Sun *Advances in Difference Equations* (2020) 2020:426
https://doi.org/10.1186/s13662-020-02812-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Some Hermite–Hadamard type inequalities for generalized h-preinvex function via local fractional integrals and their applications

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Abstract

The concept of generalized h-preinvex function on real linear fractal sets R^β ($0 < \beta \le 1$) is introduced, which extends generalized preinvex, generalized s-preinvex, generalized Godunova–Levin preinvex, and generalized P-preinvex functions. In addition, some Hermite–Hadamard type inequalities for these classes of functions involving local fractional integrals are established. Lastly, the upper bounds for generalized expectation, generalized rth moment, and generalized variance of a continuous random variable are given to illustrate the applications of the obtained results.

MSC: 26D15; 26A51; 26A33

Keywords: Generalized h-preinvex function; Hermite–Hadamard type inequalities; Local fractional integrals; Fractal sets

1 Introduction

We recall the well-known Hermite–Hadamard inequality for convex function.

Theorem 1 ([1], Hermite–Hadamard's inequality) *Let $g : I \subseteq R \to R$ be a convex function and $c, d \in I$ with $c < d$, then*

$$g\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d g(x)\,dx \le \frac{g(c)+g(d)}{2}. \tag{1.1}$$

Most of the research on this class of inequalities is related to convexity. With the improvement of the definition of convexity, some new results for Hermite–Hadamard's inequality are obtained. We refer the readers to [2–7]. We firstly recall the following well-known definitions of convexity.

Definition 1 ([8, 9]) Let $C \subseteq R^n$. The set C is said to be invex with respect to $\eta : R^n \times R^n \to R^n$ if

$$x, y \in C, \qquad 0 \le \tau \le 1 \quad \Rightarrow \quad y + \tau\eta(x,y) \in C.$$

Definition 2 ([8, 9]) Let $C \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \to R^n$. The function $g : C \to R$ is called preinvex function if

$$g(y + \tau\eta(x,y)) \le \tau g(x) + (1-\tau)g(y), \tag{1.2}$$

where $x, y \in C$, $0 \le \tau \le 1$.

Definition 3 ([5]) Let $h : J \to R$, where $(0,1) \subseteq J$, be an interval in R, and let C be an invex set with respect to $\eta(\cdot,\cdot)$. We say that $g : C \to R$ is an h-preinvex function with respect to $\eta(\cdot,\cdot)$ if for all $x, y \in C$ and $\tau \in (0,1)$ we have

$$g(y + \tau\eta(x,y)) \le h(\tau)g(x) + h(1-\tau)g(y). \tag{1.3}$$

If inequality (1.3) is reversed, then g is said to be h-preconcave with respect to $\eta(\cdot,\cdot)$.

Remark 1 Obviously, if $h(\tau) = \tau$, then h-preinvex reduces to preinvex; if $h(\tau) = \tau^s$, $s \in (0,1)$, then h-preinvex reduces to s-preinvex; if $h(\tau) = \frac{1}{\tau}$, then h-preinvex reduces to Q-preinvex; if $h(\tau) = 1$, then h-preinvex reduces to P-preinvex.

In [10], Mohan and Neogy proposed the Condition C for bifunction $\eta(\cdot,\cdot)$.

Condition C Let $I \subset R$ be an invex set, for every $x, y \in I$, $\tau \in [0,1]$, bifunction $\eta(\cdot,\cdot)$ satisfies

$$\eta(y, y + \tau\eta(x,y)) = -\tau\eta(x,y),$$
$$\eta(x, y + \tau\eta(x,y)) = (1-\tau)\eta(x,y).$$

From Condition C, we know that the following equality holds:

$$\eta(y + \tau_2\eta(x,y), y + \tau_1\eta(x,y)) = (\tau_2 - \tau_1)\eta(x,y),$$

where $x, y \in I$, $\tau_1, \tau_2 \in [0,1]$.

Recently, Riemann–Liouville and Katugampola fractional derivatives and integrals were widely used in various research works [3, 11–13]. Especially, in order to explain the phenomenon of continuous but nowhere differentiable function, Yang [14, 15] introduced the definition of local fractional calculus. Local fractional calculus theory is widely used in many fields of science and physics [16–19]. Based on Yang's theory on fractal sets, some researchers have extended the definitions of convexity to study Hermite–Hadamard type inequalities [19–29].

In [22] and [23], Sun introduced generalized preinvex function and generalized s-preinvex function on fractal sets, respectively.

Definition 4 ([22]) Let $C \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \to R^n$. The function $g : C \to R^\beta$ is called generalized preinvex function if, for every $x, y \in C$, $0 \le \tau \le 1$, we have

$$g(y + \tau\eta(x,y)) \le \tau^\beta g(x) + (1-\tau)^\beta g(y). \tag{1.4}$$

Definition 5 ([23]) Let $C \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \to R^n$. A function $g : C \to R^\beta$ is called generalized s-preinvex function if, for every $x, y \in C$, $s \in (0,1]$, $0 \leq \tau \leq 1$, we have

$$g\big(y + \tau\eta(x,y)\big) \leq \tau^{s\beta} g(x) + (1-\tau)^{s\beta} g(y). \tag{1.5}$$

In [25], Vivas et al. proposed generalized h-convex function. Sun [26] introduced another form of this definition with β-type as follows.

