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An extension of several essential numerical radius inequalities of 2×2 off-diagonal operator matrices

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Abstract

In this work, we provide upper and lower bounds for the numerical radius of an $n \times n$ off-diagonal operator matrix, which extends some results by Abu-Omar and Kittaneh (Stud. Math. 216:69–75, 2013; Linear Algebra Appl. 468:18–26, 2015; Rocky Mt. J. Math. 45(4):1055–1065, 2015), and Paul and Bag (Appl. Math. Comput. 222:2731–2943, 2013).

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1 Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H endowed with inner product $\langle \cdot, \cdot \rangle$. The numerical radius and the usual operator norm of $T \in B(H)$ are given by $w(T) = \sup |\langle Tx, x\rangle|$ and $\|T\| = \sup \sqrt{\langle Tx, Tx\rangle}$, respectively, over all the unit vectors $x \in H$. Also, the nonnegative number $m(T)$ is given by $m(T) = \inf |\langle Tx, x\rangle|$. It is well known that the numerical radius $w(T)$ defines an equivalent norm to the usual operator norm on $B(H)$ as follows [8]:

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|. \tag{1.1}$$

In [9], Kittaneh gives an improvement for the upper bound of (1.1) by using several norm inequalities

$$\frac{1}{4}\big\||T| + |T^*|\big\| \le w^2(T) \le \frac{1}{2}\big\||T| + |T^*|\big\|, \tag{1.2}$$

which has been again refined in [1] by using the concept of the generalized Aluthge transform of T as follows:

$$w(T) \le \frac{1}{2}\big(\|T\| + w(\tilde{T}_t)\big), \tag{1.3}$$

where $\tilde{T}_t = |T|^t U|T|^{1-t}$ for $t \in [0,1]$. Another improvement for the two-sided inequality (1.1) has been provided in [3] by showing that

$$\frac{1}{2}\sqrt{\alpha + 2m(T^2)} \leq w(T) \leq \frac{1}{2}\sqrt{\alpha + 2w(T^2)}, \tag{1.4}$$

where $\alpha = \| |T|^2 + |T^*|^2 \|$. This estimation has been recently improved in [12] in the following form:

$$\begin{aligned} &\frac{1}{4}c^2(T^2) + \frac{1}{8}m(T^2P + PT^2) + \frac{1}{16}\|P\|^2 \\ &\quad \leq w^4(T) \leq \frac{1}{4}w^2(T^2) + \frac{1}{8}w(T^2P + PT^2) + \frac{1}{16}\|P\|^2, \end{aligned} \tag{1.5}$$

with $P = |T|^2 + |T^*|$ and $c(T) = \inf_{\substack{\|x\|=1 \\ x\in H}} \inf_{\theta \in R} \| \mathrm{Re}(e^{i\theta} T)x\|$. Moreover, the authors in [2] gave a generalization for (1.4) in this fashion:

$$\frac{1}{2}\sqrt{\beta + 2m(CB)} \leq w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\sqrt{\beta + 2w(CB)}, \tag{1.6}$$

where $\beta = \| |B|^2 + |C^*|^2 \|$.

Let H_1, H_2 be Hilbert spaces, and let $A \in B(H_1)$, $B \in B(H_2, H_1)$, $C \in B(H_1, H_2)$, and $D \in B(H_2)$. For our purposes, we recall the following fundamental facts that are relevant to our work:

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\| = \max\{\|A\|, \|B\|\}, \tag{1.7}$$

$$\left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| = \max\{\|A\|, \|B\|\}, \tag{1.8}$$

$$w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\{w(A), w(B)\}, \tag{1.9}$$

$$w\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = w(A), \tag{1.10}$$

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \geq \max\left\{w(A), w(D), w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\}. \tag{1.11}$$

It is worth mentioning here that the inequalities (1.7), (1.8), and (1.9) remain valid for $n \times n$ operator matrices.

The aim of this paper is to give generalizations for the inequalities (1.3), (1.4), (1.5), (1.6), and (1.10).

2 Numerical radius inequalities for $n \times n$ operator matrices

In this section, we will extend several well-known numerical radius inequalities of 2×2 operator matrices. We start by the following characterization for the numerical radius of $T \in B(H)$ [13].

