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An exponential-trigonometric spline minimizing a seminorm in a Hilbert space

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Abstract

In the present paper, using the discrete analogue of the operator $d^6/dx^6 - 1$, we construct an interpolation spline that minimizes the quantity $\int_0^1 (\varphi'''(x) + \varphi(x))^2 dx$ in the Hilbert space $W_2^{(3,0)}$. We obtain explicit formulas for the coefficients of the interpolation spline. The obtained interpolation spline is exact for the exponential-trigonometric functions e^{-x} , $e^{\frac{x}{2}} \cos(\frac{\sqrt{3}}{2}x)$, and $e^{\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x)$.

Keywords: Interpolation spline; Hilbert space; The norm minimizing property; Sobolev's method; Discrete argument function

1 Introduction. Statement of the problem

One of the problems of approximation is the interpolation problem. The classical method of its solution consists of construction of an interpolation polynomial. However, it is known that the polynomial approximation is unpractical for approximating functions with finite and small smoothness, which often occurs in applications. Therefore, in practice, to approximate functions, splines are used. There are algebraic and variational approaches in the theory of splines [1]. In the algebraic approach, splines are considered as some smooth piecewise polynomial functions. In the variational approach, splines are understood as elements of a Hilbert or Banach space minimizing certain functionals. Then problems of existence, uniqueness, and convergence of splines and algorithms for their constructions are studied based on intrinsic properties of splines (see, e.g., [22]).

L -(generalized) splines are a generalization of polynomial splines. Since the results of this paper are connected with L -splines, we give some definitions following Ahlberg, Nilson, and Walsh [1].

Let L be a linear differential operator given by

$$L \equiv a_m(x)D^m + a_{m-1}(x)D^{m-1} + \dots + a_0(x),$$

where $D = d/dx$, each $a_j(x)$ ($j = 0, 1, \dots, m$) is in $C^m[a, b]$, and $a_m(x) \neq 0$ on $[a, b]$. By L^* we denote the formal adjoint of L :

$$L^* \equiv (-1)^m D^m \{a_m(x)\} + (-1)^{m-1} D^{m-1} \{a_{m-1}(x)\} + \dots - D \{a_1(x)\} + a_0(x).$$

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By $K^m(a, b)$ we denote the class of all functions $f(x)$ defined on $[a, b]$ that possess absolutely continuous $(n - 1)$ th derivatives on $[a, b]$ and have the n th derivatives in $L_2(a, b)$; it is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_a^b (Lf)(x) \cdot (Lg)(x) \, dx \tag{1}$$

if functions that differ by a solution of the equation $Lf = 0$ are identified.

If $\Delta : a = x_0 < x_1 < \dots < x_N = b$ is a mesh on $[a, b]$, then a generalized spline of deficiency k ($0 \leq k \leq m$) with respect to Δ is a function $S_\Delta(x)$ from $K^{2m-k}(a, b)$ that satisfies the differential equation

$$L^*LS_\Delta = 0$$

on each open mesh interval (x_{i-1}, x_i) ($i = 1, 2, \dots, N$) of Δ . The ordinary spline of deficiency 1 allows discontinuities in the $(2m - 1)$ th derivative, but only at mesh points.

From the results of Ahlberg, Nilson, and Walsh [1] it is known that for generalized spline of deficiency 1, the following is true: Let $\Delta : a = x_0 < x_1 < \dots < x_N = b$ and $Y = \{y_i, i = 0, 1, \dots, N\}$ be given. Then among all functions $f(x)$ in $K^m(a, b)$ such that $f(x_i) = y_i$ ($i = 0, 1, \dots, N$), the generalized spline $S_\Delta(Y; x)$, when it exists, minimizes the quantity $\int_a^b (Lf(x))^2 \, dx$.

Further, we give some results obtained from the theory of L -splines. First, contributions to the theory of splines include the works of Greville, Ahlberg, Nilson, and Walsh and of Schultz and Varga (see [22, p. 459]). These works are concentrated on natural L -splines, which appear as solutions to the corresponding best interpolation problems, and the order of approximation of generalized splines was first studied.

Trigonometric splines were first considered by Schoenberg [20]. Construction of local B -splines for general spaces of L -splines was studied in [9, 14]. Further results on the approximation of trigonometric splines were obtained in [16] based on construction of quasi-interpolation operators. In [12], control curves of trigonometric splines are constructed, and it is proved that they have similar properties as polynomial splines. Then trigonometric B -splines were studied in [8, 10, 19].

The space of natural hyperbolic splines was introduced in [22, p. 407]. In [21], it is shown that hyperbolic splines can be treated as an example of L -splines.

There are different variants of hyperbolic and trigonometric L -splines. Recently, several classes of splines whose parts are combinations of algebraic polynomials and exponential or trigonometric functions are considered. Splines whose parts are spanned by $\{1, x, \exp(\rho x), \exp(-\rho x)\}$ are called tension exponential splines. Such splines were studied in [11, 17–19]. Algebraic and exponential B -splines generated over the space spanned by $\{1, x, \dots, x^{m-3}, \cosh x, \sinh x\}$ were presented in [15]. Algebraic-trigonometric splines generated by $\{1, x, \sin x, \cos x\}$ were studied in [31, 32]. Similar results were developed in [13, 30] for splines obtained by combinations of functions $\{1, x, \dots, x^{m-3}, \cos x, \sin x\}$. In [5], using Sobolev’s method, interpolation splines that minimize the expression $\int_0^1 (\varphi^{(m)} + \omega^2 \varphi^{(m-2)}(x))^2 \, dx$ in the space $K_2(P_m)$ are constructed. Explicit formulas for the coefficients of the interpolation splines are obtained. The obtained interpolation splines are exact for monomials $1, x, x^2, \dots, x^{m-3}$ and for trigonometric functions $\sin \omega x$ and $\cos \omega x$.

In the present work, we solve the problem of construction of a natural L -spline in the case $m = 3$, $L \equiv d^3/dx^3 + 1$ on the interval $[0, 1]$, and we denote by $W_2^{(3,0)}(0, 1)$ the corresponding space, that is,

$$W_2^{(3,0)}(0, 1) = \{ \varphi : [0, 1] \rightarrow R \mid \varphi'' \text{ is absolutely continuous, and } \varphi''' \in L_2(0, 1) \}.$$

The class $W_2^{(3,0)}$, equipped with the seminorm

$$\| \varphi \|_{W_2^{(3,0)}} = \left(\int_0^1 (\varphi'''(x) + \varphi(x))^2 dx \right)^{1/2}, \tag{2}$$

is a Hilbert space if we identify functions that differ by a solution of the equation $\varphi'''(x) + \varphi(x) = 0$. A solution of the last equation has the form $\varphi(x) = d_0 e^{-x} + d_1 e^{\frac{x}{2}} \cdot \cos(\frac{\sqrt{3}}{2}x) + d_2 e^{\frac{x}{2}} \cdot \sin(\frac{\sqrt{3}}{2}x)$.

