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A new approach to interval-valued inequalities

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Abstract

The objective of this work is to advance and simplify the notion of Gronwall's inequality. By using an efficient partial order and concept of gH-differentiability on interval-valued functions, we investigate some new variants of Gronwall type inequalities on time scales.

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1 Introduction

Many problems in real life involve Gronwall's inequality [23]. It has had an important role in the research of differential and integral equations for nearly 100 years. Its first generalization proved by Richard Bellman [7] motivated many researchers to obtain various generalizations and extensions [2, 3, 6, 31, 34, 35]. The Gronwall–Bellman type inequalities enable critical insight into the uniqueness of solutions, a priori and error estimate in the Galerkin method [41, Ch. 3].

Several research papers in the interval analysis (IA) are based on the demonstration of an uncertain variable as an interval [22, 30, 32, 38]. The relevant formulations of interval calculus on time scales, including some general approaches to differential theory, have been systematized in recent paper [29]. The interval-valued functions and sequences have been recently studied by many authors in various aspects (see [16–21]).

Inequalities are used as a tool for almost all mathematical branches and other subjects of applied and engineering sciences. A detailed study of various inequalities is found in [4, 24, 28, 42]. Some of the differential integral inequalities have been prolonged into set-valued function [5, 10, 14, 15, 36]. Among the more recent investigations on interval-valued Gronwall type inequalities, let us mention the work of Younus et al. [39, 40], where the authors obtain Gronwall inequalities for the interval-valued functions under the notion of Kulish–Mirankor partial order on a set of compact intervals. However, there are many other partial orders, which cannot be covered by Kulish–Mirankor partial order.

In the study of Gronwall type inequalities, an important notion is an exponential function on time scales. A difficult situation has accrued in the case of trigonometric, exponential, hyperbolic, and parabolic functions, where Hilger's technique [25, 26] differs from

Bohner and Peterson's technique [8, 9]. A newly improved trigonometric, hyperbolic, and parabolic functions base on Cayley transformation has been defined by Cieśliński [12, 13].

In the main part of the proposed study, we firstly discuss some new variants of Gronwall type inequalities on time scale by using the concept of Cayley exponential function, which is the generalization of some inequalities from [1, 11, 27]. Also, by defining an efficient partial order on a set of compact intervals, we obtain new variants of Gronwall type inequality for interval-valued functions, which gives more general than existing results of [39, 40].

2 Preliminary notation

For time scales calculus, we refer to [8, 29].

In order to define Cayley-exponential (shortly, C-exponential) function, Cieśliński [13], redefined a notion of regressivity as follows:

$$C_{rd} := \{f : \mathbb{T} \to \mathbb{R} : f \text{ is rd-continuous } \forall\, t \in \mathbb{T}\},$$
$$\mathcal{R} := \left\{f \in C_{rd} : \mu(t)f(t) \neq \pm 2 \ \forall\, t \in \mathbb{T}^k\right\}$$

and

$$\mathcal{R}^+ := \left\{f \in C_{rd} : \left|\mu(t)f(t)\right| < 2 \ \forall\, t \in \mathbb{T}^k\right\}.$$

Under the binary operation $\oplus$, defined by $\alpha \oplus \beta = \frac{\alpha+\beta}{1+\frac{1}{4}\mu^2\alpha\beta}$, $\mathcal{R}^+$ is an Abelian group [13, Theorem 3.14]. However, the set $\mathcal{R}$ is not closed with respect to $\oplus$.

For $f \in \mathcal{R}$ and $s \in \mathbb{T}$, consider the subsequent initial value problem (IVP)

$$\begin{cases} x^{\Delta}(t) = f(t)\langle x(t)\rangle, \\ x(s) = 1, \end{cases} \tag{2.1}$$

where

$$\langle x(t)\rangle := \frac{x(t) + x(\sigma(t))}{2}.$$

For $h \in \mathbb{R}^+$, the Cayley transformation ξ_h is defined as

$$\xi_h(z) : = \begin{cases} z, & h = 0, \\ \frac{1}{h}\operatorname{Log}\left(\frac{1+\frac{zh}{2}}{1-\frac{zh}{2}}\right), & h > 0, \end{cases}$$

and the Cayley-exponential function for $f \in \mathcal{R}$ is defined by

$$E_f(t,s) : = \exp\left\{\int_s^t \xi_{\mu(\tau)}\big(f(\tau)\big)\Delta\tau\right\} \quad \text{for } s,t \in \mathbb{T}.$$

It is easy to see that $E_f(\cdot,s)$ on $\mathbb{T}$ is the unique solution of IVP (2.1).

Lemma 2.1 ([13]) *If $\alpha, \beta \in \mathcal{R}$, then the subsequent properties hold*:

1. $E_\alpha(t^\sigma, t_0) = \frac{1+\frac{1}{2}\alpha(t)\mu(t)}{1-\frac{1}{2}\alpha(t)\mu(t)}E_\alpha(t,t_0)$,

2. $(E_\alpha(t,t_0))^{-1} = E_{-\alpha}(t,t_0) = \frac{1}{E_\alpha(t,t_0)}$,
3. $\overline{E_\alpha(t,t_0)} = E_{\bar{\alpha}}(t,t_0)$,
4. $E_\alpha(t,t_0)E_\alpha(t_0,t_1) = E_\alpha(t,t_1)$,
5. $E_\alpha(t,t_0)E_\beta(t,t_0) = E_{\alpha\oplus\beta}(t,t_0)$.

Lemma 2.2 ([13]) *If* $\alpha \in \mathcal{R}^+$, *then* $E_\alpha > 0$.

