)\left(\frac{\lambda}{\|m\|}\right)\\
&=\lambda\Xi_1\psi\big(\|y-y'\|\big).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
|\mathcal{D}_qp_1-\mathcal{D}_qp_2|&\le\left[\frac{1}{\Gamma_q(\vartheta)}+\frac{[\Delta_1+(1+q)\Gamma_q(3-\sigma)](\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\Gamma_q(3-\sigma)\Gamma_q(\vartheta+\kappa+1)}\right.\\
&\quad\left.+\frac{(1+q)\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\vartheta-1)}+\frac{[(1+q)\Delta_2+\Delta_4]\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\right]\\
&\quad\times\|m\|\psi\big(\|y-y'\|\big)\left(\frac{\lambda}{\|m\|}\right)\\
&=\Xi_2\|m\|\psi\big(\|y-y'\|\big)\left(\frac{\lambda}{\|m\|}\right)\\
&=\lambda\Xi_2\psi\big(\|y-y'\|\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\big|\mathcal{D}_q^2p_1-\mathcal{D}_q^2p_2\big|&\le\left[\frac{1}{\Gamma_q(\vartheta-1)}+\frac{(1+q)(\nu^{(\vartheta+\kappa)}+1)}{\Delta_3\Gamma_q(\vartheta+\kappa+1)}\right.\\
&\quad\left.+\frac{(1+q)\Delta_2}{\Delta_5\Gamma_q(\vartheta-1)}+\frac{(1+q)\Delta_2\nu^{(\vartheta-\sigma)}}{\Delta_5\Gamma_q(\vartheta-\sigma+1)}\right]\\
&\quad\times\|m\|\psi\big(\|y-y'\|\big)\left(\frac{\lambda}{\|m\|}\right)\\
&=\Xi_3\|m\|\psi\big(\|y-y'\|\big)\left(\frac{\lambda}{\|m\|}\right)\\
&=\lambda\Xi_3\psi\big(\|y-y'\|\big),
\end{aligned}
$$

for all $t\in[0,1]$. Hence, we obtain

$$
\begin{aligned}
\|p_1-p_2\|&=\sup_{t\in[0,1]}\big|p_1(t)-p_2(t)\big|+\sup_{t\in[0,1]}\big|\mathcal{D}_qp_1(t)-\mathcal{D}_qp_2(t)\big|\\
&\quad+\sup_{t\in[0,1]}\big|\mathcal{D}_q^2p_1(t)-\mathcal{D}_q^2p_2(t)\big|\\
&\le(\Xi_1+\Xi_2+\Xi_3)\lambda\psi\big(\|y-y'\|\big)=\psi\big(\|y-y'\|\big).
\end{aligned}
$$

Thus, $\alpha(y,y')\mathcal{H}_d(\mathcal{N}(y),\mathcal{N}(y'))\le\psi(\|y-y'\|)$ holds for all $y,y'\in\mathcal{X}$ which means that $\mathcal{N}$ is an α-ψ-contractive set-valued map. Now, consider two functions $y\in\mathcal{X}$ and $y'\in\mathcal{N}(y)$ such that $\alpha(y,y') \geq 1$. In this case,

$$\chi\left(\left(y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t)\right),\left(y'(t),\mathcal{D}_q y'(t),\mathcal{D}_q^2 y'(t)\right)\right) \geq 0,$$

so there exists a function $w \in \mathcal{N}(y')$ such that

$$\chi\left(\left(y'(t),\mathcal{D}_q y'(t),\mathcal{D}_q^2 y'(t)\right),\left(w(t),\mathcal{D}_q w(t),\mathcal{D}_q^2 w(t)\right)\right) \geq 0.$$

It follows that $\alpha(y',w) \geq 1$ and so the operator $\mathcal{N}$ is α-admissible. Now, let $y_0 \in \mathcal{X}$ and $y' \in \mathcal{N}(y_0)$ be such that

$$\chi\left(\left(y_0(t),\mathcal{D}_q y_0(t),\mathcal{D}_q^2 y_0(t)\right),\left(y'(t),\mathcal{D}_q y'(t),\mathcal{D}_q^2 y'(t)\right)\right) \geq 0,$$

for all t. Then we have $\alpha(y_0,y') \geq 1$. Suppose that $\{y_n\}_{n\geq 1}$ is a sequence in $\mathcal{X}$ with $y_n \to y$ and $\alpha(y_n,y_{n+1}) \geq 1$ for all n. Then we get

$$\chi\left(\left(y_n(t),\mathcal{D}_q y_n(t),\mathcal{D}_q^2 y_n(t)\right),\left(y_{n+1}(t),\mathcal{D}_q y_{n+1}(t),\mathcal{D}_q^2 y_{n+1}(t)\right)\right) \geq 0.$$

By using (H9), there exists a subsequence $\{y_{n_j}\}_{j\geq 1}$ of $\{y_n\}$ such that

$$\chi\left(\left(y_{n_j}(t),\mathcal{D}_q y_{n_j}(t),\mathcal{D}_q^2 y_{n_j}(t)\right),\left(y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t)\right)\right) \geq 0$$

for all $t \in [0,1]$. Thus, $\alpha(y_{n_j},y) \geq 1$ for all j. This means that the Banach space $\mathcal{X}$ has the property (C_α). Now by using Theorem 5, the map $\mathcal{N}$ has a fixed point which is a solution for the q-fractional inclusion problem (3)–(4). □

Theorem 11 *Let $\mathcal{G} : [0,1] \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{P}_{cp}(\mathcal{X})$ be a set-valued map. Suppose that*:

- $(H12)$ *The non-negative function $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing upper semi-continuous map such that $\liminf_{t\to\infty}(t-\psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$.*
- $(H13)$ *The operator $\mathcal{G} : [0,1] \times \mathcal{X}^3 \to \mathcal{P}_{cp}(\mathcal{X})$ is an integrable bounded multifunction such that $\mathcal{G}(\cdot,y_1,y_2,y_3) : [0,1] \to \mathcal{P}_{cp}(\mathcal{X})$ is measurable for all $y_1,y_2,y_3 \in \mathcal{X}$.*
- $(H14)$ *There exists a non-negative function $m \in C([0,1],[0,\infty))$ such that*

$$\begin{aligned}&\mathcal{H}_d\left(\mathcal{G}(t,y_1,y_2,y_3) - \mathcal{G}\left(t,y'_1,y'_2,y'_3\right)\right)\\&\quad\leq m(t)\lambda\psi\left(\left|y_1-y'_1\right| + \left|y_2-y'_2\right| + \left|y_3-y'_3\right|\right)\end{aligned}$$

  for all $t \in [0,1]$ and $y_1,y_2,y_3,y'_1,y'_2,y'_3 \in \mathcal{X}$, where $\lambda = \frac{1}{\Sigma_1+\Sigma_2+\Sigma_3}$ and the Σ_i are given by (10).
- $(H15)$ *The operator $\mathcal{N}$ has the approximate endpoint property where $\mathcal{N}$ is defined by (13).*

Then the q-fractional inclusion problem (3)–(4) has a solution.

Proof We show that the set-valued map $\mathcal{N} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ has an endpoint. First, we prove that $\mathcal{N}(y)$ is closed for all $y \in \mathcal{X}$. Since the map $t \mapsto \mathcal{G}(t,y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t))$ is measurable and has closed values for all $y \in \mathcal{X}$, it has a measurable selection and so $\mathcal{S}_{\mathcal{G},y} \neq \emptyset$ for all $y \in \mathcal{X}$. Similar to the proof of Theorem 10, we can show that the operator $\mathcal{N}(y)$ has closed values. Also, $\mathcal{N}(y)$ is a bounded set for all $y \in \mathcal{X}$ because $\mathcal{G}$ is a compact multivalued map.