Suppose we are given the values y_β , $\beta = 0, 1, \dots, N$, at points $x_\beta \in [0, 1]$, $\beta = 0, 1, \dots, N$. Consider the following variational interpolation problem.

Problem 1 Among all functions $f(x)$ in $W_2^{(3,0)}$ satisfying the conditions

$$f(x_\beta) = \varphi(x_\beta), \quad \beta = 0, 1, \dots, N, \tag{3}$$

find a function $S(x)$ that gives the minimum of the norm (2), where $x_\beta \in [0, 1]$ are the nodes of interpolation, and $\varphi(x_\beta) = y_\beta$ are given values.

The solution $S(x)$ of Problem 1 is a ordinary generalized spline and is uniquely defined with respect to mesh $\Delta : 0 = x_0 < x_1 < \dots < x_N = 1$ on the interval $[0, 1]$ as follows:

- (i) $S(x)$ is a linear combination of functions e^{-x} , $e^{\frac{x}{2}} \cos(\frac{\sqrt{3}}{2}x)$, $e^{\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x)$, $e^{-\frac{x}{2}} \cos(\frac{\sqrt{3}}{2}x)$, and $e^{-\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x)$ on each open mesh interval $(x_\beta, x_{\beta+1})$, $\beta = 0, 1, \dots, N - 1$;
- (ii) $S(x)$ is a linear combination of functions e^{-x} , $e^{\frac{x}{2}} \cos(\frac{\sqrt{3}}{2}x)$, and $e^{\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x)$ on the intervals $(-\infty, 0)$ and $(1, \infty)$;
- (iii) $S^{(\alpha)}(x_\beta^-) = S^{(\alpha)}(x_\beta^+)$, $\alpha = 0, 1, \dots, 4$ and $\beta = 1, 2, \dots, N - 1$;
- (iv) $S(x_\beta) = \varphi(x_\beta)$, $\beta = 0, 1, \dots, N$;
- (v) $S(x)$ satisfies the following boundary conditions:

$$\begin{aligned} S'''(1) + S(1) &= 0, & S'''(0) + S(0) &= 0, \\ S^{(4)}(1) + S'(1) &= 0, & S^{(4)}(0) + S'(0) &= 0. \end{aligned}$$

We consider the fundamental solution

$$G(x) = \frac{\operatorname{sgn} x}{6} \cdot \left[\sinh(x) + e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{3}\right) + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x + \frac{2\pi}{3}\right) \right] \tag{4}$$

of the differential operator $\frac{d^6}{dx^6} - 1$, that is, the solution of the equation

$$G^{(6)}(x) - G(x) = \delta(x), \tag{5}$$

where $\delta(x)$ is the Dirac delta-function.

Remark The following rule for finding a fundamental solution of a linear differential operator

$$P\left(\frac{d}{dx}\right) := \frac{d^m}{dx^m} + a_1 \frac{d^{m-1}}{dx^{m-1}} + \dots + a_m, \tag{6}$$

where a_j are real numbers, is given in [29, pp. 88–89]: replacing $\frac{d}{dx}$ by p , we get a polynomial $P(p)$. Then we expand the expression $\frac{1}{P(p)}$ to partial fractions:

$$\frac{1}{P(p)} = \prod_j (p - \lambda)^{-k_j} = \sum_j [c_{j,k_j} (p - \lambda)^{-k_j} + \dots + c_{j,1} (p - \lambda)^{-1}], \tag{7}$$

and with every partial fraction $(p - \lambda)^{-k}$, we associate $\frac{x^{k-1} \operatorname{sgn} x}{2(k-1)!} \cdot e^{\lambda x}$.

Using this rule, we find the function $G(x)$ that is the fundamental solution of the operator $\frac{d^6}{dx^6} - 1$ and has the form (4).

It is easy to check that the fifth derivative of the function

$$G(x - x_\gamma) = \frac{\operatorname{sgn}(x - x_\gamma)}{6} \cdot \left[\sinh(x - x_\gamma) + e^{\frac{x-x_\gamma}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}(x - x_\gamma) + \frac{\pi}{3}\right) + e^{-\frac{x-x_\gamma}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}(x - x_\gamma) + \frac{2\pi}{3}\right) \right]$$

has a discontinuity equal to 1 at the point x_γ , and the third and the fourth derivatives of $G(x - x_\gamma)$ are continuous. Suppose a function $p_\gamma(x)$ coincides with the spline $S(x)$ on the interval $(x_\gamma, x_{\gamma+1})$, that is, $p_\gamma(x) := p_{\gamma-1}(x) + C_\gamma G(x - x_\gamma)$, $x \in (x_\gamma, x_{\gamma+1})$, where C_γ is the jump of the function $S^{(5)}(x)$ at x_γ :

$$C_\gamma = S^{(5)}(x_\gamma^+) - S^{(5)}(x_\gamma^-).$$

Then the spline $S(x)$ can be written in the form

$$S(x) = \sum_{\gamma=0}^N C_\gamma G(x - x_\gamma) + p_{-1}(x), \tag{8}$$

where $p_{-1}(x) = d_0 e^{-x} + d_1 e^{\frac{x}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}x\right) + d_2 e^{\frac{x}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}x\right)$, and d_0, d_1 , and d_2 are real numbers.

It is known that (see, e.g., [28]) the solution $S(x)$ of the form (4) of Problem 1 exists and is unique when $N \geq 2$, and the coefficients C_γ, d_0, d_1 , and d_2 of $S(x)$ are determined by the following system of $N + 4$ linear equations:

$$\sum_{\gamma=0}^N C_\gamma G(x_\beta - x_\gamma) + d_0 e^{-x_\beta} + d_1 e^{\frac{x_\beta}{2}} \cos\left(\frac{\sqrt{3}}{2}x_\beta\right) + d_2 e^{\frac{x_\beta}{2}} \sin\left(\frac{\sqrt{3}}{2}x_\beta\right) = \varphi(x_\beta), \quad \beta = 0, 1, \dots, N, \tag{9}$$

$$\sum_{\gamma=0}^N C_\gamma e^{-x_\gamma} = 0, \tag{10}$$

$$\sum_{\gamma=0}^N C_\gamma e^{\frac{x_\gamma}{2}} \cos\left(\frac{\sqrt{3}}{2}x_\gamma\right) = 0, \tag{11}$$

$$\sum_{\gamma=0}^N C_\gamma e^{\frac{x_\gamma}{2}} \sin\left(\frac{\sqrt{3}}{2}x_\gamma\right) = 0. \tag{12}$$

It is easy to show that the spline $S(x)$ defined by equation (8) with coefficients $C_\gamma, d_0, d_1,$ and d_2 satisfies conditions (i)–(v).