Theorem 2.1 *Let $T \in B(H)$. Then*

$$w(T) = \max_{\theta \in \mathbb{R}} \left\| \operatorname{Re}\left(e^{i\theta} T\right) \right\|.$$

We now state our main results which can be seen as a generalization of (1.10) from 2×2 operator matrices to a broad family of $n \times n$ operator matrices.

Theorem 2.2 *Let $Y \in B(H)$ and n be an odd natural number. Then, for arbitrary $\lambda_1, \ldots, \lambda_{\frac{n+1}{2}} \in \mathbb{C}$, we have*

$$w\left(\begin{bmatrix} & \mathbf{0} & & & \lambda_1 Y \\ & & & \lambda_2 Y & \\ & & & \iddots & \\ & & \lambda_{\frac{n+1}{2}} Y & & \\ & \iddots & & & \\ & \lambda_2 Y & & \mathbf{0} & \\ \lambda_1 Y & & & & \end{bmatrix}_{n \times n}\right) = \max\left\{|\lambda_i|\right\}_{i=1}^{\frac{n+1}{2}} w(Y). \tag{2.1}$$

Proof Let

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & & \mathbf{0} & & I \\ & \ddots & & \iddots & \\ & I & & I & \\ \mathbf{0} & & \sqrt{2} I & & \mathbf{0} \\ & -I & & I & \\ & \iddots & & \ddots & \\ -I & & \mathbf{0} & & I \end{bmatrix}.$$

A direct computation shows that U is unitary operator and

$$U \begin{bmatrix} & \mathbf{0} & & & \lambda_1 Y \\ & & & \lambda_2 Y & \\ & & & \iddots & \\ & & \lambda_{\frac{n+1}{2}} Y & & \\ & \iddots & & & \\ & \lambda_2 Y & & \mathbf{0} & \\ \lambda_1 Y & & & & \end{bmatrix} U^* = \begin{bmatrix} \lambda_1 Y & & & & & \\ & \lambda_2 Y & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & \lambda_{\frac{n+1}{2}} Y & & \\ & & & & -\lambda_{\frac{n-1}{2}} Y & \\ & \mathbf{0} & & & \ddots & \\ & & & & & -\lambda_1 Y \end{bmatrix}.$$

Therefore, the result follows by the weakly unitary invariance of $w(\cdot)$ and (1.9). □

As a direct consequence, we have the following generalization of (1.10).

Corollary 2.3 *Let $Y \in B(H)$. Then, for every $n \in \mathbb{N}$, we have*

$$w\left(\begin{bmatrix} \mathbf{0} & & Y \\ & Y & \\ & \iddots & \\ Y & & \mathbf{0} \end{bmatrix}_n\right) = w(Y).$$

Proof It suffices to show the result in the case when $n = 2$ and n is an odd natural number. The first case follows from (1.10), and the second case from Theorem 2.2 by letting $\lambda_i = 1$, $(i = 1, 2, \dots, \frac{n+1}{2})$. □

In our next result, we extend inequality (1.3) to arbitrary finite number of bounded linear operators on a complex Hilbert space.

Theorem 2.4 *Let $\{A_i\}_{i=1}^n \in B(H)$ and let $A_i = U_i|A_i|$ be the polar decomposition of A_i. Then*

$$w\left(\begin{bmatrix} \mathbf{0} & & & A_1 \\ & & A_2 & \\ & \iddots & & \\ A_n & & & \mathbf{0} \end{bmatrix}\right) \le \frac{1}{2}\max\left\{\|A_i\|\right\}_{i=1}^n + \frac{1}{2} w\left(\begin{bmatrix} \mathbf{0} & & & O_1 \\ & & O_2 & \\ & \iddots & & \\ O_n & & & \mathbf{0} \end{bmatrix}\right), \tag{2.2}$$

where $O_i = |A_{n+1-i}|^t U_i |A_i|^{1-t}$, $t \in [0,1]$.