Lemma 2.3 ([13]) $E_\alpha(t,t_0) = e_\beta(t,t_0)$ *if* $\alpha(t) = \frac{\beta(t)}{1+\frac{1}{2}\beta(t)\mu(t)}$, $\beta(t) = \frac{\alpha(t)}{1-\frac{1}{2}\alpha(t)\mu(t)}$, *with* $\alpha\mu \neq \pm 2$ *and* $\beta\mu \neq -1$.

These lemmas are Theorem 3.10 and 3.13 in [13] and Theorem 3.2 in [12], respectively. Let

$$\mathcal{K}_C := \big\{[a,b] : a,b \in \mathbb{R}\big\}.$$

For $[\bar{x},\underline{x}]$, $[\bar{y},\underline{y}] \in \mathcal{K}_C$,

$$[\bar{x},x] + [\bar{y},y] = [\bar{x}+\bar{y}, x+y]$$

and

$$\lambda[\bar{x},x] = \begin{cases} [\lambda\bar{x},\lambda x] & \text{if } \lambda > 0, \\ \{0\} & \text{if } \lambda = 0, \\ [\lambda x,\lambda\bar{x}] & \text{if } \lambda < 0, \end{cases}$$

respectively. By definition, we have $\lambda X = X\lambda \ \forall \lambda \in \mathbb{R}$.

Moreover,

$$[\bar{x},x] \ominus_g [\bar{y},y] = \big[\min\{\bar{x}-\bar{y}, x-y\}, \max\{\bar{x}-\bar{y}, x-y\}\big], \tag{2.2}$$

where "$\ominus_g$" is called gH-difference [30, 37].

For $X = [\bar{x},\underline{x}] \in \mathcal{K}_C$, width of X is defined as $w(X) = \underline{x} - \bar{x}$. By using $w(\cdot)$, we can write

$$X \ominus_g Y = \begin{cases} [\bar{x}-\bar{y}, \underline{x}-\underline{y}], & \text{if } w(X) \geq w(Y), \\ [\underline{x}-\underline{y}, \bar{x}-\bar{y}], & \text{if } w(X) < w(Y). \end{cases} \tag{2.3}$$

More explicitly, for $X, Y, C \in \mathcal{K}_C$, we have

$$X \ominus_g Y = C \quad \Longleftrightarrow \quad \begin{cases} X = Y + C, & \text{if } w(X) \geq w(Y), \\ Y = X + (-C), & \text{if } w(X) < w(Y). \end{cases} \tag{2.4}$$

Since $\mathcal{K}_C$ is not totally order set (e.g., see [10, 30, 33, 39]). To compare the images of IVFs in the context of inequalities, several partial order relations exist over $\mathcal{K}_C$, which are summarized as follows.

For $X, Y \in \mathcal{K}_C$, such that $X = [\bar{x},x]$, $Y = [\bar{y},y]$, we say that:

1. "$X \preceq_{\mathrm{LU}} Y$ (or $X \preceq_{\mathrm{LR}} Y$), $\Leftrightarrow$ $\bar{x} \leq \bar{y}$ and $\underline{x} \leq \underline{y}$, $X \prec_{\mathrm{LU}} Y$ if $X \preceq_{\mathrm{LU}} Y$ and $X \neq Y$".

2. "$X \preceq_{\mathrm{LC}} Y \Leftrightarrow \bar{x} \le \bar{y}$ and $m(X) \le m(Y)$, $X \prec_{\mathrm{LC}} Y$ if $X \preceq_{\mathrm{LC}} Y$ and $X \neq Y$, where $m(X) = \frac{\bar{x}+\underline{x}}{2}$".
3. "$X \preceq_{\mathrm{UC}} Y \Leftrightarrow \underline{x} \le \underline{y}$ and $m(X) \le m(Y)$, $X \prec_{\mathrm{UC}} Y$ if $X \preceq_{\mathrm{UC}} Y$ and $X \neq Y$".
4. "$X \preceq_{\mathrm{CW}} Y \Leftrightarrow m(X) \le m(Y)$ and $w(X) \le w(Y)$, $X \prec_{\mathrm{CW}} Y$ if $X \preceq_{\mathrm{CW}} Y$ and $X \neq Y$, where $w(X) = \underline{x} - \bar{x}$".
5. "$X \preceq_{\mathrm{LW}} Y \Leftrightarrow \bar{x} \le \bar{y}$ and $w(X) \le w(Y)$, $X \prec_{\mathrm{LW}} Y$ if $X \preceq_{\mathrm{LW}} Y$ and $X \neq Y$".
6. "$X \preceq_{\mathrm{UW}} Y \Leftrightarrow \underline{x} \le \underline{y}$ and $w(X) \le w(Y)$, $X \prec_{\mathrm{UW}} Y$ if $X \preceq_{\mathrm{UW}} Y$ and $X \neq Y$".

Let $\mathbb{P} = \{\preceq_{\mathrm{LU}}, \preceq_{\mathrm{LC}}, \preceq_{\mathrm{UC}}, \preceq_{\mathrm{CW}}, \preceq_{\mathrm{LW}}, \preceq_{\mathrm{UW}}\}$ be the set of these partial orders on $\mathcal{K}_C$. Some properties of these partial orders are examined in the following results.