The main aim of the present paper is to solve Problem 1, that is, to solve system (9)–(12) for equally spaced nodes $x_\beta = h\beta, \beta = 0, 1, \dots, N, h = 1/N, N \geq 2,$ and to find analytic formulas for the coefficients $C_\gamma, d_0, d_1,$ and d_2 of $S(x)$.

It should be noted that using Sobolev method, interpolation splines minimizing the seminorms in different Hilbert spaces were constructed in [2, 4, 6, 7, 23, 24]. Furthermore, connection between interpolation splines and optimal quadrature formulas in the sense of Sard in $L_2^{(m)}(0, 1)$ and $K_2(P_2)$ spaces were shown in [4] and [6].

The rest of the paper is organized as follows. In Sect. 2, we give some definitions and known results. In Sect. 3, we give an algorithm for solving system (9)–(12) when the nodes x_β are equally spaced. Using this algorithm, the coefficients of the interpolation spline $S(x)$ are computed in Sect. 4.

2 Preliminaries

In this section, we give some definitions and known results that we need to prove the main results.

We mainly use the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [25, 27]. For completeness, we give some definitions about functions of discrete argument.

Assume that the nodes x_β are equally spaced, that is, $x_\beta = h\beta, h = \frac{1}{N}, N = 1, 2, \dots$. Further, we also use the notation $[\beta] = h\beta$.

Definition 1 The function $\varphi[\beta]$ is a *function of discrete argument* if it is given on some set of integer values of β .

Definition 2 The *inner product* of two discrete-argument functions $\varphi[\beta]$ and $\psi[\beta]$ is given by

$$[\varphi[\beta], \psi[\beta]] = \sum_{\beta=-\infty}^{\infty} \varphi[\beta] \cdot \psi[\beta],$$

if the series on the right-hand side of the last equality converges absolutely.

Definition 3 The *convolution* of two functions $\varphi[\beta]$ and $\psi[\beta]$ is the inner product

$$\varphi[\beta] * \psi[\beta] = [\varphi[\gamma], \psi[\beta - \gamma]] = \sum_{\gamma=-\infty}^{\infty} \varphi[\gamma] \cdot \psi[\beta - \gamma].$$

In our computations we need the discrete analogue $D[\beta]$ of the differential operator $\frac{d^6}{dx^6} - 1,$ which satisfies the equality

$$D[\beta] * G[\beta] = \delta[\beta], \tag{13}$$

where $G[\beta]$ is the discrete-argument function corresponding to $G(x)$ defined by (4), $\delta[\beta]$ is equal to 0 when $\beta \neq 0$ and to 1 when $\beta = 0$, that is, $\delta[\beta]$ is the discrete delta-function. Equation (13) is a discrete analogue of equation (5).

In [2, 3] the discrete analogue $D[\beta]$ of the differential operator $\frac{d^6}{dx^6} - 1$, which satisfies equation (13), was constructed, and the following theorem was proved.

Theorem 1 *The discrete analogue of the differential operator $\frac{d^6}{dx^6} - 1$ satisfying equation (13) has the form*

$$D[\beta] = \frac{3}{K} \begin{cases} \sum_{k=1}^2 B_k \lambda_k^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + \sum_{k=1}^2 B_k, & |\beta| = 1, \\ K_3 - K_1 + \sum_{k=1}^2 \frac{B_k}{\lambda_k}, & \beta = 0, \end{cases} \tag{14}$$

where

$$\begin{aligned} K &= \sinh(h) + \sinh\left(\frac{h}{2}\right) \cdot \cos\left(\frac{\sqrt{3}}{2}h\right) - \sqrt{3} \cdot \cosh\left(\frac{h}{2}\right) \cdot \sin\left(\frac{\sqrt{3}}{2}h\right), \\ K_1 &= 2 \cosh(h) \\ &\quad + \frac{4 \cos\left(\frac{\sqrt{3}}{2}h\right) \cosh\left(\frac{h}{2}\right) \sinh(h) + \sinh(h) - \sqrt{3} \sin(\sqrt{3}h) - 2 \sinh(h) \cosh(h)}{\sinh(h) + \sinh\left(\frac{h}{2}\right) \cdot \cos\left(\frac{\sqrt{3}}{2}h\right) - \sqrt{3} \cdot \cosh\left(\frac{h}{2}\right) \cdot \sin\left(\frac{\sqrt{3}}{2}h\right)}, \\ K_2 &= 2 + \frac{2 \cos(\sqrt{3}h) \sinh(h) + 4 \sinh(h) \cosh(h) - 2\sqrt{3} \sin(\sqrt{3}h) \cosh(h)}{\sinh(h) + \sinh\left(\frac{h}{2}\right) \cdot \cos\left(\frac{\sqrt{3}}{2}h\right) - \sqrt{3} \cdot \cosh\left(\frac{h}{2}\right) \cdot \sin\left(\frac{\sqrt{3}}{2}h\right)}, \\ K_3 &= 2 \cdot \left(\cosh(h) + 2 \cos\left(\frac{\sqrt{3}}{2}h\right) \cosh\left(\frac{h}{2}\right) \right), \\ B_k &= \frac{(\lambda_k^2 - 2\lambda_k \cdot \cosh(h) + 1) \cdot A_4(\lambda_k)}{(\lambda_k^2 - 1) \cdot (2\lambda_k^2 - K_1 \lambda_k + 2)}, \\ A_4(\lambda_k) &= \lambda_k^4 - 4\lambda_k^3 \cdot \cos\left(\frac{\sqrt{3}}{2}h\right) \cosh\left(\frac{h}{2}\right) + 2\lambda_k^2 \cdot (1 + \cos(\sqrt{3}h) + \cosh(h)) \\ &\quad - 4\lambda_k \cdot \cos\left(\frac{\sqrt{3}}{2}h\right) \cosh\left(\frac{h}{2}\right) + 1, \\ \lambda_k &= \frac{1}{4} \cdot [K_1 + \sqrt{K_1^2 - 4 \cdot K_2 + 8} + (-1)^k \sqrt{(K_1 + \sqrt{K_1^2 - 4 \cdot K_2 + 8})^2 - 16}], \quad k = 1, 2, \end{aligned}$$

h is a small positive parameter, and λ_k are the roots of the polynomial $A_4(\lambda)$ such that $|\lambda_k| < 1$.

Furthermore, several properties of the discrete-argument function $D[\beta]$ were given in [2, 3]. Here we give the following its properties, which we need in our computations.

