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Pullback attractors of nonautonomous discrete p-Laplacian complex Ginzburg–Landau equations with fast-varying delays

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Abstract

In this paper, we consider a class of nonautonomous discrete p-Laplacian complex Ginzburg–Landau equations with time-varying delays. We prove the existence and uniqueness of pullback attractor for these equations. The existing results of studying attractors for time-varying delay equations require that the derivative of the delay term should be less than 1 (called slow-varying delay). By using differential inequality technique, our results remove the constraints on the delay derivative. So, we can deal with the equations with fast-varying delays (without any constraints on the delay derivative).

MSC: Primary 35B40; secondary 35B41; 37L30

Keywords: P-Laplacian lattice; Pullback attractor; Ginzburg–Landau equations; Fast-varying delay

1 Introduction

Due to numerous applications in physics, biology, and engineering such as pattern formation, propagation of nerve pulses, electric circuits, and so on, see, e.g., [2, 6, 7, 10, 12], lattice differential equations have become a large and growing interdisciplinary area of research. For an understanding of the dynamical behavior of dissipative infinite lattice systems, attractors are especially important because they retain most of the dynamical information. The existence of global attractors for lattice systems was initialed by Bates et al. [1], followed by extensions in [3, 8, 13, 16, 19, 24] and the references therein. Of those, the asymptotic behavior of an infinite-dimensional p-Laplacian lattice system was investigated in [8]. The dynamical behaviors for discrete complex Ginzburg–Landau equations were studied in [11, 27].

Since time-delays are frequently encountered in many practical systems, which may induce instability, oscillation, and poor performance of systems, delay lattice systems then arise naturally while these delays are taken into account. Recently, attractors of delay lattice systems have been considered in [4, 5, 9, 23, 26].

The existence and uniqueness of solutions were proved for the complex Ginzburg–Landau equation with p-Laplacian in [17, 18]. The dynamical behavior of p-Laplacian complex Ginzburg–Landau equations was considered in [25]. The existence and uniqueness of attractor for nonautonomous discrete p-Laplacian complex Ginzburg–Landau equations with fast-varying delays based on nonautonomous p-Laplacian complex Ginzburg–Landau equations with fast-varying delays are investigated in this paper. We prove the existence and uniqueness of pullback attractor for these equations. The existing results of studying attractors for time-varying delay equations require that the derivative of the delay term should be less than 1, see [14, 15, 21–23]. By using the differential inequality technique, our results remove the constrains on the delay derivative. So, we can deal with the equations with fast-varying delays.

Motivated by the discussions above, in this paper we study the dynamical behavior of nonautonomous discrete p-Laplacian complex Ginzburg–Landau equation with time-varying delays:

$$\begin{aligned}\frac{du_n}{dt} &= (\lambda + i\alpha)\big(|u_{n+1} - u_n|^{p-2}(u_{n+1} - u_n) - |u_n - u_{n-1}|^{p-2}(u_n - u_{n-1})\big) - (\kappa + i\beta)|u_n|^q u_n \\ &\quad + f_n\big(u_n\big(t - \rho_0(t)\big)\big) - (\gamma + i\delta)u_n + g_n(t), \quad n \in \mathbb{Z}, \tau \in \mathbb{R}, t > \tau, \end{aligned} \tag{1.1}$$

with the initial condition

$$u_n(\tau + s) = \varphi_n(s), \quad s \in [-\rho, 0], \tag{1.2}$$

where u_n is the unknown complex-valued function, λ, α, κ, β, γ, δ, ρ, p, q are real constants, where $\lambda, \kappa, \gamma, \rho, q > 0$ and $p \geq q+2$, f_n is a nonlinear function satisfying certain conditions, $\rho_0 \in C(\mathbb{R}, [0, \rho])$ is an adequate given delay function, $g(t) = (g_n(t))_{n\in\mathbb{Z}} \in L^2_{\text{loc}}(\mathbb{R}, l^2)$ (l^2 defined later) is a given time-dependent sequence, and $\varphi_n \in C([-\rho, 0], \mathbb{C})$.

The plan of this paper is as follows. In the next section, we establish the existence of a continuous nonautonomous dynamical system in $C([-\rho, 0], l^2)$ for nonautonomous equation (1.1) and (1.2). Section 3 contains all necessary uniform estimates of the solutions. We then prove the existence and uniqueness of a pullback attractor for the nonautonomous equations in Sect. 4.

2 Nonautonomous dynamical systems associated with nonautonomous lattice systems

In this section we show that there is a continuous nonautonomous dynamical system generated by the nonautonomous discrete p-Laplacian complex Ginzburg–Landau equations with time-varying delays.

Denote

$$l^2 = \left\{ u = \{u_n\}_{n\in\mathbb{Z}} \middle| u_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |u_n|^2 < \infty \right\},$$

and let l^2 be a Hilbert space with the inner produce and norm given by

$$(u, v) = \sum_{n\in\mathbb{Z}} u_n \overline{v_n}, \qquad \|u\|^2 = (u, u), \quad u, v \in l^2.$$

We define the linear operators $A, B, B^* : l^2 \to l^2$ as follows:

$$(Au)_n = |u_n - u_{n-1}|^{p-2}(u_n - u_{n-1}) - |u_{n+1} - u_n|^{p-2}(u_{n+1} - u_n),$$
$$(Bu)_n = u_{n+1} - u_n, \qquad (B^*u)_n = u_{n-1} - u_n, \quad n \in \mathbb{Z}, u \in l^2.$$

Then

$$\big(B^*u, v\big) = (u, Bv), \qquad (Au, v) = \big(B^*\big(|Bu|^{p-2} \otimes (Bu)\big), v\big) = \big(|Bu|^{p-2} \otimes (Bu), Bv\big),$$
$$(Au, u) = \big(|Bu|^{p-2} \otimes (Bu), Bu\big) = \|Bu\|_p^p \le \|Bu\|^p \le 2^p \|u\|^p, \quad u, v \in l^2,$$

where $u \otimes v = (u_i v_i)_{i\in\mathbb{Z}}$ and $\|u\|_p = (\sum_{n\in\mathbb{Z}} |u_n|^p)^{\frac{1}{p}}$.

