Existence results of nonlocal Robin boundary value problems for fractional (p,q)-integrodifference equations

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Abstract

The existence results of a fractional (p,q)-integrodifference equation with nonlocal Robin boundary condition are investigated by using Banach's and Schauder's fixed point theorems. Moreover, we study some properties of (p,q)-integral that will be used as a tool for our calculations.

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Keywords: Fractional (p,q)-integral; Fractional (p,q)-difference; Robin boundary value problems; Existence

1 Introduction

Along with the evolution of the theory and application of classical calculus, quantum calculus (calculus without limit) has also received more intense attention in the last three decades. In this article, we study the development of q-calculus which is one type of quantum calculus. The q-calculus was first introduced by Jackson [1, 2] in 1910. In recent years, the extension of this topic has been studied by many researchers and has many new results in [3–9] and their references. The knowledge of q-calculus was used in physical problems, see [10–27] and the references cited therein.

Later, the study of quantum calculus based on two-parameter (p,q)-integer was presented. The (p,q)-calculus was presented by Chakrabarti and Jagannathan [27]. The extension of studies of (p,q)-calculus was given in [28–39]. In addition, it is used in many branches such as physical sciences, hypergeometric series, Lie group, special functions, approximation theory, Bézier curves and surfaces, etc. [40–47].

Then, the study of fractional quantum calculus was initiated [48–50]. Agarwal [48] and Al-Salam [49] studied fractional q-calculus, whilst Díaz and Osler [50] proposed fractional difference calculus. In 2017, Brikshavana and Sitthiwirattham [51] introduced fractional Hahn difference calculus. Recently, Patanarapeelert and Sitthiwirattham [52] studied fractional symmetric Hahn difference calculus. Presently, Soontharanon and Sitthiwirattham [53] introduced the fractional (p,q)-difference operators and their properties.

There are some recent papers studying the boundary value problem for (p,q)-difference equations [54–56]. However, the boundary value problem for fractional (p,q)-difference

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equations has not been studied since fractional (p,q)-operators have been defined lately. These motivate the authors for this research. This article investigates the existence results of a fractional (p,q)-integrodifference equation with nonlocal Robin boundary value conditions of the form

$$
\begin{aligned}
& D_{p,q}^{\alpha} u(t) = F\big[t, u(t), \Psi_{p,q}^{\gamma} u(t), D_{p,q}^{\nu} u(t)\big], \quad t \in I_{p,q}^{T}, \\
& \lambda_1 u(\eta) + \lambda_2 D_{p,q}^{\beta} u(\eta) = \phi_1(u), \quad \eta \in I_{p,q}^{T} - \left\{0, \frac{T}{p}\right\}, \\
& \mu_1 u\left(\frac{T}{p}\right) + \mu_2 D_{p,q}^{\beta} u\left(\frac{T}{p}\right) = \phi_2(u),
\end{aligned}
\tag{1.1}
$$

where $I_{p,q}^{T} := \{\frac{q^k}{p^{k+1}} T : k \in \mathbb{N}_0\} \cup \{0\}; \ 0 < q < p \le 1 \ \alpha \in (1,2], \ \beta, \gamma, \nu \in (0,1], \ \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^{+}; F \in C(I_{p,q}^{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function; $\phi_1, \phi_2 : C(I_{p,q}^{T}, \mathbb{R}) \to \mathbb{R}$ are given functionals; and for $\varphi \in C(I_{p,q}^{T} \times I_{p,q}^{T}, [0, \infty))$, we define an operator of the (p,q)-integral of the product of functions φ and u as

$$
\Psi_{p,q}^{\gamma} u(t) := \big(\mathcal{I}_{p,q}^{\gamma} \varphi u\big)(t) = \frac{1}{p^{\binom{\gamma}{2}} \Gamma_{p,q}(\gamma)} \int_0^t (t - qs)_{p,q}^{\gamma-1} \varphi(t,s) u\left(\frac{s}{p^{\gamma-1}}\right) d_{p,q}s.
$$

We aim to prove the existence and uniqueness of a solution for this problem by using Banach's fixed point theorem, and the existence of at least one solution by using Schauder's fixed point theorem. In addition, we provide an example to illustrate our results.

2 Preliminaries

In this section, we recall some basic definitions, notations, and lemmas. Letting $0 < q < p \le 1$, we define the notations

$$
[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}
$$

$$
[k]_{p,q} := \begin{cases} \frac{p^k - q^k}{p-q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}
$$

$$
[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q} \cdots [1]_{p,q} = \prod_{i=1}^{k} \frac{p^i - q^i}{p-q}, & k \in \mathbb{N} \\ 1, & k = 0. \end{cases}
$$

The (p,q)-forward jump and the (p,q)-backward jump operators are defined as

$$
\sigma_{p,q}^{k}(t) := \left(\frac{q}{p}\right)^{k} t \quad \text{and} \quad \rho_{p,q}^{k}(t) := \left(\frac{p}{q}\right)^{k} t \quad \text{for } k \in \mathbb{N}, \text{ respectively.}
$$

The q-analogue of the power function $(a-b)_q^n$ with $n \in \mathbb{N}_0 := \{0,1,2,\dots\}$ is given by

$$
(a-b)_q^0 := 1, \qquad (a-b)_q^n := \prod_{i=0}^{n-1} (a - bq^i), \quad a,b \in \mathbb{R}.
$$

The (p,q)-analogue of the power function $(a - b)^n_{p,q}$ with $n \in \mathbb{N}_0$ is given by

$$(a - b)^0_{p,q} := 1, \qquad (a - b)^n_{p,q} := \prod_{k=0}^{n-1} (ap^k - bq^k), \quad a, b \in \mathbb{R}.$$

For $\alpha \in \mathbb{R}$, we define a general form:

$$(a - b)^\alpha_q = a^\alpha \prod_{i=0}^{\infty} \frac{1 - (\frac{b}{a})q^i}{1 - (\frac{b}{a})q^{\alpha+i}}, \quad a \neq 0.$$

$$(a - b)^\alpha_{p,q} = p^{\binom{\alpha}{2}}(a - b)^\alpha_{\frac{q}{p}} = a^\alpha \prod_{i=0}^{\infty} \frac{1}{p^\alpha} \left[\frac{1 - \frac{b}{a}(\frac{q}{p})^i}{1 - \frac{b}{a}(\frac{q}{p})^{i+\alpha}} \right], \quad a \neq 0.$$

Note that $a^\alpha_q = a^\alpha$, $a^\alpha_{p,q} = (\frac{a}{p})^\alpha$ and $(0)^\alpha_q = (0)^\alpha_{p,q} = 0$ for $\alpha > 0$.

