Ghanbari and Haghi *Advances in Difference Equations* (2020) 2020:247
https://doi.org/10.1186/s13662-020-02706-w

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Parameter interval of positive solutions for a system of fractional difference equation

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Abstract

This paper deals with a typical system of Caputo fractional difference equations. Using the Guo–Krasnosel'skii fixed point theorem, we find a parameter interval for which at least one positive solution of the system exists. We give two examples to illustrate the results.

MSC: 26A33; 39A06; 39A13

Keywords: Fractional difference equation; Caputo difference operator; Green's function; System of difference equations

1 Introduction

Fractional calculus, as a generalization of classical calculus, is one of those mathematical topics that received much attention. It has been shown for many years that the using of this emerging tool in modeling and design helps to improve the efficiency of various sciences. On the other hand, in recent years the fractional difference equations have been of great interest, there is much work focused on studying the existence and uniqueness of solutions [6, 19, 20]. The theory of discrete version of fractional calculus is very similar and parallel to the theory of continuous case.

Kutter was the first one studied the time differences of fractional order [16]. Diaz and Osler introduced a discrete fractional difference operator defined as an infinite series [7]. Grey and Zhang developed a fractional calculus for the discrete nabla difference operator [13]. At the same time, Miller and Ross defined a fractional sum via the solution of a linear difference equation [17]. Atici and Eloe introduced the Riemann–Liouville like fractional difference, and developed some of its properties that allow one to obtain solutions of certain fractional difference equations [5]. Ferreira introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations [8, 9]. Holm developed and applied the tools of discrete fractional calculus to the area of fractional difference equations [14, 15]. Abdeljawad obtained dual identities in fractional difference calculus which they relate the delta and nabla and the left and right fractional sums and differences [1–3]. Goodrich and Peterson develop basic theoretical results in the field of discrete fractional calculus [11]. Moreover, Goodrich studied existence of positive solutions for discrete fractional systems and geometrical properties [10, 12].

Our objective is to explore the existence and uniqueness results for the following system of fractional difference equations:

$$\begin{cases} -\Delta_c^{\alpha} y_i(t) = \lambda_i f_i(y_1(t+\alpha-1), y_2(t+\alpha-1), \dots, y_n(t+\alpha-1)), \\ y_i(\alpha-k) = a_i^0, \quad t \in [\alpha-(k-1), \alpha+T]_{\mathbb{N}_{\alpha-(k-1)}}, i = 1,2,\dots,n, \\ \Delta y_i(\alpha+T) = \Delta^j y_i(\alpha-k) = 0, \quad j = 2,3,\dots,k-1, \end{cases} \tag{1}$$

where $\alpha \in (k-1,k]$ and $a_i^0 \geq 0, T \geq k \geq 2$ are real numbers, Δ_c^{α} is the standard Caputo difference, $f_i : [0,\infty) \times [0,\infty) \times \cdots \times [0,\infty) \to [0,\infty)$ is continuous.

As we stated in the abstract, our objective is to use fixed point theory in special normed spaces to achieve an interval for parameter λ for which the problem (1) may or may not have a positive solution.

2 Preliminaries and basic notations

Here, we give some basic definitions and properties of the discrete fractional calculus theory which can be found in [11].

Definition 2.1 Let $\mathbb{N}_a := \{a, a+1, \dots\}, a \in \mathbb{R}$ and $f : \mathbb{N}_a \to \mathbb{R}$ be a real function. The difference operator Δ acts on f by

$$\Delta f(x) := f(x+1) - f(x), \quad x \in \mathbb{N}_a.$$

Definition 2.2 The falling fractional power $x^{\underline{\alpha}}$ is given by

$$x^{\underline{\alpha}} = \frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}.$$

Theorem 2.3 *According to the definition of Δ and the falling fractional power we have*

$$\Delta x^{\underline{\alpha}} = \alpha x^{\underline{\alpha-1}}.$$

Definition 2.4 The fractional sum of order α for a given function h, for $\alpha > 0$, is defined by

$$\Delta^{-\alpha} h(x) := \frac{1}{\Gamma(\alpha)} \sum_{i=a}^{x-\alpha} \bigl(x - \sigma(i)\bigr)^{\underline{\alpha-1}} h(i),$$

for $x \in \{\alpha + a, \alpha + a + 1, \dots\} := \mathbb{N}_{a+\alpha}$ and $\sigma(i) = i + 1$. The αth fractional difference for $\alpha > 0$ is defined by $\Delta^{\alpha} h(x) = \Delta^n \Delta^{\alpha-n} h(x)$, where $x \in \mathbb{N}_{a+n-\alpha}$ and $n \in \mathbb{N}$ such that $0 \leq n-1 < \alpha \leq n$.

Furthermore the Caputo fractional difference for $\alpha > 0$ is defined by

$$\Delta_c^{-\alpha} h(x) := \Delta^{-(n-\alpha)} \Delta^n h(x) = \frac{1}{\Gamma(n-\alpha)} \sum_{i=a}^{x-(n-\alpha)} \bigl(x - \sigma(i)\bigr)^{\underline{n-\alpha-1}} \Delta^n h(i), \tag{2}$$

where $0 \leq n-1 < \alpha \leq n$.

Lemma 2.5 *Let $\alpha > 0$ and h be defined on $\mathbb{N}_a$, then*

$$\Delta_{a+(n-\alpha)}^{-\alpha}\Delta_c^{\alpha}h(x) = h(x) - \sum_{i=0}^{n-1} c_i(x-a)^{\underline{i}}, \tag{3}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1$, and $n-1 < \alpha \leq n$.

3 Representation of the solution by Green's function

Now we are ready to represent the solution of the problem (1) by Green's function.

Lemma 3.1 *The discrete fractional boundary value problem*

$$\begin{cases} -\Delta_c^{\alpha}y(t) = \lambda h(t+\alpha-1), \\ y(\alpha-k) = a_0, \\ \Delta y(\alpha+T) = \Delta^j y(\alpha-k) = 0, \quad j = 2, 3, \dots, k-1, \end{cases} \tag{4}$$

has a unique solution given by

$$y(t) = \lambda \sum_{s=0}^{T+1} G(t,s)h(s+\alpha-1) + a_0, \tag{5}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\alpha-1)(t-\alpha+k)(T+\alpha-s-1)^{\underline{\alpha-2}} - (t-s-1)^{\underline{\alpha-1}}, \\ \quad 0 \leq s < t-\alpha+1, \\ (\alpha-1)(t-\alpha+k)(T+\alpha-s-1)^{\underline{\alpha-2}}, \quad 0 \leq t-\alpha+1 \leq s. \end{cases}$$

Proof Using Lemma 2.5

$$\begin{aligned} y(t) = {}& -\frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) + c_0 + c_1 t + c_2 t^{\underline{2}} \\ & + \cdots + c_{k-1}t^{\underline{k-1}}, \end{aligned}$$

for $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, k-1$.

