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Measure of noncompactness and a generalized Darbo's fixed point theorem and its applications to a system of integral equations

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Abstract
Işık et al. (Mathematics 7:862, 2019) presented an interesting generalization of the Banach contraction principle. In this paper, motivated by Işık et al., we give a new extension of the well-known Darbo inequality in a Banach space. Our results provide several generalizations of the Darbo inequality. As an application, we study the existence of solutions for a system of functional integral equations in $C[0,T]$. Finally, we expose a genuine example to support the effectiveness of our results.

1 Introduction and preliminaries

The notion of a measure of noncompactness (MNC) was introduced by Kuratowski [13] in 1930. This concept is a very useful tool in functional analysis, for example, in metric fixed point theory and operator equation theory in Banach spaces. This notion is also applied in the study of existence of solutions for ordinary and partial differential equations, integral, and integro-differential equations. For more details, we refer the reader to [2–6, 8, 9].

The aim of this paper is to generalize the Darbo's fixed point theorem in Banach space and study the existence of solutions for the following system of integral equations:

$$\begin{cases} \mu_1(\iota) = \varpi(\iota, \mu_1(\rho(\iota)), \mu_2(\rho(\iota)), \int_0^{\varrho(\iota)} g(\iota,\kappa,\mu_1(\rho(\kappa)),\mu_2(\rho(\kappa)))\,d\kappa), \\ \mu_2(\iota) = \varpi(\iota, \mu_2(\rho(\iota)), \mu_1(\rho(\iota)), \int_0^{\varrho(\iota)} g(\iota,\kappa,\mu_2(\rho(\kappa)),\mu_1(\rho(\kappa)))\,d\kappa), \end{cases} \tag{1}$$

where $\iota \in [0,T]$.

We collect some notations and definitions applied throughout this paper. Let $\mathcal{R}$ denote the set of real numbers, and let $\mathcal{R}_+ = [0,+\infty)$. Let $(\Lambda, \|\cdot\|)$ be a real Banach space. Moreover, by $\overline{B}(\iota, r)$ we denote the closed ball centered at ι with radius r, and by $\overline{B}_r$ the ball $\overline{B}(0,r)$. For a nonempty subset Δ of Λ, we denote by $\overline{\Delta}$ and $\mathrm{Conv}(\Delta)$ the closure and the closed convex hull of Δ, respectively. Furthermore, we denote by $\mathcal{M}(\Lambda)$ the family of nonempty bounded subsets of Λ and by $\mathcal{N}(\Lambda)$ its subfamily consisting of all relatively compact subsets of Λ.

Definition 1.1 ([7]) A mapping $\chi : \mathcal{M}(\Lambda) \longrightarrow \mathcal{R}_+$ is said to be a measure of noncompactness in Λ if it satisfies the following conditions:

1° The family $\ker \chi = \{\Delta \in \mathcal{M}(\Lambda) : \chi(\Delta) = 0\}$ is nonempty, and $\ker \chi \subset \mathcal{N}(\Lambda)$;

2° $\Delta \subset Y \Longrightarrow \chi(\Delta) \leq \chi(Y)$;

3° $\chi(\overline{\Delta}) = \chi(\Delta)$;

4° $\chi(\operatorname{Conv} \Delta) = \chi(\Delta)$;

5° $\chi(\lambda\Delta + (1-\lambda)Y) \leq \lambda\chi(\Delta) + (1-\lambda)\chi(Y)$ for $\lambda \in [0,1]$;

6° If $\{\Delta_n\}$ is a sequence of closed sets from $\mathcal{M}(\Lambda)$ such that $\Delta_{n+1} \subset \Delta_n$ for $n = 1, 2, \ldots$, and if $\lim_{n\to\infty} \chi(\Delta_n) = 0$, then $\Delta_\infty = \bigcap_{n=1}^{\infty} \Delta_n \neq \emptyset$.

Two important theorems having a key role in the fixed point theory are the Schauder fixed point principle and the Darbo fixed point theorem.

Theorem 1.2 ([1]) *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space Λ. Then each continuous compact map $\Upsilon : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

The following theorem is a generalization of the Schauder fixed point principle and the Darbo fixed point theorem.

Theorem 1.3 ([10]) *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space Λ, and let $\Upsilon : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $K \in [0,1)$ such that $\chi(\Upsilon\Delta) \leq K\chi(\Delta)$ for any nonempty subset Δ of Ω, where χ is an MNC defined in Λ. Then Υ has at least one fixed point in Ω.*

Işık et al. [12] introduced the following generalization of the Banach contraction principle, where substituting different functions f, we obtain a variety number of contractive inequalities.

Theorem 1.4 *Let (Δ, d) be a complete metric space, and let $T : \Delta \to \Delta$ be a continuous self-mapping. Suppose that there exists a function $f : \mathcal{R}^+ \to \mathcal{R}^+$ such that $\lim_{t\to 0^+} f(t) = 0$, $f(0) = 0$, and*

$$d(Tx, Ty) \leq f\bigl(d(x,y)\bigr) - f\bigl(d(Tx, Ty)\bigr) \tag{2}$$

for all $x, y \in \Delta$. Then T has a unique fixed point.