Denote by u_t, $t \in \mathbb{R}$, the function defined on $[-\rho, 0]$ according to the relation

$$u_t(s) = \big(u_{n,t}(s)\big)_{n\in\mathbb{Z}} = \big(u_n(t+s)\big)_{n\in\mathbb{Z}} = u(t+s), \quad s \in [-\rho, 0],$$

and let $C_\rho = C([-\rho, 0], l^2)$ with the maximum norm $\|\psi\|_\rho = \sup_{-\rho\le s\le 0} \|\psi(s)\|$, $\psi \in C_\rho$.

Then problem (1.1)–(1.2) can be written as an equation in l^2: for $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{aligned}\frac{du(t)}{dt} &= -(\lambda + i\alpha)Au - (\kappa + i\beta)|u|^q u \\ &\quad + f\big(u\big(t - \rho_0(t)\big)\big) - (\gamma + i\delta)u + g(t)\end{aligned} \tag{2.1}$$

and

$$u(\tau + s) = \varphi(s), \quad s \in [-\rho, 0], \tag{2.2}$$

where $u = (u_n)_{n\in\mathbb{Z}}$, $|u|^q u = (|u_n|^q u_n)_{n\in\mathbb{Z}}$, $f(u(t - \rho_0(t))) = (f_n(u_n(t - \rho_0(t))))_{n\in\mathbb{Z}}$, $g(t) = (g_n(t))_{n\in\mathbb{Z}}$, and $\varphi = (\varphi_n)_{n\in\mathbb{Z}}$.

We make the following assumptions on f_n. For each $n \in \mathbb{Z}$, f_n is a nonlinear function satisfying the following assumption:

(H) $f_n(0) = 0$ and $f_n(s)$ is Lipschitz continuous with respect to s, that is, there is a positive constant L such that, for all $s_1, s_2 \in \mathbb{C}$,

$$\big|f_n(s_1) - f_n(s_2)\big| \le L|s_1 - s_2|.$$

In fact, by (H) we find that

$$\big\|f(u) - f(v)\big\| \le L\|u - v\|, \quad u, v \in l^2.$$

Lemma 2.1 *For any $p > 0$ and $a, b \in \mathbb{C}$, we have that there exists $c = c(p) > 0$ such that*

$$\big||a|^p a - |b|^p b\big| \le c\big(|a|^p + |b|^p\big)|a - b|.$$

Proof Without loss of generality, we assume that $|a| \ge |b|$. By mean value theorem, we have

$$\begin{aligned}\big||a|^p - |b|^p\big| &= p\big(\theta|a| + (1-\theta)|b|\big)^{p-1}\big||a| - |b|\big| \\ &\le p\big(\theta|a| + (1-\theta)|b|\big)^{p-1}|a - b|, \quad 0 < \theta < 1.\end{aligned} \tag{2.3}$$

Then

$$\big||a|^p - |b|^p\big| \le \begin{cases} p|b|^{p-1}|a-b|, & 0 < p \le 1, \\ p|a|^{p-1}|a-b|, & p > 1. \end{cases} \tag{2.4}$$

By (2.3), (2.4), and Young's inequality, we get

$$\begin{aligned} \big||a|^p a - |b|^p b\big| &\le \big||a|^p(a-b)\big| + \big|\big(|a|^p - |b|^p\big)b\big| \\ &\le |a|^p|a-b| + |b|\big||a|^p - |b|^p\big| \\ &\le |a|^p|a-b| + p|b|\big(|a|^{p-1} + |b|^{p-1}\big)|a-b| \\ &\le c\big(|a|^p + |b|^p\big)|a-b|. \end{aligned}$$

This completes the proof. □

Lemma 2.2 *The operator $A : l^2 \to l^2$ is locally Lipschitz continuous.*

Proof Based on Lemma 2.1 we have, for any $u, v \in l^2$,

$$\begin{aligned} \|Au - Av\|^2 &= \sum_{n\in\mathbb{Z}} \big((Au)_n - (Av)_n\big)^2 \\ &\le 2\sum_{n\in\mathbb{Z}} \big(\big|(Bu)_n\big|^{p-2}(Bu)_n - \big|(Bv)_n\big|^{p-2}(Bv)_n\big)^2 \\ &\quad + 2\sum_{n\in\mathbb{Z}} \big(\big|(Bu)_{n+1}\big|^{p-2}(Bu)_{n+1} - \big|(Bv)_{n+1}\big|^{p-2}(Bv)_{n+1}\big)^2 \\ &\le 2c^2\sum_{n\in\mathbb{Z}} \big(\big|(Bu)_n\big| + \big|(Bv)_n\big|\big)^{2p-4}\big|(Bu)_n - (Bv)_n\big|^2 \\ &\quad + 2c^2\sum_{n\in\mathbb{Z}} \big(\big|(Bu)_{n+1}\big| + \big|(Bv)_{n+1}\big|\big)^{2p-4}\big|(Bu)_{n+1} - (Bv)_{n+1}\big|^2 \\ &\le 2^{2p}c^2\sum_{n\in\mathbb{Z}} \big(|u_n| + |v_n|\big)^{2p-4}|u_n - v_n|^2 \\ &\le 2^{2p}c^2\big(\|u\| + \|v\|\big)^{2p-4}\|u-v\|^2. \end{aligned}$$

This completes the proof. □

It follows from Lemma 2.2 that the right-hand side function in (2.1) is locally Lipschitz continuous from l^2 to l^2. Therefore, by the standard theory of functional differential equations, one can show that, for every $\varphi \in C_\rho$, there exists $T > 0$ such that system (2.1)–(2.2) has a unique solution $u_t(\cdot, \tau, \varphi) \in C([\tau, T), C_\rho)$. As showed below, under some conditions this local solution is actually defined for all $t > \tau$. Furthermore, one may show that $u_t(\cdot, \tau, \varphi)$ is continuous in φ with respect to the norm of C_ρ.