Definition 6 Let $h : J \to R$ be a nonnegative function, $h \not\equiv 0$, and $g : I \to R^\beta$ $(0 < \beta \leq 1)$ be a function of fractal dimension β. We say that g is a generalized h-convex function on fractal sets if g is nonnegative ($g \geq 0^\beta$) and for all $x, y \in I$ and $\tau \in (0,1)$ we have

$$g\big(\tau x + (1-\tau)y\big) \leq h^\beta(\tau) g(x) + h^\beta(1-\tau) g(y). \tag{1.6}$$

Combining the definitions of generalized preinvex function and generalized h-convex function, the main purpose of this paper is to introduce the concept of generalized h-preinvex function on fractal sets, which extends generalized h-convex function, generalized preinvex function, and generalized s-preinvex function. And we derive the definitions of generalized Godunova–Levin preinvex function and generalized P-preinvex function, which are special cases of generalized h-preinvex function. Therefore, this definition is a great generalization of convexity. Then, some generalized Hermite–Hadamard type inequalities for generalized h-preinvex function are established under certain conditions. Some upper bounds for generalized expectation, generalized rth moment, and generalized variance of a continuous random variable are discussed, which can be obtained from previous results in this paper.

2 Preliminaries

Using Yang's idea [14, 15], recall Yang's fractional sets Ω^β as follows, where the set Ω is called base set of fractional set, and β denotes the dimension of cantor set, $0 < \beta \leq 1$:

The β-type set of integers Z^β is defined by

$$Z^\beta = \{0^\beta, \pm 1^\beta, \pm 2^\beta, \pm 3^\beta, \dots\};$$

The β-type set of rational numbers Q^β is defined by

$$Q^\beta = \left\{ m^\beta = \left(\frac{r}{s}\right)^\beta : r, s \in Z, s \neq 0 \right\};$$

The β-type set of irrational numbers $\Im^\beta$ is defined by

$$\Im^\beta = \left\{ m^\beta \neq \left(\frac{r}{s}\right)^\beta : r, s \in Z, s \neq 0 \right\};$$

The β-type set of the real line numbers R^β is defined by

$$R^\beta = Q^\beta \cup \Im^\beta.$$

If $a^\beta, b^\beta, c^\beta \in R^\beta$, then

(1) $a^\beta + b^\beta \in R^\beta$, $a^\beta b^\beta \in R^\beta$,

(2) $a^\beta + b^\beta = b^\beta + a^\beta = (a+b)^\beta = (b+a)^\beta$,

(3) $a^\beta + (b^\beta + c^\beta) = (a+b)^\beta + c^\beta$,

(4) $a^\beta b^\beta = b^\beta a^\beta = (ab)^\beta = (ba)^\beta$,

(5) $a^\beta (b^\beta c^\beta) = (a^\beta b^\beta) c^\beta$,

(6) $a^\beta (b^\beta + c^\beta) = a^\beta b^\beta + a^\beta c^\beta$,

(7) $a^\beta + 0^\beta = 0^\beta + a^\beta = a^\beta$ and $a^\beta 1^\beta = 1^\beta a^\beta = a^\beta$,

(8) For each $a^\beta \in R^\beta$, its inverse element $(-a)^\beta$ may be written as $-a^\beta$; for each $b^\beta \in R^\beta \setminus 0^\beta$, its inverse element $(1/b)^\beta$ may be written as $1^\beta / b^\beta$ but not as $1/b^\beta$ [29],

(9) $a^\beta = b^\beta$ if and only if $a = b$,

(10) $a^\beta < b^\beta$ if and only if $a < b$.

The definitions of the local fractional derivative and local fractional integral on R^β are stated as follows.

Definition 7 ([14, 15]) A non-differentiable function $g : R \to R^\beta$, $x \to g(x)$ is called local fractional continuous at x_0 if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|g(x) - g(x_0)\right| < \varepsilon^\beta$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $g(x)$ is local fractional continuous on (c,d), we denote $g(x) \in C_\beta(c,d)$.

Note that the functions are local fractional continuous, then the functions are also local fractional derivable and integrable.

Definition 8 ([14, 15]) The local fractional derivative of $g(x)$ of order β at $x = x_0$ is defined by

$$g^{(\beta)}(x_0) = \left.\frac{d^\beta g(x)}{dx^\beta}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Gamma(\beta+1)(g(x) - g(x_0))}{(x-x_0)^\beta}.$$

$D_\beta(c,d)$ is called β-local fractional derivative set. If there exists $g^{((k+1)\beta)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{(n+1)\text{ times}} g(x)$ for any $x \in I \subseteq R$, then we denote $g \in D_{(n+1)\beta}(I)$, where $n = 0, 1, 2, \ldots$.

Definition 9 ([14, 15]) Let $g(x) \in C_\beta[c,d]$. The local fractional integral of function $g(x)$ of order β is defined by

$${}_cI_d^{(\beta)} g(x) = \frac{1}{\Gamma(\beta+1)} \int_c^d g(\tau)(d\tau)^\beta = \frac{1}{\Gamma(\beta+1)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} g(\tau_j)(\Delta\tau_j)^\beta,$$

where $c = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = d$, $[\tau_j, \tau_{j+1}]$ is a partition of the interval $[c,d]$, $\Delta\tau_j = \tau_{j+1} - \tau_j$, $\Delta\tau = \max\{\Delta\tau_0, \Delta\tau_1 \cdots \Delta\tau_{N-1}\}$.

Note that ${}_bI_b^{(\beta)} g(x) = 0$, and ${}_cI_d^{(\beta)} g(x) = -{}_dI_c^{(\beta)} g(x)$ if $c < d$. We denote $g(x) \in I_x^{(\beta)}[c,d]$ if there exists ${}_cI_x^{(\beta)} g(x)$ for any $x \in [c,d]$.

