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Growth of solutions for a coupled nonlinear Klein–Gordon system with strong damping, source, and distributed delay terms

Abdelaziz Rahmoune[1], Djamel Ouchenane[1], Salah Boulaaras[2,3*] and Praveen Agarwal[4,5,6,7]

*Correspondence:
S.Boularas@qu.edu.sa;
saleh_boulaares@yahoo.fr
[2]Department of Mathematics, College of Sciences and Arts, Al-Ras, Qassim University, Buraidah, Kingdom of Saudi Arabia
[3]Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Oran, Algeria
Full list of author information is available at the end of the article

Abstract

In this work, the exponential growth of solutions for a coupled nonlinear Klein–Gordon system with distributed delay, strong damping, and source terms is proved. Take into consideration some suitable assumptions.

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1 Introduction

In modeling in the biological, physical, and social sciences, it is sometimes necessary to take account of optimal control or time delays inherent in the phenomena (see for example [4, 16]). The inclusion of delays explicitly in the equations is often a simplification or idealization that is introduced because a detailed description of the underlying processes is too complicated to be modeled mathematically, or because some of the details are unknown. More generally, how does the qualitative behavior depend on the form and magnitude of the delays? In this paper we examine how we can apply the distributed delay term for knowing the behavior of growth of solutions for a coupled nonlinear Klein–Gordon system with strong damping, source terms.

We consider the following system:

$$\begin{cases} u_{tt} + m_1 u^2 - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)\,ds \\ \quad + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x,t-\varrho)\,d\varrho = f_1(u,v), \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ v_{tt} + m_2 v^2 - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)\,ds \\ \quad + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x,t-\varrho)\,d\varrho = f_2(u,v), \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ u(x,t) = 0, \qquad v(x,t) = 0, \quad x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), \qquad v_t(x,-t) = k_0(x,t) \quad (x,t) \in \Omega \times (0,\tau_2), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and the source terms are defined as follows:

$$\begin{cases} f_1(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|u|^p.u.|v|^{p+2}, \\ f_2(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|v|^p.v.|u|^{p+2} \end{cases} \tag{1.2}$$

and $m_1, m_2, \omega_1, \omega_2, \mu_1, \mu_3, a_1, b_1 > 0$, and τ_1, τ_2 are the time delay with $0 \le \tau_1 < \tau_2$, and μ_2, μ_4 are L^∞ functions, and g, h are differentiable functions.

Viscous materials are the opposite of elastic materials that possess the ability to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of natural sciences, many authors have given attention to this problem since the beginning of the new millennium.

In the case of only one equation and if $\omega_1 = 0$ (i.e., $\Delta u_t = 0$), and $\mu_1 = \mu_2 = 0$. Our problem (1.1) has been studied in [7]. By using the Galerkin method they established the local existence result. Also, they showed the local solution is global in time under suitable conditions and with the same rate of decaying (polynomial or exponential) of the kernel g. They proved that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution. Moreover, their result has been obtained under weaker conditions than those used in [11]. In [12], the authors proved the exponential decay of the following problem:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + a(x)u_t + |u|^\gamma.u = 0. \tag{1.3}$$

This later result has been improved in [7], in which they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate.

In many works on this field under assumptions of the kernel g. For problem (1.1) and with $\mu_1 \neq 0$, for example, in [18], the authors proved a blow-up result for the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)\,ds + u_t = |u|^{p-2}.u, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \end{cases} \tag{1.4}$$

where g satisfies $\int_0^\infty g(s)\,ds < (2p-4)/(2p-3)$, initial data were supported with negative energy like that $\int u_0 u_1\,dx > 0$.