Taking difference operator we find

$$\begin{aligned} \Delta y(t) = {}& -\frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{t+1-\alpha}(\alpha-1)(t-s-1)^{\underline{\alpha-2}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) + c_1 + 2c_2 t \\ & + \cdots + (k-1)c_{k-1}t^{\underline{k-2}}, \\ \Delta^2 y(t) = {}& -\frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{t+2-\alpha}(\alpha-1)(\alpha-2)(t-s-1)^{\underline{\alpha-3}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) + 2c_2 \\ & + \cdots + (k-1)(k-2)c_{k-1}t^{\underline{k-3}}, \end{aligned}$$

$$\vdots$$

$$\Delta^{k-1}y(t) = -\frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{t+k-1-\alpha}(\alpha-1)\cdots(\alpha-k+1)(t-s-1)^{\underline{\alpha-k}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr)$$
$$+ (k-1)(k-2)\cdots c_{k-1}.$$

From $\Delta^j y(\alpha-k) = 0, j = 2, 3, \ldots, k-1$, we get $c_2 = c_3 = \cdots = c_{k-1} = 0$, and by $\Delta y(\alpha+T) = 0, y(\alpha-k) = a_0$, we have

$$c_1 = \frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{T+1}(\alpha-1)(T+\alpha-s-1)^{\underline{\alpha-2}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr),$$

and $c_0 = -(\alpha-k)c_1 + a_0$, then we have

$$c_0 = \frac{-(\alpha-k)\lambda}{\Gamma(\alpha)}\sum_{s=0}^{T+1}(\alpha-1)(T+\alpha-s-1)^{\underline{\alpha-2}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) + a_0.$$

Therefore, the solution of (1) is

$$\begin{aligned} y(t) &= -\frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) \\ &\quad + \frac{-(\alpha-k)\lambda}{\Gamma(\alpha)}\sum_{s=0}^{T+1}(\alpha-1)(T+\alpha-s-1)^{\underline{\alpha-2}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) \\ &\quad + \frac{t\lambda}{\Gamma(\alpha)}\sum_{s=0}^{T+1}(\alpha-1)(T+\alpha-s-1)^{\underline{\alpha-2}}h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) + a_0 \\ &= \lambda\sum_{s=0}^{T+1}G(t,s)h\bigl(s+\alpha-1, y(s+\alpha-1)\bigr) + a_0. \end{aligned}$$

□

Lemma 3.2 *The Green's function G given in Lemma 3.1 satisfies in the following conditions*:

(i) $G(t,s) > 0, (t,s) \in [\alpha-(k-1), \alpha+T]_{\mathbb{N}_{\alpha-(k-1)}} \times [0, T+1]_{\mathbb{N}_0}$.

(ii) $\max_{t\in[\alpha-(k-1),\alpha+T]_{\mathbb{N}_{\alpha-(k-1)}}} G(t,s) = G(T+\alpha, s)$.

(iii) $\min_{t\in[\frac{(\alpha+T)}{4},\frac{3(\alpha+T)}{4}]_{\mathbb{N}_{\alpha-(k-1)}}} G(t,s) \geq \frac{1}{4}\max_{t\in[\alpha-(k-1),\alpha+T]_{\mathbb{N}_{\alpha-(k-1)}}} G(t,s) = \frac{1}{4}G(T+\alpha, s)$.

Proof The proof is similar to the proof of Lemma 2.4 in [6]. □

4 Existence of positive solutions

In this section, we define the Banach space

$$\begin{aligned} \mathcal{E}_i = \bigl\{&y_i : [\alpha-k, \alpha+T]_{\mathbb{N}_{\alpha-k}} \to \mathbb{R} | y_i(\alpha-k) = a_i^0, \\ &\Delta y_i(\alpha+T) = \Delta^j y_i(\alpha-k) = 0, j = 2, 3, \ldots, k-1\bigr\}, \end{aligned}$$

with norm

$$\|y_i\|_{\mathcal{E}_i} = \max\bigl|y_i(t)\bigr|, \quad t \in [\alpha-k, \alpha+T]_{\mathbb{N}_{\alpha-k}}.$$

It is clear that $\mathcal{E}_i$ is a Banach space. Put $\mathcal{E} := \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$. By equipping $\mathcal{E}$ with the norm

$$\|(y_1, y_2, \ldots, y_n)\| = \max\{\|y_1\|_{\mathcal{E}_1}, \|y_2\|_{\mathcal{E}_2}, \ldots, \|y_n\|_{\mathcal{E}_n}\},$$

it follows that $(\mathcal{E}, \|\cdot\|)$ is a Banach space. It is clear that $(y_1, y_2, \ldots, y_n)$ is solution of (1) if and only if y_i satisfies

$$y_i(t) = \lambda_i \sum_{s=0}^{T+1} G(t,s) f_i\bigl(y_1(s+\alpha-1), y_2(s+\alpha-1), \ldots, y_n(s+\alpha-1)\bigr) + a_i^0.$$

Let $\mathcal{T}_i : \mathcal{E} \to \mathcal{E}_i$ be the operator defined by

$$\begin{aligned} &\mathcal{T}_i(y_1, y_2, \ldots, y_n)(t) \\ &\quad = \lambda_i \sum_{s=0}^{T+1} G(t,s) f_i\bigl(y_1(s+\alpha-1), y_2(s+\alpha-1), \ldots, y_n(s+\alpha-1)\bigr) + a_i^0. \end{aligned} \tag{6}$$

Define the operator $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ by

$$\begin{aligned} &\mathcal{T}(y_1, y_2, \ldots, y_n)(t) \\ &\quad = \bigl(\mathcal{T}_1(y_1, y_2, \ldots, y_n)(t), \mathcal{T}_2(y_1, y_2, \ldots, y_n)(t), \ldots, \mathcal{T}_n(y_1, y_2, \ldots, y_n)(t)\bigr). \end{aligned} \tag{7}$$

Let $\mathcal{P}$ be a cone defined by

$$\mathcal{P} = \left\{(y_1, y_2, \ldots, y_n) \in \mathcal{E} : y_i(t) \ge 0, \min_{t \in [\frac{\alpha+T}{4}, \frac{3(\alpha+T)}{4}]} y_i \ge \frac{1}{4}\|(y_1, y_2, \ldots, y_n)\|, i = 1, 2, \ldots, n\right\}.$$

Lemma 4.1 *Assume that $\mathcal{T}$ is the operator defined in (7). Then $\mathcal{T} : \mathcal{P} \to \mathcal{P}$.*