Lemma 1 ([14, 15])

(1) *Suppose that* $u(x) = v^{(\beta)}(x) \in C_\beta[c,d]$, *then*

$$_{c}I_{d}^{(\beta)}u(x) = v(d) - v(c).$$

(2) *Suppose that* $u(x), v(x) \in D_\alpha[c,d]$ *and* $u^{(\beta)}(x), v^{(\beta)}(x) \in C_\beta[c,d]$, *then*

$$_{c}I_{d}^{(\beta)}u(x)v^{(\beta)}(x) = u(x)v(x)|_c^d - {_{c}I_{d}^{(\beta)}}u^{(\beta)}(x)v(x).$$

Lemma 2 ([14, 15])

$$\frac{d^\beta x^{s\beta}}{dx^\beta} = \frac{\Gamma(1+s\beta)}{\Gamma(1+(s-1)\beta)}x^{(s-1)\beta};$$
$$\frac{1}{\Gamma(\beta+1)}\int_c^d x^{s\beta}(dx)^\beta = \frac{\Gamma(1+s\beta)}{\Gamma(1+(s+1)\beta)}\big(d^{(s+1)\beta} - c^{(s+1)\beta}\big), \quad s > 0.$$

Lemma 3 ([14, 15])

$$_{c}I_{d}^{(\beta)}1^\beta = \frac{(d-c)^\beta}{\Gamma(1+\beta)}.$$

We recall the generalized beta function on fractal sets:

$$B_\beta(x,y) = \frac{1}{\Gamma(1+\beta)}\int_0^1 \tau^{(x-1)\beta}(1-\tau)^{(y-1)\beta}(d\tau)^\beta, \quad x > 0, y > 0.$$

3 Main results

In order to establish some new Hermite–Hadamard type inequalities, we firstly introduce the definition of generalized h-preinvex function on fractal sets. The symbol R_+ denotes an interval $(0,\infty)$ in the later sections.

Definition 10 Let $h: J \to R$ be a nonnegative mapping and $h^\beta \not\equiv 0^\beta$, where $(0,1) \subseteq J$, be an interval in R, and let C be an invex set with respect to $\eta(\cdot,\cdot)$. We say that $g: C \to R^\beta$ $(0 < \beta \leq 1)$ is a generalized h-preinvex function with respect to $\eta(\cdot,\cdot)$ if, for all $x, y \in C$ and $\tau \in (0,1)$, we have

$$g\big(y + \tau\eta(x,y)\big) \leq h^\beta(\tau)g(x) + h^\beta(1-\tau)g(y). \tag{3.1}$$

If inequality (3.1) is reversed, then f is said to be generalized h-preconcave with respect to $\eta(\cdot,\cdot)$.

Remark 2 Obviously, if $\beta = 1$, then generalized h-preinvex reduces to classical h-preinvex; if $h^\beta(\tau) = \tau^\beta$, then generalized h-preinvex reduces to generalized preinvex; and if $h^\beta(\tau) = \tau^{s\beta}$, where $s \in (0,1)$, then generalized h-preinvex is just generalized s-preinvex; if $\eta(x,y) = x - y$, then nonnegative generalized h-preinvex reduces to generalized h-convex.

If $h^\beta(\tau) = (\frac{1}{\tau})^\beta$ and $h^\beta(\tau) = 1^\beta$, we can derive generalized Godunova–Levin preinvex functions and generalized P-preinvex functions on fractal sets, respectively.

Definition 11 A function $g : C \to R^\beta$ $(0 < \beta \leq 1)$ is said to be a generalized Godunova–Levin preinvex function with respect to $\eta(\cdot,\cdot)$ if, for all $x, y \in C$ and $\tau \in (0,1)$, we have

$$g\bigl(y + \tau\eta(x,y)\bigr) \leq \frac{g(x)}{\tau^\beta} + \frac{g(y)}{(1-\tau)^\beta}. \tag{3.2}$$

Definition 12 A function $g : C \to R^\beta$ $(0 < \beta \leq 1)$ is said to be a generalized *P*- preinvex function with respect to $\eta(\cdot,\cdot)$ if, for all $x, y \in C$ and $\tau \in [0,1]$, we have

$$g\bigl(y + \tau\eta(x,y)\bigr) \leq g(x) + g(y). \tag{3.3}$$

Lemma 4 *Let g be a generalized h-preinvex function with respect to $\eta(\cdot,\cdot)$, then for any $x \in [c, c+\eta(b,c)]$, we have*

$$g\bigl(2c + \eta(b,c) - x\bigr) \leq \bigl[h^\beta(\tau) + h^\beta(1-\tau)\bigr]\bigl[g(c) + g(b)\bigr] - g(x). \tag{3.4}$$

Proof For any $x \in [c, c+\eta(b,c)]$, letting $x = c + \tau\eta(b,c)$, $\tau \in [0,1]$, then

$$\begin{aligned} g\bigl(2c + \eta(b,c) - x\bigr) &= g\bigl(c + (1-\tau)\eta(b,c)\bigr) \\ &\leq h^\beta(\tau)g(c) + h^\beta(1-\tau)g(b) \\ &= \bigl[h^\beta(\tau) + h^\beta(1-\tau)\bigr]\bigl[g(c) + g(b)\bigr] - \bigl[h^\beta(\tau)g(b) + h^\beta(1-\tau)g(c)\bigr] \\ &\leq \bigl[h^\beta(\tau) + h^\beta(1-\tau)\bigr]\bigl[g(c) + g(b)\bigr] - g\bigl(c + \tau\eta(b,c)\bigr) \\ &= \bigl[h^\beta(\tau) + h^\beta(1-\tau)\bigr]\bigl[g(c) + g(b)\bigr] - g(x). \end{aligned}$$