Let $\chi : \mathcal{M}(\Lambda) \longrightarrow \mathcal{R}_+$ be a mapping defined by $\chi(X) = \operatorname{diam} X$, where $\operatorname{diam} X = \sup\{\|x - y\| : x, y \in X\}$ is the diameter of X. We easily see that χ is a measure of noncompactness in a Banach space E (see [7]). According to this measure of noncompactness, we can easily observe that the Darbo fixed point theorem is a generalization of the Banach fixed point theorem. Using this idea, we want to generalize the result of Işık et al. to a Banach space E.

2 Main results

Now we state one of the main results in this paper, which extends and generalizes the Darbo fixed point theorem. In fact, motivated by the current work of Işık et al. [12], we give a new extension of the well-known Darbo fixed point theorem in a Banach space. Our results provide several inequalities, which all are generalizations of the Darbo fixed point theorem via substituting appropriate mappings instead of the control function f.

Theorem 2.1 *Let Ω be a nonempty, bounded, closed, and convex (NBCC) subset of a Banach space Λ, and let $\Upsilon : \Omega \to \Omega$ be a continuous operator such that*

$$\chi(\Upsilon\Delta) \le \varpi\big(\chi(\Delta)\big) - \varpi\big(\chi(\Upsilon\Delta)\big) \tag{3}$$

for all $\Delta \subseteq \Omega$, where $\varpi : [0,\infty) \to [0,\infty)$ is such that $\lim_{t\to 0^+} \varpi(t) = 0$, $\varpi(0) = 0$, and χ is an arbitrary MNC. Then Υ has at least one fixed point in Ω.

Proof 2.2 Let $\{\Omega_n\}$ be a sequence such that $\Omega_0 = \Omega$ and $\Omega_{n+1} = \overline{\mathrm{Conv}}(\Upsilon(\Omega_n))$ for all $n \in \mathcal{N}$.

If there exists an integer $N \in \mathcal{N}$ such that $\chi(\Omega_N) = 0$, then Ω_N is relatively compact, and Theorem 1.2 implies that Υ has a fixed point. So, we assume that $\chi(\Omega_N) > 0$ for each $n \in \mathcal{N}$.

It is clear that $\{\Omega_n\}_{n\in\mathcal{N}}$ is a sequence of NBCC sets such that

$$\Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_n \supseteq \Omega_{n+1}.$$

Thus the sequence $\{\varpi(\chi(\Omega_n))\}$ is nonincreasing. Since ϖ is bounded below, there exists $L \in \mathcal{R}^+$ such that $\lim_{n\to\infty} \varpi(\chi(\Omega_n)) = L$.

We know that $\{\chi(\Omega_n)\}_{n\in\mathcal{N}}$ is a positive decreasing and bounded below sequence of real numbers. Thus $\{\chi(\Omega_n)\}_{n\in\mathcal{N}}$ is a convergent sequence. Let $\lim_{n\to\infty} \chi(\Omega_n) = r$.

In view of condition (3), we have

$$\begin{aligned} 0 \le \chi(\Omega_{n+1}) = \chi(\Upsilon\Omega_n) &\le \varpi\big(\chi(\Omega_n)\big) - \varpi\big(\chi(\Upsilon\Omega_n)\big) \\ &= \varpi\big(\chi(\Omega_n)\big) - \varpi\big(\chi(\Omega_{n+1})\big). \end{aligned} \tag{4}$$

Taking the limsup in this inequality, we have

$$\limsup_{n\to\infty} \chi(\Omega_{n+1}) \le \limsup_{n\to\infty} \varpi\big(\chi(\Omega_n)\big) - \liminf_{n\to\infty} \varpi\big(\chi(\Omega_{n+1})\big).$$

Therefore $\lim_{n\to\infty} \chi(\Omega_n) = 0$. According to axiom (6°) of Definition 1.1, we have that the set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is an NBCC set and is invariant under the operator Υ and belongs to $\ker\chi$. Then in view of the Schauder theorem, Υ has a fixed point.

Remark 2.3 Note that Theorem 2.1 is a generalization of the Darbo fixed point theorem. Since $\Upsilon : \Delta \to \Delta$ is a Darbo mapping, there exists $k \in [0,1)$ such that

$$\chi(\Upsilon\Delta) \le k\chi(\Delta)$$

for all $\Delta \subseteq \Omega$. Therefore

$$\chi(\Upsilon\Delta) \le k\chi(\Delta) \le \frac{k}{1 + k - \sqrt{k}}\chi(\Delta)$$

for all $\Delta \subseteq \Omega$. Consequently,

$$k\chi(\Upsilon\Delta) + (1-\sqrt{k})\chi(\Upsilon\Delta) \le k\chi(\Delta),$$

and so

$$(1-\sqrt{k})\chi(\Upsilon\Delta) \le k\chi(\Delta) - k\chi(\Upsilon\Delta).$$

Therefore

$$\chi(\Upsilon\Delta) \le \frac{k}{1-\sqrt{k}}\chi(\Delta) - \frac{k}{1-\sqrt{k}}\chi(\Upsilon\Delta).$$

Taking $\varpi(t) = \frac{k}{1-\sqrt{k}}t$, we have $\chi(\Upsilon\Delta) \le \varpi(\chi(\Delta)) - \varpi(\chi(\Upsilon\Delta))$ for all $\Delta \subseteq \Omega$. Thus the Darbo inequality is a particular case of the contractive inequality of Theorem 2.1.