Proof By definition of $\mathcal{T}_i$, for $(y_1, y_2, \ldots, y_n) \in \mathcal{E}$, we have

$$\mathcal{T}_i(y_1, y_2, \ldots, y_n)(t) \ge 0, \quad i = 1, 2, \ldots, n.$$

We show that

$$\min_{t \in [\frac{\alpha+T}{4}, \frac{3(\alpha+T)}{4}]} \mathcal{T}_i(y_1, y_2, \ldots, y_n)(t) \ge \frac{1}{4}\|(y_1, y_2, \ldots, y_n)\|,$$

for $(y_1, y_2, \ldots, y_n) \in \mathcal{E}$. By Lemma 3.2(iii), we have

$$\begin{aligned} &\min_{t \in [\frac{\alpha+T}{4}, \frac{3(\alpha+T)}{4}]} \mathcal{T}_i(y_1, y_2, \ldots, y_n)(t) \\ &\quad = \min_{t \in [\frac{\alpha+T}{4}, \frac{3(\alpha+T)}{4}]} \lambda_i \sum_{s=0}^{T+1} G(t,s) f_i(y_1(s+\alpha-1), y_2\bigl(s+\alpha, \ldots, y_n(s+\alpha-1)\bigr) + a_i^0 \\ &\quad \ge \frac{\lambda_i}{4} \max_{t \in [\alpha-(k-1), \alpha+T]_{\mathbb{N}_{\alpha-(k-1)}}} \sum_{s=0}^{T+1} G(t,s) f_i\bigl(y_1(s+\alpha-1), y_2(s+\alpha-1), \ldots, y_n(s+\alpha-1)\bigr) \end{aligned}$$

$$+\frac{a_i^0}{4} = \frac{1}{4}\left\|\mathcal{T}_i(y_1, y_2, \ldots, y_n)\right\|.$$

This proves that $\mathcal{T} : \mathcal{P} \to \mathcal{P}$. □

Theorem 4.2 *Let $f_i : [0,\infty) \times [0,\infty) \times \cdots \times [0,\infty) \to [0,\infty)$ be given for $i = 1, 2, \ldots, n$. If $(y_1, y_2, \ldots, y_n) \in \mathcal{E}$ is a fixed point of $\mathcal{T}$. Then $(y_1, y_2, \ldots, y_n) \in \mathcal{E}$ is a solution of (1).*

Proof Let $(y_1, y_2, \ldots, y_n) \in \mathcal{E}$ be a fixed point of $\mathcal{T}$, then we have

$$y_i(t) = \mathcal{T}_i(y_1, y_2, \ldots, y_n)(t) \geq 0, \quad i = 1, 2, \ldots, n,$$

where $\mathcal{T}_i$ is defined as in (6). It is easy to see that

$$\mathcal{T}_i(y_1, y_2, \ldots, y_n)(\alpha - k) = a_i^0$$

and

$$\begin{aligned}
\Delta\mathcal{T}_i(y_1, y_2, \ldots, y_n)(\alpha + T) &= \mathcal{T}_i(y_1, y_2, \ldots, y_n)(\alpha + T + 1) - \mathcal{T}_i(y_1, y_2, \ldots, y_n)(\alpha + T) \\
&= \lambda_i \sum_{s=0}^{T+1} G(\alpha + T + 1, s) f_i\big(y_1(s + \alpha - 1), y_2(s + \alpha - 1), \ldots, y_n(s + \alpha - 1)\big) + a_i^0 \\
&\quad - \lambda_i \sum_{s=0}^{T+1} G(\alpha + T, s) f_i\big(y_1(s + \alpha - 1), y_2(s + \alpha - 1), \ldots, y_n(s + \alpha - 1)\big) + a_i^0 \\
&= \lambda_i \sum_{s=0}^{T+1} \big[G(\alpha + T + 1, s) - G(\alpha + T, s)\big] f_i\big(y_1(s + \alpha - 1), \ldots, y_n(s + \alpha - 1)\big) \\
&= \frac{\lambda_i}{\Gamma(\alpha)} \sum_{s=0}^{T+1} \big[(\alpha - 1)(\alpha + T + 1 - \alpha + k)(T + \alpha - s - 1)^{\underline{\alpha-2}} - (\alpha + T + 1 - s - 1)^{\underline{\alpha-1}} \\
&\quad - (\alpha - 1)(\alpha + T - \alpha + k)(T + \alpha - s - 1)^{\underline{\alpha-2}} + (\alpha + T - s - 1)^{\underline{\alpha-1}}\big] \\
&\quad \times f_i\big(y_1(s + \alpha - 1), y_2(s + \alpha - 1), \ldots, y_n(s + \alpha - 1)\big) \\
&= \frac{\lambda_i}{\Gamma(\alpha)} \sum_{s=0}^{T+1} \left\{(\alpha - 1)(T + \alpha - s - 1)^{\underline{\alpha-2}} - \left[\frac{\Gamma(\alpha + T - s + 1)}{\Gamma(T - s + 2)} - \frac{\Gamma(\alpha + T - s)}{\Gamma(T - s + 1)}\right]\right\} \\
&\quad \times f_i\big(y_1(s + \alpha - 1), y_2(s + \alpha - 1), \ldots, y_n(s + \alpha - 1)\big) \\
&= \frac{\lambda_i}{\Gamma(\alpha)} \sum_{s=0}^{T+1} \left\{(\alpha - 1)(T + \alpha - s - 1)^{\underline{\alpha-2}} - \left[\frac{(\alpha + T - s)\Gamma(\alpha + T - s)}{(T - s + 1)\Gamma(T - s + 1)} - \frac{\Gamma(\alpha + T - s)}{\Gamma(T - s + 1)}\right]\right\} \\
&\quad \times f_i\big(y_1(s + \alpha - 1), y_2(s + \alpha - 1), \ldots, y_n(s + \alpha - 1)\big) \\
&= \frac{\lambda_i}{\Gamma(\alpha)} \sum_{s=0}^{T+1} \left\{(\alpha - 1)(T + \alpha - s - 1)^{\underline{\alpha-2}} - \left[\frac{\Gamma(\alpha + T - s)}{\Gamma(T - s + 1)}\left(\frac{\alpha + T - s}{T - s + 1} - 1\right)\right]\right\} \\
&\quad \times f_i\big(y_1(s + \alpha - 1), y_2(s + \alpha - 1), \ldots, y_n(s + \alpha - 1)\big) \\
&= \frac{\lambda_i}{\Gamma(\alpha)} \sum_{s=0}^{T+1} \left[(\alpha - 1)(T + \alpha - s - 1)^{\underline{\alpha-2}} - \frac{(\alpha - 1)\Gamma(\alpha + T - s)}{(T - s + 1)\Gamma(T - s + 1)}\right]
\end{aligned}$$

$$\begin{aligned}
&\times f_i\big(y_1(s+\alpha-1),y_2(s+\alpha-1),\dots,y_n(s+\alpha-1)\big)\\
&=\frac{\lambda_i}{\Gamma(\alpha)}\sum_{s=0}^{T+1}\left[(\alpha-1)(T+\alpha-s-1)^{\underline{\alpha-2}}-\frac{(\alpha-1)\Gamma(\alpha+T-s)}{\Gamma(T-s+2)}\right]\\
&\times f_i\big(y_1(s+\alpha-1),y_2(s+\alpha-1),\dots,y_n(s+\alpha-1)\big)\\
&=\frac{\lambda_i}{\Gamma(\alpha)}\sum_{s=0}^{T+1}\big[(\alpha-1)(T+\alpha-s-1)^{\underline{\alpha-2}}-(\alpha-1)(T+\alpha-s-1)^{\underline{\alpha-2}}\big]\\
&\times f_i\big(y_1(s+\alpha-1),y_2(s+\alpha-1),\dots,y_n(s+\alpha-1)\big)=0.
\end{aligned}$$