In the sequence, we assume that

$$\eta = \frac{2L^2}{\gamma^2} < 1. \tag{2.5}$$

Lemma 2.3 *Assume that* (H) *and* (2.5) *hold. Then, for every* $\tau \in \mathbb{R}$, $T > 0$ *and* $\varphi \in C_\rho$, *there exists a positive constant* $c = c(\tau, T, \varphi)$ *such that the solution* u *of* (2.1)–(2.2) *satisfies*

$$\left\|u_t(\cdot, \tau, \varphi)\right\|_\rho \le c \quad \text{for } t \in [\tau, \tau + T). \tag{2.6}$$

Proof Taking the inner product of (2.1) with u in l^2 and keeping the real part, we find that for $t > \tau$,

$$\begin{aligned}\frac{1}{2}\frac{d\|u\|^2}{dt} &= -\lambda\big(|Bu|^{p-2} \otimes (Bu), Bu\big) - \kappa\|u\|_{q+2}^{q+2} + \operatorname{Re}\big(f\big(u\big(t-\rho_0(t)\big)\big), u\big) \\ &\quad - \gamma\|u\|^2 + \operatorname{Re}\big(g(t), u\big) \\ &\le -\gamma\|u\|^2 + \operatorname{Re}\big(f\big(u\big(t-\rho_0(t)\big)\big), u\big) + \operatorname{Re}\big(g(t), u\big).\end{aligned} \tag{2.7}$$

For the last two terms on the right-hand side of (2.7), by (H) we have

$$\operatorname{Re}\big(f\big(u\big(t-\rho_0(t)\big)\big), u\big) + \operatorname{Re}\big(g(t), u\big) \le \frac{L^2}{\gamma}\big\|u\big(t-\rho_0(t)\big)\big\| + \frac{\gamma}{2}\|u\|^2 + \frac{1}{\gamma}\big\|g(t)\big\|^2. \tag{2.8}$$

Consequently, it follows from (2.7)–(2.8) that

$$\frac{d\|u(t)\|^2}{dt} \le -\frac{\gamma}{2}\big\|u(t)\big\|^2 + \frac{L^2}{\gamma}\big\|u\big(t-\rho_0(t)\big)\big\| - \kappa\|u\|^{q+2} + \frac{1}{\gamma}\big\|g(t)\big\|^2, \quad t > \tau, \tag{2.9}$$

which implies that

$$\frac{d\|u(t)\|^2}{dt} \le -\frac{\gamma}{2}\big\|u(t)\big\|^2 + \frac{L^2}{\gamma}\big\|u\big(t-\rho_0(t)\big)\big\| + \frac{1}{\gamma}\big\|g(t)\big\|^2, \quad t > \tau. \tag{2.10}$$

It follows from (2.10) and Gronwall's inequality that, for $t \ge \tau$,

$$\begin{aligned}\big\|u(t)\big\| &\le \big\|u(\tau)\big\|e^{-\frac{\gamma}{2}(t-\tau)} + \frac{L^2}{\gamma}\int_\tau^t e^{-\frac{\gamma}{2}(t-s)}\big\|u\big(s-\rho_0(s)\big)\big\|\,ds \\ &\quad + \frac{1}{\gamma}\int_\tau^t e^{-\frac{\gamma}{2}(t-s)}\big\|g(s)\big\|\,ds.\end{aligned} \tag{2.11}$$

From condition (2.5), by using continuity, we obtain that there exist positive constants λ and N such that $\|\varphi\|_\rho \le N$ and

$$\frac{\|\varphi\|_\rho}{N} + e^{\lambda\rho}\frac{L^2}{(\frac{\gamma}{2}-\lambda)\gamma} < 1 \tag{2.12}$$

hold.

We will prove that, for $t \ge \tau$,

$$\big\|u(t)\big\| \le Ne^{-\lambda(t-\tau)} + (1-\eta)^{-1}I(t), \tag{2.13}$$

where $I(t) = \max_{\tau \le \xi \le t} \frac{1}{\gamma}\int_\tau^\xi e^{-\frac{\gamma}{2}(\xi-s)}\|g(s)\|\,ds$. To this end, we first prove, for any $d > 1$,

$$\big\|u(t)\big\| < dNe^{-\lambda(t-\tau)} + (1-\eta)^{-1}I(t), \quad t \ge \tau. \tag{2.14}$$

If inequality (2.14) is not true, from the fact that $\|\varphi\|_\rho \le N$ and $\|u(t)\|$ are continuous, then there must be $t^* > \tau$ such that

$$\|u(t^*)\| \ge dNe^{-\lambda(t^*-\tau)} + (1-\eta)^{-1}I(t) \tag{2.15}$$

and

$$\|u(t)\| < dNe^{-\lambda(t-\tau)} + (1-\eta)^{-1}I(t), \quad \tau-\rho \le t < t^*. \tag{2.16}$$