Finally, one can show that $\mathcal{H}_d(\mathcal{N}(y),\mathcal{N}(w)) \leq \psi(\|y-w\|)$ holds. Let $y,w \in \mathcal{X}$ and $p_1 \in \mathcal{N}(w)$. Choose $\Theta_1 \in \mathcal{S}_{\mathcal{G},w}$ such that

$$\begin{aligned} p_1(t) &= \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\Theta_1(\tau)\,\mathrm{d}_q\tau \\ &\quad + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Theta_1(\tau)\,\mathrm{d}_q\tau \\ &\quad + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\Theta_1(\tau)\,\mathrm{d}_q\tau \\ &\quad + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\Theta_1(\tau)\,\mathrm{d}_q\tau \\ &\quad + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu \frac{(\nu-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Theta_1(\tau)\,\mathrm{d}_q\tau \end{aligned}$$

for almost all $t \in [0,1]$. Since

$$\begin{aligned} &\mathcal{H}_d\big(\mathcal{G}\big(t,y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t)\big) - \mathcal{G}\big(t,w(t),\mathcal{D}_q w(t),\mathcal{D}_q^2 w(t)\big)\big) \\ &\quad \leq m(t)\lambda\psi\big(\big|y(t)-w(t)\big| + \big|\mathcal{D}_q y(t) - \mathcal{D}_q w(t)\big| + \big|\mathcal{D}_q^2 y(t) - \mathcal{D}_q^2 w(t)\big|\big) \end{aligned}$$

for all $t \in [0,1]$, there exists $\bar{\sigma} \in \mathcal{G}(t,y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t))$ such that

$$\big|\Theta_1(t) - \bar{\sigma}\big| \leq m(t)\lambda\psi\big(\big|y(t)-w(t)\big| + \big|\mathcal{D}_q y(t) - \mathcal{D}_q w(t)\big| + \big|\mathcal{D}_q^2 y(t) - \mathcal{D}_q^2 w(t)\big|\big)$$

for all $t \in [0,1]$. Now, consider the multivalued map $O : [0,1] \to \mathcal{P}(\mathcal{X})$ which is defined by

$$\begin{aligned} O(t) = \big\{\bar{\sigma} \in \mathcal{X} : \big|\Theta_1(t) - \bar{\sigma}\big| &\leq m(t)\lambda\psi\big(\big|y(t)-w(t)\big| \\ &\quad + \big|\mathcal{D}_q y(t) - \mathcal{D}_q w(t)\big| + \big|\mathcal{D}_q^2 y(t) - \mathcal{D}_q^2 w(t)\big|\big)\big\}. \end{aligned}$$

By the measurability of Θ_1 and $\varphi = m\lambda\psi(|y-w| + |\mathcal{D}_q y - \mathcal{D}_q w| + |\mathcal{D}_q^2 y - \mathcal{D}_q^2 w|)$, it is easy to see that the multifunction $O(\cdot) \cap \mathcal{G}(\cdot, y(\cdot), \mathcal{D}_q y(\cdot), \mathcal{D}_q^2 y(\cdot))$ is measurable. Now, we choose $\Theta_2(t) \in \mathcal{G}(t,y(t),\mathcal{D}_q y(t),\mathcal{D}_q^2 y(t))$ such that

$$\begin{aligned} \big|\Theta_1(t) - \Theta_2(t)\big| &\leq m(t)\psi\big(\big|y(t)-w(t)\big| + \big|\mathcal{D}_q y(t) - \mathcal{D}_q w(t)\big| \\ &\quad + \big|\mathcal{D}_q^2 y(t) - \mathcal{D}_q^2 w(t)\big|\big)\bigg[\frac{1}{\Sigma_1 + \Sigma_2 + \Sigma_3}\bigg], \end{aligned}$$

for all $t \in [0,1]$. Select $p_2 \in \mathcal{N}(y)$ such that

$$\begin{aligned} p_2(t) &= \int_0^t \frac{(t-q\tau)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}\Theta_2(\tau)\,\mathrm{d}_q\tau \\ &\quad + \frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^1 \frac{(1-q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Theta_2(\tau)\,\mathrm{d}_q\tau \\ &\quad + \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^1 \frac{(1-q\tau)^{(\vartheta-3)}}{\Gamma_q(\vartheta-2)}\Theta_2(\tau)\,\mathrm{d}_q\tau \end{aligned}$$

$$+\frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^\nu \frac{(\nu - q\tau)^{(\vartheta-\sigma-1)}}{\Gamma_q(\vartheta-\sigma)}\Theta_2(\tau)\,\mathrm{d}_q\tau$$
$$+\frac{t\Delta_1 - t^2\Gamma_q(3-\sigma)}{\Delta_3\Gamma_q(3-\sigma)}\int_0^\nu \frac{(\nu - q\tau)^{(\vartheta+\kappa-1)}}{\Gamma_q(\vartheta+\kappa)}\Theta_2(\tau)\,\mathrm{d}_q\tau$$

for all $t \in [0,1]$. Thus by using a similar method in the proof of Theorem 10, we get

$$\begin{aligned}\|p_1 - p_2\| &= \sup_{t\in[0,1]}\big|p_1(t) - p_2(t)\big| + \sup_{t\in[0,1]}\big|\mathcal{D}_q p_1(t) - \mathcal{D}_q p_2(t)\big| \\ &\quad + \sup_{t\in[0,1]}\big|\mathcal{D}_q^2 p_1(t) - \mathcal{D}_q^2 p_2(t)\big| \\ &\le (\Sigma_1 + \Sigma_2 + \Sigma_3)\lambda\psi\big(\|y - w\|\big) = \psi\big(\|y - w\|\big).\end{aligned}$$

Hence $\mathcal{H}_d(\mathcal{N}(y),\mathcal{N}(w)) \le \psi(\|y - w\|)$ for all $y, w \in \mathcal{X}$. By using $(H15)$, we find that the multifunction $\mathcal{N}$ has approximate endpoint property. Now by using Theorem 6, there exists $y^* \in \mathcal{X}$ such that $\mathcal{N}(y^*) = \{y^*\}$. This implies that y^* is a solution for the q-fractional inclusion problem (3)–(4). □

4 Examples

Here, we provide two examples to illustrate our main results. Also we present a simplified analysis that can be executed to calculate the value of the q-Gamma function, $\Gamma_q(x)$, for input q, x and different values of n. To this aim, we consider a pseudo-code description of the method for the calculated q-Gamma function of order n in Algorithm 2 (for more details, see the link https://en.wikipedia.org/wiki/Q-gamma_function).