□

Theorem 2 *Let $g : I \to R_+^\beta$ be a generalized h-preinvex function with $c < c+\eta(b,c)$, $h(\frac{1}{2}) \neq 0$, and $u : [c, c+\eta(b,c)] \to R^\beta$ be a nonnegative, integrable function and symmetric about $c + \frac{1}{2}\eta(b,c)$, $gu \in I_x^{(\beta)}[c, c+\eta(b,c)]$. If η satisfies Condition C, then*

$$\begin{aligned} &\frac{1^\beta}{2^\beta h^\beta(\frac{1}{2})} g\left(\frac{2c+\eta(b,c)}{2}\right) {}_cI_{c+\eta(b,c)}^{(\beta)} u(x) \\ &\quad\leq {}_cI_{c+\eta(b,c)}^{(\beta)} g(x)u(x) \\ &\quad\leq \frac{g(c)+g(b)}{2^\beta}\bigl[h^\beta(\tau) + h^\beta(1-\tau)\bigr] {}_cI_{c+\eta(b,c)}^{(\beta)} u(x). \end{aligned} \tag{3.5}$$

Proof Since g is a generalized h-preinvex function on $[c, c+\eta(b,c)]$ and u is nonnegative, integrable, and symmetric about $c + \frac{1}{2}\eta(b,c)$, then

$$\begin{aligned} &\frac{1^\beta}{2^\beta h^\beta(\frac{1}{2})} g\left(\frac{2c+\eta(b,c)}{2}\right) {}_cI_{c+\eta(b,c)}^{(\beta)} u(x) \\ &\quad= \frac{1^\beta}{2^\beta h^\beta(\frac{1}{2})} \frac{1}{\Gamma(1+\beta)} \int_c^{c+\eta(b,c)} g\left(\frac{2c+\eta(b,c)}{2}\right) u(x)(dx)^\beta \\ &\quad= \frac{1^\beta}{2^\beta h^\beta(\frac{1}{2})} \frac{1}{\Gamma(1+\beta)} \int_c^{c+\eta(b,c)} g\left(\frac{2c+\eta(b,c)-x+x}{2}\right) u(x)(dx)^\beta \\ &\quad\leq \frac{1^\beta}{2^\beta h^\beta(\frac{1}{2})} \frac{1}{\Gamma(1+\beta)} \int_c^{c+\eta(b,c)} h^\beta\left(\frac{1}{2}\right)\bigl(f\bigl(2c+\eta(b,c)-x\bigr) + g(x)\bigr)u(x)(dx)^\beta \end{aligned}$$

$$
\begin{aligned}
&= \frac{1^\beta}{2^\beta}\left[\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g\big(2c+\eta(b,c)-x\big)u(x)(dx)^\beta\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)u(x)(dx)^\beta\right]\\
&= \frac{1^\beta}{2^\beta}\left[\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g\big(2c+\eta(b,c)-x\big)u\big(2c+\eta(b,c)-x\big)(dx)^\beta\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)u(x)(dx)^\beta\right]\\
&= \frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)u(x)(dx)^\beta = {}_cI^{(\beta)}_{c+\eta(b,c)}g(x)u(x).
\end{aligned}
\tag{3.6}
$$

By Lemma 4, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)u(x)(dx)^\beta\\
&= \frac{1^\beta}{2^\beta}\left[\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g\big(2c+\eta(b,a)-x\big)u\big(2c+\eta(b,c)-x\big)(dx)^\beta\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)u(x)(dx)^\beta\right]\\
&= \frac{1^\beta}{2^\beta}\left[\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} f\big(2c+\eta(b,c)-x\big)u(x)(dx)^\beta\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)u(x)(dx)^\beta\right]\\
&\le \frac{1^\beta}{2^\beta}\left\{\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} \big[\big(h^\beta(\tau)+h^\beta(1-\tau)\big)\big(g(c)+g(b)\big)-g(x)\big]u(x)(dx)^\beta\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)u(x)(dx)^\beta\right\}\\
&= \frac{1^\beta}{2^\beta}\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} \big[h^\beta(\tau)+h^\beta(1-\tau)\big]\big[g(c)+g(b)\big]u(x)(dx)^\beta\\
&= \frac{[h^\beta(\tau)+h^\beta(1-\tau)][g(c)+g(b)]}{2^\beta}\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} u(x)(dx)^\beta.
\end{aligned}
\tag{3.7}
$$

Combining (3.6) and (3.7), this completes the proof. □

Remark 3 In Theorem 2, if we take $\beta = 1$, then it reduces to Theorem 3.3 given in Ref. [6].

Theorem 3 *Let $g: I \to R^\beta_+$, $\psi: I \to R^\beta_+$ be a generalized h_1-preinvex function and a generalized h_2-preinvex function respectively with $c < c+\eta(b,c)$, $g\psi \in I^{(\beta)}_x[c,b]$, $h_1^\beta h_2^\beta \in I^{(\beta)}_x[c,b]$, then*

$$
\frac{1^\beta}{\eta^\beta(b,c)}\,{}_cI^{(\beta)}_{c+\eta(b,c)}g(x)\psi(x) \le M(c,b)\,{}_0I^{(\beta)}_1 h_1^\beta(\tau)h_2^\beta(\tau) + N(c,b)\,{}_0I^{(\beta)}_1 h_1^\beta(\tau)h_2^\beta(1-\tau),
\tag{3.8}
$$

where $M(c,b) = g(c)\psi(c) + g(b)\psi(b)$ and $N(c,b) = g(c)\psi(b) + g(b)\psi(c)$.