Hence, it follows from (2.11), (2.12), (2.15), and (2.16) that

$$\begin{aligned}
\|u(t^*)\| \le{}& \|u(\tau)\|e^{-\frac{\gamma}{2}(t^*-\tau)} + \frac{L^2}{\gamma}\int_\tau^{t^*} e^{-\frac{\gamma}{2}(t^*-s)}\|u(s-\rho_0(s))\|\,ds \\
&+ \frac{1}{\gamma}\int_\tau^{t^*} e^{-\frac{\gamma}{2}(t^*-s)}\|g(s)\|\,ds \\
<{}& \|u(\tau)\|e^{-\lambda(t^*-\tau)} + \frac{L^2}{\gamma}\int_\tau^{t^*} e^{-\frac{\gamma}{2}(t^*-s)}\big(dNe^{\lambda\rho}e^{-\lambda(s-\tau)} + (1-\eta)^{-1}I(t^*)\big)\,ds \\
&+ \frac{1}{\gamma}\int_\tau^{t^*} e^{-\frac{\gamma}{2}(t^*-s)}\|g(s)\|\,ds \\
\le{}& \|u(\tau)\|e^{-\lambda(t^*-\tau)} + \frac{L^2}{\gamma}\int_\tau^{t^*} e^{-\frac{\gamma}{2}(t^*-s)}dNe^{\lambda\rho}e^{-\lambda(s-\tau)}\,ds \\
&+ \frac{L^2}{\gamma}(1-\eta)^{-1}I(t^*)\int_\tau^{t^*} e^{-\frac{\gamma}{2}(t^*-s)}\,ds + I(t^*) \\
\le{}& \left(\frac{\|u(\tau)\|}{N} + \frac{L^2}{\gamma}e^{\lambda\rho}\int_\tau^{t^*} e^{-(\frac{\gamma}{2}-\lambda)(t^*-s)}\,ds\right)dNe^{-\lambda(t^*-\tau)} + \eta(1-\eta)^{-1}I(t^*) + I(t^*) \\
\le{}& \left(\frac{\|u(\tau)\|}{N} + e^{\lambda\rho}\frac{L^2}{\gamma(\frac{\gamma}{2}-\lambda)}\right)dNe^{-\lambda(t^*-\tau)} + (1-\eta)^{-1}I(t^*) \\
\le{}& dNe^{-\lambda(t^*-\tau)} + (1-\eta)^{-1}I(t^*),
\end{aligned} \tag{2.17}$$

which contradicts inequality (2.15). So inequality (2.14) holds for all $t \ge \tau$. Letting $d \to 1$ in inequality (2.14), we have inequality (2.13). The proof is complete. □

Lemma 2.3 implies that the solution u is defined in any interval of $[\tau, \tau+T]$ for any $T > 0$. It means that this local solution is, in fact, a global one.

Given $t \in \mathbb{R}$, define a translation θ_t on $\mathbb{R}$ by

$$\theta_t(\tau) = \tau + t \quad \text{for all } \tau \in \mathbb{R}. \tag{2.18}$$

Then $\{\theta_t\}_{t\in\mathbb{R}}$ is a group acting on $\mathbb{R}$. We now define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times C_\rho \to C_\rho$ for problem (2.1)–(2.2). Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $u_\tau \in C_\rho$, let

$$\Phi(t,\tau,u_\tau) = u_{t+\tau}(\cdot,\tau,u_\tau), \tag{2.19}$$

where $u_{t+\tau}(s,\tau,u_\tau) = u(t+\tau+s,\tau,u_\tau)$, $s \in [-\rho,0]$. By the uniqueness of solutions, we find that for every $t,s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $u_\tau \in C_\rho$,

$$\Phi(t+s,\tau,u_\tau) = \Phi\bigl(t,s+\tau,\bigl(\Phi(s,\tau,u_\tau)\bigr)\bigr).$$

Then we see that Φ is a continuous nonautonomous dynamical system on C_ρ. In the following two sections, we investigate the existence of pullback attractor for Φ. To this end, we need to define an appropriate collection of families of subsets of C_ρ.

Let $B_\rho = \{B_\rho(\tau) : \tau \in \mathbb{R}\}$ be a family of nonempty subsets of C_ρ. Then B_ρ is called tempered (or subexponentially growing) if for every $c > 0$ the following holds:

$$\lim_{t\to-\infty} e^{ct}\bigl\|B_\rho(\tau+t)\bigr\|_\rho = 0,$$

where $\|B_\rho\|_\rho = \sup_{x\in B_\rho}\|x\|_\rho$. In the sequel, we denote by $\mathcal{D}_\rho$ the collection of all families of tempered nonempty subsets of C_ρ, i.e.,

$$\mathcal{D}_\rho = \bigl\{B_\rho = \bigl\{B_\rho(\tau) : \tau \in \mathbb{R}\bigr\} : B_\rho \text{ is tempered in } C_\rho\bigr\}.$$

The following condition will be needed when deriving uniform estimates of solutions:

$$\int_{-\infty}^{\tau} e^{\lambda s}\bigl\|g(s)\bigr\|^2\,ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{2.20}$$

3 Uniform estimates of solutions

In this section, we derive uniform estimates of solutions of problem (2.1)–(2.2) which are needed for proving the existence and uniqueness of a pullback attractor for problem (2.1)–(2.2). The estimates of solutions of problem (2.1)–(2.2) in C_ρ are provided below. The symbol c is a positive constant which may change its value from line to line.