Lemma 2.4 *Let $\mathbb{P}_1 := \{\preceq_{\mathrm{LU}}, \preceq_{\mathrm{LC}}, \preceq_{\mathrm{UC}}, \preceq_{\mathrm{CW}}, \preceq_{\mathrm{UW}}\}$. If $X \preceq_{\mathrm{LW}} Y$, then $X \preceq_* Y\ \forall \preceq_* \in \mathbb{P}_1$.*

Proof For $X, Y \in K_C$, with $X = [\bar{x}, \underline{x}]$, $Y = [\bar{y}, \underline{y}]$, it implies that $\bar{x} \le \bar{y}$ and $\underline{x} - \bar{x} \le \underline{y} - \bar{y}$. By adding these two inequalities, we have $\underline{x} \le \underline{y}$ and, furthermore, $m(X) \le m(Y)$. Hence $X \preceq_* Y, \forall \preceq_* \in \mathbb{P}_1$. □

Lemma 2.5 *Let $\mathbb{P}_2 := \{\preceq_{\mathrm{UC}}, \preceq_{\mathrm{UW}}\}$. If $X \preceq_{\mathrm{CW}} Y$, then $X \preceq_* Y\ \forall \preceq_* \in \mathbb{P}_2$.*

Proof For $X, Y \in K_C$, with $X = [\bar{x}, \underline{x}]$, $Y = [\bar{y}, \underline{y}]$, we have $\bar{x} + \underline{x} \le \bar{y} + \underline{y}$ and $\underline{x} - \bar{x} \le \underline{y} - \bar{y}$. By adding these two inequalities, we have $\underline{x} \le \underline{y}$. Hence $X \preceq_* Y, \forall \preceq_* \in \mathbb{P}_2$. □

Lemma 2.6 *Let $X, Y, C \in \mathcal{K}_C$. If $X \preceq_{\mathrm{LW}} Y$ and $w(X) \ge w(C)$, then $X \ominus_g C \preceq_{\mathrm{LW}} Y \ominus_g C$.*

Proof For $X, Y, C \in I_c$ with $X = [\bar{x}, \underline{x}]$, $Y = [\bar{y}, \underline{y}]$ and $C = [\bar{c}, c^+]$, LW partial order implies that $\bar{x} \le \bar{y}$ and $\underline{x} - \bar{x} \le \underline{y} - \bar{y}$. Since $w(X) \ge w(C)$, moreover $w(Y) \ge w(X) \ge w(C)$, it follows that $X \ominus_g C = [\bar{x} - \bar{c}, \underline{x} - c^+]$ and $Y \ominus_g C = [\bar{y} - \bar{c}, \underline{y} - c^+]$. By using the fact $\bar{x} \le \bar{y}$ and $\underline{x} - \bar{x} \le \underline{y} - \bar{y}$ implies that $\bar{x} - \bar{c} \le \bar{y} - \bar{c}$ and $\underline{x} - \bar{x} - (\underline{c} - \bar{c}) \le \underline{y} - \bar{y} - (\underline{c} - \bar{c})$. Hence, we obtain that $X \ominus_g C \preceq_{\mathrm{LW}} Y \ominus_g C$. □

The subsequent corollaries are direct implications of Lemma 2.4 and 2.5.

Corollary 2.7 *If $X \preceq_{\mathrm{LU}} Y$, then $X \preceq_{\mathrm{LC}} Y$ and $X \preceq_{\mathrm{UC}} Y$.*

Corollary 2.8 *If $X \preceq_{\mathrm{CW}} Y$, then $X \preceq_{\mathrm{UC}} Y$ and $X \preceq_{\mathrm{UW}} Y$.*

Corollary 2.9 *If $X \preceq_{\mathrm{UW}} Y$, then $X \preceq_{\mathrm{UC}} Y$.*

However, the converse of the above implications may not be true. To demonstrate this, we provide the following examples.

Example 2.10 For $X = [1,4]$ and $Y = [3,5]$, $X \preceq_{\mathrm{LU}} Y$, but $X \npreceq_{\mathrm{CW}} Y, X \npreceq_{\mathrm{LW}} Y$ and $X \npreceq_{\mathrm{UW}} Y$.

If $X = [1,4]$ and $Y = [3,3.5]$, then $X \preceq_{\mathrm{LC}} Y$, but $X \npreceq_* Y$ for all $\{\preceq_{\mathrm{LU}}, \preceq_{\mathrm{LW}}, \preceq_{\mathrm{UC}}, \preceq_{\mathrm{CW}}, \preceq_{\mathrm{UW}}\}$.

$[1,2] \preceq_{\mathrm{UC}} [\frac{1}{2},4]$, but $[1,2] \npreceq_{\mathrm{LU}} [\frac{1}{2},4]$ and $[1,2] \npreceq_{\mathrm{LC}} [\frac{1}{2},4]$, furthermore, $[2,\frac{7}{2}] \preceq_{\mathrm{UC}} [3,4]$, $[2,\frac{7}{2}] \npreceq_* [3,4], \forall \{\preceq_{\mathrm{LW}}, \preceq_{\mathrm{CW}}, \preceq_{\mathrm{UW}}\}$.

Moreover, for $X = [1,2]$ and $Y = [\frac{1}{2},5]$, $X \preceq_{\mathrm{CW}} Y, X \npreceq_{\mathrm{LU}} Y, X \npreceq_{\mathrm{LC}} Y$, and $X \npreceq_{\mathrm{LW}} Y$.

Finally, let $X = [3,4]$ and $Y = [\frac{1}{2},5]$, then $X \preceq_{\mathrm{UW}} Y, X \npreceq_{\mathrm{LU}} Y, X \npreceq_{\mathrm{LC}} Y$, $X \npreceq_{\mathrm{LW}} Y$, and $X \npreceq_{\mathrm{CW}} Y$.

It is noted that the partial order $\preceq_{\mathrm{LC}}$ does not imply other partial orders as shown in Example 2.10.

For the interval-valued calculus on time scales, we refer to [29].

3 Main results

Throughout this section, assume that $\varsigma_0 \in \mathbb{T}$, $\mathbb{T}_0 = [\varsigma_0, \infty) \cap \mathbb{T}$ and $\mathbb{T}_0^- = (-\infty, \varsigma_0] \cap \mathbb{T}$.

