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Dynamics of a delayed SIR model for the transmission of PRRSV among a swine population

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Abstract

The objective of this paper is to propose a delayed susceptible-infectious-recovered (SIR) model for the transmission of porcine reproductive respiratory syndrome virus (PRRSV) among a swine population, including the latent period delay of the virus and the time delay due to the period the infectious swines need to recover. By taking different combinations of the two delays as the bifurcation parameter, local stability of the disease-present equilibrium and the existence of Hopf bifurcation are analyzed. Sufficient conditions for global stability of the disease-present equilibrium are derived by constructing a suitable Lyapunov function. Directly afterwards, properties of the Hopf bifurcation such as direction and stability are studied with the aid of the normal form theory and center manifold theorem. Finally, numerical simulations are presented to justify the validity of the derived theoretical results.

Keywords: Hopf bifurcation; SIR model; Time delay; PRRSV; Periodic solutions

1 Introduction

Porcine reproductive respiratory syndrome, also known as blue ear disease and epidemic abortion respiratory syndrome, is caused by Lelystad virus. It was first reported in the United States in 1987 and isolated by Dutch scientists in 1991. Since then it has spread to most of the European pig industry and has a global trend. Porcine reproductive respiratory syndrome has a major economic impact on the swine industry with cost of about 664 million USD annually in the United States [1–3]. In the same way, porcine reproductive respiratory syndrome has also caused huge economic losses to pig industry in China. For example, the porcine reproductive respiratory syndrome epidemic in 2006 forced Chinese farmers to slaughter millions of pigs, leading to China's highest inflation rate in the past decade [4, 5].

To better understand the population dynamics of infectious diseases, mathematical models for infectious disease dynamics have been formulated and studied for a long time. For example, delayed epidemic model [6–10], stochastic epidemic model [11–16], age-structured epidemic model [17–19], and so on. Since porcine reproductive respiratory syndrome is a devastating infectious disease among a swine population, it is reasonable to

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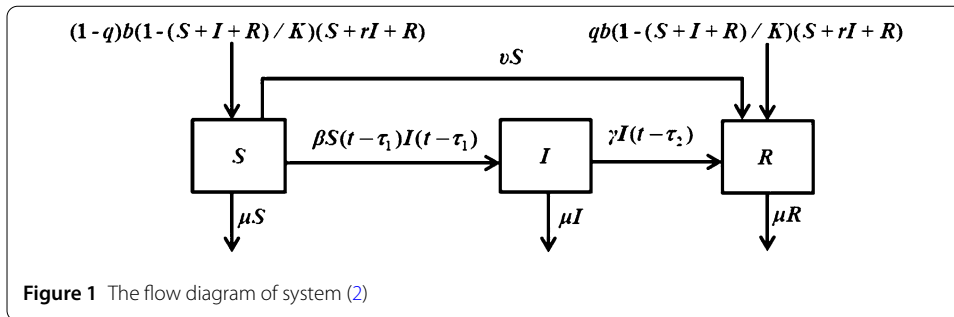
use mathematical modeling describing the propagation of porcine reproductive respiratory syndrome virus among a swine population. Arruda et al. [20–23] proposed different forms of a stochastic model to investigate transmission dynamics of porcine reproductive respiratory syndrome virus. In [24], Suksamran et al. proposed a structured model for the spread of porcine reproductive respiratory syndrome virus incorporating both time and spatial dimensions as well as the decline of infectiousness with time.

Studies have shown that porcine reproductive respiratory syndrome virus is a double-edged sword to the immune system of pigs [25]. On the one hand, porcine reproductive respiratory syndrome virus can specifically bind to immune cells, especially macrophages, and once it is widely replicated, the immune system will be inhibited, resulting in immune failure of a variety of infectious diseases vaccines; on the other hand, as virus infection stimulates the immune system of pigs, the body produces immunity, which will protect the body from secondary infection. Accordingly, vaccination is an important method used for controlling the spread of porcine reproductive respiratory syndrome virus. Based on this consideration, Phoo-ngurn et al. [26] proposed the following susceptible-infectious-recovered (SIR) model for the transmission of porcine reproductive respiratory syndrome virus among a swine population:

$$\begin{cases} \frac{dS(t)}{dt} = (1 - q)b\left(1 - \frac{S(t)+I(t)+R(t)}{K}\right)(S(t) + rI(t) + R(t)) \\ \quad - \beta S(t)I(t) - (\mu + \nu)S(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = qb\left(1 - \frac{S(t)+I(t)+R(t)}{K}\right)(S(t) + rI(t) + R(t)) \\ \quad + \gamma I(t) + \nu S(t) - \mu R(t), \end{cases} \tag{1}$$

where $S(t)$, $I(t)$, and $R(t)$ stand for the numbers of susceptible swines, infectious swines, and the recovered swines at time t , respectively. b is the birth rate of the susceptible swines; q denotes vaccination coverage; K is the swine carrying capacity; μ is the natural death rate of the swines; γ is the recovery rate of the infectious swines; ν is the vaccination rate of the susceptible swines; β is the transmission coefficient and r is the abortion proportion.

One of the significant features of porcine reproductive respiratory syndrome virus is its latent characteristic, which means that when viruses enter a pig, they hide themselves and only become active after a certain period. It is therefore easy to show that when the susceptible swines are infected by porcine reproductive respiratory syndrome virus, there will be a time delay before these swines develop themselves into the infectious ones. Likewise, a certain period of time is usually needed for the infectious swines to recover. On the other hand, many research works about a delayed dynamical system in epidemics [6, 9, 27–29], population dynamics [30–35], and neural networks [36–40] have shown that delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause the equilibrium of a dynamical model to lose its stability and cause the occurrence of Hopf bifurcation, which is not welcome for a dynamical system. Due to these facts and without loss of reality, we consider the following delayed



SIR model for the transmission of porcine reproductive respiratory syndrome virus:

$$\begin{cases} \frac{dS(t)}{dt} = (1 - q)b(1 - \frac{S(t)+I(t)+R(t)}{K})(S(t) + rI(t) + R(t)) \\ \quad - \beta S(t - \tau_1)I(t - \tau_1) - (\mu + v)S(t), \\ \frac{dI(t)}{dt} = \beta S(t - \tau_1)I(t - \tau_1) - \mu I(t) - \gamma I(t - \tau_2), \\ \frac{dR(t)}{dt} = qb(1 - \frac{S(t)+I(t)+R(t)}{K})(S(t) + rI(t) + R(t)) \\ \quad + \gamma I(t - \tau_2) + vS(t) - \mu R(t), \end{cases} \tag{2}$$

where τ_1 is latent period delay of the porcine reproductive respiratory syndrome virus; τ_2 is time delay due to the period the infectious swines need to recover. The flow diagram for system (2) is illustrated in Fig. 1.

The remainder of this paper is planned as follows. In Sect. 2, sufficient criteria to ensure local stability of the disease-present equilibrium and the existence of Hopf bifurcation of the involved model are presented. In Sect. 3, global stability of the disease-present equilibrium is analyzed. Direction and stability of the Hopf bifurcation are investigated in Sect. 4. Numerical simulations are performed to check our obtained theoretical results in Sect. 5. Finally, a brief conclusion is presented.

2 Local stability and existence of Hopf bifurcation

According to the analysis in [26], we know that system (2) has disease-present equilibrium $E^*(S^*, I^*, R^*)$ when the basic reproduction number $R_0 > 1$ where

$$\begin{aligned} S^* &= \frac{\mu + \gamma}{\beta}, \\ R^* &= \frac{(1 - q)\mu S^* + (1 - q)\gamma I^* + q\beta S^* I^* + q(\mu + v)S^*}{(1 - q)\mu}, \end{aligned}$$

where I^* satisfies

$$a_2(I^*)^2 + a_1 I^* + a_0 = 0 \tag{3}$$

with

$$\begin{aligned} a_0 &= \left(\frac{b}{\mu} - 1\right) \left(\frac{(\mu + v)(\mu + \gamma)}{\beta}\right) - \frac{b(\mu + v)^2(\mu + \gamma)^2}{K\mu^2\beta^2(1 - q)}, \\ a_1 &= \left((1 - q)b - \frac{b(\mu + v)(\mu + \gamma)}{K\mu\beta}\right) \left(r - 1 + \frac{\mu + \gamma}{(1 - q)\mu}\right) - \frac{b(\mu + v)(\mu + \gamma)^2}{K\beta\mu^2(1 - q)} - (\mu + \gamma), \end{aligned}$$