If $(w > 0)$. In [29], the authors considered the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)\,ds - \Delta u_t = |u|^{p-2}.u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases} \tag{1.5}$$

Under suitable assumptions on g that there were solutions of (1.5) with initial energy, they showed the blow-up in a finite time. For the same problem (1.5), in [30], Song et al. proved that there were solutions of (1.5) with positive initial energy that blows up in finite time. In addition, in [19] the authors showed a blow-up result if $p > m$ and established the global existence of the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)\,ds - \varepsilon_1 \Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1.6}$$

In the case of coupled of equations, in [2], the authors studied the following system of equations:

$$\begin{cases} u_{tt} - \Delta u + u_t|u_t|^{m-2} = f_1(u,v), \\ v_{tt} - \Delta v + v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{1.7}$$

with nonlinear functions f_1 and f_2 satisfying appropriate conditions. Under certain restrictions imposed on the parameters and the initial data, they obtained numerous results on the existence of weak solutions. They also showed that any weak solution with negative initial energy blows up for a finite period of time by using the same techniques as in [17].

In [6], the authors considered the system:

$$\begin{cases} u_{tt} - \Delta u + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u,v), \\ v_{tt} - \Delta v + (a|u|^\theta + b|v|^\vartheta)v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{1.8}$$

where they stated and proved the blow-up in finite time of solution under some restrictions on the initial data and (with positive initial energy) for some conditions on the functions f_1 and f_2.

Later, in [23], the authors extended the result of [6], where they considered the following nonlinear viscoelastic system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)\,ds + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^\infty h(s)\Delta v(t-s)\,ds + (a|u|^\theta + b|v|^\varrho)v_t|v_t|^{r-2} = f_2(u,v) \end{cases} \tag{1.9}$$

and proved that the solutions of the system of wave equations with viscoelastic term, degenerate damping, and strong nonlinear sources acting in both equations at the same time are globally nonexisting provided that the initial data are sufficiently large in a bounded domain of Ω.

To complement the above works, we are working to prove under appropriate assumptions that the solution of problem (1.1) grows exponentially:

$$\lim_{t\to\infty} \|u_t\|_{2(p+2)}^{2(p+2)} + \|\nabla u\|_{2(p+2)}^{2(p+2)} \quad \text{goes to } \infty. \tag{1.10}$$

The paper is organized as follows. In Sect. 2, some necessary assumptions related to the problem are given. Then, in Sect. 3, the main result is proved.

2 Assumptions

We consider the following assumptions:

(A1) $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable and decreasing functions such that

$$\begin{cases} g(t) \geq 0, & 1 - \int_0^\infty g(s)\,ds = l_1 > 0, \\ h(t) \geq 0, & 1 - \int_0^\infty h(s)\,ds = l_2 > 0. \end{cases} \tag{2.1}$$

(A2) There exist constants $\xi_1, \xi_2 > 0$ such that

$$\begin{cases} g'(t) \leq -\xi_1 g(t), & t \geq 0, \\ h'(t) \leq -\xi_2 h(t), & t \geq 0. \end{cases} \tag{2.2}$$

(A3) $\mu_2, \mu_4 : [\tau_1, \tau_2] \to \mathbb{R}$ are L^∞ functions so that, for all $\delta > \frac{1}{2}$,

$$\begin{aligned} &\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} \left|\mu_2(\varrho)\right| d\varrho < \mu_1, \\ &\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} \left|\mu_4(\varrho)\right| d\varrho < \mu_3. \end{aligned} \tag{2.3}$$

3 Main results

In this section, the blow-up result of solution of problem (1.1) is proved.

First, as in [22], we introduce the new variables:

$$\begin{aligned} &y(x, \rho, \varrho, t) = u_t(x, t - \varrho\rho), \\ &z(x, \rho, \varrho, t) = v_t(x, t - \varrho\rho), \end{aligned}$$

then

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ y(x, 0, \varrho, t) = u_t(x, t), \end{cases} \tag{3.1}$$

and

$$\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ z(x, 0, \varrho, t) = v_t(x, t). \end{cases} \tag{3.2}$$

Let us denote

$$gou = \int_\Omega \int_0^t g(t - s)\left|u(t) - u(s)\right|^2 ds\,dx. \tag{3.3}$$