Lemma 3.1 *Assume that* (H), (2.5), *and* (2.20) *hold. Then, for every* $\tau \in \mathbb{R}$ *and* $D_\rho = \{D_\rho(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_\rho$, *there exists* $T = T(\tau, D_\rho) > \rho$ *such that, for all* $t \geq T$, $0 \leq \xi \leq \rho$, *and* $\varphi \in D_\rho(\tau - t)$, *the solution* u *of* (2.1)–(2.2) *satisfies*

$$\begin{aligned}&\bigl\|u_\tau(\cdot,\tau-t,\varphi)\bigr\|_\rho^2 + \kappa e^{-\lambda(\tau+\xi)}\int_{\tau-t}^{\tau+\xi} e^{\lambda s}\bigl\|u(s,\tau-t,\varphi)\bigr\|_{q+2}^{q+2}\,ds \\ &\quad \leq 2\frac{1}{\gamma}e^{\lambda\rho}\int_{-\infty}^{0} e^{\lambda s}\bigl\|g(s+\tau)\bigr\|\,ds.\end{aligned} \tag{3.1}$$

Proof From condition (2.5), by using continuity, we obtain that there exists a positive constant λ such that

$$\lambda - \frac{\gamma}{2} + \frac{L^2}{\gamma}e^{\lambda\rho} < 0 \tag{3.2}$$

holds.

Replacing t and τ in (2.9) by ϖ and $\tau - t$, respectively, we have, for $\varpi > \tau - t$,

$$\begin{aligned}\frac{d\|u(\varpi,\tau-t,\varphi)\|^2}{dt} &\le -\frac{\gamma}{2}\big\|u(\varpi,\tau-t,\varphi)\big\|^2 + \frac{L^2}{\gamma}\big\|u\big(\varpi-\rho_0(\varpi),\tau-t,\varphi\big)\big\| \\ &\quad - \kappa\big\|u(\varpi,\tau-t,\varphi)\big\|_{q+2}^{q+2} + \frac{1}{\gamma}\big\|g(\varpi)\big\|^2. \end{aligned} \tag{3.3}$$

For simplicity, we denote $u(\varpi) = u(\varpi,\tau-t,\varphi)$. Then, let us define functions $V(\varpi) = e^{\lambda\varpi}\|u(\varpi)\|$, $\varpi \ge \tau - t - \rho$, and

$$U(\varpi) \triangleq \begin{cases} e^{\lambda(\tau-t)}\|\varphi\|_\rho, & \varpi \in [\tau-t-\rho,\tau-t), \\ e^{\lambda(\tau-t)}\|\varphi\|_\rho - \kappa\int_{\tau-t}^{\varpi} e^{\lambda s}\|u(s)\|_{q+2}^{q+2}\,ds & \\ \quad + \frac{1}{\gamma}\int_{\tau-t}^{\varpi} e^{\lambda s}\|g(s)\|\,ds, & \varpi \ge \tau - t. \end{cases}$$

Now, we claim that

$$V(\varpi) \le U(\varpi), \quad \varpi \ge \tau - t. \tag{3.4}$$

If inequality (3.4) is not true from the fact that $V(t)$ and $U(t)$ are continuous, then there must be $\varpi^* > \tau - t$ such that

$$V(\varpi) < U(\varpi), \quad \varpi \in \big[\tau-t-\rho,\varpi^*\big), \tag{3.5}$$

$$V\big(\varpi^*\big) = U\big(\varpi^*\big), \tag{3.6}$$

where

$$\varpi^* \triangleq \inf\big\{\varpi > \tau - t | V(\varpi) > U(\varpi)\big\},$$

and there is a sufficiently small positive constant $\Delta\varpi$ such that

$$V(\varpi) > U(\varpi), \quad \varpi \in \big(\varpi^*,\varpi^*+\Delta\varpi\big). \tag{3.7}$$

Calculating the upper right-hand Dini derivative of $V(t)$ at ϖ^* and considering (3.6) and (3.7), we obtain

$$\begin{aligned} D^+V\big(\varpi^*\big) &= \limsup_{h\to 0^+}\frac{V(\varpi^*+h)-V(\varpi^*)}{h} \ge \limsup_{h\to 0^+}\frac{U(\varpi^*+h)-U(\varpi^*)}{h} \\ &= -\kappa e^{\lambda\varpi^*}\big\|u\big(\varpi^*\big)\big\|^{q+2} + \frac{1}{\gamma}e^{\lambda\varpi^*}\big\|g\big(\varpi^*\big)\big\|. \end{aligned} \tag{3.8}$$

On the other hand, it follows from (3.3) that

$$\begin{aligned} D^+V\big(\varpi^*\big) &= \lambda e^{\lambda\varpi^*}\big\|u\big(\varpi^*\big)\big\| + e^{\lambda\varpi^*}D^+\big\|u\big(\varpi^*\big)\big\| \\ &\le \left(\lambda - \frac{\gamma}{2}\right)e^{\lambda\varpi^*}\big\|u\big(\varpi^*\big)\big\| + \frac{L^2}{\gamma}e^{\lambda\varpi^*}\big\|u\big(\varpi^*-\rho_0\big(\varpi^*\big)\big)\big\| \\ &\quad - \kappa e^{\lambda\varpi^*}\big\|u\big(\varpi^*\big)\big\|^{q+2} + \frac{1}{\gamma}e^{\lambda\varpi^*}\big\|g\big(\varpi^*\big)\big\|. \end{aligned} \tag{3.9}$$

Notice that $U(\varpi)$ is monotone nondecreasing on $[\tau-t-\rho,+\infty)$. This, together with (3.5) and (3.6), yields

$$V\big(\varpi^* - \rho_0\big(\varpi^*\big)\big) < U\big(\varpi^* - \rho_0\big(\varpi^*\big)\big) < U\big(\varpi^*\big) = V\big(\varpi^*\big), \tag{3.10}$$

which implies

$$\big\|u\big(\varpi^* - \rho_0\big(\varpi^*\big)\big)\big\| \le e^{\lambda\rho}\big\|u\big(\varpi^*\big)\big\|. \tag{3.11}$$