Theorem 2 *The discrete analogue $D[\beta]$ of the differential operator $\frac{d^6}{dx^6} - 1$ satisfies the following equalities:*

- (1) $D[\beta] * e^{[\beta]} = 0,$
- (2) $D[\beta] * e^{-[\beta]} = 0,$
- (3) $D[\beta] * e^{\frac{[\beta]}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) = 0,$

- (4) $D[\beta] * e^{\frac{[\beta]}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}[\beta]\right) = 0,$
- (5) $D[\beta] * e^{-\frac{[\beta]}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) = 0,$
- (6) $D[\beta] * e^{-\frac{[\beta]}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}[\beta]\right) = 0.$

3 The algorithm for computation of coefficients of interpolation spline

In the present section, we give an algorithm for solving system (9)–(12) when the nodes x_β are equally spaced, that is, $x_\beta = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, \dots$. Here we use a similar method suggested by Sobolev [26, 27] for finding the coefficients of optimal quadrature formulas in the Sobolev space $L_2^{(m)}(0, 1)$. We note that here $[\beta]$ means $(h\beta)$.

Suppose that $C[\beta] = 0$ when $\beta < 0$ and $\beta > N$. Using Definition 3, we rewrite system (9)–(12) in the convolution form:

$$G[\beta] * C[\beta] + d_0 e^{-[\beta]} + d_1 e^{\frac{[\beta]}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) + d_2 e^{\frac{[\beta]}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}[\beta]\right) = \varphi[\beta], \quad \beta = 0, 1, \dots, N, \tag{15}$$

$$\sum_{\beta=0}^N C[\beta] \cdot e^{-[\beta]} = 0, \tag{16}$$

$$\sum_{\beta=0}^N C[\beta] \cdot e^{\frac{[\beta]}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) = 0, \tag{17}$$

$$\sum_{\beta=0}^N C[\beta] \cdot e^{\frac{[\beta]}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}[\beta]\right) = 0. \tag{18}$$

Thus we have the following problem.

Problem 2 Find the coefficients $C[\beta]$ ($\beta = 0, 1, \dots, N$), d_0 , d_1 , and d_2 that satisfy system (15)–(18).

Further, we investigate Problem 2, which is equivalent to Problem 1. Instead of $C[\beta]$, we introduce the following discrete-argument functions:

$$v[\beta] = G[\beta] * C[\beta], \tag{19}$$

$$u[\beta] = v[\beta] + d_0 e^{-[\beta]} + d_1 e^{\frac{[\beta]}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) + d_2 e^{\frac{[\beta]}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}[\beta]\right). \tag{20}$$

Now we express the coefficients $C[\beta]$ by the function $u[\beta]$.

Taking into account (14), (20), and Theorems 1 and 2, for the coefficients, we have

$$C[\beta] = D[\beta] * u[\beta]. \tag{21}$$

Thus, if we find the function $u[\beta]$, then the coefficients $C[\beta]$ will be found from equality (21).

To calculate convolution (21), it is required to find the representation of the function $u[\beta]$ for all integer values of β . From equality (15) we get that $u[\beta] = \varphi[\beta]$ when $[\beta] \in [0, 1]$. Now we need to find the representation of the function $u[\beta]$ when $\beta < 0$ and $\beta > N$.

Since $C[\beta] = 0$ when $[\beta] \notin [0, 1]$, we have

$$C[\beta] = D[\beta] * u[\beta] = 0 \quad \text{for } [\beta] \notin [0, 1].$$

Now we calculate the convolution $v[\beta] = G[\beta] * C[\beta]$ when $\beta \leq 0$ and $\beta \geq N$. Suppose $\beta \leq 0$. Then taking into account equalities (8) and (16)–(18), we have

$$\begin{aligned} v[\beta] &= \sum_{\gamma=-\infty}^{\infty} C[\gamma]G[\beta - \gamma] \\ &= -\frac{1}{12} \left[-e^{-[\beta]} \sum_{\gamma=0}^N C[\gamma]e^{[\gamma]} \right. \\ &\quad \left. + \left(\sum_{\gamma=0}^N C[\gamma]e^{-\frac{[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) + \sqrt{3} \sum_{\gamma=0}^N C[\gamma]e^{-\frac{[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right) \right. \\ &\quad \left. \times e^{\frac{[\beta]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) \right. \\ &\quad \left. + \left(\sum_{\gamma=0}^N C[\gamma]e^{-\frac{[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) - \sqrt{3} \sum_{\gamma=0}^N C[\gamma]e^{-\frac{[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right) \right. \\ &\quad \left. \times e^{\frac{[\beta]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\beta]\right) \right]. \end{aligned}$$

Thus, when $\beta \leq 0$, we get

$$v[\beta] = -b_0e^{-[\beta]} - b_1e^{\frac{[\beta]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) - b_2e^{\frac{[\beta]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\beta]\right), \tag{22}$$

where

$$\begin{aligned} b_0 &= -\frac{1}{12} \sum_{\gamma=0}^N C[\gamma]e^{[\gamma]}, \\ b_1 &= \frac{1}{12} \sum_{\gamma=0}^N C[\gamma]e^{-\frac{[\gamma]}{2}} \left[\cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right], \end{aligned}$$

and

$$b_2 = \frac{1}{12} \sum_{\gamma=0}^N C[\gamma]e^{-\frac{[\gamma]}{2}} \left[\sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) - \sqrt{3} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right]. \tag{23}$$

Similarly, in the case $\beta \geq N$, for the convolution $v[\beta] = G[\beta] * C[\beta]$, we obtain

$$v[\beta] = b_0e^{-[\beta]} + b_1e^{\frac{[\beta]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) + b_2e^{\frac{[\beta]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\beta]\right). \tag{24}$$

We denote

$$d_0^- = d_0 - b_0, \quad d_1^- = d_1 - b_1, \quad d_2^- = d_2 - b_2, \tag{25}$$

$$d_0^+ = d_0 + b_0, \quad d_1^+ = d_1 + b_1, \quad d_2^+ = d_2 + b_2, \tag{26}$$

Taking into account (20), (22), and (24), we get the following problem.