Therefore, problem (1.1) takes the form

$$\begin{cases} u_{tt} + m_1 u^2 - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)\,ds \\ \quad + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)\,d\varrho = f_1(u,v), \quad x \in \Omega, t \geq 0, \\ v_{tt} + m_2 v^2 - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)\,ds \\ \quad + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z(x,1,\varrho,t)\,d\varrho = f_2(u,v), \quad x \in \Omega, t \geq 0, \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \\ \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0 \end{cases} \tag{3.4}$$

with the initial and boundary condition

$$\begin{cases} u(x,t) = 0, \qquad v(x,t) = 0, \quad x \in \partial\Omega, \\ y(x,\rho,\varrho,0) = f_0(x,\varrho\rho), \qquad z(x,\rho,\varrho,0) = k_0(x,\varrho\rho), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \end{cases} \tag{3.5}$$

where

$$(x,\rho,\varrho,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

Theorem 3.1 *Assume* (2.1), (2.2), *and* (2.3) *hold. Let*

$$\begin{cases} -1 < p < \frac{4-n}{n-2}, \quad n \geq 3, \\ p \geq -1, \qquad n = 1,2. \end{cases} \tag{3.6}$$

Then, for any initial data,

$$(u_0, u_1, v_0, v_1, f_0, k_0) \in \mathcal{H},$$

where

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2\big(\Omega \times (0,1) \times (\tau_1,\tau_2)\big) \\ \times L^2\big(\Omega \times (0,1) \times (\tau_1,\tau_2)\big),$$

problem (3.4) *has a unique solution*

$$u \in C\big([0,T];\mathcal{H}\big)$$

for some $T > 0$.

In the next theorem we give the global existence result, its proof is based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time. We will make use of arguments in [28].

Theorem 3.2 *Suppose that* (2.1), (2.2), (2.3), *and* (3.6) *hold. If* $u_0, v_0 \in W$, $u_1, v_1 \in H_0^1(\Omega)$, $y, z \in L^2(\Omega \times (0,1) \times (\tau_1, \tau_2))$, *and*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{3.7}$$

where C_* *is the best Poincare constant, then the local solution* (u, v, y, z) *is global in time.*

In order to achieve the main result, the following lemmas are needed.

Lemma 3.1 *There exists a function* $F(u, v)$ *such that*

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)}\left[uf_1(u,v) + vf_2(u,v)\right] \\ &= \frac{1}{2(\rho+2)}\left[a_1|u+v|^{2(p+2)} + 2b_1|uv|^{p+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \qquad \frac{\partial F}{\partial v} = f_2(u,v),$$

taking $a_1 = b_1 = 1$ for convenience.

Lemma 3.2 ([23]) *There exist two positive constants* c_0 *and* c_1 *such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(p+2)}\right). \tag{3.8}$$

Define the energy functional as follows.

Lemma 3.3 *Assume that* (2.1), (2.2), (2.3), *and* (3.6) *hold, let* (u, v, y, z) *be a solution of* (3.4), *then* $E(t)$ *is nonincreasing, that is,*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 + \frac{m_1}{2}\|u\|_2^2 + \frac{m_2}{2}\|v\|_2^2 + \frac{1}{2}l_1\|\nabla u\|_2^2 + \frac{1}{2}l_2\|\nabla v\|_2^2 \\ &\quad + \frac{1}{2}(go\nabla u) + \frac{1}{2}(ho\nabla v) + \frac{1}{2}K(y,z) - \int_\Omega F(u,v)\,dx \end{aligned} \tag{3.9}$$

satisfies

$$\begin{aligned} E'(t) \leq -c_3\bigg\{&\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(\varrho)\right|y^2(x,1,\varrho,t)\,d\varrho\,dx \\ &+ \int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_4(\varrho)\right|z^2(x,1,\varrho,t)\,d\varrho\,dx\bigg\} \leq 0, \end{aligned} \tag{3.10}$$

where

$$K(y,z) = \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho\left\{\left|\mu_2(\varrho)\right|y^2(x,\rho,\varrho,t) + \left|\mu_4(\varrho)\right|z^2(x,\rho,\varrho,t)\right\}d\varrho\,d\rho\,dx. \tag{3.11}$$