The (p,q)-gamma and (p,q)-beta functions are defined by

$$\Gamma_{p,q}(x) := \begin{cases} \frac{(p-q)^{x-1}_{p,q}}{(p-q)^{x-1}} = \frac{(1-\frac{q}{p})^{x-1}_{p,q}}{(1-\frac{q}{p})^{x-1}}, & x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \\ [x-1]_{p,q}!, & x \in \mathbb{N} \end{cases}$$

$$B_{p,q}(x, y) := \int_0^1 t^{x-1}(1 - qt)^{y-1}_{p,q} \, d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x)\Gamma_q p, q(y)}{\Gamma_{p,q}(x + y)},$$

respectively.

Lemma 2.1 ([53]) *For $\alpha, \beta, \gamma, \lambda \in \mathbb{R}$,*

    (a) $(\gamma\beta - \gamma\lambda)^\alpha_{p,q} = \gamma^\alpha(\beta - \lambda)^\alpha_{p,q}$,

    (b) $(\beta - \gamma)^{\alpha+\gamma}_{p,q} = \frac{1}{p^{\alpha\gamma}}(\beta - \gamma)^\alpha_{p,q}(p^\alpha\beta - q^\alpha\lambda)^\gamma_{p,q}$,

    (c) $(t - s)^\alpha_{p,q} = 0, \quad \alpha \notin \mathbb{N}_0, t \geq s,$ *and* $t, s \in I^T_{p,q}$.

Lemma 2.2 ([53]) *For $m, n \in \mathbb{N}_0, \alpha \in \mathbb{R}$, and $0 < q < p \leq 1$,*

    (a) $\left(t - \sigma^n_{p,q}(t)\right)^\alpha_{p,q} = t^\alpha \left(1 - \left(\frac{q}{p}\right)^n\right)^\alpha_{p,q}$,

    (b) $\left(\sigma^m_{p,q}(t) - \sigma^n_{p,q}(t)\right)^\alpha_{p,q} = \left(\frac{q}{p}\right)^{m\alpha} t^\alpha \left(1 - \left(\frac{q}{p}\right)^{n-m}\right)^\alpha_{p,q}$.

Definition 2.1 For $0 < q < p \leq 1$ and $f : [0, T] \to \mathbb{R}$, we define the (p,q)-difference of f as

$$D_{p,q}f(t) := \begin{cases} \frac{f(pt) - f(qt)}{(p-q)(t)}, & \text{for } t \neq 0 \\ f'(0), & \text{for } t = 0 \end{cases}$$

provided that f is differentiable at 0. f is called (p,q)-differentiable on $I^T_{p,q}$ if $D_{p,q}f(t)$ exists for all $t \in I^T_{p,q}$.

Lemma 2.3 ([31]) *Let f, g be (p,q)-differentiable on $I_{p,q}^T$. The properties of (p,q)-difference operator are as follows*:

(a) $D_{p,q}[f(t) + g(t)] = D_{p,q}f(t) + D_{p,q}g(t)$,

(b) $D_{p,q}[\alpha f(t)] = \alpha D_{p,q}f(t)$ *for* $\alpha \in \mathbb{R}$,

(c) $D_{p,q}[f(t)g(t)] = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t) = g(pt)D_{p,q}f(t) + f(qt)D_{p,q}g(t)$,

(d) $D_{p,q}\left[\dfrac{f(t)}{g(t)}\right] = \dfrac{g(qt)D_{p,q}f(t) - f(qt)D_{p,q}g(t)}{g(pt)g(qt)} = \dfrac{g(pt)D_{p,q}f(t) - f(pt)D_{p,q}g(t)}{g(pt)g(qt)}$

   for $g(pt)g(qt) \neq 0$.

Lemma 2.4 ([53]) *Let $t \in I_{p,q}^T$, $0 < q < p \leq 1$, $\alpha \geq 1$, and $a \in \mathbb{R}$. Then*

(a) $D_{p,q}(t - a)_{p,q}^{\alpha} = [\alpha]_{p,q}(pt - a)_{p,q}^{\alpha-1}$,

(b) $D_{p,q}(a - t)_{p,q}^{\alpha} = -[\alpha]_{p,q}(a - qt)_{p,q}^{\alpha-1}$.

Definition 2.2 Let I be any closed interval of $\mathbb{R}$ containing a, b, and 0. Assuming that $f : I \to \mathbb{R}$ is a given function, we define (p,q)-integral of f from a to b by

$$\int_a^b f(t)\, d_{p,q}t := \int_0^b f(t)\, d_{p,q}t - \int_0^a f(t)\, d_{p,q}t,$$

where

$$\mathcal{I}_{p,q}f(x) = \int_0^x f(t)\, d_{p,q}t = (p - q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$. f is called (p,q)-integrable on $[a, b]$ if it is (p,q)-integrable on $[a, b]$ for all $a, b \in I$.

Next, we define an operator $\mathcal{I}_{p,q}^N$ as

$$\mathcal{I}_{p,q}^0 f(x) = f(x) \quad \text{and} \quad \mathcal{I}_{p,q}^N f(x) = \mathcal{I}_{p,q}\mathcal{I}_{p,q}^{N-1}f(x), \quad N \in \mathbb{N}.$$

The relations between (p,q)-difference and (p,q)-integral operators are given by

$$D_{p,q}\mathcal{I}_{p,q}f(x) = f(x) \quad \text{and} \quad \mathcal{I}_{p,q}D_{p,q}f(x) = f(x) - f(0).$$

Lemma 2.5 ([31]) *Let $0 < q < p \leq 1$, $a, b \in I_{p,q}^T$, and f, g be (p,q)-integrable on $I_{p,q}^T$. Then the following formulas hold*:

(a) $\displaystyle\int_a^a f(t)\, d_{p,q}t = 0$,

(b) $\displaystyle\int_a^b \alpha f(t)\, d_{p,q}t = \alpha \int_a^b f(t)\, d_{p,q}t$, $\quad \alpha \in \mathbb{R}$,

(c) $\displaystyle\int_a^b f(t)\, d_{p,q}t = -\int_b^a f(t)\, d_{p,q}t$,

(d) $\displaystyle\int_a^b f(t)\,d_{p,q}t = \int_c^b f(t)\,d_{p,q}t + \int_a^c f(t)\,d_{p,q}t, \quad c \in I_{p,q}^T, a < c < b,$

(e) $\displaystyle\int_a^b \big[f(t) + g(t)\big]\,d_{p,q}t = \int_a^b f(t)\,d_{p,q}t + \int_a^b g(t)\,d_{p,q}t,$

(f) $\displaystyle\int_a^b \big[f(pt)D_{p,q}g(t)\big]\,d_{p,q}t = \big[f(t)g(t)\big]_a^b - \int_a^b \big[g(qt)D_{p,q}f(t)\big]\,d_{p,q}t.$

Lemma 2.6 ([31], Fundamental theorem of (p,q)-calculus) *Letting $f : I \to \mathbb{R}$ be continuous at 0 and*

$$F(x) := \int_0^x f(t)\,d_{p,q}t, \quad x \in I,$$

then F is continuous at 0 and $D_{p,q}F(x)$ exists for every $x \in I$ where

$$D_{p,q}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{p,q}f(t)\,d_{p,q}t = f(b) - f(a) \quad \text{for all } a, b \in I.$$

Lemma 2.7 ([53], Leibniz formula of (p,q)-calculus) *Letting $f : I_{p,q}^T \times I_{p,q}^T \to \mathbb{R}$,*

$$D_{p,q}\left[\int_0^t f(t,s)\,d_{p,q}s\right] = \int_0^{qt} {}_t D_{p,q}f(t,s)\,d_{p,q}s + f(pt,t),$$

where ${}_t D_{p,q}$ is (p,q)-difference with respect to t.