Problem 3 Find the solution of the equation

$$D[\beta] * u[\beta] = 0, \quad [\beta] \notin [0, 1], \tag{27}$$

of the form

$$u[\beta] = \begin{cases} d_0^- e^{-[\beta]} + d_1^- e^{\frac{[\beta]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) + d_2^- e^{\frac{[\beta]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\beta]\right), & \beta \leq 0, \\ \varphi[\beta], & 0 \leq \beta \leq N, \\ d_0^+ e^{-[\beta]} + d_1^+ e^{\frac{[\beta]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) + d_2^+ e^{\frac{[\beta]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\beta]\right), & \beta \geq N. \end{cases} \tag{28}$$

If we find $d_0^-, d_0^+, d_1^-, d_1^+$, and d_2^-, d_2^+ , then from (25) and (26) we have

$$\begin{aligned} d_n &= \frac{1}{2}(d_n^+ + d_n^-), \quad n = 0, 1, 2, \\ b_n &= \frac{1}{2}(b_n^+ - b_n^-), \quad n = 0, 1, 2. \end{aligned} \tag{29}$$

The unknowns $d_0^-, d_0^+, d_1^-, d_1^+$, and d_2^-, d_2^+ can be found from equation (27) using the function $D[\beta]$ defined by (14). Then we obtain an explicit form of the function $u[\beta]$, and from (21) we find the coefficients $C[\beta]$. Furthermore, from (29) we get d_0, d_1 , and d_2 .

Thus Problem 3 and, respectively, Problems 2 and 1 will be solved.

In the next section, we apply this algorithm to compute the coefficients $C[\beta]$, $\beta = 0, 1, \dots, N, d_0, d_1$, and d_2 of the interpolation spline (8) for $N \geq 2$.

4 Computation of coefficients of interpolation spline (8)

In this section, using the presented algorithm, we obtain explicit formulas for the coefficients of the interpolation spline (8), which, as we have proved in the previous section, is the solution of Problem 1.

It should be noted that the interpolation spline (8), the solution of Problem 1, is exact for functions e^{-x} , $e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$, and $e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$.

We now obtain exact formulas for the coefficients of the interpolation spline (8) in the following:

Theorem 3 *The coefficients of the interpolation spline (8) with equally spaced nodes in the space $W_2^{(3,0)}$ have the following form:*

$$\begin{aligned} C[0] &= \frac{3}{K} \left[\varphi(0)(K_3 - K_1) + \varphi(h) + d_0^- e^h + d_1^- e^{-\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) - d_2^- e^{-\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}h\right) \right. \\ &\quad \left. + \sum_{k=1}^2 \frac{B_k}{\lambda_k} \left(\sum_{\gamma=0}^N \lambda_k^\gamma \varphi(h\gamma) + M_k + \lambda_k^N \cdot N_k \right) \right], \end{aligned}$$

$$\begin{aligned}
 C[\beta] &= \frac{3}{K} \left[\varphi(h\beta)(K_3 - K_1) + \varphi(h(\beta - 1)) + \varphi(h(\beta + 1)) \right. \\
 &\quad \left. + \sum_{k=1}^2 \frac{B_k}{\lambda_k} \left(\sum_{\gamma=0}^N \lambda_k^{|\beta-\gamma|} \varphi(h\gamma) + \lambda_k^\beta \cdot M_k + \lambda_k^{N-\beta} \cdot N_k \right) \right], \quad \beta = 1, 2, \dots, N - 1, \\
 C[N] &= \frac{3}{K} \left[\varphi(1)(K_3 - K_1) + \varphi(1 - h) + d_0^+ e^{-(1+h)} + d_1^+ e^{\frac{1+h}{2}} \cos\left(\frac{\sqrt{3}}{2}(1 + h)\right) \right. \\
 &\quad \left. + d_2^+ e^{\frac{1+h}{2}} \sin\left(\frac{\sqrt{3}}{2}(1 + h)\right) + \sum_{k=1}^2 \frac{B_k}{\lambda_k} \left(\sum_{\gamma=0}^N \lambda_k^{N-\gamma} \varphi(h\gamma) + \lambda_k^N \cdot M_k + N_k \right) \right], \\
 d_n &= \frac{1}{2} (d_n^+ + d_n^-), \quad n = 0, 1, 2,
 \end{aligned}$$

where

$$M_k = \lambda_k \left[\frac{d_0^- e^h}{1 - \lambda_k e^h} - \frac{d_1^- (\lambda_k - e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h)) + d_2^- e^{\frac{h}{2}} \sin(\frac{\sqrt{3}}{2}h)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + e^h} \right], \tag{30}$$

$$\begin{aligned}
 N_k &= \lambda_k \left[\frac{d_0^+ e^{-1}}{e^h - \lambda_k} + e^{\frac{1+h}{2}} \left[\frac{d_1^+ (\cos(\frac{\sqrt{3}}{2}(1 + h)) - \lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}))}{\lambda_k^2 e^h - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + 1} \right. \right. \\
 &\quad \left. \left. + \frac{d_2^+ (\sin(\frac{\sqrt{3}}{2}(1 + h)) - \lambda_k e^{\frac{h}{2}} \sin(\frac{\sqrt{3}}{2}))}{\lambda_k^2 e^h - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + 1} \right] \right], \quad k = 1, 2, \tag{31}
 \end{aligned}$$

and K, K_1, K_3, B_k , and λ_k are defined in Theorem 1, $|\lambda_k| < 1$, d_n^- and d_n^+ , $n = 0, 1, 2$, are defined from system (32)–(33), (35)–(38).

Proof First, we find the expressions for d_0^- and d_0^+ . When $\beta = 0$ and $\beta = N$, from (28) for d_0^- and d_0^+ we get

$$d_0^- = \varphi(0) - d_1^-, \tag{32}$$

$$d_0^+ = e \cdot \varphi(1) - e^{\frac{3}{2}} \cos\frac{\sqrt{3}}{2} d_1^+ - e^{\frac{3}{2}} \sin\frac{\sqrt{3}}{2} d_2^+. \tag{33}$$

Now we have four unknowns d_1^-, d_2^-, d_1^+ , and d_2^+ .

From equation (27) for $\beta = -1, -2$ and $\beta = N + 1, N + 2$, we find d_1^-, d_2^-, d_1^+ , and d_2^+ .

Taking into account (29), (32), and (33), from (27) we get the following equation for d_1^-, d_2^-, d_1^+ , and d_2^+ :

$$\begin{aligned}
 &\sum_{\gamma=-\infty}^{-1} D[\beta - \gamma] \left[d_0^- e^{-[\beta]} + d_1^- e^{\frac{[\beta]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) + d_2^- e^{\frac{[\beta]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\beta]\right) \right] \\
 &+ \sum_{\gamma=0}^N D[\beta - \gamma] \varphi[\gamma] + \sum_{\gamma=N+1}^{\infty} D[\beta - \gamma] \left[d_0^+ e^{-[\beta]} + d_1^+ e^{\frac{[\beta]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\beta]\right) \right. \\
 &\quad \left. + d_2^+ e^{\frac{[\beta]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\beta]\right) \right] = 0, \tag{34}
 \end{aligned}$$

where $\beta < 0$ and $\beta > N$. First, we consider the cases $\beta = -1, -2$. From (34) we obtain the following system of two linear equations