Finally, when $0 < t-\alpha+1 \le s \le T+1$,

$$G(t,s)=(\alpha-1)(t-\alpha+k)(T+\alpha-s-1)^{\underline{\alpha-2}},$$

then

$$\Delta^j G(t,s)=0,\quad j=2,\dots,k-1.$$

Therefore, we conclude

$$\Delta^j \mathcal{T}_i(y_1,y_2,\dots,y_n)(\alpha-k)=0,\quad j=2,\dots,k-1,$$

which completes the proof. □

Theorem 4.3 ([20]) *Let $\mathcal{E}$ be a Banach space, and let $\mathcal{P}\subset\mathcal{E}$ be a cone in $\mathcal{E}$. Assume Ω_1,Ω_2 are open subsets of $\mathcal{E}$ with $0\in\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$ and let $S:\mathcal{P}\to\mathcal{P}$ be a completely continuous operator such that either*

(A1) $\|\mathcal{T}w\|\le\|w\|, w\in\mathcal{P}\cap\partial\Omega_1$, $\|\mathcal{T}w\|\ge\|w\|, w\in\mathcal{P}\cap\partial\Omega_2$, *or*

(A2) $\|\mathcal{T}w\|\ge\|w\|, w\in\mathcal{P}\cap\partial\Omega_1$, $\|\mathcal{T}w\|\le\|w\|, w\in\mathcal{P}\cap\partial\Omega_2$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{P}\cap(\overline{\Omega_2}\setminus\Omega_1)$.

Now we find the parameter interval for which (1) has a positive solution. We use the following notations:

$$\begin{aligned}
&F_i^0=\limsup_{y_i\to0^+}\frac{f_i(y_1,y_2,\dots,y_n)}{y_i},\\
&F_i^\infty=\limsup_{y_i\to+\infty}\frac{f_i(y_1,y_2,\dots,y_n)}{y_i},\\
&f_i^0=\liminf_{y_i\to0^+}\frac{f_i(y_1,y_2,\dots,y_n)}{y_i},\\
&f_i^\infty=\liminf_{y_i\to+\infty}\frac{f_i(y_1,y_2,\dots,y_n)}{y_i},\\
&\Omega_r=\big\{(y_1,y_2,\dots,y_n)\in\mathcal{E}:\big\|(y_1,y_2,\dots,y_n)\big\|<r\big\},\\
&K=\max G(t,s),\quad\text{for }(t,s)\in\big[\alpha-(k-1),\alpha+T\big]_{\mathbb{N}_{\alpha-(k-1)}}\times[0,T+1],
\end{aligned}$$

$$l=\left(\frac{3(\alpha+T)}{4}-\alpha+1\right)+1-\left(\frac{\alpha+T}{4}-\alpha+1\right).$$

In this section without loss of generality, we consider the operator $\mathcal{T}_i$, without a_i^0.

Theorem 4.4 *If $\frac{1}{16}f_i^\infty Kl > F_i^0 K(T+1), f_i^\infty Kl \neq 0$ hold, then for each*

$$\lambda_i \in \left(\left(\frac{1}{16}f_i^\infty Kl\right)^{-1}, \left(F_i^0 K(T+1)\right)^{-1}\right) \tag{8}$$

the problem (1) has at least one positive solution. Note that we assume $(f_i^\infty Kl)^{-1} = 0$ if $f_i^\infty = +\infty$ and $(F_i^0 K(T+1))^{-1} = +\infty$ if $F_i^0 = 0$.

Proof If λ_i satisfies in (8) and $\varepsilon > 0$ is given such that

$$\left(\frac{1}{16}(f_i^\infty - \varepsilon)Kl\right)^{-1} \leq \lambda_i \leq \left((F_i^0+\varepsilon)K(T+1)\right)^{-1}, \tag{9}$$

then, using the notation with F_i^0, there exists $r_1 > 0$ such that for $(y_1, y_2, \dots, y_n) \in \Omega_{r_1}$

$$f_i(y_1, y_2, \dots, y_n) \leq \left(F_i^0 + \varepsilon\right)y_i. \tag{10}$$

So if $(y_1, y_2, \dots, y_n) \in \partial\mathcal{P}$ with $\|(y_1, y_2, \dots, y_n)\| = r_1$ then, by (9) and (10), we have

$$\left\|\mathcal{T}_i(y_1, y_2, \dots, y_n)\right\|_{\mathcal{E}_i} \leq \lambda_i\left(F_i^0+\varepsilon\right)\left\|(y_1, y_2, \dots, y_n)\right\|K(T+1) \leq r_1 = \left\|(y_1, y_2, \dots, y_n)\right\|.$$

Hence for $(y_1, y_2, \dots, y_n) \in \mathcal{P} \cap \partial\Omega_{r_1}$

$$\left\|\mathcal{T}(y_1, y_2, \dots, y_n)\right\| = \max\left\{\left\|\mathcal{T}_i(y_1, y_2, \dots, y_n)\right\|_{\mathcal{E}_i}\right\} \leq \left\|(y_1, y_2, \dots, y_n)\right\|. \tag{11}$$

Let $r_3 > 0$ be such that for $y_i \geq r_3$

$$f_i(y_1, y_2, \dots, y_n) \geq \left(f_i^\infty - \varepsilon\right)y_i. \tag{12}$$

If $(y_1, y_2, \dots, y_n) \in \partial\mathcal{P}$ and $\|(y_1, y_2, \dots, y_n)\| = r_2 = \max\{2r_1, r_3\}$ then using (9) and (12) implies

$$\begin{aligned}
\left\|\mathcal{T}_i(y_1, y_2, \dots, y_n)\right\|_{\mathcal{E}_i} \geq \mathcal{T}_i(y_1, y_2, \dots, y_n) &= \lambda_i \sum_{s=0}^{T+1} G(t,s) f_i(y_1, y_2, \dots, y_n) \\
&\geq \lambda_i\left(f_i^\infty - \varepsilon\right) \sum_{s=(\frac{\alpha+T}{4}-\alpha+1)}^{(\frac{3(\alpha+T)}{4}-\alpha+1)} G(t,s) y_i(s+\alpha-1) \\
&\geq \frac{\frac{1}{4}(f_i^\infty - \varepsilon)\|(y_1, y_2, \dots, y_n)\|}{\frac{1}{16}(f_i^\infty - \varepsilon)Kl} \frac{1}{4}Kl \\
&= \left\|(y_1, y_2, \dots, y_n)\right\|.
\end{aligned}$$