Next we introduce fractional (p,q)-integral and fractional (p,q)-difference of Riemann–Liouville type as follows.

Definition 2.3 For $\alpha > 0$, $0 < q < p \le 1$, and f defined on $I_{p,q}^T$, the fractional (p,q)-integral is defined by

$$\begin{aligned}
\mathcal{I}_{p,q}^\alpha f(t) &:= \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{\alpha-1} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
&= \frac{(p-q)t}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}}\left(t - \left(\frac{q}{p}\right)^{k+1} t\right)_{p,q}^{\alpha-1} f\left(\frac{q^k}{p^{k+\alpha}}t\right),
\end{aligned}$$

and $(\mathcal{I}_{p,q}^0 f)(t) = f(t)$.

Definition 2.4 For $\alpha > 0$, $0 < q < p \le 1$, and f defined on $I_{p,q}^T$, the fractional (p,q)-difference operator of Riemann–Liouville type of order α is defined by

$$\begin{aligned}
D_{p,q}^\alpha f(t) &:= D_{p,q}^N \mathcal{I}_{p,q}^{N-\alpha} f(t) \\
&= \frac{1}{p^{\binom{-\alpha}{2}}\Gamma_{p,q}(-\alpha)} \int_0^t (t - qs)_{p,q}^{-\alpha-1} f\left(\frac{s}{p^{-\alpha-1}}\right) d_{p,q}s,
\end{aligned}$$

and $D_{p,q}^0 f(t) = f(t)$, where $N - 1 < \alpha < N$, $N \in \mathbb{N}$.

Lemma 2.8 ([53]) *Letting* $\alpha \in (N-1, N)$, $N \in \mathbb{N}$, $0 < q < p \le 1$, *and* $f : I_{p,q}^T \to \mathbb{R}$,

$$\mathcal{I}_{p,q}^\alpha D_{p,q}^\alpha f(t) = f(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

Lemma 2.9 ([53]) *Letting* $0 < q < p \le 1$ *and* $f : I_{p,q}^T \to \mathbb{R}$ *be continuous at* 0,

$$\int_0^x \int_0^s f(\tau)\, d_{p,q}\tau\, d_{p,q}s = \int_0^{\frac{x}{p}} \int_{pq\tau}^x f(\tau)\, d_{p,q}s\, d_{p,q}\tau.$$

Lemma 2.10 ([53]) *Let* $\alpha, \beta > 0$, $0 < q < p \le 1$. *Then*

(a) $\displaystyle \int_0^t (t - qs)_{p,q}^{\alpha-1} s^\beta\, d_{p,q}s = t^{\alpha+\beta} B_{p,q}(\beta + 1, \alpha),$

(b) $\displaystyle \int_0^t \int_0^x (t - qx)_{p,q}^{\alpha-1} (x - qs)_{p,q}^{\beta-1}\, d_{p,q}s\, d_{p,q}x = \frac{B_{p,q}(\beta + 1, \alpha)}{[\beta]_{p,q}} t^{\alpha+\beta}.$

Lemma 2.11 *Let* $\alpha, \beta > 0$, $0 < q < p \le 1$, *and* $n \in \mathbb{Z}$. *Then*

(a) $\displaystyle \int_0^t (t - qs)_{p,q}^{\alpha-1}\, d_{p,q}s = p^{\binom{\alpha}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha + 1)} t^\alpha,$

(b) $\displaystyle \int_0^t \int_0^{p^{-\beta-1}} (t - qx)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs \right)_{p,q}^{\alpha-1}\, d_{p,q}s\, d_{p,q}x = p^{\binom{\alpha}{2} + \binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha + 1)} t^{\alpha+\beta},$

(c) $\displaystyle \int_0^t (t - qs)_{p,q}^{-\beta-1} \left(\frac{s}{p^{-\beta-1}} \right)^{\alpha-n}\, d_{p,q}s = p^{\binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha - n + 1)\Gamma_{p,q}(-\beta)}{\Gamma_{p,q}(\alpha - \beta - n + 1)} t^{\alpha-\beta-n}.$

Proof From Lemma 2.10(a) and the definition of (p, q)-beta function, we have

$$\int_0^t (t - qs)_{p,q}^{\alpha-1}\, d_{p,q}s = t^\alpha B_{p,q}(1, \alpha) = p^{\binom{\alpha}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha + 1)} t^\alpha,$$

$$\int_0^t \int_0^{p^{-\beta-1}} (t - qx)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs \right)_{p,q}^{\alpha-1}\, d_{p,q}s\, d_{p,q}x$$

$$= \frac{B_{p,q}(1, \alpha)}{p^{-\alpha(\beta+1)}} \int_0^t (t - qx)_{p,q}^{-\beta-1} x^\alpha\, d_{p,q}x$$

$$= \frac{B_{p,q}(1, \alpha)}{p^{-\alpha(\beta+1)}} B_{p,q}(\alpha + 1, -\beta) t^{\alpha-\beta}$$

$$= p^{\binom{\alpha}{2} + \binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\beta)}{\Gamma_{p,q}(\alpha - \beta + 1)} t^{\alpha-\beta}.$$

For $n \in \mathbb{Z}$, we have

$$\int_0^t (t - qs)_{p,q}^{-\beta-1} \left(\frac{s}{p^{-\beta-1}} \right)^{\alpha-n}\, d_{p,q}s = \frac{B_{p,q}(\alpha, -\beta)}{p^{(\alpha-n)(-\beta-1)}} t^{\alpha-\beta-n+1}$$

$$= p^{\binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha - n + 1)\Gamma_{p,q}(-\beta)}{\Gamma_{p,q}(\alpha - \beta - n + 1)} t^{\alpha-\beta-n}.$$

The proof is complete. □

We next provide a lemma showing a result of the linear variant of problem (1.1).