It follows from (3.2), (3.9), and (3.11) that

$$\begin{aligned} D^+V\big(\varpi^*\big) &< \Big(\lambda - \frac{\gamma}{2} + \frac{L^2}{\gamma}e^{\lambda\rho}\Big)e^{\lambda\varpi^*}\big\|u\big(\varpi^*\big)\big\| - \kappa e^{\lambda\varpi^*}\big\|u\big(\varpi^*\big)\big\|_{q+2}^{q+2} + \frac{1}{\gamma}e^{\lambda\varpi^*}\big\|g\big(\varpi^*\big)\big\| \\ &< -\kappa e^{\lambda\varpi^*}\big\|u\big(\varpi^*\big)\big\|^{q+2} + \frac{1}{\gamma}e^{\lambda\varpi^*}\big\|g\big(\varpi^*\big)\big\|, \end{aligned}$$

which contradicts (3.8). Until now, (3.4) has been proven to be true. Thus, we get, for $t > \rho$ and $-\rho \le \xi \le 0$,

$$\begin{aligned} &\big\|u(\tau+\xi,\tau-t,\varphi)\big\| + \kappa e^{-\lambda(\tau+\xi)}\int_{\tau-t}^{\tau+\xi} e^{\lambda s}\big\|u(s)\big\|_{q+2}^{q+2}\,ds \\ &\quad\le \|\varphi\|_\rho e^{-\lambda(t+\xi)} + e^{-\lambda(\tau+\xi)}\frac{1}{\gamma}\int_{\tau-t}^{\tau+\xi} e^{\lambda s}\big\|g(s)\big\|\,ds \\ &\quad\le \|\varphi\|_\rho e^{\lambda\rho}e^{-\lambda t} + e^{\lambda\rho}e^{-\lambda\tau}\frac{1}{\gamma}\int_{\tau-t}^{\tau} e^{\lambda s}\big\|g(s)\big\|\,ds. \end{aligned}$$

Since $\varphi \in D(\tau-t)$ and $D \in \mathcal{D}$, we find that for every $\tau \in \mathbb{R}$ and $D \in \mathcal{D}$ there exists $T = T(\tau,D) > \rho$ such that, for all $t \ge T$ and $-\rho \le \xi \le 0$,

$$\big\|u(\tau+\xi,\tau-t,\varphi)\big\| + \kappa e^{-\lambda(\tau+\xi)}\int_{\tau-t}^{\tau+\xi} e^{\lambda s}\big\|u(s)\big\|_{q+2}^{q+2}\,ds \le 2\frac{1}{\gamma}e^{\lambda\rho}\int_{-\infty}^{0} e^{\lambda s}\big\|g(s+\tau)\big\|\,ds.$$

This completes the proof. □

Lemma 3.2 *Assume that* (*H*), (2.5), *and* (2.20) *hold. Then, for every* $\tau \in \mathbb{R}, D_\rho = \{D_\rho(\tau) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\rho$, *there exist* $T = T(\tau, D_\rho) > \rho$ *and* $N = N(\tau) > 0$ *such that, for all* $t \ge T$ *and* $\phi \in D_\rho(\tau-t)$, *the solution* u *of* (2.1)–(2.2) *satisfies*

$$\sup_{-\rho\le s\le 0}\sum_{|n|\ge N}\big|u_n(\tau+s,\tau-t,\varphi)\big|^2 \le \varepsilon. \tag{3.12}$$

Proof Let ϑ be a smooth function defined on $\mathbb{R}^+$ such that $0 \le \vartheta(s) \le 1$ for all $s \in \mathbb{R}^+$, $\vartheta(s) = 0$ for $0 \le s \le 1$ and $\vartheta(s) = 1$ for $s \ge 2$. Note that ϑ' is bounded on $\mathbb{R}^+$, i.e., there exists a constant c_0 such that $|\vartheta'(s)| \le c_0$ for $s \in \mathbb{R}^+$.

Taking the inner product of (2.1) with $x = (\vartheta(\frac{|n|}{k})u_n)_{n\in\mathbb{Z}}$ in l^2 and keeping the real part, where k is a fixed positive integer specified later, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)|u_n|^2+\operatorname{Re}(\lambda+i\alpha)(Au,x)\\&\quad\leq-\gamma\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)|u_n|^2+\operatorname{Re}\bigl(f\bigl(u\bigl(t-\rho_0(t)\bigr)\bigr),x\bigr)+\operatorname{Re}\bigl(g(t),x\bigr).\end{aligned}\tag{3.13}$$

We now estimate the terms in (3.13) as follows. First, we have

$$\begin{aligned}&\operatorname{Re}(\lambda+i\alpha)(Au,x)\\&\quad=\operatorname{Re}(\lambda+i\alpha)\bigl(|Bu|^{p-2}\otimes Bu,Bx\bigr)\\&\quad=\operatorname{Re}(\lambda+i\alpha)\sum_{n\in\mathbb{Z}}|u_{n+1}-u_n|^{p-2}(u_{n+1}-u_n)\left(\vartheta\left(\frac{|n+1|}{k}\right)\overline{u_{n+1}}-\vartheta\left(\frac{|n|}{k}\right)\overline{u_n}\right)\\&\quad=\operatorname{Re}(\lambda+i\alpha)\sum_{n\in\mathbb{Z}}\left(\vartheta\left(\frac{|n+1|}{k}\right)-\vartheta\left(\frac{|n|}{k}\right)\right)|u_{n+1}-u_n|^{p-2}(u_{n+1}-u_n)\overline{u_{n+1}}\\&\qquad+\lambda\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)|u_{n+1}-u_n|^p\\&\quad\geq\operatorname{Re}(\lambda+i\alpha)\sum_{n\in\mathbb{Z}}\left(\vartheta\left(\frac{|n+1|}{k}\right)-\vartheta\left(\frac{|n|}{k}\right)\right)|u_{n+1}-u_n|^{p-2}(u_{n+1}-u_n)\overline{u_{n+1}}.\end{aligned}$$