Proof Note that we have the following polar formulation

$$\begin{bmatrix} \mathbf{0} & & & A_1 \\ & & A_2 & \\ & \iddots & & \\ A_n & & & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & & & U_1 \\ & & U_2 & \\ & \iddots & & \\ U_n & & & \mathbf{0} \end{bmatrix} \begin{bmatrix} |A_1| & & & \mathbf{0} \\ & |A_2| & & \\ & & \ddots & \\ \mathbf{0} & & & |A_n| \end{bmatrix}.$$

Thus the result follows from inequality (1.3). □

It is worth noting here that the inequality (1.3) can now be seen as a direct consequence of Theorems 2.2 and 2.4 when $A_1 = A_2 = \cdots = A_n$.

Now, by using Theorem 2.1 together with the inequality (1.8), we have the following extension of the two-sided inequality (1.6).

Theorem 2.5 *Let $H_1, H_2, \dots, H_n$ be Hilbert spaces, and let*

$$T = \begin{bmatrix} \mathbf{0} & & & A_1 \\ & & A_2 & \\ & \iddots & & \\ A_n & & & \mathbf{0} \end{bmatrix},$$

with $\{A_m\}_{m=1}^n \in B(H_{n-m+1}, H_m)$. Set $E = \{1, 2, \dots, \frac{n}{2}\}$ and $O = \{1, 2, \dots, \frac{n-1}{2}\}$, then

$$\max_{k\in E}\{d_k\} \le 2w(T) \le \max_{k\in E}\{c_k\}, \quad \text{for } n \text{ even}, \tag{2.3}$$

$$\max_{k\in O}\{d_k\} \le 2w(T) \le \max_{k\in O}\left\{2w(A_{\frac{n+1}{2}}), c_k\right\}, \quad \text{for } n \text{ odd}, \tag{2.4}$$

where

$$c_k = \sqrt{\left\| |A_k|^2 + \left|A_{n-k+1}^*\right|^2 \right\| + 2w(A_{n-k+1}A_k)}$$

and

$$d_k = \sqrt{\left\| |A_k|^2 + \left|A_{n-k+1}^*\right|^2 \right\| + 2m(A_{n-k+1}A_k)}.$$

Proof Let n be an even number. Then

$$\left\|\operatorname{Re}\left(e^{i\theta}T\right)\right\| = \frac{1}{2}\max_{k\in E}\left\{\left\|e^{i\theta}A_k + e^{-i\theta}A_{n-k+1}^*\right\|\right\}.$$

Now,

$$\begin{aligned}\left\|e^{i\theta}A_k + e^{-i\theta}A_{n-k+1}^*\right\| &= \left\|\left(e^{-i\theta}A_k^* + e^{i\theta}A_{n-k+1}\right)\left(e^{i\theta}A_k + e^{-i\theta}A_{n-k+1}^*\right)\right\|^{\frac{1}{2}} \\ &= \left\||A_k|^2 + \left|A_{n-k+1}^*\right|^2 + 2\operatorname{Re}\left(e^{2i\theta}A_{n-k+1}A_k\right)\right\|^{\frac{1}{2}} \\ &\leq c_k.\end{aligned}$$

Therefore,

$$\begin{aligned}w(T) &= \sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}T\right)\right\| \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\{\max_{k\in E}\left\{\left\|e^{i\theta}A_k + e^{-i\theta}A_{n-k+1}\right\|\right\}\right\} \\ &\leq \frac{1}{2}\max_{k\in E}\{c_k\},\end{aligned}$$

which shows the upper bound of inequality (2.3). To show the lower bound of inequality (2.3), let $\psi \in \mathbb{R}$ be such that $e^{2i\psi}\langle A_{n-s+1}A_s x, x\rangle = |\langle A_{n-s+1}A_s x, x\rangle|$ for any unit vector $x \in H_{n-s+1}$ where $s \in E$. Then

$$\begin{aligned}w(T) &\geq \left\|\operatorname{Re}\left(e^{i\psi}T\right)\right\| \\ &\geq \frac{1}{2}\left\|e^{i\psi}A_s + e^{-i\psi}A_{n-s+1}^*\right\| \\ &\geq \frac{1}{2}\sqrt{\left|\left\langle\left(|A_s|^2 + \left|A_{n-s+1}^*\right|^2\right)x, x\right\rangle + 2\operatorname{Re}\left(e^{2i\psi}\langle A_{n-s+1}A_s x, x\rangle\right)\right|} \\ &= \frac{1}{2}\sqrt{\left\langle\left(|A_s|^2 + \left|A_{n-s+1}^*\right|^2\right)x, x\right\rangle + 2\left|\langle A_{n-s+1}A_s x, x\rangle\right|} \\ &\geq \frac{1}{2}\sqrt{\left\langle\left(|A_s|^2 + \left|A_{n-s+1}^*\right|^2\right)x, x\right\rangle + 2m(A_{n-s+1}A_s)}.\end{aligned}$$