$$a_2 = \frac{-b(\mu + \gamma)}{K\mu} \left(r - 1 + \frac{\mu + \gamma}{(1 - q)\mu} \right)$$

and

$$R_0 = (1 - q) \frac{\beta(1 - \mu/b)\mu K}{(\mu + \nu)(\mu + \gamma)}.$$

For the distribution of positive roots of Eq. (3), we have Lemma 1.

Lemma 1 (i) *If the coefficients in Eq. (3) satisfy one of the conditions in (a)–(d), then Eq. (3) has one positive root. (a): if $a_2 = 0$ and $a_0a_1 < 0$; (b): if $\sqrt{a_1^2 - 4a_2a_0} > 0$ and $a_0a_2 < 0$; (c): if $\sqrt{a_1^2 - 4a_2a_0} > 0$, $a_0 = 0$, and $a_1a_2 < 0$; (d): if $\sqrt{a_1^2 - 4a_2a_0} = 0$ and $a_1a_2 < 0$; (ii) if the coefficients in Eq. (3) satisfy the condition in (e): $\sqrt{a_1^2 - 4a_2a_0} > 0$, $a_1a_2 < 0$, and $a_0a_2 > 0$, then Eq. (3) has two positive roots.*

The linear section of system (2) at E^* is

$$\begin{cases} \frac{dS(t)}{dt} = p_{11}S(t) + p_{12}I(t) + p_{13}R(t) + q_{11}S(t - \tau_1) + q_{12}I(t - \tau_1), \\ \frac{dI(t)}{dt} = p_{22}I(t) + q_{21}S(t - \tau_1) + q_{22}I(t - \tau_1) + s_{22}I(t - \tau_2), \\ \frac{dR(t)}{dt} = p_{31}S(t) + p_{32}I(t) + p_{33}R(t) + s_{32}I(t - \tau_2), \end{cases} \tag{4}$$

with

$$\begin{aligned} p_{11} &= (1 - q)b - (\mu + \nu) - \frac{(1 - q)b(2S^* + (1 + r)I^* + 2R^*)}{K}, \\ p_{12} &= (1 - q)br - \frac{(1 - q)b((1 + r)S^* + 2rI^* + (1 + r)R^*)}{K}, \\ p_{13} &= (1 - q)b - \frac{(1 - q)b(2S^* + (1 + r)I^* + 2R^*)}{K}, \quad p_{22} = -\mu, \\ p_{31} &= qb + \nu - \frac{qb(2S^* + (1 + r)I^* + 2R^*)}{K}, \\ p_{32} &= qbr - \frac{qb((1 + r)S^* + 2rI^* + (1 + r)R^*)}{K}, \\ p_{33} &= qb - \mu - \frac{qb(2S^* + (1 + r)I^* + 2R^*)}{K}, \\ q_{11} &= -\beta I^*, \quad q_{12} = -\beta S^*, \quad q_{21} = \beta I^*, \quad q_{22} = \beta S^*, \quad s_{22} = -\gamma, \quad s_{32} = \gamma. \end{aligned}$$

Thus, the associated characteristic equation is

$$\begin{aligned} \lambda^3 + P_2\lambda^2 + P_1\lambda + P_0 + (Q_2\lambda^2 + Q_1\lambda + Q_0)e^{-\lambda\tau_1} \\ + (S_2\lambda^2 + S_1\lambda + S_0)e^{-\lambda\tau_2} + (T_1\lambda + T_0)e^{-\lambda(\tau_1 + \tau_2)} = 0, \end{aligned} \tag{5}$$

where

$$\begin{aligned} P_0 &= p_{22}(p_{13}p_{31} - p_{11}p_{33}), \\ P_1 &= p_{11}p_{22} + p_{11}p_{33} + p_{22}p_{33} - p_{13}p_{31}, \end{aligned}$$