In the following corollaries, we provide examples of the function ϖ for equation (3) in Theorem 2.1 (the contractive inequality from Theorem 2.1) that have no Darbo constant k.

Taking $\varpi(t) = te^t$, for all $t \ge 0$, we deduce the following corollary.

Corollary 2.4 *Let Ω be an NBCC subset of a Banach space Λ, and let $\Upsilon : \Omega \to \Omega$ be a continuous operator such that*

$$\frac{\chi(\Upsilon\Delta)(1+e^{\chi(\Upsilon\Delta)})}{\chi(\Delta)e^{\chi(\Delta)}} \le 1 \tag{5}$$

for all $\Delta \subseteq \Omega$, where χ is an arbitrary MNC. Then Υ has at least one fixed point in Ω.

Taking $\varpi(t) = \sinh t$ for $t \ge 0$, we deduce the following corollary.

Corollary 2.5 *Let Ω be an NBCC subset of a Banach space Λ, and let $\Upsilon : \Omega \to \Omega$ be a continuous operator such that*

$$\chi(\Upsilon\Delta) \le 2\cosh\frac{\chi(\Delta)+\chi(\Upsilon\Delta)}{2}\sinh\frac{\chi(\Delta)-\chi(\Upsilon\Delta)}{2} \tag{6}$$

for all $\Delta \subseteq \Omega$, where χ is an arbitrary MNC. Then Υ has at least one fixed point in Ω.

Taking $\varpi(t) = \cosh t - 1$ for $t \ge 0$, we deduce the following corollary.

Corollary 2.6 *Let Ω be an NBCC subset of a Banach space Λ, and let $\Upsilon : \Omega \to \Omega$ be a continuous operator such that*

$$\chi(\Upsilon\Delta) \le 2\sinh\frac{\chi(\Delta)+\chi(\Upsilon\Delta)-2}{2}\sinh\frac{\chi(\Delta)-\chi(\Upsilon\Delta)}{2} \tag{7}$$

for all $\Delta \subseteq \Omega$, where χ is an arbitrary MNC. Then Υ has at least one fixed point in Ω.

Taking $\varpi(t) = \ln(1+t)$ for $t \ge 0$, we deduce the following corollary.

Corollary 2.7 *Let Ω be an NBCC subset of a Banach space Λ, and let $\Upsilon : \Omega \to \Omega$ be a continuous operator such that*

$$e^{\chi(\Upsilon\Delta)} \leq \frac{\chi(\Delta)+1}{\chi(\Upsilon\Delta)+1} \tag{8}$$

for all $\Delta \subseteq \Omega$, where χ is an arbitrary MNC. Then Υ has at least one fixed point in Ω.

3 Coupled fixed point

Bhaskar and Lakshmikantham [11] have introduced the notion of a coupled fixed point and proved some coupled fixed point theorems for some mappings and discussed the existence and uniqueness of solutions for periodic boundary value problems.

Definition 3.1 ([11]) An element $(\iota,\kappa) \in \Lambda^2$ is called a coupled fixed point of a mapping $\Upsilon : \Lambda \times \Lambda \to \Lambda$ if $\Upsilon(\iota,\kappa) = \iota$ and $\Upsilon(\kappa,\iota) = \kappa$.

Theorem 3.2 ([8]) *Suppose that $\chi_1, \chi_2, \dots, \chi_n$ are measures of noncompactness in Banach spaces $\Lambda_1, \Lambda_2, \dots, \Lambda_n$, respectively. Moreover, assume that a function $\Upsilon : [0,\infty)^n \longrightarrow [0,\infty)$ is convex and such that $\Upsilon(\iota_1,\dots,\iota_n) = 0$ if and only if $\iota_i = 0$ for $i = 1,2,\dots,n$. Then*

$$\tilde{\chi}(\Delta) = \Upsilon\big(\chi_1(\Delta_1), \chi_2(\Delta_2), \dots, \chi_n(\Delta_n)\big)$$

defines a measure of noncompactness in $\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$, where Δ_i denotes the natural projection of Δ into Λ_i for $i = 1,2,\dots,n$.

Theorem 3.3 *Let Ω be an NBCC subset of a Banach space Λ, and let $\Upsilon : \Omega \times \Omega \to \Omega$ be a continuous function such that*

$$\chi\big(\Upsilon(\Delta_1 \times \Delta_2)\big) \leq \frac{1}{2}\big[\varpi\big(\chi(\Delta_1) + \chi(\Delta_2)\big)\big] - \varpi\big(\chi\big(\Upsilon(\Delta_1 \times \Delta_2)\big) \tag{9}$$

for any subset Δ_1, Δ_2 of Ω, where χ is an arbitrary MNC, and ϖ is as in Theorem 2.1. In addition, we assume that ϖ is a subadditive mapping. Then Υ has at least one coupled fixed point.