Lemma 3.1 ([40]) *Let $f, x \in C_{rd}$ and $a \in \mathcal{R}^+$. Then*

$$x^{\Delta}(\varsigma) \le a(\varsigma)\langle x(\varsigma)\rangle + f(\varsigma) \quad \forall \varsigma \in \mathbb{T}_0 \tag{3.1}$$

implies

$$x(\varsigma) \le x(\varsigma_0)E_a(\varsigma, \varsigma_0) + \int_{\varsigma_0}^{\varsigma} f(s)\langle E_{-a}(s, \varsigma)\rangle \Delta s \tag{3.2}$$

$\forall \varsigma \in \mathbb{T}_0$.

Lemma 3.2 ([40]) *Let $f, x \in C_{rd}$ and $a \in \mathcal{R}^+$. Then*

$$x^{\Delta}(\varsigma) \le -a(\varsigma)\langle x(\varsigma)\rangle + f(\varsigma) \quad \forall \varsigma \in \mathbb{T}_0 \tag{3.3}$$

implies

$$x(\varsigma) \le x(\varsigma_0)E_{-a}(\varsigma, \varsigma_0) + \int_{\varsigma_0}^{\varsigma} f(s)\langle E_a(s, \varsigma)\rangle \Delta s \quad \forall \varsigma \in \mathbb{T}_0 \tag{3.4}$$

and

$$x^{\Delta}(\varsigma) \le -a(\varsigma)\langle x(\varsigma)\rangle + f(\varsigma) \quad \forall \varsigma \in \mathbb{T}_0^- \tag{3.5}$$

implies

$$x(\varsigma) \ge x(\varsigma_0)E_{-a}(\varsigma, \varsigma_0) + \int_{\varsigma_0}^{\varsigma} f(s)\langle E_a(s, \varsigma)\rangle \Delta s \quad \forall \varsigma \in \mathbb{T}_0^-. \tag{3.6}$$

Theorem 3.3 ([40]) *Suppose that $f, x \in C_{rd}$, $a \in \mathcal{R}^+$, and $a \ge 0$. Then*

$$x(\varsigma) \le f(\varsigma) + \int_{\varsigma_0}^{\varsigma} a(s)\langle x(s)\rangle \Delta s \quad \forall \varsigma \in \mathbb{T}_0 \tag{3.7}$$

implies

$$x(\varsigma) \le f(\varsigma) + \int_{\varsigma_0}^{\varsigma} a(s)\langle f(s)\rangle\langle E_{-a}(s, \varsigma)\rangle \Delta s \quad \forall \varsigma \in \mathbb{T}_0. \tag{3.8}$$

Corollary 3.4 ([40]) *Suppose that $x \in C_{rd}$, $a \in \mathcal{R}^+$, and $a \ge 0$. Then*

$$x(\varsigma) \le \int_{\varsigma_0}^{\varsigma} a(s)\langle x(s)\rangle \Delta s \quad \forall \varsigma \in \mathbb{T}_0 \tag{3.9}$$

implies

$$x(\varsigma) \leq 0 \quad \forall\, \varsigma \in \mathbb{T}_0. \tag{3.10}$$

Corollary 3.5 ([40]) *Suppose that $x \in C_{rd}, f_0 \in \mathcal{R}$, $a \in \mathcal{R}^+$, and $a \geq 0$. Then*

$$x(\varsigma) \leq f_0 + \int_{\varsigma_0}^{\varsigma} a(s)\langle x(s)\rangle \Delta s \quad \forall\, \varsigma \in \mathbb{T}_0 \tag{3.11}$$

implies

$$x(\varsigma) \leq f_0 E_a(\varsigma, \varsigma_0) \quad \forall\, \varsigma \in \mathbb{T}_0. \tag{3.12}$$

Corollary 3.6 ([40]) *If $a, q \in \mathcal{R}^+$ with $a(\varsigma) \leq q(\varsigma)$ $\forall\, \varsigma \in \mathbb{T}$, then*

$$E_a(\varsigma, \varsigma_0) \leq E_q(\varsigma, \varsigma_0) \quad \forall\, \varsigma \in \mathbb{T}_0. \tag{3.13}$$

Moreover,

$$\langle E_a(\varsigma, \varsigma_0)\rangle \leq \langle E_q(\varsigma, \varsigma_0)\rangle \quad \forall\, \varsigma \in \mathbb{T}_0. \tag{3.14}$$

Similar to Theorem 3.3, one can get the following results.

Theorem 3.7 ([40]) *Suppose that $f, g, x \in C_{rd}$, $\alpha_0 \in \mathcal{R}$, $q \in \mathcal{R}^+$, and $q \geq 0$. Then*

$$x(\varsigma) \leq f(\varsigma) + \alpha_0 \int_{\varsigma_0}^{\varsigma} \big[q(s)\langle x(s)\rangle + g(s)\big] \Delta s \quad \forall\, \varsigma \in \mathbb{T}_0 \tag{3.15}$$

implies

$$x(\varsigma) \leq f(\varsigma) + \alpha_0 \int_{\varsigma_0}^{\varsigma} \big[q(s)\langle f(s)\rangle + g(s)\big]\langle E_{-\alpha_0 q}(s, \varsigma)\rangle \Delta s \quad \forall\, \varsigma \in \mathbb{T}_0. \tag{3.16}$$

An important consequence of Lemma 3.2 is as follows.