Then for $(y_1, y_2, \dots, y_n) \in \mathcal{P} \cap \partial \Omega_{r_2}$

$$\left\| \mathcal{T}(y_1, y_2, \dots, y_n) \right\| = \max\left\{ \left\| \mathcal{T}_i(y_1, y_2, \dots, y_n) \right\|_{\mathcal{E}_i} \right\} \geq \left\| (y_1, y_2, \dots, y_n) \right\|. \tag{13}$$

Now, from (11), (13), and Theorem 4.3, we see that $\mathcal{T}$ has a fixed point $(y_1, y_2, \dots, y_n) \in \mathcal{P} \cap (\overline{\Omega_{r_2}} \setminus \Omega_{r_1})$, where $r_1 \leq \|(y_1, y_2, \dots, y_n)\| \leq r_2$, and clearly $(y_1, y_2, \dots, y_n)$ is a positive solution of the problem (1). □

Theorem 4.5 *If* $\frac{1}{16} f_i^0 Kl > F_i^\infty K(T+1), f_i^0 Kl \neq 0$ *hold, then for each*

$$\lambda_i \in \left(\left(\frac{1}{16} f_i^0 Kl \right)^{-1}, \left(F_i^\infty K(T+1) \right)^{-1} \right) \tag{14}$$

the problem (1) *has at least one positive solution. Note that we assume* $(f_i^0 Kl)^{-1} = 0$ *if* $f_i^0 = +\infty$ *and* $(F_i^\infty K(T+1))^{-1} = +\infty$ *if* $F_i^\infty = 0$.

Proof Suppose λ_i satisfies in (14) and $\varepsilon > 0$ is such that

$$\left(\frac{1}{16} (f_i^0 - \varepsilon) Kl \right)^{-1} \leq \lambda_i \leq \left((F_i^\infty + \varepsilon) K(T+1) \right)^{-1}. \tag{15}$$

By using the notation of f_i^0, there exists $r_1 > 0$ such that for $(y_1, y_2, \dots, y_n) \in \Omega_{r_1}$

$$f_i(y_1, y_2, \dots, y_n) \geq (f_i^0 - \varepsilon) y_i. \tag{16}$$

So if $(y_1, y_2, \dots, y_n) \in \partial \mathcal{P}$ with $\|(y_1, y_2, \dots, y_n)\| = r_1$ then analogous to Theorem 4.4, we deduce

$$\left\| \mathcal{T}_i(y_1, y_2, \dots, y_n) \right\|_{\mathcal{E}_i} \geq \left\| (y_1, y_2, \dots, y_n) \right\|.$$

Hence, for $(y_1, y_2, \dots, y_n) \in \mathcal{P} \cap \partial \Omega_{r_1}$

$$\left\| \mathcal{T}(y_1, y_2, \dots, y_n) \right\| \geq \left\| (y_1, y_2, \dots, y_n) \right\|. \tag{17}$$

Next, we may choose $R_1 > 0$ such that for $y_i \geq R_1$

$$f_i(y_1, y_2, \dots, y_n) \leq \left(F_i^\infty + \varepsilon \right) y_i. \tag{18}$$

Case 1. If f_i is bounded, then, for some $N_i > 0$, we have

$$f_i(y_1, y_2, \dots, y_n) \leq N_i \quad \text{for } y_i \in [0, +\infty).$$

Now let $r_3 = \max\{2r_1, \lambda N_i K(T+1)\}$ and $(y_1, y_2, \dots, y_n) \in \mathcal{P}$ with $\|(y_1, y_2, \dots, y_n)\| = r_3$, thus

$$\left\| \mathcal{T}_i(y_1, y_2, \dots, y_n) \right\|_{\mathcal{E}_i} \leq \lambda_i \sum_{s=0}^{T+1} G(\alpha + T, s) f_i(y_1, y_2, \dots, y_n)$$

$$\leq \lambda_i N_i \sum_{s=0}^{T+1} G(\alpha+T,s) = \lambda_i K(T+1)N_i \leq r_3$$
$$= \|(y_1,y_2,\dots,y_n)\|.$$

Hence, for $(y_1,y_2,\dots,y_n) \in \partial\Omega_{r_3}$,

$$\|\mathcal{T}(y_1,y_2,\dots,y_n)\| \leq \|(y_1,y_2,\dots,y_n)\|. \tag{19}$$

Case 2. If f_i is not bounded. Then for some $r_4 > \max\{2r_1,R_1\}$ we have

$$f_i(y_1,y_2,\dots,y_n) \geq f_i(r_4,r_4,\dots,r_4) \quad \text{for } (y_1,y_2,\dots,y_n) \in \Omega_{r_4}.$$

If $(y_1,y_2,\dots,y_n) \in \mathcal{P}$ with $\|(y_1,y_2,\dots,y_n)\| = r_4$, then, by (15) and (18), we have

$$\begin{aligned}\|\mathcal{T}_i(y_1,y_2,\dots,y_n)\|_{\mathcal{E}_i} &\leq \lambda_i \sum_{s=0}^{T+1} G(\alpha+T,s)\big(F_i^\infty+\varepsilon\big)y_i(s+\alpha-1)\\ &\leq \lambda_i\big(F_i^\infty+\varepsilon\big)\|(y_1,y_2,\dots,y_n)\|K(T+1)\\ &= \lambda_i K(T+1)\big(F_i^\infty+\varepsilon\big)\|(y_1,y_2,\dots,y_n)\|\\ &\leq \|(y_1,y_2,\dots,y_n)\|.\end{aligned}$$

Thus (19) holds.