Lemma 2.12 *Let $\Omega \neq 0$, $\alpha \in (1,2]$, $\beta \in (0,1]$, $0 < q < p \leq 1$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$, $h \in C(I_{p,q}^T, \mathbb{R})$ is a given function; $\phi_1, \phi_2 : C(I_{p,q}^T, \mathbb{R}) \to \mathbb{R}$ are given functionals. The linear variant problem of* (1.1)

$$
D_{p,q}^\alpha u(t) = h(t), \quad t \in I_{p,q}^T,
$$

$$
\lambda_1 u(\eta) + \lambda_2 D_{p,q}^\beta u(\eta) = \phi_1(u), \quad \eta \in I_{p,q}^T - \left\{ \omega_0, \frac{T}{p} \right\},
$$

$$
\mu_1 u\left(\frac{T}{p}\right) + \mu_2 D_{p,q}^\beta u\left(\frac{T}{p}\right) = \phi_2(u) \tag{2.1}
$$

has the unique solution

$$
\begin{aligned}
u(t) &= \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{\alpha-1} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
&\quad - \frac{t^{\alpha-1}}{\Omega} \left\{ \mathbf{B}_T \Phi_\eta[\phi_1, h] - \mathbf{B}_\eta \Phi_T[\phi_2, h] \right\} \\
&\quad + \frac{t^{\alpha-2}}{\Omega} \left\{ \mathbf{A}_T \Phi_\eta[\phi_1, h] - \mathbf{A}_\eta \Phi_T[\phi_2, h] \right\},
\end{aligned} \tag{2.2}
$$

where the functionals $\Phi_\eta[\phi_1, h]$, $\Phi_T[\phi_2, h]$ are defined by

$$
\begin{aligned}
\Phi_\eta[\phi_1, h] &:= \phi_1(u) - \frac{\lambda_1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^\eta (\eta - qs)_{p,q}^{\alpha-1} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
&\quad - \frac{\lambda_2}{p^{\binom{\alpha}{2} + \binom{-\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(-\beta)} \\
&\quad \times \int_0^\eta \int_0^{\frac{x}{p^{-\beta-1}}} (\eta - qx)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs\right)_{p,q}^{\alpha-1} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \, d_{p,q}x,
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
\Phi_T[\phi_2, h] &:= \phi_2(u) - \frac{\mu_1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{\alpha-1} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
&\quad - \frac{\mu_2}{p^{\binom{\alpha}{2} + \binom{-\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(-\beta)} \\
&\quad \times \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{-\beta-1}}} \left(\frac{T}{p} - qx\right)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs\right)_{p,q}^{\alpha-1} \\
&\quad \times h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \, d_{p,q}x,
\end{aligned} \tag{2.4}
$$

and the constants $\mathbf{A}_\eta$, $\mathbf{A}_T$, $\mathbf{B}_\eta$, $\mathbf{B}_T$, and Ω are defined by

$$
\begin{aligned}
\mathbf{A}_\eta &:= \lambda_1 \eta^{\alpha-1} + \frac{\lambda_2}{p^{\binom{-\beta}{2}} \Gamma_{p,q}(-\beta)} \int_0^\eta (\eta - qs)_{p,q}^{-\beta-1} \left(\frac{s}{p^{-\beta-1}}\right)^{\alpha-1} d_{p,q}s \\
&= \eta^{\alpha-1} \left(\lambda_1 + \frac{\lambda_2 \Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha - \beta)} \eta^{-\beta} \right),
\end{aligned} \tag{2.5}
$$

$$\mathbf{A}_T := \mu_1 \left(\frac{T}{p}\right)^{\alpha-1} + \frac{\mu_2}{p^{\binom{-\beta}{2}}\Gamma_{p,q}(-\beta)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{-\beta-1} \left(\frac{s}{p^{-\beta-1}}\right)^{\alpha-1} d_{p,q}s$$

$$= \left(\frac{T}{p}\right)^{\alpha-1} \left(\mu_1 + \frac{\mu_2 \Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha-\beta)}\left(\frac{T}{p}\right)^{-\beta}\right), \tag{2.6}$$

$$\mathbf{B}_\eta := \lambda_1 \eta^{\alpha-2} + \frac{\lambda_2}{p^{\binom{-\beta}{2}}\Gamma_{p,q}(-\beta)} \int_0^\eta (\eta - qs)_{p,q}^{-\beta-1} \left(\frac{s}{p^{-\beta-1}}\right)^{\alpha-2} d_{p,q}s$$

$$= \eta^{\alpha-2} \left(\lambda_1 + \frac{\lambda_2 \Gamma_{p,q}(\alpha-1)}{\Gamma_{p,q}(\alpha-\beta-1)}\eta^{-\beta}\right), \tag{2.7}$$

$$\mathbf{B}_T := \mu_1 \left(\frac{T}{p}\right)^{\alpha-2} + \frac{\mu_2}{p^{\binom{-\beta}{2}}\Gamma_{p,q}(-\beta)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{-\beta-1} \left(\frac{s}{p^{-\beta-1}}\right)^{\alpha-2} d_{p,q}s$$

$$= \left(\frac{T}{p}\right)^{\alpha-2} \left(\mu_1 + \frac{\mu_2 \Gamma_{p,q}(\alpha-1)}{\Gamma_{p,q}(\alpha-\beta-1)}\left(\frac{T}{p}\right)^{-\beta}\right), \tag{2.8}$$

$$\Omega := \mathbf{A}_T \mathbf{B}_\eta - \mathbf{A}_\eta \mathbf{B}_T. \tag{2.9}$$

Proof To obtain the solution, we first take a fractional (p, q)-integral of order α for (2.1). Then we have

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \mathcal{I}_{p,q}^\alpha h(t)$$

$$= C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{\alpha-1} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s. \tag{2.10}$$

Next, we take fractional p, q-difference of order β for (2.10) to get

$$D_{p,q}^\beta u(t) = \frac{1}{p^{\binom{-\beta}{2}}\Gamma_{p,q}(-\beta)} (t - qs)_{p,q}^{\alpha-1} \left[C_1 \left(\frac{s}{p^{\alpha-1}}\right)^{\alpha-1} + C_2 \left(\frac{s}{p^{\alpha-1}}\right)^{\alpha-2}\right] d_{p,q}s$$

$$+ \frac{1}{p^{\binom{\alpha}{2}+\binom{-\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\beta)}$$

$$\times \int_0^t \int_0^{\frac{x}{p^{-\beta-1}}} (t - qx)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs\right)_{p,q}^{\alpha-1} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x. \tag{2.11}$$

Substituting $t = \eta$ into (2.10) and (2.11) and employing the first condition of (2.1), we have

$$\mathbf{A}_\eta C_1 + \mathbf{B}_\eta C_2 = \Phi_\eta[\phi_1, h]. \tag{2.12}$$

Taking $t = T$ into (2.10) and (2.11) and employing the second condition of (2.1), we have

$$\mathbf{A}_T C_1 + \mathbf{B}_T C_2 = \Phi_T[\phi_2, h]. \tag{2.13}$$

Solving (2.12) − −(2.13), we find that

$$C_1 = \frac{\mathbf{B}_\eta \Phi_T - \mathbf{B}_T \Phi_\eta}{\Omega} \quad \text{and} \quad C_2 = \frac{\mathbf{A}_T \Phi_\eta - \mathbf{A}_\eta \Phi_T}{\Omega},$$

where $\Phi_\eta[\phi_1, h]$, $\Phi_T[\phi_2, h]$, $\mathbf{A}_\eta$, $\mathbf{A}_T$, $\mathbf{B}_\eta$, $\mathbf{B}_T$, Ω are defined by (2.3)–(2.9), respectively.