Proof By multiplying the first and the second equation in (3.4) respectively by u_t, v_t and integrating over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|v_t\|_2^2+\frac{m_1}{2}\|u\|_2^2+\frac{m_2}{2}\|v\|_2^2+\frac{1}{2}l_1\|\nabla u\|_2^2+\frac{1}{2}l_2\|\nabla v\|_2^2+\frac{1}{2}(go\nabla u)\right.\\
&\quad\left.+\frac{1}{2}(ho\nabla v)-\int_\Omega F(u,v)\,dx\right\}\\
&\quad=-\mu_1\|u_t\|_2^2-m_1\|u\|_2^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}\bigl|\mu_2(\varrho)\bigr|y(x,1,\varrho,t)\,d\varrho\,dx\\
&\quad\quad-\mu_3\|v_t\|_2^2-m_2\|v\|_2^2-\int_\Omega v_t\int_{\tau_1}^{\tau_2}\bigl|\mu_4(\varrho)\bigr|z(x,1,\varrho,t)\,d\varrho\,dx\\
&\quad\quad+\frac{1}{2}\bigl(g'o\nabla u\bigr)-\frac{1}{2}g(t)\|\nabla u\|_2^2-\omega_1\|\nabla u_t\|_2^2\\
&\quad\quad+\frac{1}{2}\bigl(h'o\nabla v\bigr)-\frac{1}{2}h(t)\|\nabla v\|_2^2-\omega_2\|\nabla v_t\|_2^2,
\end{aligned}\tag{3.12}$$

and, from the initial and boundary condition in (3.4)

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho\bigl|\mu_2(\varrho)\bigr|y^2(x,\rho,\varrho,t)\,d\varrho\,d\rho\,dx\\
&\quad=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2\bigl|\mu_2(\varrho)\bigr|yy_\rho\,d\varrho\,d\rho\,dx\\
&\quad=\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\bigl|\mu_2(\varrho)\bigr|y^2(x,0,\varrho,t)\,d\varrho\,dx\\
&\quad\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\bigl|\mu_2(\varrho)\bigr|y^2(x,1,\varrho,t)\,d\varrho\,dx\\
&\quad=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)\,d\varrho\right)\|u_t\|_2^2\\
&\quad\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\bigl|\mu_2(\varrho)\bigr|y^2(x,1,\varrho,t)\,d\varrho\,dx
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho\bigl|\mu_4(\varrho)\bigr|z^2(x,\rho,\varrho,t)\,d\varrho\,d\rho\,dx\\
&\quad=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2\bigl|\mu_4(\varrho)\bigr|zz_\rho\,d\varrho\,d\rho\,dx\\
&\quad=\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\bigl|\mu_4(\varrho)\bigr|z^2(x,0,\varrho,t)\,d\varrho\,dx\\
&\quad\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\bigl|\mu_4(\varrho)\bigr|z^2(x,1,\varrho,t)\,d\varrho\,dx\\
&\quad=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)\,d\varrho\right)\|v_t\|_2^2\\
&\quad\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\bigl|\mu_4(\varrho)\bigr|z^2(x,1,\varrho,t)\,d\varrho\,dx,
\end{aligned}\tag{3.14}$$

then

$$\begin{aligned}\frac{d}{dt}E(t) &= -\mu_1\|u_t\|_2^2 - m_1\|u\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(\varrho)\big|u_t y(x,1,\varrho,t)\,d\varrho\,dx + \frac{1}{2}\big(g'o\nabla u\big)\\ &\quad - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega_1\|\nabla u_t\|_2^2 + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)\,d\varrho\right)\|u_t\|_2^2\\ &\quad - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(\varrho)\big|y^2(x,1,\varrho,t)\,d\varrho\,dx\\ &\quad - \mu_3\|v_t\|_2^2 - m_2\|v\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_4(\varrho)\big|v_t z(x,1,\varrho,t)\,d\varrho\,dx + \frac{1}{2}\big(h'o\nabla v\big)\\ &\quad - \frac{1}{2}h(t)\|\nabla v\|_2^2 - \omega_2\|\nabla v_t\|_2^2 + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)\,d\varrho\right)\|v_t\|_2^2\\ &\quad - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_4(\varrho)\big|z^2(x,1,\varrho,t)\,d\varrho\,dx. \end{aligned} \tag{3.15}$$