Example 1 Consider the q-fractional boundary value problem

$$\begin{aligned}{}^c\mathcal{D}_q^{\frac{7}{3}} y(t) &= \frac{t|\sin t|}{16(1+t^2)} + \frac{t|\cos(y(t))|}{16(1+|\cos(y(t))|)} + \frac{t^2|\mathcal{D}_q y(t)|}{16(1+|\mathcal{D}_q y(t)|)} \\ &\quad + \frac{t|\tan^{-1}(\mathcal{D}_q^2 y(t))|}{16(1+16|\tan^{-1}(\mathcal{D}_q^2 y(t))|)},\end{aligned} \tag{15}$$

with the three-point boundary value conditions

$$\begin{cases} y(0) = 0, \\ \mathcal{D}_q y(0) + {}^c\mathcal{D}_q^{\frac{8}{7}} y(\frac{3}{5}) + \mathcal{D}_q^2 y(1) = 0, \\ \mathcal{J}_q^3 y(0) + \mathcal{J}_q^3 y(\frac{3}{5}) + \mathcal{J}_q^3 y(1) = 0, \end{cases} \tag{16}$$

where $0 < q < 1$ and $0 < t < 1$. Put $\vartheta = \frac{7}{3}$, $\nu = \frac{3}{5}$ and $\sigma = \frac{8}{7}$ belonging to $(2,3]$, $(0,1)$ and $(1,2)$, respectively, and $\kappa = 3$. Here, ${}^c\mathcal{D}_q^{\frac{7}{3}}$ denotes the fractional q-derivative of the Caputo type of order $\frac{7}{3}$ and $\mathcal{J}_q^3$ denotes the fractional q-integral of the Riemann–Liouville type of order 3. Define the continuous map $\Phi : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi\big(t, x(t), y(t), z(t)\big) &= \frac{t|\sin t|}{16(1+t^2)} + \frac{t|\cos(x(t))|}{16(1+|\cos(x(t))|)} \\ &\quad + \frac{t^2|y(t)|}{16(1+|y(t)|)} + \frac{t|\tan^{-1} z(t)|}{16(1+|z(t)|)}.\end{aligned}$$

For each $x_1, x_2, y_1, y_2, z_1.z_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\Phi\big(t, x_1(t), y_1(t), z_1(t)\big) - \Phi\big(t, x_2(t), y_2(t), z_2(t)\big)\right| \\
&\quad \le \frac{t}{16}\Big(\left|\cos x_1(t) - \cos x_2(t)\right| + \left|y_1(t) - y_2(t)\right| \\
&\qquad + \left|\tan^{-1}\big(z_1(t)\big) - \tan^{-1}\big(z_2(t)\big)\right|\Big) \\
&\quad \le \frac{t}{16}\Big(\left|x_1(t) - x_2(t)\right| + \left|y_1(t) - \mathcal{D}_q y_2(t)\right| + \left|z_1(t) - z_2(t)\right|\Big).
\end{aligned}$$

Put $L(t) = \frac{t}{16}$ for all t. Then $\|L\| = \frac{t}{16}$. Consider the continuous and nondecreasing function $\psi : [0,1] \to \mathbb{R}^+$ defined by $\psi(x) = x$ for all $x \in \mathbb{R}^+$. Then we have

$$\begin{aligned}
\Phi\big(t, y(t), \mathcal{D}_q y(t), \mathcal{D}_q^2 y(t)\big) &\le \frac{t}{16}\big(|y| + |\mathcal{D}_q y| + \left|\mathcal{D}_q^2 y\right|\big) \\
&= \frac{t}{16}\psi\big(|y| + |\mathcal{D}_q y| + \left|\mathcal{D}_q^2 y\right|\big).
\end{aligned}$$

It is clear that $\zeta : [0,1] \to \mathbb{R}^+$ defined by $\zeta = \frac{t}{10}$ is continuous function. Now by using (6), we have

$$\begin{aligned}
\Delta_1 &= 2\left(\frac{3}{5}\right)^{2-\frac{8}{7}} + (1+q)\Gamma_q\left(3 - \frac{8}{7}\right) = 2\left(\frac{3}{5}\right)^{\frac{6}{7}} + (1+q)\Gamma_q\left(\frac{13}{7}\right), \\
\Delta_2 &= \left[1 + \left(\frac{3}{5}\right)^{3+1}\right]\Gamma_q\left(3 - \frac{8}{7}\right) = \left[1 + \left(\frac{3}{5}\right)^{4}\right]\Gamma_q\left(\frac{13}{7}\right), \\
\Delta_3 &= \left|\frac{-\Gamma_q(6)[(\frac{3}{5})^4 + 1]\Delta_1 + \Gamma_q(\frac{13}{7})\Gamma_q(5)(1+q)[(\frac{3}{5})^5 + 1]}{\Gamma_q(5)\Gamma_q(6)\Gamma_q(\frac{13}{7})}\right|, \\
\Delta_4 &= \left[1 + \left(\frac{3}{5}\right)^{4}\right]\Delta_1 + \Delta_3\Gamma_q(5)\Gamma_q\left(\frac{13}{7}\right), \\
\Delta_5 &= \Delta_3\Gamma_q\left(3 - \frac{8}{7}\right)\Gamma_q(3+2) = \Delta_3\Gamma_q\left(\frac{13}{7}\right)\Gamma_q(5),
\end{aligned}$$

and by applying (9), we obtain

$$\begin{aligned}
\Delta^{(1)} &= \frac{(\Delta_1 + \Gamma_q(3 - \frac{8}{7}))((\frac{3}{5})^{(\frac{7}{3}+3)} + 1)}{\Delta_3\Gamma_q(3 - \frac{8}{7})\Gamma_q(\frac{7}{3} + 3 + 1)} + \frac{\Delta_2 + \Delta_4}{\Delta_5\Gamma_q(\frac{7}{3} - 1)} \\
&\quad + \frac{(\Delta_2 + \Delta_4)(\frac{3}{5})^{(\frac{7}{3} - \frac{8}{7})}}{\Delta_5\Gamma_q(\frac{7}{3} - \frac{8}{7} + 1)}, \\
&= \frac{(\Delta_1 + \Gamma_q(\frac{13}{7}))((\frac{3}{5})^{\frac{16}{3}} + 1)}{\Delta_3\Gamma_q(\frac{13}{7})\Gamma_q(\frac{19}{3})} + \frac{\Delta_2 + \Delta_4}{\Delta_5\Gamma_q(\frac{4}{3})} + \frac{(\Delta_2 + \Delta_4)(\frac{3}{5})^{\frac{25}{21}}}{\Delta_5\Gamma_q(\frac{46}{21})}, \\
\Delta^{(2)} &= \frac{[\Delta_1 + (1+q)\Gamma_q(3 - \frac{8}{7})]((\frac{3}{5})^{(\frac{7}{3}+3)} + 1)}{\Delta_3\Gamma_q(3 - \frac{8}{7})\Gamma_q(\frac{7}{3} + 3 + 1)} + \frac{(1+q)\Delta_2 + \Delta_4}{\Delta_5\Gamma_q(\frac{7}{3} - 1)} \\
&\quad + \frac{[(1+q)\Delta_2 + \Delta_4](\frac{3}{5})^{(\frac{7}{3} - \frac{8}{7})}}{\Delta_5\Gamma_q(\frac{7}{3} - \frac{8}{7} + 1)},
\end{aligned}$$

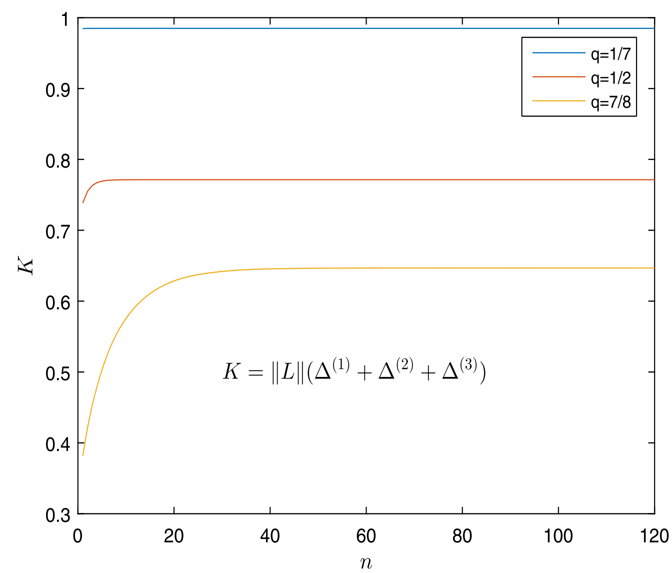

Figure 1 Numerical results of K for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$ in Example 1