$$\begin{aligned} & \sum_{\gamma=1}^{\infty} D[\gamma - 1] \cdot \left[d_1^- \left(e^{-\frac{[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) - e^{[\gamma]} \right) - d_2^- e^{-\frac{[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right] \\ & + \sum_{\gamma=1}^{\infty} D[N + \gamma + 1] \cdot \left[d_1^+ \left(e^{\frac{1+[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right. \\ & \left. + d_2^+ \left(e^{\frac{1+[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right] \\ & = - \left[\varphi(0) \sum_{\gamma=1}^{\infty} D[\gamma - 1] e^{[\gamma]} + \varphi(1) \sum_{\gamma=1}^{\infty} D[N + \gamma + 1] e^{-[\gamma]} + \sum_{\gamma=0}^N D[\gamma + 1] \varphi[\gamma] \right], \end{aligned} \tag{35}$$

$$\begin{aligned} & \sum_{\gamma=1}^{\infty} D[\gamma - 2] \cdot \left[d_1^- \left(e^{-\frac{[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) - e^{[\gamma]} \right) - d_2^- e^{-\frac{[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right] \\ & + \sum_{\gamma=1}^{\infty} D[N + \gamma + 2] \cdot \left[d_1^+ \left(e^{\frac{1+[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right. \\ & \left. + d_2^+ \left(e^{\frac{1+[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right] \\ & = - \left[\varphi(0) \sum_{\gamma=1}^{\infty} D[\gamma - 2] e^{[\gamma]} + \varphi(1) \sum_{\gamma=1}^{\infty} D[N + \gamma + 2] e^{-[\gamma]} + \sum_{\gamma=0}^N D[\gamma + 2] \varphi[\gamma] \right]. \end{aligned} \tag{36}$$

Now we consider the cases $\beta = N + 1, N + 2$. From (34) we obtain

$$\begin{aligned} & \sum_{\gamma=1}^{\infty} D[N + \gamma + 1] \cdot \left[d_1^- \left(e^{-\frac{[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) - e^{[\gamma]} \right) - d_2^- e^{-\frac{[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right] \\ & + \sum_{\gamma=1}^{\infty} D[\gamma - 1] \cdot \left[d_1^+ \left(e^{\frac{1+[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right. \\ & \left. + d_2^+ \left(e^{\frac{1+[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right] \\ & = - \left[\varphi(0) \sum_{\gamma=1}^{\infty} D[N + \gamma + 1] e^{[\gamma]} + \varphi(1) \sum_{\gamma=1}^{\infty} D[\gamma - 1] e^{-[\gamma]} \right. \\ & \left. + \sum_{\gamma=0}^N D[N + 1 - \gamma] \varphi[\gamma] \right], \end{aligned} \tag{37}$$

$$\begin{aligned} & \sum_{\gamma=1}^{\infty} D[N + \gamma + 2] \cdot \left[d_1^- \left(e^{-\frac{[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) - e^{[\gamma]} \right) - d_2^- e^{-\frac{[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right] \\ & + \sum_{\gamma=1}^{\infty} D[\gamma - 2] \cdot \left[d_1^+ \left(e^{\frac{1+[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right. \\ & \left. + d_2^+ \left(e^{\frac{1+[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) - e^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\right) e^{-[\gamma]} \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= - \left[\varphi(0) \sum_{\gamma=1}^{\infty} D[N + \gamma + 2]e^{[\gamma]} + \varphi(1) \sum_{\gamma=1}^{\infty} D[\gamma - 2]e^{-[\gamma]} \right. \\
 &\quad \left. + \sum_{\gamma=0}^N D[N + 2 - \gamma]\varphi[\gamma] \right]. \tag{38}
 \end{aligned}$$

Thus for unknowns d_1^-, d_1^+, d_2^- , and d_2^+ we get system (35)–(38) consisting of four linear equations. Since the interpolation problem has a unique solution, the main matrix of this system is nonsingular. To find the unknowns d_1^-, d_1^+, d_2^- , and d_2^+ , we rewrite system (35)–(38) as follows:

$$\begin{cases} M_{11}d_1^- + M_{12}d_2^- + M_{13}d_1^+ + M_{14}d_2^+ = T_1, \\ M_{21}d_1^- + M_{22}d_2^- + M_{23}d_1^+ + M_{24}d_2^+ = T_2, \\ M_{31}d_1^- + M_{32}d_2^- + M_{33}d_1^+ + M_{34}d_2^+ = T_3, \\ M_{41}d_1^- + M_{42}d_2^- + M_{43}d_1^+ + M_{44}d_2^+ = T_4, \end{cases} \tag{39}$$

where

$$\begin{aligned}
 M_{11} &= \frac{3}{K} \left[(K_3 - K_1) \left(e^{-\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) - e^h \right) + e^{-h} \cos(\sqrt{3}h) - e^{2h} \right. \\
 &\quad \left. - \sum_{k=1}^2 \frac{B_k}{\lambda_k} \left[\frac{\lambda_k - e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + e^h} + \frac{e^h}{1 - \lambda_k e^h} \right] \right], \\
 M_{12} &= -\frac{3}{K} \left[(K_3 - K_1) \cdot e^{-\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}h\right) + e^{-h} \sin(\sqrt{3}h) \right. \\
 &\quad \left. + \sum_{k=1}^2 \frac{B_k}{\lambda_k} \frac{e^{\frac{h}{2}} \sin(\frac{\sqrt{3}}{2}h)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + e^h} \right], \\
 M_{13} &= -\frac{3e^{\frac{1}{2}}}{K} \sum_{k=1}^2 B_k \lambda_k^{N+1} \left[\frac{e^h \cos(\frac{\sqrt{3}}{2})\lambda_k - e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}(1+h))}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + 1} - \frac{\cos(\frac{\sqrt{3}}{2})}{\lambda_k - e^h} \right], \\
 M_{14} &= -\frac{3e^{\frac{1}{2}}}{K} \sum_{k=1}^2 B_k \lambda_k^{N+1} \left[\frac{e^h \sin(\frac{\sqrt{3}}{2})\lambda_k - e^{\frac{h}{2}} \sin(\frac{\sqrt{3}}{2}(1+h))}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + 1} - \frac{\sin(\frac{\sqrt{3}}{2})}{\lambda_k - e^h} \right], \\
 M_{21} &= -\frac{3}{K} \sum_{k=1}^2 B_k \lambda_k^{N+1} \left[\frac{\lambda_k - e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + e^h} + \frac{e^h}{1 - e^h \lambda_k} \right], \\
 M_{22} &= -\frac{3}{K} \sum_{k=1}^2 B_k \lambda_k^{N+1} \frac{e^{\frac{h}{2}} \sin(\frac{\sqrt{3}}{2}h)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + e^h}, \\
 M_{23} &= \frac{3e^{\frac{1}{2}}}{K} \left[(K_3 - K_1) \left(e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}(1+h)\right) - \cos\left(\frac{\sqrt{3}}{2}\right)e^{-h} \right) + e^h \cos\left(\frac{\sqrt{3}}{2}(1+2h)\right) \right. \\
 &\quad \left. - \cos\left(\frac{\sqrt{3}}{2}\right)e^{-2h} - \sum_{k=1}^2 \frac{B_k}{\lambda_k} \left[\frac{e^h \cos(\frac{\sqrt{3}}{2})\lambda_k - e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}(1+h))}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos(\frac{\sqrt{3}}{2}h) + 1} - \frac{\cos(\frac{\sqrt{3}}{2})}{\lambda_k - e^h} \right] \right],
 \end{aligned}$$