By the property of the function ϑ and Young's inequality, we have

$$\begin{aligned}&\left|\operatorname{Re}(\lambda+i\alpha)\sum_{n\in\mathbb{Z}}\left(\vartheta\left(\frac{|n+1|}{k}\right)-\vartheta\left(\frac{|n|}{k}\right)\right)|u_{n+1}-u_n|^{p-2}(u_{n+1}-u_n)\overline{u_{n+1}}\right|\\&\quad\leq\sqrt{\lambda^2+\alpha^2}\sum_{n\in\mathbb{Z}}\frac{|\vartheta'(\xi_n)|}{k}|u_{n+1}-u_n|^{p-1}|u_{n+1}|\\&\quad\leq\frac{\sqrt{\lambda^2+\alpha^2}c_0}{k}\sum_{n\in\mathbb{Z}}|u_{n+1}-u_n|^{p-1}|u_{n+1}|\leq\frac{c}{k}\|u\|_p^p,\end{aligned}$$

which implies that

$$-\operatorname{Re}(\lambda+i\alpha)(Au,x)\leq\frac{c}{k}\|u\|_p^p.\tag{3.14}$$

For the last two terms on the right-hand side of (3.13), we get from (H)

$$\begin{aligned}&\operatorname{Re}\bigl(f\bigl(u\bigl(t-\rho_0(t)\bigr)\bigr),x\bigr)+\operatorname{Re}\bigl(g(t),x\bigr)\\&\quad\leq\frac{L^2}{\gamma}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\bigl|u_n\bigl(t-\rho_0(t)\bigr)\bigr|^2\\&\qquad+\frac{\gamma}{2}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)|u_n|^2+\frac{1}{\gamma}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\bigl|g_n(t)\bigr|^2.\end{aligned}\tag{3.15}$$

Then, by (3.13)–(3.15) we have

$$\frac{d}{dt}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\left|u_n(t)\right|^2 \le -\frac{\gamma}{2}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\left|u_n(t)\right|^2 + \frac{L^2}{\gamma}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\left|u_n\bigl(t-\rho_0(t)\bigr)\right|^2 + \frac{c}{k}\left\|u(t)\right\|_p^p + \frac{1}{\gamma}\sum_{|n|\ge k}\left|g_n(t)\right|^2. \tag{3.16}$$

By a similar argument as in Lemma 3.1, we get from (3.16) for any $t > \rho$ and $-\rho \le \xi \le 0$

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\left|u_n(\tau+\xi,\tau-t,\varphi)\right|^2 &\le \|\varphi\|_\rho^2 e^{-\lambda(t+\xi)} + \frac{c}{k}e^{-\lambda(\tau+\xi)}\int_{\tau-t}^{\tau+\xi} e^{\lambda s}\left\|u(s,\tau-t,\varphi)\right\|_p^p\,dr \\ &\quad + \frac{1}{\gamma}e^{-\lambda(\tau+\xi)}\int_{\tau-t}^{\tau+\xi} e^{-\lambda s}\sum_{|n|\ge k}\left|g_n(s)\right|^2\,dr.\end{aligned} \tag{3.17}$$

It follows from Lemma 3.1 and the relation $l^p \subseteq l^{q+2}$, $p \ge q+2$, that for any $\tau \in \mathbb{R}$, $\varphi \in D_\rho$ and $\varepsilon > 0$, there exist $T = T(\tau, D_\rho) > \rho$ and $K_1 = K_1(\tau, D_\rho, \varepsilon)$ such that, for $k > K_1$, $t > T$ and $-\rho \le \xi \le 0$,

$$\frac{c}{k}e^{-\lambda(\tau+\xi)}\int_{\tau-t}^{\tau+\xi} e^{\lambda s}\left\|u(s,\tau-t,\varphi)\right\|_p^p\,dr \le \varepsilon, \tag{3.18}$$

which together with (3.17) implies

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\left|u_n(\tau+\xi,\tau-t,\varphi)\right|^2 &\le \|\varphi\|_\rho^2 e^{-\lambda(t+\xi)} + \frac{\varepsilon}{3} + \frac{1}{\gamma}e^{-\lambda(\tau+\xi)}\int_{\tau-t}^{\tau+\xi} e^{-\lambda s}\sum_{|n|\ge k}\left|g_n(s)\right|^2\,dr.\end{aligned} \tag{3.19}$$

We have from $\varphi \in D_\rho(\tau - t)$ that there exists $T_1 = T_1(\tau, D_\rho, \varepsilon) > 0$ such that, for all $t \ge T_1$ and $-\rho \le \xi \le 0$,

$$\|\varphi\|_\rho^2 e^{-\lambda(t+\xi)} \le \|\varphi\|_\rho^2 e^{\lambda\rho}e^{-\lambda t} \le \frac{\varepsilon}{3}. \tag{3.20}$$

We have from (2.20) that there is $N_1 = N_1(\tau, \varepsilon) > 0$ such that, for all $k \ge N_1$,

$$\frac{1}{\gamma}e^{\lambda\rho}e^{-\lambda\tau}\int_{-\infty}^{0} e^{-\lambda r}\sum_{|n|\ge k}\left|g_n(s+\tau)\right|^2\,dr \le \frac{\varepsilon}{3}. \tag{3.21}$$

Note that

$$\sup_{-\rho\le\xi\le 0}\sum_{|n|\ge 2k}\left|u_n(\tau+\xi,\tau-t,\varphi)\right|^2 \le \sup_{-\rho\le\xi\le 0}\sum_{n\in\mathbb{Z}}\vartheta\left(\frac{|n|}{k}\right)\left|u_n(\tau+\xi,\tau-t,\varphi)\right|^2,$$

which along with (3.19)–(3.21) concludes the proof. □

4 Existence of pullback attractors

In this section, we establish the existence of $\mathcal{D}_\rho$-pullback attractor for the nonautonomous dynamical system Φ associated with problem (2.1)–(2.2).