Proof 3.4 We define the mapping $\widetilde{\Upsilon} : \Omega^2 \to \Omega^2$ by

$$\widetilde{\Upsilon}(\iota,\kappa) = \big(\Upsilon(\iota,\kappa), \Upsilon(\kappa,\iota)\big).$$

It is clear that $\widetilde{\Upsilon}$ is continuous. We show that $\widetilde{\Upsilon}$ satisfies all the conditions of Theorem 2.1. Let $\Delta \subset \Omega^2$ be a nonempty subset. We know that $\widetilde{\chi}(\Delta) = \chi(\Delta_1) + \chi(\Delta_2)$ is an MNC [7], where Δ_1 and Δ_2 denote the natural projections of Δ into Λ. From (9) we have

$$\begin{aligned}
\widetilde{\chi}\big(\widetilde{\Upsilon}(\Delta)\big) &\leq \widetilde{\chi}\big(\Upsilon(\Delta_1 \times \Delta_2) \times \Upsilon(\Delta_2 \times \Delta_1)\big) \\
&= \chi\big(\Upsilon(\Delta_1 \times \Delta_2)\big) + \chi\big(\Upsilon(\Delta_2 \times \Delta_1)\big) \\
&\leq \frac{1}{2}\big[\varpi\big(\chi(\Delta_1) + \chi(\Delta_2)\big)\big] - \varpi\big(\chi\big(\Upsilon(\Delta_1 \times \Delta_2)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\big[\varpi\big(\chi(\Delta_2)+\chi(\Delta_1)\big)\big]-\varpi\big(\chi\big(\Upsilon(\Delta_2\times\Delta_1)\big)\big)\\
&\leq \varpi\big(\chi(\Delta_1)+\chi(\Delta_2)\big)-\big[\varpi\big(\chi\big(\Upsilon(\Delta_1\times\Delta_2)\big)+\chi\big(\Upsilon(\Delta_2\times\Delta_1)\big)\big)\big]\\
&=\varpi\big(\widetilde{\chi}(\Delta)\big)-\varpi\big(\widetilde{\chi}\big(\widetilde{\Upsilon}(\Delta)\big)\big).
\end{aligned}
$$

Now, from Theorem 2.1 we deduce that $\widetilde{\Upsilon}$ has at least one fixed point, which implies that Υ has at least one coupled fixed point.

4 Application

In this section, as an application of Theorem 3.3, we study the existence of solutions for the system of functional integral equations (1).

Let $C[0,T]$ be the space of all real bounded continuous functions on the interval $\mathcal{J}=[0,T]$ equipped with the standard norm

$$\|\iota\|=\sup\big\{\big|\iota(t)\big|:t\in[0,T]\big\}.$$

Recall that the modulus of continuity of a function $\iota\in C[0,T]$ is defined by

$$\omega(\iota,\epsilon)=\sup\big\{\big|\iota(t)-\iota(s)\big|:t,s\in[0,T],|t-s|\leq\epsilon\big\}.$$

The Hausdorff measure of noncompactness for all bounded sets Ω of $C[0,T]$ is defined as

$$\chi(\Omega)=\lim_{\epsilon\to 0}\Big\{\sup_{\iota\in\Delta}\omega(\iota,\epsilon)\Big\}.$$

(For more detail, see [8].)

Theorem 4.1 *Suppose that the following assumptions are satisfied*:

(i) ρ *and* $\varrho:[0,T]\longrightarrow[0,T]$ *are continuous functions.*

(ii) *The function* $\varpi:[0,T]\times\mathcal{R}^3\longrightarrow\mathcal{R}$ *is continuous, and*

$$
\begin{aligned}
&\max\big\{\big|\varpi(\iota,\mu_1,\mu_2,\kappa)-\varpi(\iota,\nu_1,\nu_2,z)\big|,\Gamma\big(\big|\varpi(\iota,\mu_1,\mu_2,\kappa)-\varpi(\iota,\nu_1,\nu_2,z)\big|\big)\big\}\\
&\quad\leq\frac{1}{4}\Gamma\big(|\mu_1-\nu_1|+|\mu_2-\nu_2|\big)+|\kappa-z|,
\end{aligned}
$$

where $\Gamma:[0,\infty)\to[0,\infty)$ *is a continuous strictly increasing subadditive mapping such that* $\lim_{t\to 0^+}\Gamma(t)=0$ *and* $\Gamma(0)=0$.

(iii) $N:=\sup\{|\varpi(\iota,0,0,0)|:\iota\in[0,T]\}$.

(iv) $g:[0,T]\times[0,T]\times\mathcal{R}^2\longrightarrow\mathcal{R}$ *is continuous, and*

$$
\begin{aligned}
G:=\ &\sup\bigg\{\bigg|\int_0^{\varrho(\iota)}g\big(\iota,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)\,d\kappa\bigg|:\\
&\iota,\kappa\in[0,T],\mu_1,\mu_2,\in C\big([0,T]\big)\bigg\}.
\end{aligned}
$$

(v) *There exists a positive solution r_0 to the inequality*

$$\left(\frac{\Gamma(2r)}{4}+G\right)+N\leq r.$$

Then the system of integral equations (1) *has at least one solution in the space* $(C[0,T])^2$.