$$= \frac{[\Delta_1 + (1+q)\Gamma_q(\frac{13}{7})]((\frac{3}{5})^{\frac{16}{3}} + 1)}{\Delta_3 \Gamma_q(\frac{13}{7})\Gamma_q(\frac{19}{3})} + \frac{(1+q)\Delta_2 + \Delta_4}{\Delta_5 \Gamma_q(\frac{4}{3})}$$

$$+ \frac{[(1+q)\Delta_2 + \Delta_4](\frac{3}{5})^{\frac{25}{21}}}{\Delta_5 \Gamma_q(\frac{46}{21})},$$

$$\Delta^{(3)} = \frac{(1+q)((\frac{3}{5})^{(\frac{7}{3}+3)} + 1)}{\Delta_3 \Gamma_q(\frac{7}{3} + 3 + 1)} + \frac{(1+q)\Delta_2}{\Delta_5 \Gamma_q(\frac{7}{3} - 1)} + \frac{(1+q)\Delta_2(\frac{3}{5})^{(\frac{7}{3} - \frac{8}{7})}}{\Delta_5 \Gamma_q(\frac{7}{3} - \frac{8}{7} + 1)},$$

$$= \frac{(1+q)((\frac{3}{5})^{\frac{19}{3}} + 1)}{\Delta_3 \Gamma_q(\frac{19}{3})} + \frac{(1+q)\Delta_2}{\Delta_5 \Gamma_q(\frac{4}{3})} + \frac{(1+q)\Delta_2(\frac{3}{5})^{\frac{25}{21}}}{\Delta_5 \Gamma_q(\frac{46}{21})}.$$

In the last section, according to Table 4, we obtain $\Delta_1 \approx 2.4176, 2.7391, 3.0751$, $\Delta_2 \approx 1.1136, 1.0906, 1.0749$, $\Delta_3 \approx 1.1057, 0.4816, 0.1911$, $\Delta_4 \approx 4.4208, 5.3826, 6.4537$, $\Delta_5 \approx 1.6900, 2.885, 2.9801$ for $q = \frac{1}{7}, \frac{1}{2}, \frac{7}{8}$, respectively. Also, Table 5 shows the values of the $\Delta^{(i)}$ as follows: $\Delta^{(1)} \approx 6.8888, 5.2250, 4.3006$, $\Delta^{(2)} \approx 7.1089, 5.6979, 4.8565$, $\Delta^{(3)} \approx 1.7608, 1.4188, 1.1913$ for $q = \frac{1}{7}, \frac{1}{2}, \frac{7}{8}$, respectively, and values of K in (11) for $q = \frac{1}{7}, \frac{1}{2}$ and $\frac{7}{8}$ are shown in Table 6 as $K_{q_1} := \|L\|(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}) = 0.9849 < 1$, $K_{q_2} := \|L\|(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}) = 0.7714 < 1$ and $K_{q_3} := \|L\|(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}) = 0.6468 < 1$, respectively (Algorithm 6). Now by using Theorem 9, the q-fractional boundary value problem (15)–(16) has a solution.

Example 2 Consider the q-fractional inclusion problem

$$^{c}\mathcal{D}_q^{\frac{14}{5}} y(t) \in \left[0, \frac{t \sin^2 y(t)}{25(1+2t^2)} + \frac{2(t+1)|\cos(\mathcal{D}_q y(t))|}{75(3 + |\cos(\mathcal{D}_q y(t))|)} + \frac{t|\mathcal{D}_q^2 y(t)|}{25(2 + |\mathcal{D}_q^2 y(t)|)}\right], \quad (17)$$

Table 1 Some numerical results for calculation of $\Gamma_q(x)$ with constant $q = \frac{1}{3}$, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \ldots, 15$ of Algorithm 2

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	2.340263	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	11.257095	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	64.350881
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table 2 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

Table 3 Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	11.257095	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	49.065751	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	259.967394
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

Table 4 Some numerical results of the Δ_i in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{7}{8}$

n	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
$q = \frac{1}{7}$					
1	2.4207	1.1167	1.0666	4.0611	1.6404
1	2.4207	1.1167	1.0993	4.425	1.6906
2	2.418	1.1141	1.1047	4.4214	1.6901
3	2.4176	1.1137	1.1055	4.4209	1.69
4	2.4176	1.1136	1.1056	4.4208	1.69
5	2.4176	1.1136	1.1057	4.4208	1.69
6	2.4176	1.1136	1.1057	4.4208	1.69
7	2.4176	1.1136	1.1057	4.4208	1.69
$q = \frac{1}{2}$					
1	2.9236	1.2296	0.3506	5.7039	2.4014
2	2.8254	1.1556	0.4124	5.5329	2.3413
3	2.7809	1.1221	0.4461	5.4554	2.3141
⋮	⋮	⋮	⋮	⋮	⋮
10	2.7394	1.0909	0.4813	5.3831	2.2887
11	2.7393	1.0907	0.4814	5.3828	2.2886
12	2.7392	1.0907	0.4815	5.3827	2.2885
13	2.7391	1.0907	0.4815	5.3826	2.2885
14	2.7391	1.0906	0.4816	5.3826	2.2885
15	2.7391	1.0906	0.4816	5.3826	2.2885
16	2.7391	1.0906	0.4816	5.3826	2.2885
$q = \frac{7}{8}$					
1	5.978	2.8238	0.0053	12.7182	5.9654
2	5.1343	2.3155	0.0102	10.9018	5.102
3	4.6238	2.0079	0.0168	9.8013	4.5783
⋮	⋮	⋮	⋮	⋮	⋮
71	3.0752	1.075	0.191	6.4539	2.9802
72	3.0751	1.075	0.191	6.4539	2.9802
73	3.0751	1.075	0.191	6.4539	2.9802
74	3.0751	1.0749	0.191	6.4538	2.9802
75	3.0751	1.0749	0.191	6.4538	2.9802
76	3.0751	1.0749	0.191	6.4538	2.9802
77	3.0751	1.0749	0.191	6.4538	2.9802
78	3.0751	1.0749	0.191	6.4538	2.9802
79	3.0751	1.0749	0.191	6.4538	2.9801
80	3.0751	1.0749	0.1911	6.4538	2.9801
81	3.0751	1.0749	0.1911	6.4538	2.9801
82	3.0751	1.0749	0.1911	6.4538	2.9801
83	3.0751	1.0749	0.1911	6.4538	2.9801
84	3.0751	1.0749	0.1911	6.4537	2.9801
85	3.0751	1.0749	0.1911	6.4537	2.9801
86	3.0751	1.0749	0.1911	6.4537	2.9801

with three-point boundary value conditions

$$\begin{cases} y(0) = 0, \\ \mathcal{D}_q y(0) + {}^c\mathcal{D}_q^{\frac{7}{4}} y(\frac{1}{6}) + \mathcal{D}_q^2 y(1) = 0, \\ \mathcal{J}_q^4 y(0) + \mathcal{J}_q^4 y(\frac{1}{6}) + \mathcal{J}_q^4 y(1) = 0, \end{cases} \tag{18}$$

where $0 < q < 1$ and $t \in [0,1]$. Put $\vartheta = \frac{14}{5}$, $\nu = \frac{1}{6}$ and $\sigma = \frac{7}{4}$ belongs to $(2,3]$, $(0,1)$ and $(1,2)$, respectively, and $\kappa = 4$. Here, ${}^c\mathcal{D}_q^{\frac{14}{5}}$ denotes the fractional q-derivative of the Caputo type and $\mathcal{J}_q^4$ is the fractional q-integral of the Riemann–Liouville type. Now, define the