Proof Since g is a nonnegative generalized h_1-preinvex function and ψ is a nonnegative generalized h_2-preinvex function, for all $\tau \in [0,1]$, we have

$$\begin{aligned} & g\bigl(c+\tau\eta(b,c)\bigr)\psi\bigl(c+\tau\eta(b,c)\bigr) \\ & \quad \le h_1^\beta(\tau)h_2^\beta(\tau)g(b)\psi(b) + h_1^\beta(\tau)h_2^\beta(1-\tau)g(b)\psi(c) \\ & \qquad + h_2^\beta(\tau)h_1^\beta(1-\tau)g(c)\psi(b) + h_1^\beta(1-\tau)h_2^\beta(1-\tau)g(c)\psi(c). \end{aligned}$$

Integrating both sides of the above inequality with respect to τ over $[0,1]$, letting $c + \tau\eta(b,c) = x$, we get

$$\begin{aligned} & \frac{1^\beta}{\eta^\beta(b,c)}\, {}_cI_{c+\eta(b,c)}^{(\beta)} g(x)\psi(x) \\ & \quad = \frac{1}{\Gamma(1+\beta)}\int_0^1 g\bigl(c+\tau\eta(b,c)\bigr)\psi\bigl(c+\tau\eta(b,c)\bigr)(d\tau)^\beta \\ & \quad \le \bigl[g(c)\psi(c)+g(b)\psi(b)\bigr]\frac{1}{\Gamma(1+\beta)}\int_0^1 h_1^\beta(t)h_2^\beta(\tau)(d\tau)^\beta \\ & \qquad + \bigl[g(c)\psi(b)+g(b)\psi(c)\bigr]\frac{1}{\Gamma(1+\beta)}\int_0^1 h_1^\beta(\tau)h_2^\beta(1-\tau)(d\tau)^\beta \\ & \quad = \bigl[g(c)\psi(c)+g(b)\psi(b)\bigr]{}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(\tau) + \bigl[g(c)\psi(b)+g(b)\psi(c)\bigr]{}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(1-\tau). \end{aligned}$$

This completes the proof. □

Remark 4 In Theorem 3, if we take $\beta = 1$, then it reduces to Theorem 3.5 given in Ref. [6].

Corollary 1 *In Theorem 3, if we take $\eta(b,c) = b - c$, then inequality (3.8) reduces to the following inequality*:

$$\frac{1^\beta}{(b-c)^\beta}\, {}_cI_b^{(\beta)} g(x)\psi(x) \le M(c,b)\, {}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(\tau) + N(c,b)\, {}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(1-\tau). \tag{3.9}$$

Remark 5 Inequality (3.9) is just the result of Theorem 9 in Ref. [26].

Corollary 2 *In Corollary 1, if we take $h_1^\beta(\tau) = h_2^\beta(\tau) = \tau^\beta$, then inequality (3.9) reduces to the following inequality for generalized convex functions*:

$$\frac{1^\beta}{(b-c)^\beta}\, {}_cI_b^{(\beta)} g(x)\psi(x) \le M(c,b)\frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} + N(c,b)B_\beta(2,2), \tag{3.10}$$

where $B_\beta(x,y)$ denotes the generalized gamma function on fractal sets.

Proof By $h_1^\beta(\tau) = h_2^\beta(\tau) = \tau^\beta$, we have

$$\begin{aligned} & {}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(\tau) = \frac{1}{\Gamma(1+\beta)}\int_0^1 \tau^{2\beta}(d\tau)^\beta = \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)}, \\ & {}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(1-\tau) = \frac{1}{\Gamma(1+\beta)}\int_0^1 \tau^\beta(1-\tau)^\beta(d\tau)^\beta = B_\beta(2,2). \end{aligned}$$

This completes the proof. □

Theorem 4 *Let $g: I \to R_{+}^{\beta}$, $\psi: I \to R_{+}^{\beta}$ be a generalized h_1-preinvex function and a generalized h_2- preinvex function respectively with $c < c+\eta(b,c)$, $g\psi \in I_x^{(\beta)}[c,b]$, $h_1^{\beta}h_2^{\beta} \in I_x^{(\beta)}[c,b]$, $h_1(\frac{1}{2}) \neq 0$, $h_2(\frac{1}{2}) \neq 0$. If η satisfies Condition C, then*

$$\begin{aligned}&\frac{1^{\beta}}{2^{\beta}h_1^{\beta}(\frac{1}{2})h_2^{\beta}(\frac{1}{2})\Gamma(1+\beta)}g\left(c+\frac{1}{2}\eta(b,c)\right)\psi\left(c+\frac{1}{2}\eta(b,c)\right)-\frac{1^{\beta}}{\eta^{\beta}(b,c)}{}_cI_{c+\eta(b,c)}^{(\beta)}g(x)\psi(x)\\&\quad\leq M(c,b)_0I_1^{(\beta)}h_1^{\beta}(\tau)h_2^{\beta}(1-\tau)+N(c,b)_0I_1^{(\beta)}h_1^{\beta}(\tau)h_2^{\beta}(\tau),\end{aligned}\tag{3.11}$$

where $M(c,b)$ and $N(c,b)$ are the same as Theorem 3.

Proof Since g is a nonnegative generalized h_1-preinvex function and ψ is a nonnegative generalized h_2-preinvex function, for all $\tau \in [0,1]$, using Condition C, we have