By (3.12)–(3.14), we get (3.9). Also, by using Young's inequality, (2.1), (2.2), and (2.3) in (3.15), we obtain (3.10). □

Now, we define the functional

$$\begin{aligned}\mathbb{H}(t) = -E(t) &= -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|v_t\|_2^2 - \frac{m_1}{2}\|u\|_2^2 - \frac{m_2}{2}\|v\|_2^2 - \frac{1}{2}l_1\|\nabla u\|_2^2\\ &\quad - \frac{1}{2}l_2\|\nabla v\|_2^2 - \frac{1}{2}(go\nabla u) - \frac{1}{2}(ho\nabla v) - \frac{1}{2}K(y,z)\\ &\quad + \frac{1}{2(p+2)}\big[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\big].\end{aligned} \tag{3.16}$$

Theorem 3.3 *Assume that (2.1)–(2.3) and (3.6) hold. Assume further that $E(0) < 0$, then the solution of problem (3.4) grows exponentially.*

Proof From (3.9 we have

$$E(t) \le E(0) \le 0. \tag{3.17}$$

Therefore,

$$\begin{aligned}\mathbb{H}'(t) &= -E'(t)\\ &\ge c_3\bigg(\|u_t\|_2^2 + \|u\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(\varrho)\big|y^2(x,1,\varrho,t)\,d\varrho\,dx\\ &\quad + \|v_t\|_2^2 + \|v\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_4(\varrho)\big|z^2(x,1,\varrho,t)\,d\varrho\,dx\bigg).\end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned}&\mathbb{H}'(t) \ge c_3\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(\varrho)\big|y^2(x,1,\varrho,t)\,d\varrho\,dx \ge 0,\\ &\mathbb{H}'(t) \ge c_3\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(\varrho)\big|z^2(x,1,\varrho,t)\,d\varrho\,dx \ge 0,\end{aligned} \tag{3.19}$$

and

$$0 \le \mathbb{H}(0) \le \mathbb{H}(t) \le \frac{1}{2(p+2)}\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right] \\ \le \frac{c_1}{2(p+2)}\left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right]. \tag{3.20}$$

Setting

$$\mathcal{K}(t) = \mathbb{H} + \varepsilon \int_\Omega (uu_t + vv_t)\,dx + \frac{\varepsilon}{2}\int_\Omega \left(\mu_1 u^2 + \mu_3 v^2\right)dx \\ + \frac{\varepsilon}{2}\int_\Omega \left(\omega_1(\nabla u)^2 + \omega_2(\nabla v)^2\right)dx, \tag{3.21}$$

where $\varepsilon > 0$ to be assigned later.

By multiplying the first and second equation on (3.4) respectively by u, v and with a derivative of (3.21), we get

$$\mathcal{K}'(t) = \mathbb{H}'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\right) - \varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ + \varepsilon\int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s)\,ds\,dx + \varepsilon\int_\Omega \nabla v\int_0^t h(t-s)\nabla v(s)\,ds\,dx \\ - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(\varrho)\right| uy(x,1,\varrho,t)\,d\varrho\,dx - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_4(\varrho)\right| vz(x,1,\varrho,t)\,d\varrho\,dx \\ + \varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right]. \tag{3.22}$$

Using Young's inequality, we get

$$\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(\varrho)\right| uy(x,1,\varrho,t)\,d\varrho\,dx \\ \le \varepsilon\left\{\delta_1\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(\varrho)\right|d\varrho\right)\|u\|_2^2 \right. \\ \left. + \frac{1}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(\varrho)\right| y^2(x,1,\varrho,t)\,d\varrho\,dx\right\}. \tag{3.23}$$

Thus

$$\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_4(\varrho)\right| vz(x,1,\varrho,t)\,d\varrho\,dx \\ \le \varepsilon\left\{\delta_2\left(\int_{\tau_1}^{\tau_2}\left|\mu_4(\varrho)\right|d\varrho\right)\|v\|_2^2 \right. \\ \left. + \frac{1}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_4(\varrho)\right| z^2(x,1,\varrho,t)\,d\varrho\,dx\right\}. \tag{3.24}$$