Table 5 Some numerical results for calculation of the $\Delta^{(i)}$ in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{7}{8}$

	$q = \frac{1}{7}$			$q = \frac{1}{2}$			$q = \frac{7}{8}$		
n	$\Delta^{(1)}$	$\Delta^{(2)}$	$\Delta^{(3)}$	$\Delta^{(1)}$	$\Delta^{(2)}$	$\Delta^{(3)}$	$\Delta^{(1)}$	$\Delta^{(2)}$	$\Delta^{(3)}$
1	6.8851	7.1054	1.7625	4.9587	5.4326	1.4216	2.4252	2.8232	0.8529
2	6.8883	7.1084	1.7611	5.0932	5.5666	1.4203	2.701	3.1283	0.9157
3	6.8888	7.1089	1.7609	5.1594	5.6326	1.4196	2.9255	3.3747	0.9626
4	6.8888	7.1089	1.7608	5.1922	5.6653	1.4192	3.1128	3.5791	0.9992
5	6.8888	7.1089	1.7608	5.2086	5.6816	1.419	3.2714	3.7514	1.0286
6	6.8888	7.1089	1.7608	5.2168	5.6898	1.4189	3.4068	3.8981	1.0527
7	6.8888	7.1089	1.7608	5.2209	5.6938	1.4188	3.5232	4.0238	1.0726
8	6.8888	7.1089	1.7608	5.2229	5.6959	1.4188	3.6237	4.132	1.0893
9	6.8888	7.1089	1.7608	5.2239	5.6969	1.4188	3.7106	4.2256	1.1034
10	6.8888	7.1089	1.7608	5.2245	5.6974	1.4188	3.7861	4.3066	1.1154
11	6.8888	7.1089	1.7608	5.2247	5.6976	1.4188	3.8516	4.3769	1.1256
12	6.8888	7.1089	1.7608	5.2248	5.6978	1.4188	3.9086	4.438	1.1343
13	6.8888	7.1089	1.7608	5.2249	5.6978	1.4188	3.9583	4.4911	1.1418
14	6.8888	7.1089	1.7608	5.2249	5.6979	1.4188	4.0015	4.5374	1.1483
15	6.8888	7.1089	1.7608	5.2249	5.6979	1.4188	4.0393	4.5778	1.1539
16	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.0722	4.6129	1.1587
17	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.101	4.6437	1.1629
18	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.1261	4.6705	1.1666
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
65	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3002	4.8562	1.1912
66	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3003	4.8562	1.1913
67	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3003	4.8562	1.1913
68	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3003	4.8563	1.1913
69	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3004	4.8563	1.1913
70	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3004	4.8563	1.1913
71	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3004	4.8563	1.1913
72	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3004	4.8564	1.1913
73	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8564	1.1913
74	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8564	1.1913
75	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8564	1.1913
76	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8564	1.1913
77	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8564	1.1913
78	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8564	1.1913
79	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8564	1.1913
80	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
81	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
82	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
83	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
84	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
85	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
86	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
87	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3005	4.8565	1.1913
88	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3006	4.8565	1.1913
89	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3006	4.8565	1.1913
90	6.8888	7.1089	1.7608	5.225	5.6979	1.4188	4.3006	4.8565	1.1913

set-valued map $\mathcal{G} : [0,1] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{G}\big(t, x(t), y(t), z(t)\big) = \left[0, \frac{t \sin^2 x(t)}{25(1+2t^2)} + \frac{2(t+1)|\cos(y(t))|}{75(3+|\cos(y(t))|)} + \frac{t|z(t)|}{25(2+|z(t)|)}\right],$$

for all $t \in [0,1]$. Choose the non-negative function $m \in C([0,1],[0,\infty))$ defined by $m(t) = \frac{t}{25}$ for all t. Then $\|m\| = \frac{1}{25}$. Also, consider the non-negative and nondecreasing upper semi-continuous function $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = \frac{t}{3}$ for almost all $t > 0$. It is clear that $\liminf_{t\to\infty}(t - \psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$. On the other hand, by applying

Table 6 Some numerical results of the $\Delta^{(i)}$ and K in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{7}{8}$

n	$\Delta^{(1)}$	$\Delta^{(2)}$	$\Delta^{(3)}$	K
$q = \frac{1}{7}$				
1	6.8851	7.1054	1.7625	0.9846
2	6.8883	7.1084	1.7611	0.9849
3	6.8888	7.1089	1.7609	0.9849
4	6.8888	7.1089	1.7608	0.9849
5	6.8888	7.1089	1.7608	0.9849
6	6.8888	7.1089	1.7608	0.9849
$q = \frac{1}{2}$				
1	4.9587	5.4326	1.4216	0.7383
2	5.0932	5.5666	1.4203	0.755
3	5.1594	5.6326	1.4196	0.7632
4	5.1922	5.6653	1.4192	0.7673
5	5.2086	5.6816	1.419	0.7693
6	5.2168	5.6898	1.4189	0.7703
7	5.2209	5.6938	1.4188	0.7708
8	5.2229	5.6959	1.4188	0.7711
9	5.2239	5.6969	1.4188	0.7712
10	5.2245	5.6974	1.4188	0.7713
11	5.2247	5.6976	1.4188	0.7713
12	5.2248	5.6978	1.4188	0.7713
13	5.2249	5.6978	1.4188	0.7713
14	5.2249	5.6979	1.4188	0.7713
15	5.2249	5.6979	1.4188	0.7714
16	5.225	5.6979	1.4188	0.7714
17	5.225	5.6979	1.4188	0.7714
18	5.225	5.6979	1.4188	0.7714
$q = \frac{7}{8}$				
1	2.4252	2.8232	0.8529	0.3813
2	2.701	3.1283	0.9157	0.4216
3	2.9255	3.3747	0.9626	0.4539
⋮	⋮	⋮	⋮	⋮
64	4.3002	4.8561	1.1912	0.6467
65	4.3002	4.8562	1.1912	0.6467
66	4.3003	4.8562	1.1913	0.6467
67	4.3003	4.8562	1.1913	0.6467
68	4.3003	4.8563	1.1913	0.6467
69	4.3004	4.8563	1.1913	0.6467
70	4.3004	4.8563	1.1913	0.6467
71	4.3004	4.8563	1.1913	0.6468
72	4.3004	4.8564	1.1913	0.6468
73	4.3005	4.8564	1.1913	0.6468
74	4.3005	4.8564	1.1913	0.6468
75	4.3005	4.8564	1.1913	0.6468

Eq. (8), we have

$$\begin{aligned}
\Delta_1 &= 2\left(\frac{1}{6}\right)^{(2-\frac{7}{4})} + (1+q)\Gamma_q\left(3-\frac{7}{4}\right) = 2\left(\frac{1}{6}\right)^{\frac{1}{4}} + (1+q)\Gamma_q\left(\frac{5}{4}\right), \\
\Delta_2 &= \left(1+\left(\frac{1}{6}\right)^{4+1}\right)\Gamma_q\left(3-\frac{7}{4}\right) = \left(1+\left(\frac{1}{6}\right)^{5}\right)\Gamma_q\left(\frac{5}{4}\right), \\
\Delta_3 &= \left|\frac{-\Gamma_q(7)((\frac{1}{6})^5+1)\Delta_1 + \Gamma_q(\frac{5}{4})\Gamma_q(6)(1+q)[(\frac{1}{6})^6+1]}{\Gamma_q(6)\Gamma_q(7)\Gamma_q(\frac{5}{4})}\right|, \\
\Delta_4 &= \left(1+\left(\frac{1}{6}\right)^{5}\right)\Delta_1 + \Delta_3\Gamma_q(6)\Gamma_q\left(\frac{5}{4}\right),
\end{aligned}$$