$$\begin{aligned}&g\left(c+\frac{1}{2}\eta(b,c)\right)\psi\left(c+\frac{1}{2}\eta(b,c)\right)\\&\quad=g\left(c+(1-\tau)\eta(b,c)+\frac{1}{2}\eta\big(c+\tau\eta(b,c),c+(1-\tau)\eta(b,c)\big)\right)\\&\qquad\times\psi\left(c+(1-\tau)\eta(b,c)+\frac{1}{2}\eta\big(c+\tau\eta(b,c),c+(1-\tau)\eta(b,c)\big)\right)\\&\quad\leq h_1^{\beta}\left(\frac{1}{2}\right)h_2^{\beta}\left(\frac{1}{2}\right)\big[g\big(c+\tau\eta(b,c)\big)+g\big(c+(1-\tau)\eta(b,c)\big)\big]\\&\qquad\times\big[\psi\big(c+\tau\eta(b,c)\big)+\psi\big(c+(1-\tau)\eta(b,c)\big)\big]\\&\quad=h_1^{\beta}\left(\frac{1}{2}\right)h_2^{\beta}\left(\frac{1}{2}\right)\big[g\big(c+\tau\eta(b,c)\big)\psi\big(c+\tau\eta(b,c)\big)\\&\qquad+g\big(c+(1-\tau)\eta(b,c)\big)\psi\big(c+(1-\tau)\eta(b,c)\big)\\&\qquad+g\big(c+\tau\eta(b,c)\big)\psi\big(c+(1-\tau)\eta(b,c)\big)+g\big(c+(1-\tau)\eta(b,c)\big)\psi\big(c+\tau\eta(b,c)\big)\big]\\&\quad\leq h_1^{\beta}\left(\frac{1}{2}\right)h_2^{\beta}\left(\frac{1}{2}\right)\big[g\big(c+\tau\eta(b,c)\big)\psi\big(c+\tau\eta(b,c)\big)\\&\qquad+g\big(c+(1-\tau)\eta(b,c)\big)\psi\big(c+(1-\tau)\eta(b,c)\big)\big]\\&\qquad+h_1^{\beta}\left(\frac{1}{2}\right)h_2^{\beta}\left(\frac{1}{2}\right)\big\{\big[h_1^{\beta}(\tau)h_2^{\beta}(1-\tau)+h_1^{\beta}(1-\tau)h_2^{\beta}(\tau)\big]M(c,b)\\&\qquad+\big[h_1^{\beta}(\tau)h_2^{\beta}(\tau)+h_1^{\beta}(1-\tau)h_2^{\beta}(1-\tau)\big]N(c,b)\big\}.\end{aligned}$$

Integrating both sides of the above inequality with respect to t over $[0,1]$, letting $c+\tau\eta(b,c)=x$, we get

$$\begin{aligned}&\frac{g(c+\frac{1}{2}\eta(b,c))\psi(c+\frac{1}{2}\eta(b,c))}{\Gamma(1+\beta)}-\frac{2^{\beta}h_1^{\beta}(\frac{1}{2})h_2^{\beta}(\frac{1}{2})}{\eta^{\beta}(b,c)}{}_cI_{c+\eta(b,c)}^{(\beta)}g(x)\psi(x)\\&\quad\leq 2^{\beta}h_1^{\beta}\left(\frac{1}{2}\right)h_2^{\beta}\left(\frac{1}{2}\right)\big[M(c,b)_0I_1^{(\beta)}h_1^{\beta}(\tau)h_2^{\beta}(1-\tau)+N(c,b)_0I_1^{(\beta)}h_1^{\beta}(\tau)h_2^{\beta}(\tau)\big].\end{aligned}$$

This completes the proof. □

Remark 6 In Theorem 4, if we take $\beta=1$, then it reduces to Theorem 2.5 given in Ref. [5].

Corollary 3 *In Theorem 4, if we take $\eta(b,c) = b - c$, then inequality (3.11) reduces to the following inequality*:

$$\frac{1^\beta}{2^\beta h_1^\beta(\frac{1}{2})h_2^\beta(\frac{1}{2})\Gamma(1+\beta)} f\left(\frac{c+b}{2}\right)g\left(\frac{c+b}{2}\right) - \frac{1^\beta}{(b-c)^\beta}\, {}_cI_b^{(\beta)} g(x)\psi(x) \\ \leq M(c,b)\, {}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(1-\tau) + N(c,b)\, {}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(\tau). \tag{3.12}$$

Corollary 4 *In Theorem 4, if we take $h_1^\beta(t) = h_2^\beta(\tau) = \tau^\beta$, then inequality (3.11) reduces to the following inequality*:

$$\frac{2^\beta}{\Gamma(1+\beta)} g\left(c+\frac{1}{2}\eta(b,c)\right)\psi\left(c+\frac{1}{2}\eta(b,c)\right) - \frac{1^\beta}{\eta^\beta(b,c)}\, {}_cI_{c+\eta(b,c)}^{(\beta)} g(x)\psi(x) \\ \leq M(c,b)B_\beta(2,2) + N(c,b)\frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)}, \tag{3.13}$$

where $B_\beta(x,y)$ denotes the generalized gamma function on fractal sets.

Proof By $h_1^\beta(t) = h_2^\beta(t) = t^\beta$, we have

$${}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(\tau) = \frac{1}{\Gamma(1+\beta)}\int_0^1 \tau^{2\beta}(d\tau)^\beta = \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)}.$$

Obviously, the result holds. □

Corollary 5 *In Theorem 4, if we take $h_1^\beta(\tau) = \tau^\beta$, $h_2^\beta(\tau) = \tau^{s\beta}$, $s \in (0,1]$, then inequality (3.11) reduces to the following inequality*:

$$\frac{2^{s\beta}}{\Gamma(1+\beta)} g\left(c+\frac{1}{2}\eta(b,c)\right)\psi\left(c+\frac{1}{2}\eta(b,c)\right) - \frac{1^\beta}{\eta^\beta(b,c)}\, {}_cI_{c+\eta(b,c)}^{(\beta)} g(x)\psi(x) \\ \leq M(c,b)B_\beta(2,s+1) + N(c,b)\frac{\Gamma(1+(s+1)\beta)}{\Gamma(1+(s+2)\beta)}, \tag{3.14}$$

where $B_\beta(x,y)$ denotes the generalized gamma function on fractal sets.