Also

$$\varepsilon\int_0^t g(t-s)\,ds\int_\Omega \nabla u.\nabla u(s)\,dx\,ds$$

$$
\begin{aligned}
&= \varepsilon \int_0^t g(t-s)\,ds \int_\Omega \nabla u.\big(\nabla u(s) - \nabla u(t)\big)\,dx\,ds \\
&\quad + \varepsilon \int_0^t g(s)\,ds \|\nabla u\|_2^2 \\
&\geq \frac{\varepsilon}{2} \int_0^t g(s)\,ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u),
\end{aligned} \tag{3.25}
$$

so

$$
\begin{aligned}
&\varepsilon \int_0^t h(t-s)\,ds \int_\Omega \nabla v.\nabla v(s)\,dx\,ds \\
&\quad= \varepsilon \int_0^t h(t-s)\,ds \int_\Omega \nabla v.\big(\nabla v(s) - \nabla v(t)\big)\,dx\,ds \\
&\qquad + \varepsilon \int_0^t h(s)\,ds \|\nabla v\|_2^2 \\
&\quad\geq \frac{\varepsilon}{2} \int_0^t h(s)\,ds \|\nabla v\|_2^2 - \frac{\varepsilon}{2}(ho\nabla v).
\end{aligned} \tag{3.26}
$$

From (3.22)

$$
\begin{aligned}
\mathcal{K}'(t) \geq{}& \mathbb{H}'(t) + \varepsilon\big(\|u_t\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\big) \\
& - \varepsilon\left(\left(1 - \frac{1}{2}\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1 - \frac{1}{2}\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2\right) \\
& - \varepsilon\delta_1\left(\int_{\tau_1}^{\tau_2} \big|\mu_2(\varrho)\big|\,d\varrho\right)\|u\|_2^2 - \varepsilon\delta_2\left(\int_{\tau_1}^{\tau_2} \big|\mu_4(\varrho)\big|\,d\varrho\right)\|v\|_2^2 \\
& - \frac{\varepsilon}{2}(go\nabla u) - \frac{\varepsilon}{4\delta_1}\int_\Omega \int_{\tau_1}^{\tau_2} \big|\mu_2(\varrho)\big| y^2(x,1,\varrho,t)\,d\varrho\,dx \\
& - \frac{\varepsilon}{2}(ho\nabla v) - \frac{\varepsilon}{4\delta_2}\int_\Omega \int_{\tau_1}^{\tau_2} \big|\mu_4(\varrho)\big| z^2(x,1,\varrho,t)\,d\varrho\,dx \\
& + \varepsilon\big[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\big].
\end{aligned} \tag{3.27}
$$

Therefore, using (3.19) and by setting δ_1, δ_1 so that $\frac{1}{4\delta_1 c_3} = \frac{\kappa}{2}$ and $\frac{1}{4\delta_2 c_3} = \frac{\kappa}{2}$, substituting in (3.27), we get

$$
\begin{aligned}
\mathcal{K}'(t) \geq{}& [1 - \varepsilon\kappa]\mathbb{H}'(t) + \varepsilon\big(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\big) \\
& - \varepsilon\left[\left(1 - \frac{1}{2}\int_0^t g(s)\,ds\right)\right]\|\nabla u\|_2^2 - \varepsilon\left[\left(1 - \frac{1}{2}\int_0^t h(s)\,ds\right)\right]\|\nabla v\|_2^2 \\
& - \varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2} \big|\mu_2(\varrho)\big|\,d\varrho\right)\|u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u) \\
& - \varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2} \big|\mu_4(\varrho)\big|\,d\varrho\right)\|v\|_2^2 - \frac{\varepsilon}{2}(ho\nabla v) \\
& + \varepsilon\big[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\big].
\end{aligned} \tag{3.28}
$$