Table 7 Some numerical results for calculation of the $\varXi$ in Example 2 for $q = \frac{1}{7}, \frac{1}{2}, \frac{7}{8}$

	$q=\frac{1}{7}$			$q=\frac{1}{2}$			$q=\frac{7}{8}$		
n	$\varXi_1$	$\varXi_2$	$\varXi_3$	$\varXi_1$	$\varXi_2$	$\varXi_3$	$\varXi_1$	$\varXi_2$	$\varXi_3$
1	6.4372	3.9299	1.2544	6.7478	4.5268	1.5052	2.5011	1.9273	0.7942
2	6.446	4.1478	1.5216	6.757	4.7764	1.8348	2.5041	2.0191	0.9492
3	6.4473	4.257	1.7524	6.7583	4.9013	2.1214	2.5045	2.0646	1.0795
4	6.4475	4.3116	1.9534	6.7585	4.9637	2.372	2.5046	2.0872	1.1902
5	6.4475	4.339	2.1292	6.7585	4.995	2.5916	2.5046	2.0985	1.2851
6	6.4475	4.3527	2.2834	6.7585	5.0106	2.7842	2.5046	2.1041	1.3667
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	6.4475	4.3661	2.8856	6.7585	5.0259	3.5351	2.5046	2.1096	1.6718
13	6.4475	4.3662	2.9479	6.7585	5.0261	3.6124	2.5046	2.1097	1.7021
14	6.4475	4.3663	3.0025	6.7585	5.0261	3.6801	2.5046	2.1097	1.7285
15	6.4475	4.3663	3.0505	6.7585	5.0262	3.7395	2.5046	2.1097	1.7515
16	6.4475	4.3663	3.0926	6.7585	5.0262	3.7916	2.5046	2.1097	1.7717
17	6.4475	4.3663	3.1295	6.7585	5.0262	3.8372	2.5046	2.1097	1.7892
18	6.4475	4.3663	3.1619	6.7585	5.0262	3.8772	2.5046	2.1097	1.8046
19	6.4475	4.3664	3.1903	6.7585	5.0262	3.9122	2.5046	2.1097	1.8179
20	6.4475	4.3664	3.2152	6.7585	5.0262	3.9428	2.5046	2.1097	1.8296
21	6.4475	4.3664	3.237	6.7585	5.0262	3.9697	2.5046	2.1097	1.8399
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
66	6.4475	4.3664	3.3901	6.7585	5.0262	4.1579	2.5046	2.1097	1.9109
67	6.4475	4.3664	3.3902	6.7585	5.0262	4.158	2.5046	2.1097	1.911
68	6.4475	4.3664	3.3902	6.7585	5.0262	4.158	2.5046	2.1097	1.911
69	6.4475	4.3664	3.3902	6.7585	5.0262	4.1581	2.5046	2.1097	1.911
70	6.4475	4.3664	3.3903	6.7585	5.0262	4.1581	2.5046	2.1097	1.911
71	6.4475	4.3664	3.3903	6.7585	5.0262	4.1582	2.5046	2.1097	1.911
72	6.4475	4.3664	3.3903	6.7585	5.0262	4.1582	2.5046	2.1097	1.911
73	6.4475	4.3664	3.3903	6.7585	5.0262	4.1582	2.5046	2.1097	1.9111
74	6.4475	4.3664	3.3904	6.7585	5.0262	4.1582	2.5046	2.1097	1.9111
75	6.4475	4.3664	3.3904	6.7585	5.0262	4.1583	2.5046	2.1097	1.9111
76	6.4475	4.3664	3.3904	6.7585	5.0262	4.1583	2.5046	2.1097	1.9111
77	6.4475	4.3664	3.3904	6.7585	5.0262	4.1583	2.5046	2.1097	1.9111

$$\Delta_5 = \Delta_3 \varGamma_q\left(3 - \frac{7}{4}\right)\varGamma_q(4+2) = \Delta_3 \varGamma_q\left(\frac{5}{4}\right)\varGamma_q(6),$$

and by using Eq. (9), we obtain

$$\begin{aligned}
\varXi_1 &= \frac{1}{\varGamma_q(\frac{14}{5}+1)} + \frac{(\Delta_1 + \varGamma_q(3-\frac{7}{4}))((\frac{1}{6})^{(\frac{14}{5}+4)}+1)}{\Delta_3\varGamma_q(3-\frac{7}{4})\varGamma_q(\frac{14}{5}+4+1)} \\
&\quad + \frac{\Delta_2+\Delta_4}{\Delta_5\varGamma_q(\frac{14}{5}-1)} + \frac{(\Delta_2+\Delta_4)(\frac{1}{6})^{(\frac{14}{5}-\frac{7}{4})}}{\Delta_5\varGamma_q(\frac{14}{5}-\frac{7}{4}+1)} \\
&= \frac{1}{\varGamma_q(\frac{19}{5})} + \frac{(\Delta_1+\varGamma_q(\frac{5}{4}))((\frac{1}{6})^{(\frac{34}{5})}+1)}{\Delta_3\varGamma_q(\frac{5}{4})\varGamma_q(\frac{39}{5})} \\
&\quad + \frac{\Delta_2+\Delta_4}{\Delta_5\varGamma_q(\frac{9}{5})} + \frac{(\Delta_2+\Delta_4)(\frac{1}{6})^{\frac{21}{20}}}{\Delta_5\varGamma_q(\frac{41}{20})}, \\
\varXi_2 &= \frac{1}{\varGamma_q(\frac{14}{5})} + \frac{[\Delta_1+(1+q)\varGamma_q(3-\frac{7}{4})]((\frac{1}{6})^{(\frac{14}{5}+4)}+1)}{\Delta_3\varGamma_q(3-\frac{7}{4})\varGamma_q(\frac{14}{5}+4+1)}
\end{aligned}$$

Algorithm 6 The proposed method for calculated Δ_i and $\Delta^{(i)}$ in Example 1

```
function  [Deltausubi, Xi, Deltausupi]= solveinclusionproblem(q, ...
    nu, sigma, kappa, k, vartheta, supL)
[xq yq]=size(q);
for n=1:k
    Deltausubi(n,1)=n;
    Xi(n,1)=n;
    Deltausupi(n,1)=n;
end;
%$\Delta_i$
column=2;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=2* power(nu, 2-sigma)+ (1+ q(s)) * ...
            qGamma(q(s), 3-sigma, n);
    end;
    column=column+1;
end;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=(1+  power(nu, kappa+1)) * ...
            qGamma(q(s), 3-sigma, n);
    end;
    column=column+1;
end;
%column=2*(1+yq);
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=abs(( (-1)*qGamma(q(s), kappa+3, ...
            n)*(1+  power(nu, kappa+1))*Deltausubi(n, column-2*yq) ...
            + qGamma(q(s), 3-sigma, n)* qGamma(q(s), kappa+2, ...
            n)*(1+q(s))*(1+power(nu, kappa+2)))/(qGamma(q(s), ...
            kappa+2, n) * qGamma(q(s), kappa+3, n)* qGamma(q(s), ...
            3-sigma, n)));
    end;
    column=column+1;
end;
%column=2+3*yq;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=(1+  power(nu, ...
            kappa+1))*Deltausubi(n, column-3*yq) + Deltausubi(n, ...
            column-yq)* qGamma(q(s), kappa+2, n)* qGamma(q(s), ...
            3-sigma, n);
    end;
    column=column+1;
end;
%column=2+4*yq;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=Deltausubi(n, column-2*yq)* ...
            qGamma(q(s), 3-sigma, n)* qGamma(q(s), kappa+2, n);
    end;
    column=column+1;
end;
%$\Delta^{(i)}$
column=2;
for s=1:yq
    for n=1:k
        Deltausupi(n, column)=((Deltausubi(n, column) + ...
            qGamma(q(s), 3-sigma, n))*(power(nu, ...
            vartheta+kappa)+1))/(Deltausubi(n, ...
            column+2*yq)*qGamma(q(s), 3-sigma, n)*qGamma(q(s), ...
            vartheta+kappa+1, n)) + (Deltausubi(n, column+yq) + ...
            Deltausubi(n, column+3*yq))/(Deltausubi(n, column+4*yq) ...
            * qGamma(q(s), vartheta-1, n)) + ((Deltausubi(n, ...
            column+yq) + Deltausubi(n, column+3*yq))* power(nu, ...
            vartheta-sigma))/ (Deltausubi(n, column+4*yq) * ...
            qGamma(q(s), vartheta-sigma +1, n));
    end;
    column=column+1;
end;
%column=5
for s=1:yq
    for n=1:k
```