Theorem 3.8 ([40]) *Suppose that $f, g, x \in C_{rd}$, $\alpha_0 \in \mathcal{R}$, $q \in \mathcal{R}^+$, and $q \geq 0$. Then*

$$x(\varsigma) \leq f(\varsigma) + \alpha_0 \int_{\varsigma}^{\varsigma_0} \big[q(s)\langle x(s)\rangle + g(s)\big] \Delta s \quad \forall\, \varsigma \in \mathbb{T}_0^- \tag{3.17}$$

implies

$$x(\varsigma) \leq f(\varsigma) + \alpha_0 \int_{\varsigma}^{\varsigma_0} \big[q(s)\langle f(s)\rangle + g(s)\big]\langle E_{\alpha_0 q}(s, \varsigma)\rangle \Delta s \quad \forall\, \varsigma \in \mathbb{T}_0^-. \tag{3.18}$$

3.1 Interval-valued case

For IVF $F : \mathbb{T} \to \mathcal{K}_C$, define

$$\langle F(\varsigma)\rangle = \frac{F(\varsigma) + F(\varsigma^\sigma)}{2}. \tag{3.19}$$

If $F : \mathbb{T} \to \mathcal{K}_C$ such that $F(\varsigma) = [f^-(\varsigma), f^+(\varsigma)]$, then (3.19) implies that

$$\langle F(\varsigma)\rangle = \big[\langle f^-(\varsigma)\rangle, \langle f^+(\varsigma)\rangle\big]. \tag{3.20}$$

By the definition of midpoint function, we can get

$$m\big(\langle F(\varsigma)\rangle\big) = \big\langle m\big(F(\varsigma)\big)\big\rangle. \tag{3.21}$$

By using "$\preceq_{\mathrm{LC}}$" and (3.19), one can easily get the following result.

Lemma 3.9 *Let $F, G : \mathbb{T} \to \mathcal{K}_C$. If $F(\varsigma) \preceq_{\mathrm{LC}} G(\varsigma)\ \forall\ \varsigma \in \mathbb{T}$, then $\langle F(\varsigma)\rangle \preceq_{\mathrm{LC}} \langle G(\varsigma)\rangle$.*

Let us start this section with comparison results for IVFs under LC-partial order. For further discussion, let us consider some function classes:

$$C_{rd}^{\mathcal{K}_C} := \{F : \mathbb{T} \to \mathcal{K}_C : F \text{ is rd-continuous } \forall\ t \in \mathbb{T}\},$$
$$C_{gH}^{1,1st} := \big\{F : \mathbb{T} \to \mathcal{K}_C : F \text{ is } \Delta_{1,gH}\text{-differentiable } \forall\ t \in \mathbb{T}_0^k\big\},$$
$$C_{gH}^{1,2nd} := \big\{F : \mathbb{T} \to \mathcal{K}_C : F \text{ is } \Delta_{2,gH}\text{-differentiable } \forall\ t \in \mathbb{T}_0^k\big\}.$$

Lemma 3.10 *Let $F, X \in C_{rd}^{\mathcal{K}_C}$ and $a \in \mathcal{R}^+$. (a) If $X \in C_{gH}^{1,1st} \ni$*

$$X^\Delta(\varsigma) \preceq_{\mathrm{LC}} a(\varsigma)\big\langle X(\varsigma)\big\rangle + F(\varsigma) \quad \forall\ \varsigma \in \mathbb{T}_0, \tag{3.22}$$

then

$$X(\varsigma) \preceq_{\mathrm{LC}} E_a(\varsigma, \varsigma_0)X(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{-a}(\tau, \varsigma)\big\rangle F(\tau)\Delta\tau \tag{3.23}$$

$\forall\ \varsigma \in \mathbb{T}_0$.

(b) If $X \in C_{gH}^{1,2nd} \ni$

$$-X^\Delta(\varsigma) \preceq_{\mathrm{LC}} a(\varsigma)\big\langle X(\varsigma)\big\rangle + F(\varsigma) \quad \forall\ \varsigma \in \mathbb{T}_0, \tag{3.24}$$

then

$$X(\varsigma) \succeq_{\mathrm{LC}} E_{-a}(\varsigma, \varsigma_0)X(\varsigma_0) - \int_{\varsigma_0}^{\varsigma} \big\langle E_a(\tau, \varsigma)\big\rangle F(\tau)\Delta\tau \tag{3.25}$$

$\forall\ \varsigma \in \mathbb{T}_0$.

Proof Let $F, X \in C_{rd}^{\mathcal{K}_C}$ with $X(\varsigma) = [\bar{x}(\varsigma), x(\varsigma)]$ and $F(\varsigma) = [f^-(\varsigma), f^+(\varsigma)]$.

(*a*) If $X \in C_{gH}^{1,1st}$, then $X^\Delta(\varsigma) = [(\bar{x})^\Delta(\varsigma), (x)^\Delta(\varsigma)]$. First, we consider the case if $a(\varsigma) \geq 0$ on $\mathbb{T}_0$, we have $a(\varsigma)\langle X(\varsigma)\rangle = [a(\varsigma)\langle \bar{x}(\varsigma)\rangle, a(\varsigma)\langle x(\varsigma)\rangle]$. By using inequality (3.22), we obtain

$$\big[(\bar{x})^\Delta(\varsigma), (x)^\Delta(\varsigma)\big] \preceq_{\mathrm{LC}} \big[a(\varsigma)\big\langle \bar{x}(\varsigma)\big\rangle + f^-(\varsigma), a(\varsigma)\big\langle x(\varsigma)\big\rangle + f^+(\varsigma)\big].$$

Applying LC-order, we obtain

$$(\bar{x})^\Delta(\varsigma) \leq a(\varsigma)\big\langle \bar{x}(\varsigma)\big\rangle + f^-(\varsigma) \tag{3.26}$$

and

$$\big(m\big(X(\varsigma)\big)\big)^{\Delta} \le a(\varsigma)\big\langle m\big(X(\varsigma)\big)\big\rangle + m\big(F(\varsigma)\big). \tag{3.27}$$