Therefore,

$$w(T) \geq \frac{1}{2}\sup_{\|x\|=1}\sqrt{\left\langle\left(|A_s|^2 + \left|A_{n-s+1}^*\right|^2\right)x, x\right\rangle + 2m(A_{n-s+1}A_s)} = \frac{1}{2}d_s.$$

As $s \in E$ is arbitrary, $2w(T) \geq \max_{k\in E}\{d_k\}$, which completes the proof of inequality (2.3).

On the other hand, if n is an odd number, then

$$\|\operatorname{Re}(e^{i\theta}T)\| = \frac{1}{2}\max_{k\in O}\left\{2\left\|\operatorname{Re}\left(e^{i\theta}A_{\frac{n+1}{2}}\right)\right\|, \left\|e^{i\theta}A_k + e^{-i\theta}A^*_{n-k+1}\right\|\right\}.$$

Now, arguing in a like fashion for the remaining steps completes the proof of inequality (2.4). □

It is clear that the two-sided inequality (1.6) can be obtained as a special case of Theorem 2.5 by taking $n = 2$ in (2.3). Further, the two-sided inequality (1.4) can be viewed as a consequence of Corollary 2.3 together with Theorem 2.5 when $H_1 = H_2 = \cdots = H_n = H$.

Theorem 2.6 *Under the same assumption of the previous theorem with* $B_k = |A_k|^2 + |A^*_{n-k+1}|^2$, *we have*

$$\max_{k\in E}\{b_k\} \le 16w^4(T) \le \max_{k\in E}\{a_k\}, \quad n \text{ even}, \tag{2.5}$$

$$\max_{k\in O}\{b_k\} \le 16w^4(T) \le \max_{k\in O}\left\{32w(A_{\frac{n+1}{2}}), a_k\right\}, \quad n \text{ odd}, \tag{2.6}$$

where

$$a_k = \|B_k\|^2 + 4w^2(A_{n-k+1}A_k) + 2w(A_{n-k+1}A_kB_k + B_kA_{n-k+1}A_k),$$
$$b_k = \|B_k\|^2 + 4c^2(A_{n-k+1}A_k) + 2m(A_{n-k+1}A_kB_k + B_kA_{n-k+1}A_k),$$

and

$$c(A_{n-k+1}A_k) = \inf_{\substack{\|x\|=1\\ x\in H_{n-k+1}}} \inf_{\theta\in R}\left\|\operatorname{Re}\left(e^{i\theta}A_{n-k+1}A_k\right)x\right\|.$$

Proof Let n be an even number. Then

$$16\left\|\operatorname{Re}\left(e^{i\theta}T\right)\right\|^4 = \max_{k\in E}\left\{\left\|e^{i\theta}A_k + e^{-i\theta}A^*_{n-k+1}\right\|^4\right\}.$$

Now,

$$\begin{aligned}\left\|e^{i\theta}A_k + e^{-i\theta}A^*_{n-k+1}\right\|^4 &= \left\|B_k + 2\operatorname{Re}\left(e^{2i\theta}A_{n-k+1}A_k\right)\right\|^2\\ &= \left\|\left(B_k + 2\operatorname{Re}\left(e^{2i\theta}A_{n-k+1}A_k\right)\right)^2\right\|\\ &\le \|B_k\|^2 + 4\left\|\operatorname{Re}\left(e^{2i\theta}A_{n-k+1}A_k\right)\right\|^2\\ &\quad + 2\left\|\operatorname{Re}\left(e^{2i\theta}(A_{n-k+1}A_kB_k + B_kA_{n-k+1}A_k)\right)\right\|\\ &\le \|B_k\|^2 + 4w^2(A_{n-k+1}A_k) + 2w(A_{n-k+1}A_kB_k + B_kA_{n-k+1}A_k).\end{aligned}$$