Lemma 4.1 *Assume that* (H), (2.5), *and* (2.20) *hold. Then, for every* $\tau \in \mathbb{R}$ *and* $D_\rho = \{D_\rho(\tau) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\rho$, *there exists* $T = T(\tau, D_\rho) > \rho$ *such that the solution* u *of* (2.1)–(2.2) *satisfies that* $u_\tau(\cdot, \tau - t, \varphi)$ *is equicontinuous in* l^2.

Proof Denote by $P_k u = (u_1, u_2, \ldots u_k, 0, 0, \ldots)$ for $u \in l^2$ and $k \in \mathbb{N}$. By Lemma 3.2, for $\varepsilon > 0$, there exist $T = T(\tau, \varepsilon) > \rho$ and large enough integer $N = N(\tau, \varepsilon)$ such that, for all $t \geq T$,

$$\max_{-\rho \leq s \leq 0} \left\| (I - P_N) u(\tau + s, \tau - t, \varphi) \right\| < \frac{\varepsilon}{3}. \tag{4.1}$$

Let $u_1 = P_N u$. By Lemma 3.1, it follows from (2.1) and the equivalence of norm in a finite dimensional space that there exists $T = T(\tau) > \rho$ such that, for all $t \geq T$,

$$\int_{\tau-\rho}^{\tau} \left\| \frac{d}{dr} u_1(r, \tau - t, \varphi) \right\|^2 dr \leq c, \tag{4.2}$$

where $c = c(\tau)$ is a positive number. Without loss of generality, we assume that $s_1, s_2 \in [-\rho, 0]$ with $0 < s_1 - s_2 < 1$. Then, for any fixed $\tau \in \mathbb{R}$,

$$\begin{aligned}
&\left\| u_1(\tau + s_1, \tau - t, \varphi) - u_1(\tau + s_2, \tau - t, \varphi) \right\| \\
&\quad \leq \int_{\tau+s_2}^{\tau+s_1} \left\| \frac{du_1(r, \tau - t, \varphi)}{dr} \right\| dr \\
&\quad \leq \left(\int_{\tau-\rho}^{\tau} \left\| \frac{du_1(r, \tau - t, \varphi)}{dr} \right\|^2 dr \right)^{\frac{1}{2}} |s_2 - s_1|^{\frac{1}{2}} \leq c |s_2 - s_1|^{\frac{1}{2}},
\end{aligned} \tag{4.3}$$

which implies that there exists a constant $\varrho = \varrho(\varepsilon) > 0$ such that if $|s_1 - s_2| < \varrho$, then

$$\left\| u_1(\tau + s_2, \tau - t, \varphi) - u_1(\tau + s_1, \tau - t, \varphi) \right\| < \frac{\varepsilon}{3},$$

which along with (4.1) implies that, for all $t \geq T$ and $|s_1 - s_2| < \varrho$,

$$\begin{aligned}
&\left\| u(\tau + s_2, \tau - t, \varphi) - u(\tau + s_1, \tau - t, \varphi) \right\| \\
&\quad \leq \left\| P_N u(\tau + s_2, \tau - t, \varphi) - P_N u\big(u(\tau + s_1, \tau - t, \varphi)\big) \right\| \\
&\qquad + \left\| (I - P_N) u(\tau + s_2, \tau - t, \varphi) \right\| \\
&\qquad + \left\| (I - P_N) u(\tau + s_1, \tau - t, \varphi) \right\| \\
&\quad \leq \varepsilon.
\end{aligned}$$

This completes the proof. □

As for the compactness in l^2 in [19], one can easily verify the following compactness criteria in C_ρ by means of uniform tail estimates.

Lemma 4.2 *Let $\{u^n\}_{n=1}^{\infty} = \{(u_i^n)_{i\in\mathbb{Z}}\}_{n=1}^{\infty} \subseteq C_\rho$. Then $\{u^n\}_{n=1}^{\infty}$ is relative compact in C_ρ if and only if the following conditions are satisfied*:

(i) $\{u^n\}_{n=1}^{\infty}$ *is bounded in* C_ρ;

(ii) $\{u^n\}_{n=1}^{\infty}$ *is equicontinuous*;

(iii) $\limsup_{k\to\infty} \limsup_{n\to\infty} \sup_{-\rho\le s\le 0} \sum_{|i|\ge k} |u_i^n|^2 = 0$.

Theorem 4.1 *Assume that* (H), (2.5), *and* (2.20) *hold. Then the nonautonomous dynamical system Φ has a unique $\mathcal{D}_\rho$-pullback attractor $\mathcal{A}_\rho = \{\mathcal{A}_\rho(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_\rho$ in C_ρ.*

Proof For $\tau \in \mathbb{R}$, denote by

$$K(\tau) = \left\{ u \in C_\rho : \|u\|_\rho^2 \le 2\frac{1}{\gamma} e^{\lambda\rho} \int_{-\infty}^{0} e^{\lambda s} \|g(s+\tau)\| \, ds \right\}.$$

First, we know from Lemma 3.1 that Φ has a $\mathcal{D}_\rho$-pullback absorbing set $K(\tau)$. Second, since Lemmas 3.1, 3.2, and 4.1 coincide with all the conditions of Lemma 4.2, Φ is $\mathcal{D}_\rho$-pullback asymptotically compact in C_ρ. Hence, the existence of a unique $\mathcal{D}_\rho$-pullback attractor for the nonautonomous dynamical system Φ follows from Proposition 2.7. in [20] immediately. □

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Authors' contributions

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