By using Lemma 3.1 on (3.26) and (3.27) respectively, we obtain

$$\bar{x}(\varsigma) \le E_a(\varsigma,\varsigma_0)\bar{x}(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{-a}(\tau,\varsigma)\big\rangle f^-(\tau)\Delta\tau \tag{3.28}$$

and

$$m\big(X(\varsigma)\big) \le E_a(\varsigma,\varsigma_0)m\big(X(\varsigma_0)\big) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{-a}(\tau,\varsigma)\big\rangle m\big(F(\tau)\big)\Delta\tau. \tag{3.29}$$

Inequalities (3.28) and (3.29) yield (3.23). $a(\varsigma) < 0$ on $\mathbb{T}_0$ implies that $a(\varsigma)X(\varsigma) = [a(\varsigma)\langle x(\varsigma)\rangle, a(\varsigma)\langle\bar{x}(\varsigma)\rangle]$. By using inequality (3.22), we obtain

$$\big[(\bar{x})^{\Delta}(\varsigma), (x)^{\Delta}(\varsigma)\big] \preceq_{\mathrm{LC}} \big[a(\varsigma)\big\langle x(\varsigma)\big\rangle + f^-(\varsigma), a(\varsigma)\big\langle\bar{x}(\varsigma)\big\rangle + f^+(\varsigma)\big].$$

Applying LC-order, we have

$$(\bar{x})^{\Delta}(\varsigma) \le a(\varsigma)\big\langle x(\varsigma)\big\rangle + f^-(\varsigma) \le a(\varsigma)\big\langle \bar{x}(\varsigma)\big\rangle + f^-(\varsigma) \tag{3.30}$$

and

$$\big(m\big(X(\varsigma)\big)\big)^{\Delta} \le \big(-a(\varsigma)\big)\big\langle m\big(X(\varsigma)\big)\big\rangle + m\big(F(\varsigma)\big). \tag{3.31}$$

By using Lemma 3.1 on (3.30) and (3.31) respectively, we obtain

$$\bar{x}(\varsigma) \le E_a(\varsigma,\varsigma_0)\bar{x}(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{-a}(\tau,\varsigma)\big\rangle f^-(\tau)\Delta\tau \tag{3.32}$$

and

$$m\big(X(\varsigma)\big) \le E_{-a}(\varsigma,\varsigma_0)m\big(X(\varsigma_0)\big) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{a}(\tau,\varsigma)\big\rangle m\big(F(\tau)\big)\Delta\tau. \tag{3.33}$$

Since $a(\varsigma) < 0$ and $a \in \mathcal{R}^+$, it follows that $(-a) \in \mathcal{R}^+$ and $a \le -a$. Therefore, Lemma 2.1 and Corollary 3.6 imply that

$$\begin{aligned}\bar{x}(\varsigma) &\le E_a(\varsigma,\varsigma_0)\bar{x}(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{-a}(\tau,\varsigma)\big\rangle f^-(\tau)\Delta\tau \\ &\le E_{-a}(\varsigma,\varsigma_0)\bar{x}(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{a}(\tau,\varsigma)\big\rangle f^-(\tau)\Delta\tau.\end{aligned} \tag{3.34}$$

Combining (3.33) and (3.34), we get

$$X(\varsigma) \preceq_{\mathrm{LC}} E_{-a}(\varsigma,\varsigma_0)X(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \big\langle E_{a}(\tau,\varsigma)\big\rangle F(\tau)\Delta\tau.$$

(b) If $X \in C_{gH}^{1,2nd}$, then $X^{\Delta}(\varsigma) = [(x)^{\Delta}(\varsigma), (\bar{x})^{\Delta}(\varsigma)]$ and for $a(\varsigma) \geq 0$, so we have $a(\varsigma)X(\varsigma) = [a(\varsigma)\langle\bar{x}(\varsigma)\rangle, a(\varsigma)\langle x(\varsigma)\rangle]$. Inequality (3.24) implies that

$$(-\bar{x})^{\Delta}(\varsigma) \leq a(\varsigma)\big\langle\bar{x}(\varsigma)\big\rangle + f^{-}(\varsigma) = \big(-a(\varsigma)\big)\big\langle-\bar{x}(\varsigma)\big\rangle + f^{-}(\varsigma)$$

and

$$\big(-m\big(X(\varsigma)\big)\big)^{\Delta} \leq \big(-a(\varsigma)\big)\big\langle-m\big(X(\varsigma)\big)\big\rangle + m\big(F(\varsigma)\big).$$

It follows that

$$\bar{x}(\varsigma) \geq E_{-a}(\varsigma, \varsigma_0)\bar{x}(\varsigma_0) - \int_{\varsigma_0}^{\varsigma} \big\langle E_a(\tau, \varsigma)\big\rangle f^{-}(\tau)\Delta\tau \tag{3.35}$$

and

$$m\big(X(\varsigma)\big) \geq E_{-a}(\varsigma, \varsigma_0)m\big(X(\varsigma_0)\big) - \int_{\varsigma_0}^{\varsigma} \big\langle E_{-a}(\tau, \varsigma)\big\rangle m\big(F(\tau)\big)\Delta\tau. \tag{3.36}$$

By using (3.35) and (3.36) in LC order, we can get (3.25). For $a(\varsigma) < 0$, similar to the second inequality of part (a), we can obtain (3.25). □

One of the consequences of Lemma 3.9 and Lemma 3.10 is as follows.

Lemma 3.11 *Let $F, X \in C_{rd}^{\mathcal{K}_C}$ and $a \in \mathcal{R}^{+}$.*

(a) If $X \in C_{gH}^{1,1st}$ ∋

$$X^{\Delta}(\varsigma) \preceq_{\mathrm{LC}} a(\varsigma)\big\langle X(\varsigma)\big\rangle + F(\varsigma) \quad \forall\, \varsigma \in \mathbb{T}_0,$$

then

$$\begin{aligned}\big\langle X(\varsigma)\big\rangle \preceq_{\mathrm{LC}} \big\langle E_a(\varsigma, \varsigma_0)\big\rangle X(\varsigma_0) &+ \int_{\varsigma_0}^{\varsigma} \big\langle\big\langle E_{-a}(\tau, \varsigma)\big\rangle\big\rangle F(\tau)\Delta\tau \\ &+ \frac{1}{2}\mu(\varsigma)\big\langle E_{-a}\big(\varsigma, \varsigma^{\sigma}\big)\big\rangle F(\varsigma)\end{aligned}$$

$\forall\, \varsigma \in \mathbb{T}_0$.