Thus,

$$\begin{aligned}16w^4(T) &= \sup_{\theta\in\mathbb{R}} 16\left\|\operatorname{Re}\left(e^{i\theta}T\right)\right\|^4\\ &\le \max_{k\in E}\{a_k\},\end{aligned}$$

which shows the upper bound of inequality (2.5). To show the lower bound of inequality (2.5), let $\psi \in \mathbb{R}$ be such that $e^{2i\psi}\langle A_{n-s+1}A_s x, x\rangle = |\langle A_{n-s+1}A_s x, x\rangle|$ for any unit vector $x \in H_{n-s+1}$ where $s \in E$. Then

$$\begin{aligned}
16w^4(T) &\geq 16\left\|\operatorname{Re}\left(e^{i\psi}T\right)\right\|^4 \\
&\geq \left\|e^{i\psi}A_s + e^{-i\psi}A_{n-s+1}^*\right\|^4 \\
&\geq \left|\left\langle B_s^2 x, x\right\rangle + 4\left\langle \operatorname{Re}\left(e^{2i\psi}A_{n-s+1}A_s\right)^2 x, x\right\rangle \right. \\
&\quad \left. + 2e^{i\psi}\left\langle \operatorname{Re}(A_{n-s+1}A_sB_s + B_sA_{n-s+1}A_s)x, x\right\rangle\right| \\
&\geq \|B_s x\|^2 + 4c^2(A_{n-s+1}A_s) + 2m(A_{n-s+1}A_sB_s + B_sA_{n-s+1}A_s).
\end{aligned}$$

Now, applying the supremum over all the unit vectors $x \in H_{n-s+1}$ yields

$$\begin{aligned}
16w^4(T) &= \sup_{\theta\in R}\left\|\operatorname{Re}\left(e^{i\theta}T\right)\right\| \\
&\geq b_s,
\end{aligned}$$

for any $s \in E$. Thus, we have the lower bound of (2.5).

Following the same argument as above, one can easily show that the inequality (2.6) holds when n is an odd number. □

It is clear to see that the lower bound provided in inequality (2.5) is preferable over the corresponding one of (1.2). Also, the upper bound of inequality (2.5) is better than the upper bound of inequality (1.11). To justify this, we need first to recall the following lemma from [7].

Lemma 2.7 *Let $T_1, T_2 \in B(H)$ such that $\|T_2\| \leq 1$. Then $w(T_1T_2 + T_2^*T_1) \leq 2w(T)$.*

Now, by Theorem 2.6 with $n = 2$ and the last lemma, we have

$$\begin{aligned}
w\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) &\leq \frac{1}{2}\sqrt[4]{a_1} \\
&= \frac{1}{2}\sqrt[4]{\|B_1\|^2 + 4w^2(A_2A_1) + 2w\left(A_2A_1\frac{B_1}{\|B_1\|} + \frac{B_1}{\|B_1\|}A_2A_1\right)\|B_1\|} \\
&\leq \frac{1}{2}\sqrt[4]{\|B_1\|^2 + 4w^2(A_2A_1) + 4w(A_2A_1)\|B_1\|} \\
&= \frac{1}{2}\sqrt{\|B_1\| + 2w(A_2A_1)}.
\end{aligned}$$

3 Conclusion

In the current work, novel upper and lower bounds for the numerical radius of $n \times n$ off-diagonal operator matrices have been provided. The obtained numerical radius inequalities generalize several well-known related results in the literature. As an application, these numerical radius inequalities can be naturally utilized to provide new bounds for the zeros of polynomials over the complex field as in [4, 5, 10].

Lastly, it is worth mentioning here that several mathematical concepts have been recently modified in the sense of fractional calculus; see for example [6, 11, 14–16]. Our aim in the future is to extend the ideas that we have discussed here into the fractional sense.

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The authors declare that this study was accomplished in collaboration with the same responsibility. All authors read and approved the final manuscript.

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