Proof 4.2 Let us consider the operator

$$\Upsilon : C[0,T]\times C[0,T]\longrightarrow C[0,T]$$

defined by

$$\Upsilon(\mu_1,\mu_2)(\iota)=\varpi\left(\iota,\mu_1\big(\rho(\iota)\big),\mu_2\big(\rho(\iota)\big),\int_0^{\varrho(\iota)}g\big(\iota,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)\,d\kappa\right).$$

We observe that for any $\iota\in[0,T]$, the function $\Upsilon(\mu_1,\mu_2)(\iota)$ is continuous. For arbitrary fixed $\iota\in[0,T]$, by assumptions (i)–(iii) we have

$$\begin{aligned}
&\big|\Upsilon(\mu_1,\mu_2)(\iota)\big|\\
&\quad\leq\left|\varpi\left(\iota,\mu_1\big(\rho(\iota)\big),\mu_2\big(\rho(\iota)\big),\int_0^{\varrho(\iota)}g\big(\iota,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)\,d\kappa\right)-\varpi(\iota,0,0,0)\right|\\
&\qquad+\big|\varpi(\iota,0,0,0)\big|\\
&\quad\leq\frac{\Gamma(|\mu_1(\rho(\iota))|+|\mu_2(\rho(\iota))|)}{4}+\left|\int_0^{\varrho(\iota)}g\big(\iota,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)\,d\kappa\right|+\big|\varpi(\iota,0,0,0)\big|\\
&\quad\leq\frac{\Gamma(\|\mu_1\|+\|\mu_2\|)}{4}+G+N.
\end{aligned}$$

Therefore

$$\big\|\Upsilon(\mu_1,\mu_2)\big\|\leq\frac{\Gamma(\|\mu_1\|+\|\mu_2\|)}{4}+G+N. \tag{10}$$

By inequality (10) and (iv), Υ is a function from $(\bar{B}_{r_0})^2$ into $(\bar{B}_{r_0})$.

Now we prove that the operator Υ is a continuous operator on $(\bar{B}_{r_0})^2$. Let us fix arbitrary $\varepsilon>0$ and take $(\mu_1,\mu_2),(\nu_1,\nu_2)\in(\bar{B}_{r_0})^2$ such that $\max\{\|\mu_1-\nu_1\|,\|\mu_2-\nu_2\|\}<\varepsilon$. Then for all $\iota\in\mathcal{J}$, we have

$$\begin{aligned}
&\big|\Upsilon(\mu_1,\mu_2)(\iota)-\Upsilon(\nu_1,\nu_2)(\iota)\big|\\
&\quad=\left|\varpi\left(\iota,\mu_1\big(\rho(\iota)\big),\mu_2\big(\rho(\iota)\big),\int_0^{\varrho(\iota)}g\big(\iota,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)\,d\kappa\right)\right.\\
&\qquad\left.-f\left(\iota,\nu_1\big(\rho(\iota)\big),\nu_2\big(\rho(\iota)\big),\int_0^{\varrho(\iota)}g\big(\iota,\kappa,\nu_1\big(\rho(\kappa)\big),\nu_2\big(\rho(\kappa)\big)\big)\,d\kappa\right)\right|\\
&\quad\leq\frac{\Gamma(|\mu_1(\rho(\iota))-\nu_1(\rho(\iota))|+|\mu_2(\rho(\iota))-\nu_2(\rho(\iota))|)}{4}\\
&\qquad+\int_0^{\varrho(\iota)}\big|g\big(\iota,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)-g\big(\iota,\kappa,\nu_1\big(\rho(\kappa)\big),\nu_2\big(\rho(\kappa)\big)\big)\big|\,d\kappa
\end{aligned}$$

$$\leq \left(\frac{\Gamma(2\varepsilon)}{4} + \varrho(T)\omega^T(g,\varepsilon)\right),$$

where

$$\begin{aligned}\omega^T(g,\varepsilon) = \sup\bigl\{\bigl|g(\iota,\kappa,\mu_1,\mu_2) - g(\iota,\kappa,\nu_1,\nu_2)\bigr| : \iota \in \mathcal{J}, \kappa \in \bigl[0,\varrho(T)\bigr],\\ \mu_1,\mu_2,\nu_1,\nu_2 \in [-r_0,r_0], \max\bigl\{\|\mu_1-\nu_1\|,\|\mu_2-\nu_2\|\bigr\} < \varepsilon\bigr\}\end{aligned}$$

and

$$\varrho(T) = \sup\bigl\{\varrho(\iota) : \iota \in \mathcal{J}\bigr\}.$$

Applying the continuity of g on $\mathcal{J} \times [0,\varrho(T)] \times [-r_0,r_0]^2$, we have $\omega^T(g,\varepsilon) \to 0$ as $\varepsilon \to 0$, which implies that Υ is a continuous function on $(\bar{B}_{r_0})^2$.