Algorithm 6 *(Continued)*

```
        Deltausupi(n, column)=(Deltausubi(n, column-yq)+(1+q(s))*  ...
            qGamma(q(s), 3-sigma, n))*(power(nu, ...
            vartheta+kappa)+1)/(Deltausubi(n, ...
            column+yq)*qGamma(q(s), 3-sigma, n)*qGamma(q(s), ...
            vartheta+kappa+1, n)) + ((1+q(s))*Deltausubi(n, column) ...
            + Deltausubi(n, column+2*yq))/(Deltausubi(n, ...
            column+3*yq) * qGamma(q(s), vartheta-1, n)) + ...
            (((1+q(s))* Deltausubi(n, column) + Deltausubi(n, ...
            column+2*yq))* power(nu, ...
            vartheta-sigma))/(Deltausubi(n, column+3*yq) * ...
            qGamma(q(s), vartheta-sigma +1, n));
    end;
    column=column+1;
end;
%column=8
for s=1:yq
    for n=1:k
        Deltausupi(n, column)=( (1+q(s))*(power(nu, vartheta+kappa) ...
            + 1 ) )/(Deltausubi(n, column)*qGamma(q(s), ...
            vartheta+kappa+1, n)) + ((1+q(s))*Deltausubi(n, ...
            column-yq))/(Deltausubi(n, column+2*yq)*qGamma(q(s), ...
            vartheta-1, n)) + ( (1+q(s))* Deltausubi(n, ...
            column-yq)*power(nu, vartheta-sigma))/(Deltausubi(n, ...
            column+2*yq) * qGamma(q(s), vartheta-sigma +1, n));
    end;
    column=column+1;
end;
%column=11;
for s=1:yq
    for n=1:k
        Deltausupi(n, column)=supL * (Deltausupi(n, ...
            column-3*yq)+Deltausupi(n, column-2*yq) + Deltausupi(n, ...
            column-yq));
    end;
    column=column+1;
end;
end
```

$$
\begin{aligned}
&+\frac{(1+q)\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\frac{14}{5}-1)}+\frac{[(1+q)\Delta_2+\Delta_4](\frac{1}{6})^{(\frac{14}{5}-\frac{7}{4})}}{\Delta_5\Gamma_q(\frac{14}{5}-\frac{7}{4}+1)}\\
&=\frac{1}{\Gamma_q(\frac{14}{5})}+\frac{[\Delta_1+(1+q)\Gamma_q(\frac{5}{4})]((\frac{1}{6})^{\frac{34}{5}}+1)}{\Delta_3\Gamma_q(\frac{5}{4})\Gamma_q(\frac{39}{5})}\\
&\quad+\frac{(1+q)\Delta_2+\Delta_4}{\Delta_5\Gamma_q(\frac{9}{5})}+\frac{[(1+q)\Delta_2+\Delta_4](\frac{1}{6})^{(\frac{21}{20})}}{\Delta_5\Gamma_q(\frac{41}{20})},
\end{aligned}
$$

$$
\begin{aligned}
\Xi_3&=\frac{1}{\Gamma_q(\frac{14}{5}-1)}+\frac{(1+q)((\frac{1}{6})^{(\frac{14}{5}+4)}+1)}{\Delta_3\Gamma_q(\frac{14}{5}+4+1)}\\
&\quad+\frac{(1+q)\Delta_2}{\Delta_5\Gamma_q(\frac{14}{5}-1)}+\frac{(1+q)\Delta_2(\frac{1}{6})^{(\frac{14}{5}-\frac{7}{4})}}{\Delta_5\Gamma_q(\frac{14}{5}-\frac{7}{4}+1)}\\
&=\frac{1}{\Gamma_q(\frac{9}{5})}+\frac{(1+q)((\frac{1}{6})^{(\frac{34}{5})}+1)}{\Delta_3\Gamma_q(\frac{39}{5})}+\frac{(1+q)\Delta_2}{\Delta_5\Gamma_q(\frac{9}{5})}+\frac{(1+q)\Delta_2(\frac{1}{6})^{\frac{21}{20}}}{\Delta_5\Gamma_q(\frac{41}{20})},
\end{aligned}
$$

$\Sigma_1=\frac{1}{25}\Xi_1$, $\Sigma_2=\frac{1}{25}\Xi_2$ and $\Sigma_3=\frac{1}{25}\Xi_3$. Table 7 shows the numerical results of the Ξ_i for $q=\frac{1}{7}$, $\frac{1}{2}$ and $\frac{7}{8}$, respectively, as follows: $\Xi_1\approx 6.4475, 6.7585, 2.5046$, $\Xi_2\approx 4.3664, 5.0262, 2.1097$ and $\Xi_3\approx 3.3904, 4.1583, 1.19111$ (Algorithm 7). For each $x_1,x_2,y_1,y_2,z_1,z_2\in\mathbb{R}$, we have

$$
\mathcal{H}_d\big(\mathcal{G}\big(t,x_1(t),y_1(t),z_1(t)\big)-\mathcal{G}\big(t,x_2(t),y_2(t),z_2(t)\big)\big)
$$