$$M_{24} = \frac{3e^{\frac{1}{2}}}{K} \left[(K_3 - K_1) \left(e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}(1+h)\right) - \sin\left(\frac{\sqrt{3}}{2}\right) e^{-h} \right) + e^h \sin\left(\frac{\sqrt{3}}{2}(1+2h)\right) - \sin\left(\frac{\sqrt{3}}{2}\right) e^{-2h} - \sum_{k=1}^2 \frac{B_k}{\lambda_k} \left[\frac{e^h \sin\left(\frac{\sqrt{3}}{2}\right) \lambda_k - e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}(1+h)\right)}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + 1} - \frac{\sin\left(\frac{\sqrt{3}}{2}\right)}{\lambda_k - e^h} \right] \right],$$

$$M_{31} = \frac{3}{K} \left[(K_3 - K_1) (e^{-h} \cos(\sqrt{3}h) - e^{2h}) + e^{-\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) - e^h + e^{-\frac{3}{2}h} \cos\left(\frac{3\sqrt{3}}{2}h\right) - e^{3h} + \left(e^{-\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) - e^h \right) \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} (\lambda_k^2 - 1) - \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} \left[\frac{\lambda_k - e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + e^h} + \frac{e^h}{1 - e^h \lambda_k} \right] \right],$$

$$M_{32} = -\frac{3}{K} \left[(K_3 - K_1) e^{-h} \sin(\sqrt{3}h) + e^{-\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}h\right) + e^{-\frac{3}{2}h} \sin\left(\frac{3\sqrt{3}}{2}h\right) + e^{-\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}h\right) \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} (\lambda_k^2 - 1) + \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} \frac{e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}h\right)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + e^h} \right],$$

$$M_{33} = -\frac{3e^{\frac{1}{2}}}{K} \sum_{k=1}^2 B_k \lambda_k^{N+2} \left[\frac{e^h \cos\left(\frac{\sqrt{3}}{2}\right) \lambda_k - e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}(1+h)\right)}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + 1} - \frac{\cos\left(\frac{\sqrt{3}}{2}\right)}{\lambda_k - e^h} \right],$$

$$M_{34} = -\frac{3e^{\frac{1}{2}}}{K} \sum_{k=1}^2 B_k \lambda_k^{N+2} \left[\frac{e^h \sin\left(\frac{\sqrt{3}}{2}\right) \lambda_k - e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}(1+h)\right)}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + 1} - \frac{\sin\left(\frac{\sqrt{3}}{2}\right)}{\lambda_k - e^h} \right],$$

$$M_{41} = -\frac{3}{K} \sum_{k=1}^2 B_k \lambda_k^{N+2} \left[\frac{\lambda_k - e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + e^h} + \frac{e^h}{1 - e^h \lambda_k} \right],$$

$$M_{42} = -\frac{3}{K} \sum_{k=1}^2 B_k \lambda_k^{N+2} \frac{e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}h\right)}{\lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + e^h},$$

$$M_{43} = \frac{3e^{\frac{1}{2}}}{K} \left[(K_3 - K_1) \left(e^h \cos\left(\frac{\sqrt{3}}{2}(1+2h)\right) - \cos\left(\frac{\sqrt{3}}{2}\right) e^{-2h} \right) + e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}(1+h)\right) - \cos\left(\frac{\sqrt{3}}{2}\right) e^{-h} + e^{\frac{3}{2}h} \cos\left(\frac{\sqrt{3}}{2}(1+3h)\right) - \cos\left(\frac{\sqrt{3}}{2}\right) e^{-3h} + \left(e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}(1+h)\right) - \cos\left(\frac{\sqrt{3}}{2}\right) e^{-h} \right) \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} (\lambda_k^2 - 1) - \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} \left[\frac{e^h \cos\left(\frac{\sqrt{3}}{2}\right) \lambda_k - e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}(1+h)\right)}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + 1} - \frac{\cos\left(\frac{\sqrt{3}}{2}\right)}{\lambda_k - e^h} \right] \right],$$

$$M_{44} = \frac{3e^{\frac{1}{2}}}{K} \left[(K_3 - K_1) \left(e^h \sin\left(\frac{\sqrt{3}}{2}(1+2h)\right) - \sin\left(\frac{\sqrt{3}}{2}\right) e^{-2h} \right) + e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}(1+h)\right) - \sin\left(\frac{\sqrt{3}}{2}\right) e^{-h} + e^{\frac{3}{2}h} \sin\left(\frac{\sqrt{3}}{2}(1+3h)\right) - \sin\left(\frac{\sqrt{3}}{2}\right) e^{-3h} + \left(e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}(1+h)\right) - \sin\left(\frac{\sqrt{3}}{2}\right) e^{-h} \right) \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} (\lambda_k^2 - 1) \right]$$