Substituting the constants C_1, C_2 into (2.10), we obtain (2.2). This completes the proof. □

3 Existence and uniqueness result

In this section, we use Banach's fixed point theorem to prove the existence and uniqueness result for problem (1.1). Let $\mathcal{C} = C(I_{p,q}^T, \mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \max_{t \in I_{p,q}^T} \{|u(t)|, |D_{p,q}^\nu u(t)|\},$$

where $\alpha \in (1,2]$, $\beta, \gamma, \nu \in (0,1]$, $0 < q < p \le 1$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$. Define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$
\begin{aligned}
(\mathcal{F}u)(t) := {} & \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{\alpha-1} \\
& \times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \Psi_{p,q}^\gamma u\left(\frac{s}{p^{\alpha-1}}\right), D_{p,q}^\nu u\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s \\
& - \frac{t^{\alpha-1}}{\Omega} \{\mathbf{B}_T \Phi_\eta^*[\phi_1, F_u] - \mathbf{B}_\eta \Phi_T^*[\phi_2, F_u]\} \\
& + \frac{t^{\alpha-2}}{\Omega} \{\mathbf{A}_T \Phi_\eta^*[\phi_1, F_u] - \mathbf{A}_\eta \Phi_T^*[\phi_2, F_u]\},
\end{aligned}
\tag{3.1}
$$

where the functionals $\Phi_\eta^*[\phi_1, F_u]$, $\Phi_T^*[\phi_2, F_u]$ are defined by

$$
\begin{aligned}
\Phi_\eta^*[\phi_1, F_u] := {} & \phi_1(u) - \frac{\lambda_1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^\eta (\eta - qs)_{p,q}^{\alpha-1} \\
& \times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \Psi_{p,q}^\gamma u\left(\frac{s}{p^{\alpha-1}}\right), D_{p,q}^\nu u\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s \\
& - \frac{\lambda_2}{p^{\binom{\alpha}{2}+\binom{-\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(-\beta)} \\
& \times \int_0^\eta \int_0^{\frac{x}{p^{-\beta-1}}} (\eta - qx)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs\right)_{p,q}^{\alpha-1} \\
& \times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \Psi_{p,q}^\gamma u\left(\frac{s}{p^{\alpha-1}}\right), D_{p,q}^\nu u\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s \, d_{p,q}x, \quad (3.2) \\
\Phi_T^*[\phi_2, F_u] := {} & \phi_2(u) - \frac{\mu_1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{\alpha-1} \\
& \times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \Psi_{p,q}^\gamma u\left(\frac{s}{p^{\alpha-1}}\right), D_{p,q}^\nu u\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s \\
& - \frac{\mu_2}{p^{\binom{\alpha}{2}+\binom{-\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(-\beta)} \\
& \times \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{-\beta-1}}} \left(\frac{T}{p} - qx\right)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs\right)_{p,q}^{\alpha-1} \\
& \times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \Psi_{p,q}^\gamma u\left(\frac{s}{p^{\alpha-1}}\right), D_{p,q}^\nu u\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s \, d_{p,q}x, \quad (3.3)
\end{aligned}
$$

and the constants $\mathbf{A}_\eta$, $\mathbf{A}_T$, $\mathbf{B}_\eta$, $\mathbf{B}_T$, Ω are defined by (2.5)–(2.9), respectively.

Theorem 3.1 *Assume that $F : I_{p,q}^T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi : I_{p,q}^T \times I_{p,q}^T \to [0, \infty)$ is continuous with $\varphi_0 = \max\{\varphi(t,s) : (t,s) \in I_{p,q}^T \times I_{p,q}^T\}$. Suppose that the following conditions hold*:

(H_1) *There exist constants $\ell_1, \ell_2, \ell_3 > 0$ such that, for each $t \in I_{p,q}^T$ and $u_i, v_i \in \mathbb{R}, i = 1, 2, 3$,*

$$\left| F[t, u_1, u_2, u_3] - F[t, v_1, v_2, v_3] \right| \leq \ell_1 |u_1 - v_1| + \ell_2 |u_2 - v_2| + \ell_3 |u_3 - v_3|.$$

(H_2) *There exist constants $\omega_1, \omega_2 > 0$ such that, for each $u, v \in \mathcal{C}$,*

$$\left| \phi_1(u) - \phi_1(v) \right| \leq \omega_1 \|u - v\|_\mathcal{C} \quad and \quad \left| \phi_2(u) - \phi_2(v) \right| \leq \omega_2 \|u - v\|_\mathcal{C}.$$

(H_3) $\mathcal{X} := (\mathcal{L} + \ell_3)\Theta + \omega_1 \Upsilon_T + \omega_2 \Upsilon_\eta < 1,$

where

$$\mathcal{L} := \ell_1 + \ell_2 \varphi_0 \frac{(\frac{T}{p})^\gamma}{\Gamma_{p,q}(\gamma + 1)}, \tag{3.4}$$

$$\Theta := \frac{(\frac{T}{p})^\alpha}{\Gamma_{p,q}(\alpha + 1)} + \mathcal{O}_1 \Upsilon_T + \mathcal{O}_2 \Upsilon_\eta. \tag{3.5}$$

$$\Upsilon_T := \frac{1}{|\Omega|}\left[|\mathbf{B}_T|\left(\frac{T}{p}\right)^{\alpha-1} + |\mathbf{A}_T|\left(\frac{T}{p}\right)^{\alpha-2} \right], \tag{3.6}$$

$$\Upsilon_\eta := \frac{1}{|\Omega|}\left[|\mathbf{B}_\eta|\left(\frac{T}{p}\right)^{\alpha-1} + |\mathbf{A}_\eta|\left(\frac{T}{p}\right)^{\alpha-2} \right], \tag{3.7}$$

$$\mathcal{O}_1 := \frac{\lambda_1 \eta^\alpha}{\Gamma_{p,q}(\alpha + 1)} + \frac{\lambda_2 \eta^{\alpha-\beta}}{\Gamma_{p,q}(\alpha - \beta + 1)}, \tag{3.8}$$

$$\mathcal{O}_2 := \frac{\lambda_1 (\frac{T}{p})^\alpha}{\Gamma_{p,q}(\alpha + 1)} + \frac{\lambda_2 (\frac{T}{p})^{\alpha-\beta}}{\Gamma_{p,q}(\alpha - \beta + 1)}. \tag{3.9}$$