Algorithm 7 The proposed method for calculated Δ_i and $\varXi_i$ in Example 2

```
function  [Deltausubi, Xi]= solveinclusionproblem2(q, nu, sigma, ...
    kappa, k, vartheta, supm)
[xq yq]=size(q);
for n=1:k
    Deltausubi(n,1)=n;
    Xi(n,1)=n;
end;
%$\Delta_i$
column=2;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=2* power(nu, 2-sigma)+ (1+ q(s)) * ...
            qGamma(q(s), 3-sigma, n);
    end;
    column=column+1;
end;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=(1+  power(nu, kappa+1)) * ...
            qGamma(q(s), 3-sigma, n);
    end;
    column=column+1;
end;
%column=2*(1+yq);
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=abs(( (-1)*qGamma(q(s), kappa+3, ...
            n)*(1+  power(nu, kappa+1))*Deltausubi(n, column-2*yq) ...
            + qGamma(q(s), 3-sigma, n)* qGamma(q(s), kappa+2, ...
            n)*(1+q(s))*(1+power(nu, kappa+2)))/(qGamma(q(s), ...
            kappa+2, n) * qGamma(q(s), kappa+3, n)* qGamma(q(s), ...
            3-sigma, n)));
    end;
    column=column+1;
end;
%column=2+3*yq;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=(1+  power(nu, ...
            kappa+1))*Deltausubi(n, column-3*yq) + Deltausubi(n, ...
            column-yq)* qGamma(q(s), kappa+2, n)* qGamma(q(s), ...
            3-sigma, n);
    end;
    column=column+1;
end;
%column=2+4*yq;
for s=1:yq
    for n=1:k
        Deltausubi(n, column)=Deltausubi(n, column-2*yq)* ...
            qGamma(q(s), 3-sigma, n)* qGamma(q(s), kappa+2, n);
    end;
    column=column+1;
end;
%$\Xi_i$
column=2;
for s=1:yq
    for n=1:k
        Xi(n, column)=1/qGamma(q(s), vartheta+1, n)+ ...
            ((Deltausubi(n, column) + qGamma(q(s), 3-sigma, ...
            n))*(power(nu, vartheta+kappa)+1))/(Deltausubi(n, ...
            column+2*yq)*qGamma(q(s), 3-sigma, n)*qGamma(q(s), ...
            vartheta+kappa+1, n)) + (Deltausubi(n, column+yq) + ...
            Deltausubi(n, column+3*yq))/(Deltausubi(n, column+4*yq) ...
            * qGamma(q(s), vartheta-1, n)) + ((Deltausubi(n, ...
            column+yq) + Deltausubi(n, column+3*yq))* power(nu, ...
            vartheta-sigma))/ (Deltausubi(n, column+4*yq) * ...
            qGamma(q(s), vartheta-sigma +1, n));
    end;
    column=column+1;
end;
%column=5
for s=1:yq
    for n=1:k
        Xi(n, column)=1/qGamma(q(s), vartheta, n) + (Deltausubi(n, ...
```

Algorithm 7 *(Continued)*

```
                column-yq)+(1+q(s))*  qGamma(q(s), 3-sigma, ...
                n))*(power(nu, vartheta+kappa)+1)/(Deltausubi(n, ...
                column+yq)*qGamma(q(s), 3-sigma, n)*qGamma(q(s), ...
                vartheta+kappa+1, n)) + ((1+q(s))*Deltausubi(n, column) ...
                + Deltausubi(n, column+2*yq))/(Deltausubi(n, ...
                column+3*yq) * qGamma(q(s), vartheta-1, n)) + ...
                (((1+q(s))* Deltausubi(n, column) + Deltausubi(n, ...
                column+2*yq))* power(nu, ...
                vartheta-sigma))/(Deltausubi(n, column+3*yq) * ...
                qGamma(q(s), vartheta-sigma +1, n));
        end;
        column=column+1;
    end;
    %column=8
    for s=1:yq
        for n=1:k
            Xi(n, column)=1/qGamma(q(s), vartheta-1, n) + ( ...
                (1+q(s))*(power(nu, vartheta+kappa) + 1 ) ...
                )/(Deltausubi(n, column)*qGamma(q(s), vartheta+kappa+1, ...
                n)) + ((1+q(s))*Deltausubi(n, ...
                column-yq))/(Deltausubi(n, column+2*yq)*qGamma(q(s), ...
                vartheta-1, n)) + ( (1+q(s))* Deltausubi(n, ...
                column-yq)*power(nu, vartheta-sigma))/(Deltausubi(n, ...
                column+2*yq) * qGamma(q(s), vartheta-sigma +1, n));
        end;
        column=column+1;
    end;
    end
```

$$
\begin{aligned}
&\leq \frac{t}{25}\cdot\frac{1}{3}\big(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\big)\\
&= \frac{t}{25}\psi\big(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\big)\\
&\leq m(t)\psi\big(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\big)\left[\frac{1}{\Sigma_1+\Sigma_2+\Sigma_3}\right].
\end{aligned}
$$

Consider the operator $\mathcal{N}:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ defined by

$$
\mathcal{N}(u)=\big\{h\in\mathcal{X}:\ \text{there exists } \Theta\in\mathcal{S}_{\mathcal{G},u} \text{ such that } h(t)=w(t) \text{ for all } t\in[0,1]\big\},
$$

where

$$
\begin{aligned}
w(t) &= \int_0^t \frac{(t-q\tau)^{(\frac{14}{5}-1)}}{\Gamma_q(\frac{14}{5})}\Theta(\tau)\,\mathrm{d}_q\tau\\
&\quad+\frac{t\Delta_1-t^2\Gamma_q(3-\frac{7}{4})}{\Delta_3\Gamma_q(3-\frac{7}{4})}\int_0^1\frac{(1-q\tau)^{(\frac{14}{5}+4-1)}}{\Gamma_q(\frac{14}{5}+4)}\Theta(\tau)\,\mathrm{d}_q\tau\\
&\quad+\frac{t^2\Delta_2-t\Delta_4}{\Delta_5}\int_0^1\frac{(1-q\tau)^{(\frac{14}{5}-3)}}{\Gamma_q(\frac{14}{5}-2)}\Theta(\tau)\,\mathrm{d}_q\tau\\
&\quad+\frac{t^2\Delta_2-t\Delta_4}{\Delta_5}\int_0^{\frac{1}{6}}\frac{(\frac{1}{6}-q\tau)^{(\frac{14}{5}-\frac{7}{4}-1)}}{\Gamma_q(\frac{14}{5}-\frac{7}{4})}\Theta(\tau)\,\mathrm{d}_q\tau\\
&\quad+\frac{t\Delta_1-t^2\Gamma_q(3-\frac{7}{4})}{\Delta_3\Gamma_q(3-\frac{7}{4})}\int_0^{\frac{1}{6}}\frac{(\frac{1}{6}-q\tau)^{(\frac{14}{5}+4-1)}}{\Gamma_q(\frac{14}{5}+4)}\Theta(\tau)\,\mathrm{d}_q\tau\\
&= \int_0^t\frac{(t-q\tau)^{\frac{9}{5}}}{\Gamma_q(\frac{14}{5})}\Theta(\tau)\,\mathrm{d}_q\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{t\Delta_1 - t^2\Gamma_q(\frac{5}{4})}{\Delta_3\Gamma_q(\frac{5}{4})}\int_0^1 \frac{(1-q\tau)^{\frac{29}{5}}}{\Gamma_q(\frac{34}{5})}\Theta(\tau)\,\mathrm{d}_q\tau \\
&+ \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^1 \frac{(1-q\tau)^{-\frac{1}{5}}}{\Gamma_q(\frac{4}{5})}\Theta(\tau)\,\mathrm{d}_q\tau \\
&+ \frac{t^2\Delta_2 - t\Delta_4}{\Delta_5}\int_0^{\frac{1}{6}} \frac{(\frac{1}{6}-q\tau)^{\frac{1}{20}}}{\Gamma_q(\frac{21}{20})}\Theta(\tau)\,\mathrm{d}_q\tau \\
&+ \frac{t\Delta_1 - t^2\Gamma_q(\frac{5}{4})}{\Delta_3\Gamma_q(\frac{5}{4})}\int_0^{\frac{1}{6}} \frac{(\frac{1}{6}-q\tau)^{\frac{29}{5}}}{\Gamma_q(\frac{34}{5})}\Theta(\tau)\,\mathrm{d}_q\tau.
\end{aligned}
$$

Now by using Theorem 11, the q-fractional inclusion problem (17)–(18) has a solution.

5 Conclusion

It is important that we increase our abilities from different points of view for studying distinct fractional integro-differential equations and inclusions. In this way, we should try to use modern and new techniques in our investigations. It would be significant if we could add numerical calculations in our results. In the present work, we studied the existence of solutions for a three-point nonlinear q-fractional differential equation and its related inclusion. In this way, we used α-ψ-contractions and multifunctions. We provided two examples to illustrate our main results. Finally, by providing some algorithms and tables, we gave some numerical computations for the results.

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