For $0 < a < 1$, from (3.16)

$$\begin{aligned}
\varepsilon\big[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\big] &= \varepsilon a\big[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\big] \\
&\quad + \varepsilon 2(p+2)(1-a)\mathbb{H}(t) \\
&\quad + \varepsilon(p+2)(1-a)\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) \\
&\quad + \varepsilon(p+2)(1-a)\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 \\
&\quad + \varepsilon(p+2)(1-a)\left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\
&\quad - \varepsilon(p+2)(1-a)(go\nabla u) \\
&\quad - \varepsilon(p+2)(1-a)(ho\nabla v) \\
&\quad + \varepsilon(p+2)(1-a)K(y,z).
\end{aligned} \tag{3.29}$$

Substituting in (3.28), we get

$$\begin{aligned}
\mathcal{K}'(t) &\geq [1-\varepsilon\kappa]\mathbb{H}'(t) + \varepsilon\big[(p+2)(1-a)+1\big]\big(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\big) \\
&\quad + \varepsilon\left[(p+2)(1-a)\left(1-\int_0^t g(s)\,ds\right) - \left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\right]\|\nabla u\|_2^2 \\
&\quad + \varepsilon\left[(p+2)(1-a)\left(1-\int_0^t h(s)\,ds\right) - \left(1-\frac{1}{2}\int_0^t h(s)\,ds\right)\right]\|\nabla v\|_2^2 \\
&\quad - \varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}\big|\mu_2(\varrho)\big|\,d\varrho\right)\|u\|_2^2 - \varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}\big|\mu_4(\varrho)\big|\,d\varrho\right)\|v\|_2^2 \\
&\quad + \varepsilon(p+2)(1-a)K(y,z) + \varepsilon\left[(p+2)(1-a) - \frac{1}{2}\right](go\nabla u + ho\nabla v) \\
&\quad + \varepsilon a\big[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\big] + \varepsilon 2(p+2)(1-a)\mathbb{H}(t).
\end{aligned} \tag{3.30}$$

Using Poincare's inequality, we obtain

$$\begin{aligned}
\mathcal{K}'(t) &\geq [1-\varepsilon\kappa]\mathbb{H}'(t) + \varepsilon\big[(p+2)(1-a)+1\big]\big(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\big) \\
&\quad + \varepsilon\left\{\big[(p+2)(1-a)-1\big] - \left(\int_0^t g(s)\,ds\right)\left[(p+2)(1-a) - \frac{1}{2}\right]\right. \\
&\quad \left. - \frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}\big|\mu_2(s)\big|\,ds\right)\right\}\|\nabla u\|_2^2 \\
&\quad + \varepsilon\left\{\big[(p+2)(1-a)-1\big] - \left(\int_0^t h(s)\,ds\right)\left[(p+2)(1-a) - \frac{1}{2}\right]\right. \\
&\quad \left. - \frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}\big|\mu_4(s)\big|\,ds\right)\right\}\|\nabla v\|_2^2 \\
&\quad + \varepsilon(p+2)(1-a)K(y,z) + \varepsilon\left[(p+2)(1-a) - \frac{1}{2}\right](go\nabla u + ho\nabla v) \\
&\quad + \varepsilon c_0 a\big[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\big] \\
&\quad + \varepsilon 2(p+2)(1-a)\mathbb{H}(t).
\end{aligned} \tag{3.31}$$

In this stage, we take $a > 0$ small enough so that

$$\alpha_1 = (p+2)(1-a) - 1 > 0,$$

and we assume

$$\max\left\{\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right\} < \frac{(p+2)(1-a)-1}{((p+2)(1-a)-\frac{1}{2})} = \frac{2\alpha_1}{2\alpha_1+1}. \tag{3.32}$$

Then we choose κ so large that

$$\begin{aligned}\alpha_2 &= \left\{(p+2)(1-a)-1) - \int_0^t g(s)\,ds\left((p+2)(1-a)-\frac{1}{2}\right)\right.\\ &\quad\left. - \frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds\right)\right\} > 0,\\ \alpha_3 &= \left\{(p+2)(1-a)-1) - \int_0^t h(s)\,ds\left((p+2)(1-a)-\frac{1}{2}\right)\right.\\ &\quad\left. - \frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}\left|\mu_4(s)\right|ds\right)\right\} > 0.\end{aligned}$$