Now we prove that Υ satisfies the conditions of Theorem 3.3. To this end, let Δ_1 and Δ_2 are nonempty and bounded subsets of $\bar{B}_{r_0}$ and let $\varepsilon > 0$ be an arbitrary constant. Let $\iota_1,\iota_2 \in [0,T]$ with $|\iota_2-\iota_1| \leq \varepsilon$, and let $(\mu_1,\mu_2) \in \Delta_1 \times \Delta_2$. Then we have

$$\begin{aligned}&\bigl|\Upsilon(\mu_1,\mu_2)(\iota_1) - \Upsilon(\mu_1,\mu_2)(\iota_2)\bigr|\\ &\quad\leq \biggl|\varpi\biggl(\iota_1,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\ &\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|, \end{aligned}\tag{11}$$

where

$$\begin{aligned}&\biggl|\varpi\biggl(\iota_1,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\ &\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\\ &\quad\leq \biggl|\varpi\biggl(\iota_1,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\ &\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\\ &\qquad + \biggl|\varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\ &\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\\ &\qquad + \biggl|\varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\ &\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\\ &\qquad + \biggl|\varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\ &\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|. \end{aligned}\tag{12}$$

By condition (ii) we have

$$\begin{aligned}
&\bigl|\Upsilon(\mu_1,\mu_2)(\iota_1)-\Upsilon(\nu_1,\nu_2)(\iota_2)\bigr| \\
&\quad\le \omega_{r_0}(\varpi,\epsilon)+\frac{\Gamma(|\mu_1(\rho(\iota_1))-\mu_1(\rho(\iota_2))|+|\mu_2(\rho(\iota_1))-\mu_2(\rho(\iota_2))|)}{4} \\
&\qquad+\left|\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa-\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\right| \\
&\qquad+\left|\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa-\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\right| \\
&\quad\le \omega_{r_0,G}(\varpi,\epsilon)+\frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon))+\omega(\mu_2,\omega(\rho,\epsilon)))}{4} \\
&\qquad+\left|\int_{\varrho(\iota_1)}^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\right| \\
&\qquad+\int_0^{\varrho(\iota_2)}\bigl|g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)-g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\bigr|\,d\kappa \\
&\quad\le \omega_{r_0,G}(\varpi,\epsilon)+\frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon))+\omega(\mu_2,\omega(\rho,\epsilon))}{4}+\varrho(T)\omega_{r_0}(g,\epsilon) \\
&\qquad+U_{r_0}\omega(\varrho,\epsilon).
\end{aligned}\tag{13}$$

By condition (ii) we have

$$\begin{aligned}
&\bigl|\Upsilon(\mu_1,\mu_2)(\iota_1)-\Upsilon(\nu_1,\nu_2)(\iota_2)\bigr| \\
&\quad\le \omega_{r_0}(\varpi,\epsilon)+\frac{\Gamma(|\mu_1(\rho(\iota_1))-\mu_1(\rho(\iota_2))|+|\mu_2(\rho(\iota_1))-\mu_2(\rho(\iota_2))|)}{4} \\
&\qquad+\left|\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa-\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\right| \\
&\qquad+\left|\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa-\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\right| \\
&\quad\le \omega_{r_0,G}(\varpi,\epsilon)+\frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon))+\omega(\mu_2,\omega(\rho,\epsilon)))}{4} \\
&\qquad+\left|\int_{\varrho(\iota_1)}^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\right| \\
&\qquad+\int_0^{\varrho(\iota_2)}\bigl|g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)-g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\bigr|\,d\kappa \\
&\quad\le \omega_{r_0,G}(\varpi,\epsilon)+\frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon))+\omega(\mu_2,\omega(\rho,\epsilon))}{4}+\varrho(T)\omega_{r_0}(g,\epsilon) \\
&\qquad+U_{r_0}\omega(\varrho,\epsilon),
\end{aligned}\tag{14}$$

where

$$\begin{aligned}
&\omega(\rho,\epsilon)=\sup\bigl\{\bigl|\rho(\iota_2)-\rho(\iota_1)\bigr|:\iota_1,\iota_2\in\mathcal{J},|\iota_2-\iota_1|\le\epsilon\bigr\}, \\
&\omega\bigl(\mu_1,\omega(\rho,\epsilon)\bigr)=\sup\bigl\{\bigl|\mu_1(\iota_2)-\mu_1(\iota_1)\bigr|:\iota_1,\iota_2\in\mathcal{J},|\iota_2-\iota_1|\le\omega(\rho,\epsilon)\bigr\}, \\
&\varrho(T)=\sup\bigl\{\varrho(\iota):\iota\in\mathcal{J}\bigr\},
\end{aligned}$$