Proof By $h_1^\beta(\tau) = \tau^\beta$, $h_2^\beta(\tau) = \tau^{s\beta}$, $s \in (0,1]$, we have

$${}_0I_1^{(\beta)} h_1^\beta(\tau)h_2^\beta(1-\tau) = \frac{1}{\Gamma(1+\beta)}\int_0^1 \beta^\beta(1-\tau)^{s\beta}(d\tau)^\beta = B_\beta(2,s+1),$$

$${}_0I_1^{(\beta)} h_1^\beta(t)h_2^\beta(\tau) = \frac{1}{\Gamma(1+\beta)}\int_0^1 \tau^{(s+1)\beta}(d\tau)^\beta = \frac{\Gamma(1+(s+1)\beta)}{\Gamma(1+(s+2)\beta)}.$$

Obviously, the result holds. □

Theorem 5 *Let $g: I \to R^\beta$ be a generalized h-preinvex function with $c < c + \eta(b,c)$, and $\psi: I \to R^\beta$ be nonnegative, symmetric with respect to $c + \frac{1}{2}\eta(b,c)$, then*

$$\frac{1^\beta}{\eta^\beta(b,c)}\, {}_cI_{c+\eta(b,c)}^{(\beta)} g(x)\psi(x) \leq \big[g(c)+g(b)\big]\, {}_0I_1^{(\beta)} h^\beta(\tau)\psi\big(c+\tau\eta(b,c)\big). \tag{3.15}$$

Proof Since g is a generalized h-preinvex function and ψ is a nonnegative function, we have

$$g\big(c+\tau\eta(b,c)\big)\psi\big(c+\tau\eta(b,c)\big)\le\big[h^{\beta}(1-\tau)g(c)+h^{\beta}(\tau)g(b)\big]\psi\big(c+\tau\eta(b,c)\big),$$

$$g\big(c+(1-\tau)\eta(b,c)\big)\psi\big(c+(1-\tau)\eta(b,c)\big)$$
$$\le\big[h^{\beta}(\tau)g(c)+h^{\beta}(1-\tau)g(b)\big]\psi\big(c+(1-\tau)\eta(b,c)\big).$$

Adding the above two inequalities and integrating with respect to τ over $[0,1]$, and using the symmetricity of the ψ, we get

$$\begin{aligned}
&\frac{1}{\Gamma(1+\beta)}\int_0^1 g\big(c+\tau\eta(b,c)\big)\psi\big(c+\tau\eta(b,c)\big)(d\tau)^{\beta}\\
&\quad+\frac{1}{\Gamma(1+\beta)}\int_0^1 g\big(c+(1-\tau)\eta(b,c)\big)\psi\big(c+(1-\tau)\eta(b,c)\big)(d\tau)^{\beta}\\
&\le\frac{1}{\Gamma(1+\beta)}\int_0^1\big\{g(c)\big[h^{\beta}(1-\tau)\psi\big(c+\tau\eta(b,c)\big)+h^{\beta}(\tau)\psi\big(c+(1-\tau)\eta(b,c)\big)\big]\\
&\quad+g(b)\big[h^{\beta}(\tau)\psi\big(c+\tau\eta(b,c)\big)+h^{\beta}(1-\tau)\psi\big(c+(1-\tau)\eta(b,c)\big)\big]\big\}(d\tau)^{\beta}\\
&=2^{\beta}g(c)\frac{1}{\Gamma(1+\beta)}\int_0^1 h^{\beta}(\tau)\psi\big(c+(1-\tau)\eta(b,c)\big)(d\tau)^{\beta}\\
&\quad+2^{\beta}g(b)\frac{1}{\Gamma(1+\beta)}\int_0^1 h^{\beta}(\tau)\psi\big(c+\tau\eta(b,c)\big)(d\tau)^{\beta}\\
&=2^{\beta}\big[g(c)+g(b)\big]\frac{1}{\Gamma(1+\beta)}\int_0^1 h^{\beta}(\tau)\psi\big(c+\tau\eta(b,c)\big)(d\tau)^{\tau}.
\end{aligned}$$

Letting $c+\tau\eta(b,c)=x$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\beta)}\int_0^1 g\big(c+\tau\eta(b,c)\big)\psi\big(c+\tau\eta(b,c)\big)(d\tau)^{\beta}\\
&=\frac{1}{\Gamma(1+\beta)}\int_0^1 g\big(c+(1-\tau)\eta(b,c)\big)\psi\big(c+(1-\tau)\eta(b,c)\big)(d\tau)^{\beta}\\
&=\frac{1^{\beta}}{\eta^{\beta}(b,c)}\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)\psi(x)(d\tau)^{\beta}.
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{1^{\beta}}{\eta^{\beta}(b,c)}\frac{1}{\Gamma(1+\beta)}\int_c^{c+\eta(b,c)} g(x)\psi(x)(d\tau)^{\beta}\\
&=\big[g(c)+g(b)\big]\frac{1}{\Gamma(1+\beta)}\int_0^1 h^{\beta}(\tau)\psi\big(c+\tau\eta(b,c)\big)(d\tau)^{\beta}.
\end{aligned}$$

This completes the proof. □

Remark 7 In Theorem 5, if we take $\beta=1$, then it reduces to Theorem 2.6 given in Ref. [5].

Remark 8

1. If $\eta(b,c)=b-c$, then our results reduce to some results for generalized h-convexity.

2. If $h^\beta(\tau) = \tau^\beta$, $h^\beta(\tau) = \tau^{s\beta}$, $h^\beta(\tau) = (\frac{1}{\tau})^\beta$, and $h^\beta(\tau) = 1^\beta$, then our results reduce to some results for generalized preinvex, generalized s-preinvex, generalized Godunova–Levin preinvex, and generalized P-preinvex, respectively.
3. If $\eta(b,c) = b - c$ and $h^\beta(\tau) = \tau^\beta$, $h^\beta(\tau) = \tau^{s\beta}$, $h^\beta(\tau) = (\frac{1}{\tau})^\beta$, and $h^\beta(\tau) = 1^\beta$, then our results reduce to some results for generalized convex, generalized s-convex, generalized Godunova–Levin convex, and generalized P-convex, respectively.