$$\begin{aligned}
 P_2 &= -(p_{11} + p_{22} + p_{33}), \\
 Q_0 &= p_{33}(p_{12}q_{21} - p_{22}q_{11} - p_{11}q_{22}) - p_{13}(p_{31}q_{22} + p_{32}q_{21}), \\
 Q_1 &= q_{11}(p_{22} + p_{33}) + q_{22}(p_{11} + p_{33}) - p_{12}q_{21}, \\
 Q_2 &= -(q_{11} + q_{22}), \\
 S_0 &= s_{22}(p_{13}p_{31} - p_{11}p_{33}), \\
 S_1 &= s_{22}(p_{11} + p_{33}), \quad S_2 = -s_{22}, \\
 T_0 &= -(p_{13}q_{21}s_{32} + p_{33}q_{11}s_{22}), \quad T_1 = q_{11}s_{22}.
 \end{aligned}$$

Case 1. $\tau_1 = \tau_2 = 0$. Equation (5) becomes

$$\lambda^3 + P_{12}\lambda^2 + P_{11}\lambda + P_{10} = 0, \tag{6}$$

where

$$P_{10} = P_0 + Q_0 + S_0 + T_0, \quad P_{11} = P_1 + Q_1 + S_1 + T_1, \quad P_{12} = P_2 + Q_2 + S_2.$$

Thus, in view of Routh–Hurwitz criteria, if condition (H_1) : $P_{10} > 0$, $P_{12} > 0$, and $P_{12}P_{11} > P_{10}$ holds, then all roots of Eq. (6) have a negative real part.

Case 2. $\tau_1 > 0$, $\tau_2 = 0$. Equation (5) becomes

$$\lambda^3 + P_{22}\lambda^2 + P_{21}\lambda + P_{20} + (Q_{22}\lambda^2 + Q_{21}\lambda + Q_{20})e^{-\lambda\tau_1} = 0, \tag{7}$$

where

$$\begin{aligned}
 P_{20} &= P_0 + S_0, & P_{21} &= P_1 + S_1, & P_{22} &= P_2 + S_2, \\
 Q_{20} &= Q_0 + T_0, & Q_{21} &= Q_1 + T_1, & Q_{22} &= Q_2.
 \end{aligned}$$

Let $\lambda = i\omega_1$ be the root of Eq. (7). Then

$$\begin{cases}
 Q_{21}\omega_1 \sin \tau_1\omega_1 + (Q_{20} - Q_{22}\omega_1^2) \cos \tau_1\omega_1 = P_{22}\omega_1^2 - P_{20}, \\
 Q_{21}\omega_1 \cos \tau_1\omega_1 - (Q_{20} - Q_{22}\omega_1^2) \sin \tau_1\omega_1 = \omega_1^3 - P_{21}\omega_1.
 \end{cases} \tag{8}$$

Then

$$\omega_1^6 + U_{22}\omega_1^4 + U_{21}\omega_1^2 + U_{20} = 0 \tag{9}$$

with

$$\begin{aligned}
 U_{20} &= P_{20}^2 - Q_{20}^2, \\
 U_{21} &= P_{21}^2 - 2P_{20}P_{22} - Q_{21}^2 + 2Q_{20}Q_{22}, \\
 U_{22} &= P_{22}^2 - 2P_{21} - Q_{22}^2.
 \end{aligned}$$

Let $\varpi_1 = \omega_1^2$, then Eq. (9) becomes

$$\varpi_1^3 + U_{22}\varpi_1^2 + U_{21}\varpi_1 + U_{20} = 0. \tag{10}$$

For the distribution of roots of Eq. (10), we have Lemma 2.

Lemma 2 (i) If $U_{20} < 0$, then Eq. (10) has at least one positive root; (ii) If $U_{20} \geq 0$ and $U_{22}^2 - 3U_{21} \leq 0$, then Eq. (10) has no positive root; (iii) If $U_{20} \geq 0$ and $U_{22}^2 - 3U_{21} > 0$, then Eq. (10) has positive roots if and only if $U_{2*} = \frac{-U_{22} + \sqrt{U_{22}^2 - 3U_{21}}}{3} > 0$ and $f_1(U_{2*}) \leq 0$, where $f_1(\varpi_1) = \varpi_1^3 + U_{22}\varpi_1^2 + U_{21}\varpi_1 + U_{20}$.