For $(y_1,y_2,\dots,y_n) \in \mathcal{P}\cap\partial\Omega_{r_2}$, we have

$$\|\mathcal{T}(y_1,y_2,\dots,y_n)\| \leq \|(y_1,y_2,\dots,y_n)\|. \tag{20}$$

Theorem 4.3 implies that $\mathcal{T}$ has a fixed point $(y_1,y_2,\dots,y_n) \in \mathcal{P}\cap(\overline{\Omega_{r_2}}\setminus\Omega_{r_1})$, where $r_1 \leq \|(y_1,y_2,\dots,y_n)\| \leq r_2$, and easily $(y_1,y_2,\dots,y_n)$ is a positive solution of (1). □

Theorem 4.6 *Suppose there exist $r_2 > r_1 > 0$ such that, for $\lambda_i > 0$,*

$$\max_{0\leq y_i\leq r_2} f_i(y_1,y_2,\dots,y_n) \leq \frac{r_2}{\lambda_i K(T+1)}, \qquad \min_{0\leq y_i\leq r_1} f_i(y_1,y_2,\dots,y_n) \geq \frac{r_1}{\frac{1}{4}\lambda_i Kl}.$$

Then (1) has a positive solution $(y_1,y_2,\dots,y_n) \in \mathcal{P}$, where $r_1 \leq \|(y_1,y_2,\dots,y_n)\| \leq r_2$.

Proof If $(y_1,y_2,\dots,y_n) \in \mathcal{P}\cap\partial\Omega_{r_1}$, we have

$$\begin{aligned}\|\mathcal{T}_i(y_1,y_2,\dots,y_n)\|_{\mathcal{E}_i} &\geq \mathcal{T}_i(y_1,y_2,\dots,y_n) = \lambda_i \sum_{s=0}^{T+1} G(t,s) f_i(y_1,y_2,\dots,y_n)\\ &\geq \lambda_i \sum_{s=(\frac{\alpha+T}{4}-\alpha+1)}^{(\frac{3(\alpha+T)}{4}-\alpha+1)} G(\alpha+T,s)\min_{0\leq y_i\leq r_1} f_i(y_1,y_2,\dots,y_n)\\ &\geq \lambda_i\frac{1}{4}Kl\frac{r_1}{\lambda_i\frac{1}{4}Kl} = r_1 = \|(y_1,y_2,\dots,y_n)\|.\end{aligned}$$

That is, for $(y_1, y_2, \dots, y_n) \in \mathcal{P} \cap \partial\Omega_{r_1}$,

$$\|\mathcal{T}(y_1, y_2, \dots, y_n)\| \geq \|(y_1, y_2, \dots, y_n)\|.$$

Also, for $(y_1, y_2, \dots, y_n) \in \mathcal{P} \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}\|\mathcal{T}_i(y_1, y_2, \dots, y_n)\|_{\mathcal{E}_i} &\leq \lambda_i \sum_{s=0}^{T+1} G(\alpha + T, s) \max_{0 \leq y_i \leq r_2} f_i(y_1, y_2, \dots, y_n) \\ &\leq \lambda_i K(T+1) \frac{r_2}{\lambda_i K(T+1)} = r_2 = \|(y_1, y_2, \dots, y_n)\|.\end{aligned}$$

That is, for $(y_1, y_2, \dots, y_n) \in \mathcal{P} \cap \partial\Omega_{r_2}$,

$$\|\mathcal{T}(y_1, y_2, \dots, y_n)\| \leq \|(y_1, y_2, \dots, y_n)\|.$$

Thus, by Theorem 4.3, the problem (1) has a positive solution $(y_1, y_2, \dots, y_n) \in \mathcal{P}$, where $r_1 \leq \|(y_1, y_2, \dots, y_n)\| \leq r_2$. □

5 Nonexistence results

In order to find some nonexistence results for problem (1), we consider the following condition:

(H) $\sup_{r>0} \min_{y_i \in (0,r)} f_i(y_1, y_2, \dots, y_n) > 0$.

Theorem 5.1 *Assume the condition* (H) *is true. If* $F_i^0 < +\infty$ *and* $F_i^\infty < +\infty$, *then there is a positive real number* $\lambda_i^0 > 0$ *such that for* $0 < \lambda_i < \lambda_i^0$ *the problem* (1) *does not have a positive solution.*

Proof Since F_i^0, F_i^∞ are finite, we can find positive real numbers l_i^1, l_i^2, r_1, r_2, where $r_1 < r_2$ and

$$\begin{aligned} f_i(y_1, y_2, \dots, y_n) &\leq l_i^1 y_i, \quad \text{for } y_i \in [0, r_1], \\ f_i(y_1, y_2, \dots, y_n) &\leq l_i^2 y_i, \quad \text{for } y_i \in [r_2, +\infty). \end{aligned}$$

Let $L_i = \max\{l_i^1, l_i^2, \max_{r_1 \leq y_i \leq r_2}\{\frac{f_i(y_1, y_2, \dots, y_n)}{y_i}\}\}$. Then we have

$$f_i(y_1, y_2, \dots, y_n) \leq L_i y_i, \quad \text{for } y_i \in [0, +\infty).$$

Assume $(w_1, w_2, \dots, w_n)(t)$ is a positive solution of (1). We find a contradiction for $0 < \lambda_i < \lambda_i^0 := (L_i K(T+1))^{-1}$. Since $\mathcal{T}(w_1, w_2, \dots, w_n)(t) = (w_1, w_2, \dots, w_n)(t)$ for $t \in [\alpha - k, \alpha + T]_{\mathbb{Z}_{\alpha-k}}$,

$$\begin{aligned}\|(w_1, w_2, \dots, w_n)\| &= \|\mathcal{T}(w_1, w_2, \dots, w_n)(t)\| = \max\{\|\mathcal{T}_i(w_1, w_2, \dots, w_n)\|_{\mathcal{E}_i}\} \\ &\leq \lambda_i K(T+1) L_i \|(w_1, w_2, \dots, w_n)\| < \|(w_1, w_2, \dots, w_n)\|,\end{aligned}$$

which is a contradiction. □

Theorem 5.2 *Suppose* (H) *holds. If* $f_i^0 > 0$ *and* $f_i^\infty > 0$, *then there is a real number* $\lambda_i^0 > 0$ *such that for* $\lambda_i > \lambda_i^0$ *the problem* (1) *does not have a positive solution.*

Proof Since f_i^0, f_i^∞ are positive, there exist $\gamma_i^1, \gamma_i^2, r_1, r_2 > 0$, where $r_1 < r_2$, and

$$f_i(y_1, y_2, \dots, y_n) \geq \gamma_i^1 y_i, \quad \text{for } y_i \in [0, r_1],$$
$$f_i(y_1, y_2, \dots, y_n) \geq \gamma_i^2 y_i, \quad \text{for } y_i \in [r_2, +\infty).$$

Let $\gamma_i = \min\{\gamma_i^1, \gamma_i^2, \min_{r_1 \leq y_i \leq r_2}\{\frac{f_i(y_1,y_2,\dots,y_n)}{y_i}\}\} > 0$. Then we get

$$f_i(y_1, y_2, \dots, y_n) \geq \gamma_i y_i, \quad \text{for } y_i \in [0, +\infty).$$