$$\begin{aligned}
 & - \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} \left[\frac{e^h \sin\left(\frac{\sqrt{3}}{2}\right)\lambda_k - e^{\frac{h}{2}} \sin\left(\frac{\sqrt{3}}{2}\right)(1+h)}{e^h \lambda_k^2 - 2\lambda_k e^{\frac{h}{2}} \cos\left(\frac{\sqrt{3}}{2}h\right) + 1} - \frac{\sin\left(\frac{\sqrt{3}}{2}\right)}{\lambda_k - e^h} \right], \\
 T_1 = & -\frac{3}{K} \left[\sum_{k=1}^2 B_k \sum_{\gamma=0}^N \lambda_k^\gamma \varphi(h\gamma) + \varphi(0) \left[1 + (K_3 - K_1)e^h + e^{2h} + \sum_{k=1}^2 \frac{B_k e^h}{\lambda_k - e^h \lambda_k^2} \right] \right. \\
 & \left. + \varphi(1) \sum_{k=1}^2 \frac{B_k \lambda_k^{N+1}}{e^h - \lambda_k} \right], \\
 T_2 = & -\frac{3}{K} \left[\sum_{k=1}^2 B_k \sum_{\gamma=0}^N \lambda_k^{N-\gamma} \varphi(h\gamma) + \varphi(0) \sum_{k=1}^2 \frac{e^h B_k \lambda_k^{N+1}}{1 - e^h \lambda_k} + \varphi(1) \left[1 + (K_3 - K_1)e^{-h} \right. \right. \\
 & \left. \left. + e^{-2h} - \sum_{k=1}^2 \frac{B_k}{\lambda_k^2 - e^h \lambda_k} \right] \right], \\
 T_3 = & -\frac{3}{K} \left[\sum_{k=1}^2 B_k \sum_{\gamma=0}^N \lambda_k^{\gamma+1} \varphi(h\gamma) + \varphi(0) \left[(K_3 - K_1)e^{2h} + e^h + e^{3h} \right. \right. \\
 & \left. \left. + e^h \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} (\lambda_k^2 - 1) + \sum_{k=1}^2 \frac{B_k e^h}{\lambda_k^2 - e^h \lambda_k^3} \right] + \varphi(1) \sum_{k=1}^2 \frac{B_k \lambda_k^{N+2}}{e^h - \lambda_k} \right], \\
 T_4 = & -\frac{3}{K} \left[\sum_{k=1}^2 B_k \sum_{\gamma=0}^N \lambda_k^{N+1-\gamma} \varphi(h\gamma) + \varphi(0) \sum_{k=1}^2 \frac{e^h B_k \lambda_k^{N+2}}{1 - e^h \lambda_k} + \varphi(1) \left[(K_3 - K_1)e^{-2h} \right. \right. \\
 & \left. \left. + e^{-h} + e^{-3h} + e^{-h} \sum_{k=1}^2 \frac{B_k}{\lambda_k^2} (\lambda_k^2 - 1) - \sum_{k=1}^2 \frac{B_k}{\lambda_k^3 - e^h \lambda_k^2} \right] \right],
 \end{aligned}$$

where $|\lambda_k| < 1$.

Solving system (39), we find $d_1^-, d_2^-, d_1^+, d_2^+$. Then using (32) and (33), we find d_0^- and d_0^+ . From (21), for $\beta = 0, 1, \dots, N$, we deduce

$$\begin{aligned}
 C[\beta] = & \sum_{\gamma=0}^N D[\beta - \gamma] \varphi[\gamma] + \sum_{\gamma=1}^{\infty} D[\beta + \gamma] \left[d_0^- e^{[\gamma]} + d_1^- e^{-\frac{[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right. \\
 & \left. - d_2^- e^{-\frac{[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}[\gamma]\right) \right] + \sum_{\gamma=1}^{\infty} D[N + \gamma - \beta] \left[d_0^+ e^{-(1+[\gamma])} \right. \\
 & \left. + d_1^+ e^{\frac{1+[\gamma]}{2}} \cos\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) + d_2^+ e^{\frac{1+[\gamma]}{2}} \sin\left(\frac{\sqrt{3}}{2}(1 + [\gamma])\right) \right]
 \end{aligned}$$

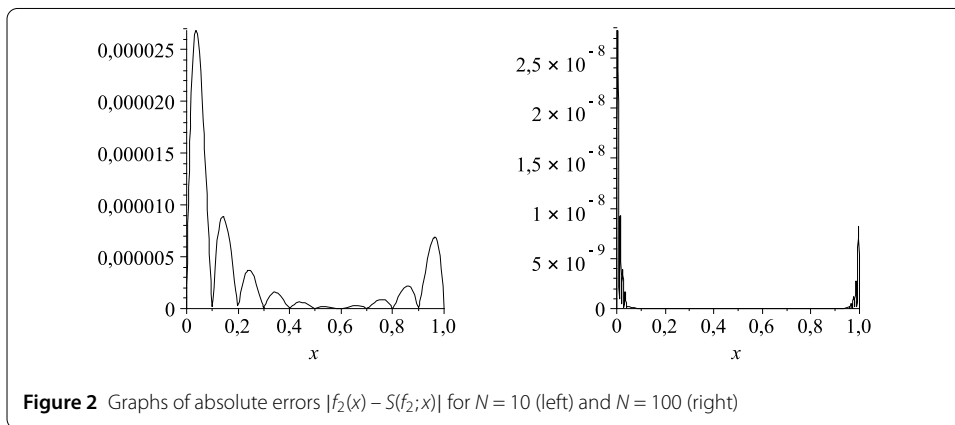
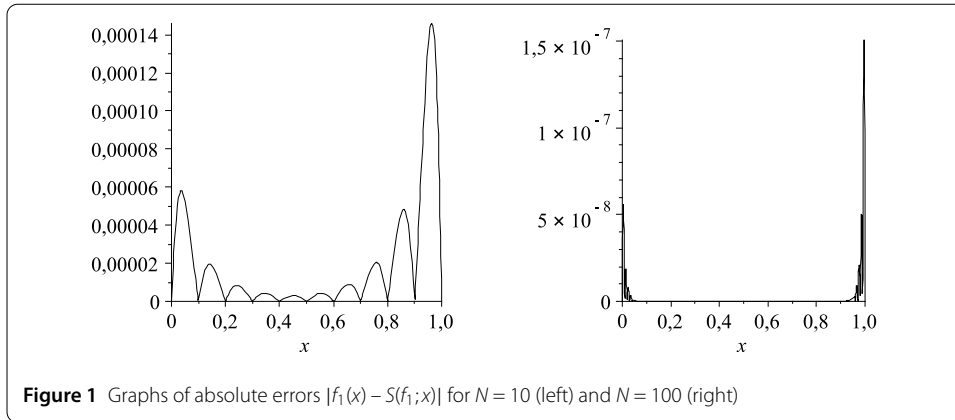
from which, using (14) and taking into account (30) and (31), after some calculations, we arrive at the expressions for the coefficients $C[\beta]$, $\beta = 0, 1, \dots, N$, which are given in the statement of the theorem.

Theorem 3 is proved. □

5 Numerical results

In this section, we give some numerical results using Theorem 3.

In numerical examples, we consider the functions $f_1(x) = e^x$ and $f_2(x) = \sin x$. We denote the corresponding interpolation splines of the form (8) by $S(f_1; x)$ and $S(f_2; x)$, respectively.



Applying $S(x)$ with $N = 10$ and $N = 100$ and using Theorem 3 for the functions $f_1(x)$ and $f_2(x)$, we obtain the absolute errors. The graphs of the corresponding absolute errors are displayed in Figs. 1 and 2. We can see that by increasing the value of N the absolute errors between interpolation splines and the given functions decrease and the order of convergence of the interpolation formula (8) in these examples is $O(h^3)$.

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Authors' contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the manuscript.

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