Assume that (H_{21}) Eq. (10) has at least one positive root. With loss of generality, we suppose that Eq. (10) has three positive roots denoted by ϖ_{11}, ϖ_{12} , and ϖ_{13} , respectively. Then Eq. (9) has positive roots $\omega_{1i} = \sqrt{\varpi_{1i}}, i = 1, 2, 3$. For ω_{1i} ,

$$\tau_{1i}^{(j)} = \frac{1}{\omega_{1i}} \left\{ \arccos \left[\frac{(Q_{21} - P_{22}Q_{22})\omega_{1i}^4 + (P_{20}Q_{22} - P_{21}Q_{21} + P_{22}Q_{20})\omega_{1i}^2 - P_{20}Q_{20}}{Q_{22}^2\omega_{1i}^4 + (Q_{21}^2 - 2Q_{20}Q_{22})\omega_{1i}^2 + Q_{20}^2} \right] + 2j\pi \right\},$$

where $i = 1, 2, 3$ and $j = 0, 1, 2, \dots$. Define

$$\tau_{10} = \tau_{1i0}^{(0)} = \min\{\tau_{1i}^{(0)}\}, \quad \omega_{10} = \omega_{1i0}, \quad i = 1, 2, 3.$$

It follows from Eq. (7) that

$$\left[\frac{d\lambda(\tau_1)}{d\tau_1} \right]^{-1} = -\frac{3\lambda^2 + 2P_{22}\lambda + P_{21}}{\lambda(\lambda^3 + P_{22}\lambda^2 + P_{21}\lambda + P_{20})} + \frac{2Q_{22}\lambda + Q_{21}}{\lambda(Q_{22}\lambda^2 + Q_{21}\lambda + Q_{20})} - \frac{\tau_1}{\lambda}.$$

Further,

$$\operatorname{Re} \left[\frac{d\lambda(\tau_1)}{d\tau_1} \right]_{\tau_1=\tau_{10}}^{-1} = \frac{f_1'(\varpi_{10})}{Q_{22}^2\omega_{10}^4 + (Q_{21}^2 - 2Q_{20}Q_{22})\omega_{10}^2 + Q_{20}^2},$$

where $f_1(\varpi_1) = \varpi_1^3 + U_{22}\varpi_1^2 + U_{21}\varpi_1 + U_{20}$ and $\varpi_{10} = \omega_{10}^2$. Then, if $(H_{22}): f_1'(\varpi_{10}) \neq 0$ holds,

$$\operatorname{sign} \left\{ \frac{d(\operatorname{Re}(\lambda))}{d\tau_1} \right\}_{\tau_1=\tau_{10}} = \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau_1} \right) \right\}_{\tau_1=\tau_{10}} \neq 0.$$

Theorem 1 If $R_0 > 1$ and conditions $(H_1), (H_{21}),$ and (H_{22}) hold, then the disease-present equilibrium $E^*(S^*, I^*, R^*)$ of system (2) is locally asymptotically stable when $\tau_1 \in [0, \tau_{10})$; system (2) undergoes a Hopf bifurcation when $\tau_1 = \tau_{10}$ and a family of periodic solutions bifurcate from the disease-present equilibrium $E^*(S^*, I^*, R^*)$ near $\tau_1 = \tau_{10}$.

Case 3. $\tau_1 = 0, \tau_2 > 0$. Equation (5) becomes

$$\lambda^3 + P_{32}\lambda^2 + P_{31}\lambda + P_{30} + (S_{32}\lambda^2 + S_{31}\lambda + S_{30})e^{-\lambda\tau_2} = 0, \tag{11}$$

where

$$\begin{aligned} P_{30} &= P_0 + Q_0, & P_{31} &= P_1 + Q_1, & P_{32} &= P_2 + Q_2, \\ S_{30} &= S_0 + T_0, & S_{31} &= S_1 + T_1, & S_{32} &= S_2. \end{aligned}$$