(b) If $X \in C_{gH}^{1,2nd}$ ∋

$$-X^{\Delta}(\varsigma) \preceq_{\mathrm{LC}} a(\varsigma)\big\langle X(\varsigma)\big\rangle + F(\varsigma) \quad \forall\, \varsigma \in \mathbb{T}_0,$$

then

$$\begin{aligned}\big\langle X(\varsigma)\big\rangle \succeq_{\mathrm{LC}} \big\langle E_{-a}(\varsigma, \varsigma_0)\big\rangle X(\varsigma_0) &- \int_{\varsigma_0}^{\varsigma} \big\langle\big\langle E_a(\tau, \varsigma)\big\rangle\big\rangle F(\tau)\Delta\tau \\ &- \frac{1}{2}\mu(\varsigma)\big\langle E_{-a}\big(\varsigma, \varsigma^{\sigma}\big)\big\rangle F(\varsigma)\end{aligned}$$

$\forall\, \varsigma \in \mathbb{T}_0$.

Similar to Lemma 3.10, by applying Lemma 3.2, we get the subsequent result.

Lemma 3.12 *Let* $F, X \in C_{rd}^{\mathcal{K}_C}$ *and* $a \in \mathcal{R}^+$.
(*a*) *If* $X \in C_{gH}^{1,1st} \ni$

$$X^{\Delta}(\varsigma) \preceq_{\mathrm{LC}} -a(\varsigma)\langle X(\varsigma)\rangle + F(\varsigma) \quad \forall\, \varsigma \in \mathbb{T}_0, \tag{3.37}$$

then

$$X(\varsigma) \preceq_{\mathrm{LC}} E_{-a}(\varsigma,\varsigma_0)X(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \langle E_a(\tau,\varsigma)\rangle F(\tau)\Delta\tau \tag{3.38}$$

$\forall\, \varsigma \in \mathbb{T}_0$.
(*b*) *If* $X \in C_{gH}^{1,2nd} \ni$

$$-X^{\Delta}(\varsigma) \preceq_{\mathrm{LC}} -a(\varsigma)\langle X(\varsigma)\rangle + F(\varsigma) \quad \forall\, \varsigma \in \mathbb{T}_0, \tag{3.39}$$

then

$$X(\varsigma) \succeq_{\mathrm{LC}} E_a(\varsigma,\varsigma_0)X(\varsigma_0) - \int_{\varsigma_0}^{\varsigma} \langle E_{-a}(\tau,\varsigma)\rangle F(\tau)\Delta\tau \tag{3.40}$$

$\forall\, \varsigma \in \mathbb{T}_0$.

Lemma 3.13 *Let* $F, X \in C_{rd}^{\mathcal{K}_C}$ *and* $a \in \mathcal{R}^+$.
(*a*) *If* $X \in C_{gH}^{1,1st} \ni$

$$X^{\Delta}(\varsigma) \preceq_{\mathrm{LC}} a(\varsigma)\langle X(\varsigma)\rangle + F(\varsigma) \quad \forall\, \varsigma \in \mathbb{T}_0,$$

then

$$\begin{aligned}\langle X(\varsigma)\rangle \preceq_{\mathrm{LC}} &\langle E_{-a}(\varsigma,\varsigma_0)\rangle X(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \langle\langle E_a(\tau,\varsigma)\rangle\rangle F(\tau)\Delta\tau \\ &+ \frac{1}{2}\mu(\varsigma)\langle E_a(\varsigma,\varsigma^{\sigma})\rangle F(\varsigma)\end{aligned}$$

$\forall\, \varsigma \in \mathbb{T}_0$.
(*b*) *If* $X \in C_{gH}^{1,2nd} \ni$

$$-X^{\Delta}(\varsigma) \preceq_{\mathrm{LC}} a(\varsigma)\langle X(\varsigma)\rangle + F(\varsigma) \quad \forall\, \varsigma \in \mathbb{T}_0,$$

then

$$\begin{aligned}\langle X(\varsigma)\rangle \succeq_{\mathrm{LC}} &\langle E_a(\varsigma,\varsigma_0)\rangle X(\varsigma_0) - \int_{\varsigma_0}^{\varsigma} \langle\langle E_{-a}(\tau,\varsigma)\rangle\rangle F(\tau)\Delta\tau \\ &- \frac{1}{2}\mu(\varsigma)\langle E_{-a}(\varsigma,\varsigma^{\sigma})\rangle F(\varsigma)\end{aligned}$$

$\forall\, \varsigma \in \mathbb{T}_0$.

Remark 3.14 It is noted that in Lemmas 3.10 and 3.12 we get a more simple and relaxed condition as compared to the main results in [40].

Now onward, we are assuming that all functions are bounded.

Theorem 3.15 *Let* $F, X \in C_{rd}^{\mathcal{K}_C}$ *and* $a \in \mathcal{R}^+$, $a(\varsigma) \geq 0\ \forall\ \varsigma \in \mathbb{T}_0$,

$$X(\varsigma) \preceq_{\mathrm{LC}} F(\varsigma) + \int_{\varsigma_0}^{\varsigma} a(s)\big\langle X(s)\big\rangle \Delta s \tag{3.41}$$

holds $\forall\ \varsigma \in \mathbb{T}_0$. *Then*

$$X(\varsigma) \preceq_{\mathrm{LC}} F(\varsigma) + \int_{\varsigma_0}^{\varsigma} a(s)\big\langle E_a(\varsigma, s)\big\rangle\big\langle F(s)\big\rangle \Delta s \tag{3.42}$$

$\forall\ \varsigma \in \mathbb{T}_0$.