We fixed κ and a, we appoint ε small enough so that

$$\alpha_4 = 1 - \varepsilon\kappa > 0,$$

and from (3.21) we get

$$\begin{aligned}\mathcal{K}(t) &\leq \frac{1}{2(p+2)}\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right]\\ &\leq \frac{c_1}{2(p+2)}\left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right].\end{aligned} \tag{3.33}$$

Thus, for some $\beta > 0$, estimate (3.31) becomes

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta\left\{\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2\right.\\ &\quad + (go\nabla u) + (ho\nabla v) + K(y,z)\\ &\quad \left.+ \left[\|u\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}\right]\right\}.\end{aligned} \tag{3.34}$$

By (3.8), for some $\beta_1 > 0$,

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta_1\left\{\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2\right.\\ &\quad + (go\nabla u) + (ho\nabla v) + K(y,z)\\ &\quad \left.+ \left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right]\right\}\end{aligned} \tag{3.35}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{3.36}$$

Next, using Young's and Poincare's inequalities ([12]), thus from (3.21), we have

$$\begin{aligned}
\mathcal{K}(t) &= \biggl(\mathbb{H}^{1-\alpha} + \varepsilon\int_\Omega (uu_t + vv_t)\,dx + \frac{\varepsilon}{2}\int_\Omega \bigl(\mu_1 u^2 + \mu_3 v^2\bigr)\,dx \\
&\quad + \frac{\varepsilon}{2}\int_\Omega \bigl(\omega_1 \nabla u^2 + \omega_2 \nabla v^2\bigr)\,dx\biggr) \\
&\le c\biggl\{\mathbb{H}(t) + \biggl|\int_\Omega (uu_t + vv_t)\,dx\biggr| + \|u\|_2 + \|\nabla u\|_2 \\
&\quad + \|v\|_2 + \|\nabla v\|_2\biggr\} \\
&\le c[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\
&\le c\bigl[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (go\nabla u) \\
&\quad + (ho\nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\bigr]
\end{aligned} \tag{3.37}$$

for some $c > 0$. From inequalities (3.34) and (3.37) we obtain the differential inequality

$$\mathcal{K}'(t) \ge \lambda\mathcal{K}(t), \tag{3.38}$$

where $\lambda > 0$, depending only on β and c.

A simple integration of (3.38) gives

$$\mathcal{K}(t) \ge \mathcal{K}(0)e^{(\lambda t)} \quad \text{for any } t > 0. \tag{3.39}$$

From (3.21) and (3.33), then

$$\mathcal{K}(t) \le \frac{c_1}{2(p+2)}\bigl[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\bigr]. \tag{3.40}$$

By (3.39) and (3.40) we have

$$\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \ge Ce^{(\lambda t)}, \quad \forall t > 0.$$

Therefore, we conclude that the solution grows exponentially. This completes the proof. □

4 Conclusion

In this work, the growth of solutions for a coupled nonlinear Klein–Gordon system with strong damping, source, and distributed delay terms was studied. Next, motivated by last works in [1, 3, 5, 8–10, 13–15, 20, 21, 24–27, 31], and [16], we obtained the growth and blow-up for the studied problem (1.1) by constructing a type of cross-constrained variational problem and establishing so-called cross-invariant manifolds of the evolution flow. Then, the result of how small the initial data for which the solution exists globally was proved by using the scaling argument.

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Author details
[1]Laboratory of Pure and Applied Mathematics, Amar Teledji Laghouat University, Laghouat, Algeria. [2]Department of Mathematics, College of Sciences and Arts, Al-Ras, Qassim University, Buraidah, Kingdom of Saudi Arabia. [3]Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Oran, Algeria. [4]Department of Mathematics, Anand International College of Engineering, Jaipur, India. [5]International Center for Basic and Applied Sciences, Jaipur, 302029, India. [6]Department of Mathematics, Harish-Chandra Research Institute, Allahabad, 211019, India. [7]Department of Mathematics, Netaji Subhas University of Technology Dwarka Sector-3, Dwarka, Delhi, 110078, India.

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