Assume $(w_1, w_2, \dots, w_n)$ is a positive solution of (1). We find a contradiction for $\lambda_i > \lambda_i^0 := (\frac{1}{16}\gamma_i K l)^{-1}$. Since $\mathcal{T}(w_1, w_2, \dots, w_n)(t) = (w_1, w_2, \dots, w_n)(t)$ for $t \in [\alpha - k, \alpha + T]_{\mathbb{Z}_{\alpha-k}}$,

$$\begin{aligned} \|(w_1, w_2, \dots, w_n)\| &= \|\mathcal{T}(w_1, w_2, \dots, w_n)(t)\| = \max\{\|\mathcal{T}_i(w_1, w_2, \dots, w_n)\|_{\mathcal{E}_i}\} \\ &\geq \lambda_i \gamma_i \frac{1}{4}\|(w_1, w_2, \dots, w_n)\|\frac{1}{4}Kl > \|(w_1, w_2, \dots, w_n)\|, \end{aligned}$$

which is a contradiction. □

6 Uniqueness

Theorem 6.1 *Assume that the Banach space* $\mathcal{E} := \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ *is endowed with the norm*

$$\|(y_1, y_2, \dots, y_n)\| = \max\{\|y_1\|_{\mathcal{E}_1}, \|y_2\|_{\mathcal{E}_2}, \dots, \|y_n\|_{\mathcal{E}_n}\},$$

and f_i *satisfying the Lipschitz condition*

$$\|(f_i(y_1, y_2, \dots, y_n) - f_i(w_1, w_2, \dots, w_n))\| \leq L_i\|(y_1, y_2, \dots, y_n) - (w_1, w_2, \dots, w_n)\|, \tag{21}$$

where $y_i, w_i \in \mathcal{E}_i, L_i > 0$. *Then the problem* (1) *has exactly one positive solution provided*

$$L(T+1)K\|(\lambda_1, \dots, \lambda_n)\| < 1,$$

where $L = \max\{L_i, i = 0, 1, 2, \dots, n\}$, $K = \max G(t, s)$ *and* $\|(\lambda_1, \dots, \lambda_n)\| = \max\{|\lambda_i|, i = 0, 1, 2, \dots, n\}$.

Proof For any $(y_1, y_2, \dots, y_n), (w_1, w_2, \dots, w_n) \in \mathcal{E}$, using the assumption (21), we have

$$\begin{aligned} &\|(\mathcal{T}_i(y_1, y_2, \dots, y_n) - \mathcal{T}_i(w_1, w_2, \dots, w_n))\|_{\mathcal{E}_i} \\ &\quad \leq \lambda_i \sum_{s=0}^{T+1} G(\alpha + T, s)\|(f_i(y_1, y_2, \dots, y_n) - f_i(w_1, w_2, \dots, w_n))\| \\ &\quad \leq L_i \lambda_i (T+1)K\|(y_1, y_2, \dots, y_n) - (w_1, w_2, \dots, w_n)\| \\ &\quad \leq L(T+1)K\|(\lambda_1, \dots, \lambda_n)\|\|(y_1, y_2, \dots, y_n) - (w_1, w_2, \dots, w_n)\|, \end{aligned}$$

$i = 1, 2, \dots, n$. That is,

$$\begin{aligned}
&\bigl\|\mathcal{T}(y_1,y_2,\dots,y_n)-\mathcal{T}(w_1,w_2,\dots,w_n)\bigr\| \\
&\quad=\bigl\|\bigl(\mathcal{T}_1(y_1,y_2,\dots,y_n),\dots,\mathcal{T}_n(y_1,y_2,\dots,y_n)\bigr) \\
&\qquad-\bigl(\mathcal{T}_1(w_1,w_2,\dots,w_n),\dots,\mathcal{T}_n(w_1,w_2,\dots,w_n)\bigr)\bigr\| \\
&\quad=\bigl\|\bigl(\mathcal{T}_1(y_1,y_2,\dots,y_n)-\mathcal{T}_1(w_1,w_2,\dots,w_n)\bigr),\dots, \\
&\qquad\bigl(\mathcal{T}_n(y_1,y_2,\dots,y_n)-\mathcal{T}_n(w_1,w_2,\dots,w_n)\bigr)\bigr\| \\
&\quad=\max\bigl\{\bigl\|\bigl(\mathcal{T}_i(y_1,y_2,\dots,y_n)-\mathcal{T}_i(w_1,w_2,\dots,w_n)\bigr)\bigr\|_{\mathcal{E}_i}, i=0,1,\dots,n\bigr\} \\
&\quad\le L(T+1)K\bigl\|(\lambda_1,\dots,\lambda_n)\bigr\|\bigl\|(y_1,y_2,\dots,y_n)-(w_1,w_2,\dots,w_n)\bigr\|.
\end{aligned}$$

Since $L = \max\{L_i, i = 0, 1, 2, \dots, n\}$ and $\|(\lambda_1, \dots, \lambda_n)\| = \max\{|\lambda_i|, i = 0, 1, 2, \dots, n\}$, for $L(T + 1)K\|(\lambda_1, \dots, \lambda_n)\| < 1$, the operator $\mathcal{T}$ is the contraction mapping. Therefore, the problem (1) has exactly one positive solution. □

7 Example

Example 7.1 Suppose that $\alpha = 1.8$, $T = 10$, $\lambda_i > 0$, and $a_i^0 > 0$ is an integer and

$$\begin{cases}
\Delta_c^\alpha y_1(t)+\lambda_1 f_1(y_1,y_2)=0, & y_1(-0.2)=a_1^0, \quad \Delta y_1(11.8)=0,\\
\Delta_c^\alpha y_2(t)+\lambda_2 f_2(y_1,y_2)=0, & y_2(-0.2)=a_2^0, \quad \Delta y_2(11.8)=0,
\end{cases} \tag{22}$$

where

$$\begin{aligned}
f_1(y_1,y_2)&=\frac{(y_1^2+y_2^2+y_1)(4+\cos(y_1y_2))}{300y_1+12y_2+1},\\
f_2(y_1,y_2)&=\frac{(y_1^2+y_2^2+y_2)(4+\sin(y_1y_2))}{20y_1+200y_2+1}.
\end{aligned}$$

Clearly $f_i(y_1, y_2) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Moreover, it is easy to prove that

$$\begin{aligned}
F_1^\infty&=\limsup_{y_1\to\infty}\frac{f_1(y_1,y_2)}{y_1}=\frac{1}{60}, \qquad F_2^\infty=\limsup_{y_2\to\infty}\frac{f_2(y_1,y_2)}{y_2}=\frac{1}{40},\\
f_1^0&=\liminf_{y_1\to 0^+}\frac{f_1(y_1,y_2)}{y_1}=4, \qquad f_2^0=\liminf_{y_2\to 0^+}\frac{f_2(y_1,y_2)}{y_2}=4.
\end{aligned}$$

Therefore, from Theorem 4.5, for each

$$\lambda_1\in(0.366,3.4441), \qquad \lambda_2\in(0.366,2.2961),$$

the boundary value problem (22) has at least one positive solution.