Proof Consider $Z(\varsigma) = \int_{\varsigma_0}^{\varsigma} a(\tau)\langle X(\tau)\rangle \Delta\tau$. Since $a(\tau)$, $\langle X(\tau)\rangle$ are bounded and belong to C_{rd} class, therefore it follows that $Z \in C_{gH}^{1,1st}$ and $Z^{\Delta}(\varsigma) = a(\varsigma)\langle X(\varsigma)\rangle$, $\varsigma \in \mathbb{T}_0$. From inequality (3.41), we can see that $\langle X(\varsigma)\rangle \preceq_{\mathrm{LC}} \langle F(\varsigma)\rangle + \langle Z(\varsigma)\rangle$. Clearly,

$$Z^{\Delta}(\varsigma) \preceq_{\mathrm{LC}} a(\varsigma)\big\langle Z(\varsigma)\big\rangle + a(\varsigma)\big\langle F(\varsigma)\big\rangle.$$

Part (a) in Lemma 3.10 and $Z(\varsigma_0) = \{0\}$ implies that

$$Z(\varsigma) \preceq_{\mathrm{LC}} \int_{\varsigma_0}^{\varsigma} a(s)\big\langle E_a(\varsigma, s)\big\rangle\big\langle F(s)\big\rangle \Delta s,$$

and hence assertion (3.42) follows by inequality (3.41). □

Corollary 3.16 *Let* $X \in C_{rd}^{\mathcal{K}_C}$, $a \in \mathcal{R}^+$, $a \geq 0$, *and* $X_0 \in \mathcal{K}_C$. *If*

$$X(\varsigma) \preceq_{\mathrm{LC}} X_0 + \int_{\varsigma_0}^{\varsigma} a(s)\big\langle X(s)\big\rangle \Delta s \quad \forall\ \varsigma \in \mathbb{T}_0, \tag{3.43}$$

then

$$X(\varsigma) \preceq_{\mathrm{LC}} X_0 E_a(\varsigma, \varsigma_0) \quad \forall\ \varsigma \in \mathbb{T}_0. \tag{3.44}$$

Proof In Theorem 3.15, if we take $F(\varsigma) = X_0$, we can get (3.44). □

Corollary 3.17 *Let* $X \in C_{rd}^{\mathcal{K}_C}$, $a \in \mathcal{R}^+$, $a \geq 0 \ni$

$$X(\varsigma) \preceq_{\mathrm{LC}} \int_{\varsigma_0}^{\varsigma} X(\varsigma) a(\varsigma) \Delta\varsigma \quad \forall\ \varsigma \in \mathbb{T}_0,$$

then

$$X(\varsigma) \preceq_{\mathrm{LC}} \{0\}$$

$\forall\ \varsigma \in \mathbb{T}_0$.

Similar to Theorem 3.15, we derive the subsequent theorem.

Theorem 3.18 *Let* $F, Q, X \in C_{rd}^{\mathcal{K}_C}$, $a \in \mathcal{R}^+$, $a \geq 0$ $b_0 \in \mathcal{R}^+$ *such that*

$$X(\varsigma) \preceq_{\mathrm{LC}} F(\varsigma) + b_0 \int_{\varsigma_0}^{\varsigma} \big[a(\tau)X(\tau) + Q(\tau)\big]\Delta\tau \quad \forall\, \varsigma \in \mathbb{T}_0,$$

then

$$X(\varsigma) \preceq_{\mathrm{LC}} F(\varsigma) + b_0 \int_{\varsigma_0}^{\varsigma} \big(a(\tau)\langle F(\tau)\rangle + Q(\tau)\big)\langle E_{ab_0}(\varsigma,\tau)\rangle\Delta\tau$$

$\forall\, \varsigma \in \mathbb{T}_0$.

If we take $F(\varsigma) = Q(\varsigma) = 0$ in Theorem 3.18, then one can get the following.

Corollary 3.19 *Suppose* $X(\varsigma) \in C_{rd}^{\mathcal{K}_C}$ *and* $a \in \mathcal{R}^+$, $a \geq 0$ $b_0 \in \mathcal{R}^+$ *such that*

$$X(\varsigma) \preceq_{\mathrm{LC}} b_0 \int_{\varsigma_0}^{\varsigma} X(\tau)a(\tau)\Delta\tau \quad \forall\, \varsigma \in \mathbb{T}_0,$$

then

$$X(\varsigma) \preceq_{\mathrm{LC}} \{0\}$$

$\forall\, \varsigma \in \mathbb{T}_0$.

Remark 3.20 If $b_0 = 1$ in Corollary 3.19, then we get Corollary 3.17.

4 Conclusions

In this paper, we presented certain results of Gronwall type inequalities concerning interval-valued functions under $\preceq_{\mathrm{LC}}$. These inequalities render explicit bounds of unknown functions. By using $\preceq_{\mathrm{LC}}$the assumptions in the main results become more relaxed compared to the main results in [40]. The results can be more beneficial in the subject of the uniqueness of solution for interval-valued differential equations or interval-valued integrodifferential equations. Moreover, we will extend these inequalities to fuzzy-interval-valued functions in our forthcoming work. This research also points out that Gronwall's inequality for interval-valued functions can be reduced to a family of classical Gronwall's inequality for real-valued functions. The interval versions of Gronwall's inequality exhibited in this study are tools to work in an uncertain environment. Furthermore, as these inequalities are given by applying different assumptions than those used in the earlier research articles, our results are new.

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