Then problem (1.1) *has a unique solution in $I_{p,q}^T$.*

Proof For each $t \in I_{p,q}^T$ and $u, v \in \mathcal{C}$, we have

$$\left| \Psi_{p,q}^\gamma u - \Psi_{p,q}^\gamma v \right| \leq \frac{\varphi_0}{p^{\binom{\gamma}{2}} \Gamma_{p,q}(\gamma)} \int_0^t (t - qs)_{p,q}^{\frac{\gamma-1}{p,q}} \left| u\left(\frac{s}{p^{\gamma-1}}\right) - v\left(\frac{s}{p^{\gamma-1}}\right) \right| d_{p,q}s.$$

$$\leq \frac{\varphi_0}{p^{\binom{\gamma}{2}} \Gamma_{p,q}(\gamma)} |u - v| \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{\gamma-1} d_{p,q}s.$$

$$= \frac{\varphi_0 (\frac{T}{p})^\gamma}{\Gamma_{p,q}(\gamma + 1)} |u - v|.$$

Denote that

$$\mathcal{H}|u - v|(t) := \left| F\left[t, u(t), \Psi_{p,q}^\gamma u(t), D_{p,q}^\nu u(t)\right] - F\left[t, v(t), \Psi_{p,q}^\gamma v(t), D_{p,q}^\nu v(t)\right] \right|.$$

Then we obtain

$$\left|\Phi_\eta^*[\phi_1, F_u] - \Phi_\eta^*[\phi_1, F_v]\right|$$

$$\leq \left|\phi_1(u) - \phi_1(v)\right| + \frac{\lambda_1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^\eta (\eta - qs)_{p,q}^{\alpha-1} \mathcal{H}|u - v|\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \frac{\lambda_2}{p^{\binom{\alpha}{2}+\binom{-\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\beta)} \int_0^\eta \int_0^{\frac{x}{p^{-\beta-1}}} (\eta - qx)_{p,q}^{-\beta-1}\left(\frac{x}{p^{-\beta-1}} - qs\right)_{p,q}^{\alpha-1},$$

$$\mathcal{H}|u - v|\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x$$

$$\leq \omega_1\|u - v\|_{\mathcal{C}} + \left(\ell_1|u - v| + \ell_2\left|\Psi_{p,q}^\gamma u - \Psi_{p,q}^\gamma v\right| + \ell_3\left|D_{p,q}^\nu u - D_{p,q}^\nu v\right|\right)$$

$$\times \left|\frac{\lambda_1\eta^\alpha}{\Gamma_{p,q}(\alpha+1)} + \frac{\lambda_2\eta^{\alpha-\beta}}{\Gamma_{p,q}(\alpha-\beta+1)}\right|$$

$$\leq \omega_1\|u - v\|_{\mathcal{C}} + \left(\left[\ell_1 + \ell_2\varphi_0\frac{(\frac{T}{p})^\gamma}{\Gamma_{p,q}(\gamma+1)}\right]|u - v| + \ell_3\left|D_{p,q}^\nu u - D_{p,q}^\nu v\right|\right)\mathcal{O}_1$$

$$\leq \left[\omega_1 + (\mathcal{L} + \ell_3)\mathcal{O}_1\right]\|u - v\|_{\mathcal{C}}.$$

Similarly,

$$\left|\Phi_T^*[\phi_2, F_u] - \Phi_T^*[\phi_2, F_v]\right| \leq \left[\omega_2 + (\mathcal{L} + \ell_3)\mathcal{O}_2\right]\|u - v\|_{\mathcal{C}}.$$

Next, we have

$$\left|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\right|$$

$$\leq \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{\alpha-1} \mathcal{H}|u - v|\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \frac{(\frac{T}{p})^{\alpha-1}}{|\Omega|}\left\{|\mathbf{B}_T|\left|\Phi_\eta^*[\phi_1, F_u] - \Phi_\eta^*[\phi_1, F_v]\right| + |\mathbf{B}_\eta|\left|\Phi_T^*[\phi_2, F_u] - \Phi_T^*[\phi_2, F_v]\right|\right\}$$

$$+ \frac{(\frac{T}{p})^{\alpha-2}}{|\Omega|}\left\{|\mathbf{A}_T|\left|\Phi_\eta^*[\phi_1, F_u] - \Phi_\eta^*[\phi_1, F_v]\right| + |\mathbf{A}_\eta|\left|\Phi_T^*[\phi_2, F_u] - \Phi_T^*[\phi_2, F_v]\right|\right\}$$

$$\leq \left[\frac{(\mathcal{L} + \ell_3)(\frac{T}{p})^\alpha}{\Gamma_{p,q}(\alpha+1)} + \frac{[\omega_1 + (\mathcal{L} + \ell_3)\mathcal{O}_1]}{\Omega}\left\{|\mathbf{B}_T|\left(\frac{T}{p}\right)^{\alpha-1} + |\mathbf{A}_T|\left(\frac{T}{p}\right)^{\alpha-2}\right\}\right.$$

$$\left. + \frac{[\omega_2 + (\mathcal{L} + \ell_3)\mathcal{O}_2]}{|\Omega|}\left\{|\mathbf{B}_\eta|\left(\frac{T}{p}\right)^{\alpha-1} + |\mathbf{A}_\eta|\left(\frac{T}{p}\right)^{\alpha-2}\right\}\right]\|u - v\|_{\mathcal{C}}$$

$$= \mathcal{X}\|u - v\|_{\mathcal{C}}. \tag{3.10}$$

Taking fractional (p, q)-difference of order ν for (3.1), we get

$$\left(D_{p,q}^\nu \mathcal{F}u\right)(t)$$

$$= \frac{1}{p^{\binom{\alpha}{2}+\binom{-\nu}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\nu)} \int_0^t \int_0^{\frac{x}{p^{-\nu-1}}} (t - qx)_{p,q}^{-\nu-1}\left(\frac{x}{p^{-\nu-1}} - qs\right)_{p,q}^{\alpha-1}$$

$$\times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \Psi_{p,q}^{\gamma} u\left(\frac{s}{p^{\alpha-1}}\right), D_{p,q}^{\nu} u\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s\, d_{p,q}x$$

$$- \frac{1}{\Omega p^{\binom{-\nu}{2}} \Gamma_{p,q}(-\nu)}\left\{\mathbf{B}_T \Phi_{\eta}^*[\phi_1, F_u] - \mathbf{B}_{\eta} \Phi_T^*[\phi_2, F_u]\right\}$$

$$\times \int_0^t (t - qs)_{p,q}^{-\nu-1}\left(\frac{s}{p^{-\nu-1}}\right)^{\alpha-1} d_{p,q}s$$

$$+ \frac{1}{\Omega p^{\binom{-\nu}{2}} \Gamma_{p,q}(-\nu)}\left\{\mathbf{A}_T \Phi_{\eta}^*[\phi_1, F_u] - \mathbf{A}_{\eta} \Phi_T^*[\phi_2, F_u]\right\}$$

$$\times \int_0^t (t - qs)_{p,q}^{-\nu-1}\left(\frac{s}{p^{-\nu-1}}\right)^{\alpha-2} d_{p,q}s. \tag{3.11}$$

Similarly, we have

$$\left|\left(D_{p,q}^{\nu} \mathcal{F} u\right)(t) - \left(D_{p,q}^{\nu} \mathcal{F} v\right)(t)\right| < \mathcal{X} \|u - v\|_{\mathcal{C}}. \tag{3.12}$$

From (3.10) and (3.12), we obtain

$$\|\mathcal{F} u - \mathcal{F} v\|_{\mathcal{C}} \leq \mathcal{X} \|u - v\|_{\mathcal{C}}.$$

By (H_3) we can conclude that $\mathcal{F}$ is a contraction. Therefore, by using Banach's fixed point theorem, $\mathcal{F}$ has a fixed point which is a unique solution of problem (1.1) on $I_{p,q}^T$. □

4 Existence of at least one solution

In this section, we present the existence of a solution to (1.1) by using Schauder's fixed point theorem.