Let $\lambda = i\omega_2$ be the root of Eq. (11). Then

$$\begin{cases} S_{31}\omega_2 \sin \tau_2\omega_2 + (S_{30} - S_{32}\omega_2^2) \cos \tau_2\omega_2 = P_{32}\omega_2^2 - P_{30}, \\ S_{31}\omega_2 \cos \tau_2\omega_2 - (S_{30} - S_{32}\omega_2^2) \sin \tau_2\omega_2 = \omega_2^3 - P_{31}\omega_2. \end{cases} \tag{12}$$

Then

$$\omega_2^6 + U_{32}\omega_2^4 + U_{31}\omega_2^2 + U_{30} = 0, \tag{13}$$

where

$$\begin{aligned} U_{30} &= P_{30}^2 - S_{30}^2, \\ U_{31} &= P_{31}^2 - 2P_{30}P_{32} - S_{31}^2 + 2S_{30}S_{32}, \\ U_{32} &= P_{32}^2 - 2P_{31} - S_{32}^2. \end{aligned}$$

Let $\varpi_2 = \omega_2^2$, then Eq. (13) becomes

$$\varpi_2^3 + U_{32}\varpi_2^2 + U_{31}\varpi_2 + U_{30} = 0. \tag{14}$$

For the distribution of roots of Eq. (14), we have Lemma 3.

Lemma 3 (i) If $U_{30} < 0$, then Eq. (14) has at least one positive root; (ii) If $U_{30} \geq 0$ and $U_{32}^2 - 3U_{31} \leq 0$, then Eq. (14) has no positive root; (iii) If $U_{30} \geq 0$ and $U_{32}^2 - 3U_{31} > 0$, then Eq. (10) has positive roots if and only if $U_{3*} = \frac{-U_{32} + \sqrt{U_{32}^2 - 3U_{31}}}{3} > 0$ and $f_2(U_{3*}) \leq 0$, where $f_2(\varpi_2) = \varpi_2^3 + U_{32}\varpi_2^2 + U_{31}\varpi_2 + U_{30}$.

Similar to Case 2, assume that (H_{31}) Eq. (14) has three positive roots denoted by ϖ_{21} , ϖ_{22} , and ϖ_{23} , respectively. Then Eq. (13) has positive roots $\omega_{2i} = \sqrt{\varpi_{2i}}$, $i = 1, 2, 3$. For ω_{2i} ,

$$\tau_{2i}^{(j)} = \frac{1}{\omega_{2i}} \left\{ \arccos \left[\frac{(S_{31} - P_{32}S_{32})\omega_{2i}^4 + (P_{30}S_{32} - P_{31}S_{31} + P_{32}S_{30})\omega_{2i}^2 - P_{30}S_{30}}{S_{32}^2\omega_{2i}^4 + (S_{31}^2 - 2S_{30}S_{32})\omega_{2i}^2 + S_{30}^2} \right] + 2j\pi \right\},$$

where $i = 1, 2, 3$ and $j = 0, 1, 2, \dots$. Define

$$\tau_{20} = \tau_{2i0}^{(0)} = \min \{ \tau_{2i}^{(0)} \}, \quad \omega_{20} = \omega_{2i0}, \quad i = 1, 2, 3.$$

Then

$$\left[\frac{d\lambda(\tau_2)}{d\tau_2} \right]^{-1} = -\frac{3\lambda^2 + 2P_{32}\lambda + P_{31}}{\lambda(\lambda^3 + P_{32}\lambda^2 + P_{31}\lambda + P_{30})} + \frac{2S_{32}\lambda + S_{31}}{\lambda(S_{32}\lambda^2 + S_{31}\lambda + S_{30})} - \frac{\tau_2}{\lambda}.$$

Further,

$$\operatorname{Re} \left[\frac{d\lambda(\tau_2)}{d\tau_2} \right]_{\tau_2=\tau_{20}}^{-1} = \frac{f_2'(\varpi_{20})}{S_{32}^2 \omega_{20}^4 + (S_{31}^2 - 2S_{30}S_{32})\omega_{20}^2 + Q_{30}^2},$$

where $f_2(\varpi_2) = \varpi_2^3 + U_{32}\varpi_2^2 + U_{31}\varpi_2 + U_{30}$ and $\varpi_{20} = \omega_{20}^2$. Then, if $(H_{32}): f_2'(\varpi_{20}) \neq 0$ holds,

$$\operatorname{sign} \left\{ \frac{d(\operatorname{Re}(\lambda))}{d\tau_2} \right\}_{\tau_2=\tau_{20}} = \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau_2} \right) \right\}_{\tau_2=\tau_{20}} \neq 0.$$