4 Applications for random variables

Let X be a continuous random variable having generalized probability density function $\psi : [c,b] \to R_+^\beta$. The generalized expectation (or generalized mean) of X is defined as follows:

$$\mu_\beta = E_\beta(X) = \frac{1}{\Gamma(1+\beta)} \int_c^b x^\beta \psi(x)(dx)^\beta.$$

The generalized rth moment and the generalized variance of X are defined as

$$E_\beta^r(X) = \frac{1}{\Gamma(1+\beta)} \int_c^b x^{r\beta} \psi(x)(dx)^\beta, \quad r > 0,$$
$$\mathrm{Var}_\beta(X) = \sigma_\beta^2 = \frac{1}{\Gamma(1+\beta)} \int_c^b (x-\mu_\beta)^{2\beta} \psi(x)(dx)^\beta.$$

Proposition 1
(1) *Let $g(x) = x^{r\beta}$, $x > 0$. If $r \geq 1$, then $g(x)$ is a generalized convex function* (*see* [19]);
(2) *Let $g(x) = (x-a)^{2\beta}$, then $g(x)$ is a generalized convex function.*

Proof (2) Since $g^{(2\beta)}(x) = \Gamma(1+2\beta) > 0$, from Corollary 10 in Reference [19], we know that $g(x) = (x-a)^{2\beta}$ is a generalized convex function. □

Theorem 6 *Let X be a random variable having generalized probability density function $\psi : [c,b] \to R_+^\beta$. $\psi(x) \in I_x^{(\beta)}[c,b]$, $c < b$. Then, for $x > 0$, we obtain the upper bound for the generalized expectation of a random variable X as follows*:

$$E_\beta(X) \leq (b-c)^\beta \left[\left(c^\beta \psi(c) + b^\beta \psi(b)\right) \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} + \left(c^\beta \psi(b) + b^\beta \psi(c)\right) B_\beta(2,2) \right]. \tag{4.1}$$

Proof Since $g(x) = x^\beta$ is a generalized convex function, choosing $g(x) = x^\beta$ in (3.10), where $M(c,b) = g(c)\psi(c) + g(b)\psi(b)$ and $N(c,b) = g(c)\psi(b) + g(b)\psi(c)$, we have

$$\begin{aligned} \frac{1^\beta}{(b-c)^\beta} E_\beta(X) &= \frac{1^\beta}{(b-c)^\beta} {}_cI_b^{(\beta)} x^\beta \psi(x) \leq M(c,b) \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} + N(c,b) B_\beta(2,2) \\ &= \left[c^\beta \psi(c) + b^\beta \psi(b)\right] \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} + \left[c^\beta \psi(b) + b^\beta \psi(c)\right] B_\beta(2,2). \end{aligned}$$

From Proposition 1, choosing $g(x) = x^{r\beta}$ and $g(x) = (x-\mu_\beta)^{2\beta}$ in (3.10), we can obtain the upper bounds for the generalized rth moment and the generalized variance of a random variable X, respectively. □

Theorem 7 *Let X be a random variable having generalized probability density function $\psi : [c,b] \to R_+^\beta$. $\psi(x) \in I_x^{(\beta)}[c,b]$, $c < b$. Then, for $x > 0$, $r \geq 1$, we obtain the upper bound for the generalized rth moment of a random variable X as follows*:

$$E_\beta^r(X) \leq (b-c)^\beta \left[\left(c^{r\beta}\psi(c) + b^{r\beta}\psi(b) \right) \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} + \left(c^{r\beta}\psi(b) + b^{r\beta}\psi(c) \right) B_\beta(2,2) \right]. \quad (4.2)$$

Proof Choosing $g(x) = x^{r\beta}$ in (3.10), the proof is similar to that of Theorem 6. □

Theorem 8 *Let X be a random variable having generalized probability density function $\psi : [c,b] \to R_+^\beta$. $\psi(x) \in I_x^{(\beta)}[c,b]$, $c < b$. The symbol μ_β denotes the generalized expectation of X.Then we obtain the upper bound for the generalized variance of a random variable X as follows*:

$$\begin{aligned} \operatorname{Var}_\beta(X) &= \sigma_\beta^2(X) \\ &\leq (b-c)^\beta \left[\left((c-\mu_\beta)^{2\beta}\psi(c) + (b-\mu_\beta)^{2\beta}\psi(b) \right) \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} \right. \\ &\quad \left. + \left((c-\mu_\beta)^{2\beta}\psi(b) + (b-\mu_\beta)^{2\beta}\psi(c) \right) B_\beta(2,2) \right]. \end{aligned} \quad (4.3)$$

Proof Choosing $g(x) = (x-\mu_\beta)^{2\beta}$ in (3.10), the proof is similar to that of Theorem 6. □

Acknowledgements
The author is also indebted to the anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.

Funding
This work is supported by the Natural Science Foundation of Hunan Province (No. 2020JJ4554; No. 2020JJ5514) and the Scientific Research Project of Hunan Provincial Education Department (No. 18B433).

Availability of data and materials
Not applicable.

Competing interests
The author declares that there are no competing interests.

Authors' contributions
The single author is responsible for the complete manuscript. The author read and approved the final manuscript.

Publisher's Note

Received: 27 April 2020 Accepted: 1 July 2020 Published online: 17 August 2020

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