Lemma 4.1 ([57]) (*Arzelá–Ascoli theorem*) *A collection of functions in $C[a,b]$ with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 4.2 ([57]) *If a set is closed and relatively compact, then it is compact.*

Lemma 4.3 ([58] (Schauder's fixed point theorem)) *Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Theorem 4.1 *Suppose that (H_1) and (H_3) hold. Then problem (1.1) has at least one solution on $I_{p,q}^T$.*

Proof We organize the proof into three steps as follows.

Step I. Verify that $\mathcal{F}$ maps bounded sets into bounded sets in $B_L = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq L\}$. Set $\max_{t \in I_{p,q}^T} |F(t,0,0,0)| = M$, $\sup_{u \in \mathcal{C}} |\phi_1(u)| = N_1$, $\sup_{u \in \mathcal{C}} |\phi_2(u)| = N_2$ and choose a constant

$$L \geq \frac{M\Theta + N_1 \Upsilon_T + N_2 \Upsilon_{\eta}}{1 - (\mathcal{L} + \ell_3)\Theta}. \tag{4.1}$$

Denote that $|\mathcal{S}(t, u, 0)| = |F[t, u(t), \Psi_{p,q}^{\gamma} u(t), D_{p,q}^{\nu} u(t)] - F[t, 0, 0, 0]| + |F[t, 0, 0, 0]|$. For each $t \in I_{p,q}^{T}$ and $u \in B_L$, we obtain

$$\left| \Phi_{\eta}^{*}[\phi_1, F_u] \right|$$

$$\leq N_1 + \frac{\lambda_1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^{\eta} (\eta - qs)_{p,q}^{\alpha-1} \left| \mathcal{S}(s, u, 0) \right| d_{p,q}s + \frac{\lambda_2}{p^{\binom{\alpha}{2}+\binom{-\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(-\beta)}$$

$$\times \int_0^{\eta} \int_0^{\frac{x}{p^{-\beta-1}}} (\eta - qx)_{p,q}^{-\beta-1} \left(\frac{x}{p^{-\beta-1}} - qs \right)_{p,q}^{\alpha-1} \left| \mathcal{S}(s, u, 0) \right| d_{p,q}s \, d_{p,q}x$$

$$\leq N_1 + \left(\left[\ell_1 + \ell_2 \varphi_0 \frac{(\frac{T}{p})^{\gamma}}{\Gamma_{p,q}(\gamma + 1)} \right] |u| + \ell_3 |D_{p,q}^{\nu} u| + M \right) \mathcal{O}_1$$

$$\leq N_1 + M\mathcal{O}_1 + (\mathcal{L} + \ell_3)\mathcal{O}_1 \|u\|_{\mathcal{C}}$$

$$\leq N_1 + \left[M + (\mathcal{L} + \ell_3)L \right] \mathcal{O}_1. \tag{4.2}$$

Similarly,

$$\left| \Phi_{T}^{*}[\phi_2, F_u] \right| \leq N_2 + \left[M + (\mathcal{L} + \ell_3)L \right] \mathcal{O}_2. \tag{4.3}$$

From (4.2)–(4.3), we find that

$$|(\mathcal{F}u)(t)| \leq \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs \right)_{p,q}^{\alpha-1} \left| \mathcal{S}(t, u, 0) \right| \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s$$

$$+ \frac{(\frac{T}{p})^{\alpha-1}}{|\Omega|} \left\{ |\mathbf{B}_T| \left| \Phi_{\eta}^{*}[\phi_1, F_u] \right| + |\mathbf{B}_\eta| \left| \Phi_{T}^{*}[\phi_2, F_u] \right| \right\}$$

$$+ \frac{(\frac{T}{p})^{\alpha-2}}{|\Omega|} \left\{ |\mathbf{A}_T| \left| \Phi_{\eta}^{*}[\phi_1, F_u] \right| + |\mathbf{A}_\eta| \left| \Phi_{T}^{*}[\phi_2, F_u] \right| \right\}$$

$$\leq \Theta \left[L(\mathcal{L} + \ell_3) + M \right] + N_1 \Upsilon_T + N_2 \Upsilon_\eta$$

$$\leq L. \tag{4.4}$$

In addition, we obtain

$$\left| (D_{p,q}^{\nu} \mathcal{F}u)(t) \right| < L. \tag{4.5}$$

Therefore, $\|\mathcal{F}u\|_{\mathcal{C}} \leq L$, which implies that $\mathcal{F}$ is uniformly bounded.

Step II. The operator $\mathcal{F}$ is continuous on B_L because of the continuity of F.

Step III. We examine that $\mathcal{F}$ is equicontinuous on B_L. For any $t_1, t_2 \in I_{p,q}^{T}$ with $t_1 < t_2$, we have

$$|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)| \leq \frac{\|F\|}{\Gamma_{p,q}(\alpha + 1)} \left| t_2^{\alpha} - t_1^{\alpha} \right|$$

$$+ \frac{|t_2^{\alpha-1} - t_1^{\alpha-1}|}{|\Omega|} \left\{ |\mathbf{B}_T| \Phi_{\eta}^{*}[\phi_1, F_u] + |\mathbf{B}_\eta| \Phi_{T}^{*}[\phi_2, F_u] \right\}$$

$$+ \frac{|t_2^{\alpha-2} - t_1^{\alpha-2}|}{|\Omega|} \left\{ |\mathbf{A}_T| \Phi_{\eta}^{*}[\phi_1, F_u] + |\mathbf{A}_\eta| \Phi_{T}^{*}[\phi_2, F_u] \right\} \tag{4.6}$$

and

$$
\begin{aligned}
&\left|\left(D_{p,q}^{\nu}\mathcal{F}u\right)(t_2) - \left(D_{p,q}^{\nu}\mathcal{F}u\right)(t_1)\right| \\
&\quad \le \frac{\|F\|}{\Gamma_{p,q}(\alpha - \nu + 1)}\left|t_2^{\alpha-\nu} - t_1^{\alpha-\nu}\right| \\
&\qquad + \frac{\Gamma_{p,q}(\alpha)}{|\Omega|\Gamma_{p,q}(\alpha - \nu)}\left\{|\mathbf{B}_T|\Phi_\eta^*[\phi_1, F_u] + |\mathbf{B}_\eta|\Phi_T^*[\phi_2, F_u]\right\}\left|t_2^{\alpha-\nu-1} - t_1^{\alpha-\nu-1}\right| \\
&\qquad + \frac{\Gamma_{p,q}(\alpha - 1)}{|\Omega|\Gamma_{p,q}(\alpha - \nu - 1)}\left\{|\mathbf{A}_T|\Phi_\eta^*[\phi_1, F_u] + |\mathbf{A}_\eta|\Phi_T^*[\phi_2, F_u]\right\}\left|t_2^{\alpha-\nu-2} - t_1^{\alpha-\nu-2}\right|. \quad (4.7)
\end{aligned}
$$

Since the right-hand side of (4.6) and (4.7) tends to be zero when $|t_2 - t_1| \to 0$, $\mathcal{F}$ is relatively compact on B_L.