Example 7.2 In this example we focus on the linearized system as follows:

$$\begin{cases} \Delta_c^\alpha y_1(t) + \frac{\lambda_1}{1000} y_1 = 0, \qquad y_1(-0.02) = a_1^0, \\ \Delta^2 y_1(-0.02) = \Delta y_1(12.98) = 0, \\ \Delta_c^\alpha y_2(t) + \frac{\lambda_2}{1000} y_2 = 0, \qquad y_2(-0.02) = a_2^0, \\ \Delta^2 y_2(-0.02) = \Delta y_2(12.98) = 0, \\ \Delta_c^\alpha y_3(t) + \frac{\lambda_3}{1000} y_3 = 0, \\ y_3(-0.02) = a_3^0, \qquad \Delta^2 y_3(-0.02) = \Delta y_3(12.98) = 0, \\ \Delta_c^\alpha y_4(t) + \frac{\lambda_4}{1000} y_4 = 0, \\ y_4(-0.02) = a_4^0, \qquad \Delta^2 y_4(-0.02) = \Delta y_4(12.98) = 0, \end{cases} \tag{23}$$

where $\alpha = 2.98, T = 10, \lambda_i > 0$, and $a_i^0 > 0$ is an integer and

$$f_1(y_1,y_2,y_3,y_4) = \frac{1}{1000} y_1, \qquad f_2(y_1,y_2,y_3,y_4) = \frac{1}{1000} y_2,$$
$$f_3(y_1,y_2,y_3,y_4) = \frac{1}{1000} y_3, \qquad f_4(y_1,y_2,y_3,y_4) = \frac{1}{1000} y_4.$$

Clearly $f_i(y_1,y_2,y_3,y_4) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Take $l_i^1 = l_i^2 = l_i^3 = l_i^4 = \frac{1}{1000}$. It is easy to prove that

$$F_1^\infty = \limsup_{y_1 \to \infty} \frac{f_1(y_1,y_2,y_3,y_4)}{y_1} = \frac{1}{1000}, \qquad F_2^\infty = \limsup_{y_2 \to \infty} \frac{f_2(y_1,y_2,y_3,y_4)}{y_2} = \frac{1}{1000},$$
$$F_3^\infty = \limsup_{y_3 \to \infty} \frac{f_3(y_1,y_2,y_3,y_4)}{y_3} = \frac{1}{1000}, \qquad F_4^\infty = \limsup_{y_4 \to \infty} \frac{f_4(y_1,y_2,y_3,y_4)}{y_4} = \frac{1}{1000},$$
$$F_1^0 = \limsup_{y_1 \to 0^+} \frac{f_1(y_1,y_2,y_3,y_4)}{y_1} = \frac{1}{1000}, \qquad F_2^0 = \limsup_{y_2 \to 0^+} \frac{f_2(y_1,y_2,y_3,y_4)}{y_2} = \frac{1}{1000},$$
$$F_3^0 = \limsup_{y_3 \to 0^+} \frac{f_3(y_1,y_2,y_3,y_4)}{y_3} = \frac{1}{1000}, \qquad F_4^0 = \limsup_{y_4 \to 0^+} \frac{f_4(y_1,y_2,y_3,y_4)}{y_4} = \frac{1}{1000},$$
$$L_i = \max\left\{ l_i^1, l_i^2, l_i^3, l_i^4, \max_{r_1 \le y \le r_2} \frac{f_i(y_1,y_2,y_3,y_4)}{y_i} \right\} = \frac{1}{1000}, \qquad \lambda_i^0 \approx 0.6078.$$

Thus, the conditions of Theorem 5.1 are satisfied. Therefore the problem (23) does not have a positive solution for $0 < \lambda_i < \lambda_i^0$. Moreover, we have

$$f_1^\infty = \liminf_{y_1 \to \infty} \frac{f_1(y_1,y_2,y_3,y_4)}{y_1} = \frac{1}{1000}, \qquad f_2^\infty = \liminf_{y_2 \to \infty} \frac{f_2(y_1,y_2,y_3,y_4)}{y_2} = \frac{1}{1000},$$
$$f_3^\infty = \liminf_{y_3 \to \infty} \frac{f_3(y_1,y_2,y_3,y_4)}{y_3} = \frac{1}{1000}, \qquad f_4^\infty = \liminf_{y_4 \to \infty} \frac{f_4(y_1,y_2,y_3,y_4)}{y_4} = \frac{1}{1000},$$
$$f_1^0 = \liminf_{y_1 \to 0^+} \frac{f_1(y_1,y_2,y_3,y_4)}{y_1} = \frac{1}{1000}, \qquad f_2^0 = \liminf_{y_2 \to 0^+} \frac{f_2(y_1,y_2,y_3,y_4)}{y_2} = \frac{1}{1000},$$
$$f_3^0 = \liminf_{y_3 \to 0^+} \frac{f_3(y_1,y_2,y_3,y_4)}{y_3} = \frac{1}{1000}, \qquad f_3^0 = \liminf_{y_3 \to 0^+} \frac{f_4(y_1,y_2,y_3,y_4)}{y_4} = \frac{1}{1000},$$
$$l = 6.4, \qquad \gamma_i = \min\left\{ \gamma_i^1, \gamma_i^2, \gamma_i^3, \gamma_i^4, \min_{r_1 \le y \le r_2} \frac{f_i(t,y_1,y_2,y_3,y_4)}{y_i} \right\} = \frac{1}{1000},$$

$$\lambda_i^0 \approx 16.4842.$$

Thus, the conditions of Theorem 5.2 are satisfied. Therefore the problem (23) does not have a positive solution for $\lambda_i > \lambda_i^0$.

8 Conclusion

In this research we consider a typical system of Caputo fractional difference equations of the form (1). Using the Guo–Krasnosel'skii fixed point theorem, we find a parameter interval for existence and nonexistence of positive solutions dependent on the parameter λ and two examples are given to illustrate the main results.

In this paper we used Caputo discrete fractional operators on the time scale $\mathbb{Z}$. It could be interesting to extend this work to the time scale $h\mathbb{Z}$. Working on $h\mathbb{Z}, 0 < h < 1$ rather than on $\mathbb{Z}$ makes it possible to guarantee the convergence of solutions for a larger class of fractional difference initial value problems [4, 18].

Acknowledgements
Not applicable.

Funding
This research has been supported by Sahand University of Technology under the title of "Ph.D. Thesis".

Availability of data and materials
Not applicable.

Competing interests
The author declares that he/she has no competing interests.

Authors' contributions
The contributions of both authors have been joint discussion, proving the results and providing the numerical examples. All authors read and approved the final manuscript.

Publisher's Note

Received: 14 December 2019 Accepted: 18 May 2020 Published online: 27 May 2020

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