$$\begin{aligned}
&U_{r_0} = \sup\bigl\{\bigl|g(\iota,\kappa,\mu_1,\mu_2)\bigr| : \iota \in \mathcal{J}, \kappa \in \bigl[0,\varrho(T)\bigr], \mu_1,\mu_2 \in [-r_0,r_0]\bigr\},\\
&G = \varrho(T)\sup\bigl\{\bigl|g(\iota,\kappa,\mu_1,\mu_2)\bigr| : \iota \in \mathcal{J}, \kappa \in \bigl[0,\varrho(T)\bigr], \mu_1,\mu_2 \in [-r_0,r_0]\bigr\},\\
&\omega_{r_0,G}(\varpi,\epsilon) = \sup\bigl\{\bigl|f(\iota_2,\mu_1,\mu_2,z) - f(\iota_1,\mu_1,\mu_2,z)\bigr| : \iota_1,\iota_2 \in \mathcal{J},\\
&\qquad |\iota_2-\iota_1| \le \epsilon, \mu_1,\mu_2 \in [-r_0,r_0], z \in [-G,G]\bigr\},\\
&\omega_{r_0}(g,\epsilon) = \sup\bigl\{\bigl|g(\iota_2,\kappa,\mu_1,\mu_2) - g(\iota_1,\kappa,\mu_1,\mu_2)\bigr| : \iota_1,\iota_2 \in \mathcal{J},\\
&\qquad |\iota_2-\iota_1| \le \epsilon, \mu_1,\mu_2 \in [-r_0,r_0], \kappa \in \bigl[0,\varrho(T)\bigr]\bigr\}.
\end{aligned}$$

Since (μ_1,μ_2) was an arbitrary element of $\Delta_1 \times \Delta_2$ in (11), we have

$$\begin{aligned}
&\omega\bigl(\Upsilon(\Delta_1\times\Delta_2),\epsilon\bigr)\\
&\quad\le \omega_{r_0,G}(\varpi,\epsilon) + \frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon)) + \omega(\mu_2,\omega(\rho,\epsilon))}{4} + \varrho(T)\omega_{r_0}(g,\epsilon) + U_{r_0}\omega(\varrho,\epsilon).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Gamma\bigl(\bigl|\Upsilon(\mu_1,\mu_2)(\iota_1) - \Upsilon(\mu_1,\mu_2)(\iota_2)\bigr|\bigr)\\
&\quad\le \Gamma\biggl(\biggl|\varpi\biggl(\iota_1,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\
&\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\biggr),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
&\Gamma\biggl(\biggl|\varpi\biggl(\iota_1,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\
&\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\biggr)\\
&\quad\le \Gamma\biggl(\biggl|\varpi\biggl(\iota_1,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\
&\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\biggr)\\
&\qquad + \Gamma\biggl(\biggl|\varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_1)\bigr),\mu_2\bigl(\rho(\iota_1)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\
&\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\biggr)\\
&\qquad + \Gamma\biggl(\biggl|\varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_1)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\
&\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\biggr)\\
&\qquad + \Gamma\biggl(\biggl|\varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_1,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\\
&\qquad - \varpi\biggl(\iota_2,\mu_1\bigl(\rho(\iota_2)\bigr),\mu_2\bigl(\rho(\iota_2)\bigr),\int_0^{\varrho(\iota_2)} g\bigl(\iota_2,\kappa,\mu_1\bigl(\rho(\kappa)\bigr),\mu_2\bigl(\rho(\kappa)\bigr)\bigr)\,d\kappa\biggr)\biggr|\biggr).
\end{aligned} \tag{16}$$

By condition (ii) we have

$$\begin{aligned}
&\Gamma\big(\big|\Upsilon(\mu_1,\mu_2)(\iota)-\Upsilon(\nu_1,\nu_2)(\iota)\big|\big)\\
&\quad\le \Gamma\big(\omega_{r_0,G}(\varpi,\epsilon)\big)+\frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon))+\omega(\mu_2,\omega(\rho,\epsilon)))}{4}\\
&\qquad+\left|\int_{\varrho(\iota_1)}^{\varrho(\iota_2)} g\big(\iota_1,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)\,d\kappa\right|\\
&\qquad+\int_0^{\varrho(\iota_2)}\big|g\big(\iota_1,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)-g\big(\iota_2,\kappa,\mu_1\big(\rho(\kappa)\big),\mu_2\big(\rho(\kappa)\big)\big)\big|\,d\kappa\\
&\quad\le \Gamma\big(\omega_{r_0,G}(\varpi,\epsilon)\big)+\frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon))+\omega(\mu_2,\omega(\rho,\epsilon))}{4}+\varrho(T)\omega_{r_0}(g,\epsilon)\\
&\qquad+U_{r_0}\omega(\varrho,\epsilon).
\end{aligned}\tag{17}$$

Therefore we find that

$$\begin{aligned}
&\Gamma\big(\omega\big(\Upsilon(\Delta_1\times\Delta_2),\epsilon\big)\big)\\
&\quad\le \Gamma\big(\omega_{r_0,G}(\varpi,\epsilon)\big)+\frac{\Gamma(\omega(\mu_1,\omega(\rho,\epsilon))+\omega(\mu_2,\omega(\rho,\epsilon))}{4}+\varrho(T)\omega_{r_0}(g,\epsilon)+U_{r_0}\omega(\varrho,\epsilon).
\end{aligned}$$

According to the uniform continuity of f and g on the compact sets

$$[0,T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-G,G]$$

and

$$[0,T]\times\big[0,\varrho(T)\big]\times[-r_0,r_0]\times[-r_0,r_0],$$

respectively, we infer that $\omega_{r_0,G}(f,\epsilon)\longrightarrow 0$, $\omega_{r_0}(g,\epsilon)\longrightarrow 0$ and $\omega(\varrho,\epsilon)\longrightarrow 0$ as $\epsilon\longrightarrow 0$.