This implies that $\mathcal{F}(B_L)$ is an equicontinuous set. From Steps I to III together with the Arzelá–Ascoli theorem, we see that $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous. By Schauder's fixed point theorem, we can conclude that problem (1.1) has at least one solution. $\square$

5 An example

Consider the following fractional (p,q)-integrodifference equation:

$$
\begin{aligned}
D_{\frac{2}{3},\frac{1}{2}}^{\frac{4}{3}}u(t) = {}&\frac{1}{(2000e^3 + t^3)(1 + |u(t)|)}\Big[e^{-2t}\left(u^2 + 2|u|\right) + e^{-(2\pi + \cos^2 \pi t)}\big|\Psi_{\frac{2}{3},\frac{1}{2}}^{\frac{1}{2}}u(t)\big| \\
&+ e^{-(2+\sin^2 \pi t)}\big|D_{\frac{2}{3},\frac{1}{2}}^{\frac{2}{5}}u(t)\big|\Big], \quad t \in I_{\frac{2}{3},\frac{1}{2}}^{10} = \left\{\frac{10(\frac{1}{2})^k}{(\frac{2}{3})^{k+1}} : k \in \mathbb{N}_0\right\} \cup \{0\}
\end{aligned}
$$

with the nonlocal Robin boundary condition

$$
\frac{1}{20e}u\left(\frac{1215}{256}\right) + 200eD_{\frac{2}{3},\frac{1}{2}}^{\frac{3}{4}}u\left(\frac{1215}{256}\right) = \sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1 + |u(t_i)|}, \quad t_i = 10\frac{(\frac{1}{2})^i}{(\frac{2}{3})^{i+1}},
$$

$$
100\pi u(15) + \frac{1}{10\pi}D_{\frac{2}{3},\frac{1}{2}}^{\frac{3}{4}}u(15) = \sum_{i=0}^{\infty}\frac{D_i|u(t_i)|}{1 + |u(t_i)|}, \quad t_i = 10\frac{(\frac{1}{2})^i}{(\frac{2}{3})^{i+1}},
$$

where $\varphi(t,s) = \frac{e^{-|s-t|}}{(t+20)^3}$ and C_i, D_i are given constants with $\frac{1}{500} \le \sum_{i=0}^{\infty}C_i \le \frac{\pi}{1000}$ and $\frac{1}{1000} \le \sum_{i=0}^{\infty}D_i \le \frac{\pi}{2000}$.

Letting $\alpha = \frac{4}{3}$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{2}$, $\nu = \frac{2}{5}$, $p = \frac{2}{3}$, $q = \frac{1}{2}$, $T = 10$, $\eta = 10\frac{(\frac{1}{2})^4}{(\frac{2}{3})^5} = \frac{1215}{256}$, $\lambda_1 = \frac{1}{20e}$, $\lambda_2 = 200e$, $\mu_1 = 100\pi$, $\mu_2 = \frac{1}{10\pi}$, $\phi_1(u) = \sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|}$, $\phi_2 = \sum_{i=0}^{\infty}\frac{D_i|u(t_i)|}{1+|u(t_i)|}$ and $F[t, u(t), \Psi_{p,q}^{\gamma}u(t), D_{p,q}^{\nu}u(t)] = \frac{1}{(2000e^3+t^3)(1+|u(t)|)}[e^{-2t}(u^2 + 2|u|) + e^{-(2\pi+\cos^2 \pi t)}|\Psi_{\frac{2}{3},\frac{1}{2}}^{\frac{1}{2}}u(t)| + e^{-(2+\sin^2 \pi t)}|D_{\frac{2}{3},\frac{1}{2}}^{\frac{2}{5}}u(t)|]$, we find that

$$|\mathbf{A}_\eta| = 574.6570, \qquad |\mathbf{A}_T| = -23.8344, \qquad |\mathbf{B}_\eta| = 774.8145, \qquad |\mathbf{B}_T| = 51.6518,$$

$$\varphi_0 = 0.000125, \quad \text{and} \quad |\Omega| = -48{,}149.3072.$$

For all $t \in I_{\frac{2}{3}, \frac{1}{2}}^{10}$ and $u, v \in \mathbb{R}$, we have

$$
\left| F\left[t, u, \Psi_{p,q}^{\gamma} u, D_{p,q}^{v} u \right] - F\left[t, v, \Psi_{p,q}^{\gamma} v, D_{p,q}^{v} v \right] \right|
$$
$$
\leq \frac{1}{2000 e^2} |u - v| + \frac{1}{2000 e^{2\pi + 2}} \left| \Psi_{p,q}^{\gamma} u - \Psi_{p,q}^{\gamma} v \right| + \frac{1}{2000 e^4} \left| D_{p,q}^{v} u - D_{p,q}^{v} v \right|.
$$

Thus, (H_1) holds with $\ell_1 = 6.767 \times 10^{-5}$, $\ell_2 = 1.264 \times 10^{-7}$, and $\ell_3 = 9.158 \times 10^{-6}$. For all $u, v \in \mathcal{C}$,

$$
\left| \phi_1(u) - \phi_1(v) \right| \leq \frac{\pi}{1000} \| u - v \|_{\mathcal{C}},
$$
$$
\left| \phi_2(u) - \phi_2(v) \right| \leq \frac{e}{2000} \| u - v \|_{\mathcal{C}}.
$$

So, (H_2) holds with $\omega_1 = 0.003142$ and $\omega_2 = 0.001571$. Since

$$
\mathcal{L} = 0.0000677, \qquad \mathcal{O}_1 = 1415.89969, \qquad \mathcal{O}_2 = 2770.8547,
$$
$$
\Upsilon_T = 0.005291, \qquad \Upsilon_\eta = 0.003183, \quad \text{and} \quad \Theta = 51{,}3459,
$$

(H_3) holds with

$$
\mathcal{X} \approx 0.00397 < 1.
$$

Hence, by Theorem 3.1 this problem has a unique solution. Moreover, by Theorem 4.1 this problem has at least one solution.

Availability of data and materials
Not applicable.

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The authors declare that they carried out all the work in this manuscript and read and approved the final manuscript.

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