From the above inequalities, according to the subadditivity of Γ, we obtain that

$$\Gamma\big(\chi\big[\Upsilon(\Delta_1\times\Delta_2)\big]\big)+\chi\big[\Upsilon(\Delta_1\times\Delta_2)\big]\le\frac{\Gamma(\chi(\Delta_1)+\chi(\Delta_2))}{2}.$$

Thus from Theorem 3.3 we obtain that the operator Υ has a coupled fixed point. Therefore the system of functional integral equations (1) has at least one solution in $(C[0,T])^2$.

5 Example

Example 5.1 Consider the following system of integral equations:

$$\begin{cases}
\iota(t)=-2+\frac{1}{2}e^{-t^2}+\frac{\arctan(\iota(t)+\kappa(t))}{8\pi+t^8}+\frac{1}{8}\int_0^t e^{-2t+s}\arctan(\iota(s)+\kappa(s))\,ds,\\
\kappa(t)=-2+\frac{1}{2}e^{-t^2}+\frac{\arctan(\iota(t)+\kappa(t))}{8\pi+t^8}+\frac{1}{8}\int_0^t e^{-2t+s}\arctan(\iota(s)+\kappa(s))\,ds.
\end{cases}\tag{18}$$

We observe that this system of integral equations (18) is a particular case of (1) with

$$\rho(t)=\varrho(t)=t,\quad t\in[0,1],$$

$$\begin{aligned}
&\varpi(t,\iota,\kappa,p) = -2 + \frac{1}{2}e^{-t^2} + \frac{\arctan(\iota(t)+\kappa(t))}{8\pi + t^8} + \frac{p}{8},\\
&g(t,s,\iota,\kappa) = e^{-2t+s}\arctan\bigl(\iota(s)+\kappa(s)\bigr).
\end{aligned}$$

To solve this system, we need to verify conditions (i)–(v) of Theorem 4.1.

Condition (i) is clearly evident since

$$\begin{aligned}
\bigl|\varpi(t,\iota,\kappa,m) - \varpi(t,u,v,n)\bigr| &\le \frac{|\arctan(\iota+\kappa) - \arctan(u+v)|}{8\pi + t^8} + \frac{|m-n|}{8}\\
&\le \frac{\arctan|(\iota+\kappa)-(u+v)|}{8\pi + t^8}) + |m-n|\\
&\le \frac{\arctan(|\iota-u| + |\kappa - v|)}{4} + |m-n|\\
&= \frac{\Gamma(|\iota-u| + |\kappa - v|)}{4} + |m-n| \qquad (19)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma\bigl(\bigl|\varpi(t,\iota,\kappa,m) - \varpi(t,u,v,n)\bigr|\bigr) &= \arctan\biggl(\frac{|\arctan(\iota+\kappa) - \arctan(u+v)|}{8\pi + t^8} + \frac{|m-n|}{8}\biggr)\\
&\le \arctan\biggl(\frac{\arctan|(\iota+\kappa)-(u+v)|}{8\pi + t^8}\biggr) + |m-n|\\
&\le \frac{\arctan(|\iota-u| + |\kappa - v|)}{4} + |m-n|\\
&= \frac{\Gamma(|\iota-u| + |\kappa - v|)}{4} + |m-n|. \qquad (20)
\end{aligned}$$

So we can find that ϖ satisfies condition (ii) of Theorem 4.1 with $\Gamma(t) = \arctan(t)$. Also,

$$N = \sup\bigl\{\bigl|\varpi(t,0,0,0)\bigr| : t \in [0,1]\bigr\} = \sup\biggl\{-2 + \frac{1}{2}e^{-t^2} : t\in[0,1]\biggr\} = -1.5.$$

Moreover, g is continuous on $[0,T]\times[0,T]\times\mathcal{R}^2$, and

$$\begin{aligned}
G = {} & \sup\biggl\{\biggl|\int_0^t e^{-2t+s}\arctan\bigl(\iota(s)+\kappa(s)\bigr)\,ds\biggr| :\\
& t,s\in[0,1], \iota,\kappa \in C[0,1]\biggr\}\\
< {} & \sup 1.1 \times 1 \times \bigl(e^t - 1\bigr) \simeq 1.8901100113.
\end{aligned}$$

Furthermore, it is easy to see that any $r \ge 0.6238$ satisfies the inequality in condition (iv), that is,

$$\frac{\arctan(2r)}{4} + G + N = \frac{\arctan(2r)}{4} + 1.9 - 1.5 \le r.$$

Consequently, all the conditions of Theorem 4.1 are satisfied. Hence the system of integral equations (18) has at least one solution that belongs to the space $C[0,1]\times C[0,1]$.

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