Theorem 2 *If $R_0 > 1$ and conditions (H_1) , (H_{31}) , and (H_{32}) hold, then the disease-present equilibrium $E^*(S^*, I^*, R^*)$ of system (2) is locally asymptotically stable when $\tau_2 \in [0, \tau_{20})$; system (2) undergoes a Hopf bifurcation when $\tau_2 = \tau_{20}$ and a family of periodic solutions bifurcate from the disease-present equilibrium $E^*(S^*, I^*, R^*)$ near $\tau_2 = \tau_{20}$.*

Case 4. $\tau_1 = \tau_2 = \tau > 0$. Equation (5) becomes

$$\lambda^3 + P_{42}\lambda^2 + P_{41}\lambda + P_{40} + (Q_{42}\lambda^2 + Q_{41}\lambda + Q_{40})e^{-\lambda\tau} + (T_{41}\lambda + T_{40})e^{-2\lambda\tau} = 0, \tag{15}$$

where

$$\begin{aligned} P_{40} &= P_0, & P_{41} &= P_1, & P_{42} &= P_2, \\ Q_{40} &= Q_0 + S_0, & Q_{41} &= Q_1 + S_1, & Q_{42} &= Q_2 + S_2, \\ T_{40} &= T_0, & T_{41} &= T_1. \end{aligned}$$

Multiplying by $e^{\lambda\tau}$, Eq. (15) becomes

$$Q_{42}\lambda^2 + Q_{41}\lambda + Q_{40} + (\lambda^3 + P_{42}\lambda^2 + P_{41}\lambda + P_{40})e^{\lambda\tau} + (T_{41}\lambda + T_{40})e^{-\lambda\tau} = 0. \tag{16}$$

Let $\lambda = i\omega$ be the root of Eq. (16). Then

$$\begin{cases} (P_{40} - T_{40} - P_{42}\omega^2) \sin \tau\omega + ((P_{41} + T_{41})\omega - \omega^3) \cos \tau\omega = -Q_{41}\omega, \\ (P_{40} + T_{40} - P_{42}\omega^2) \cos \tau\omega - ((P_{41} - T_{41})\omega - \omega^3) \sin \tau\omega = Q_{42}\omega^2 - Q_{40}. \end{cases} \tag{17}$$

Thus, one can obtain the expressions of $\cos \tau\omega$ and $\sin \tau\omega$ as follows:

$$\cos \tau\omega = \frac{L_{24}\omega^4 + L_{22}\omega^2 + L_{20}}{\omega^6 + L_{14}\omega^4 + L_{12}\omega^2 + L_{10}}, \quad \sin \tau\omega = \frac{L_{25}\omega^5 + L_{23}\omega^3 + L_{21}\omega}{\omega^6 + L_{14}\omega^4 + L_{12}\omega^2 + L_{10}},$$

with

$$\begin{aligned} L_{10} &= -T_{40}^2, & L_{12} &= P_{42}^2 - 2P_{40}P_{42} - T_{41}^2, & L_{14} &= P_{42}^2 - 2P_{41}, \\ L_{20} &= -(P_{40} - Q_{40}), & L_{21} &= (P_{41} + T_{41})Q_{40} - (P_{40} + T_{40})Q_{41}, \\ L_{22} &= (P_{40} - T_{40})Q_{40} - (P_{41} - T_{41})Q_{41} + P_{42}Q_{40}, \\ L_{23} &= P_{42}Q_{41} - Q_{40} - (P_{41} + T_{41})Q_{42}, & L_{24} &= Q_{41} - P_{42}Q_{42}, & L_{25} &= Q_{42}. \end{aligned}$$

Further one can obtain the equation with respect to ω as follows:

$$\omega^{12} + L_{35}\omega^{10} + L_{34}\omega^8 + L_{33}\omega^6 + L_{32}\omega^4 + L_{31}\omega^2 + L_{30} = 0 \tag{18}$$

with

$$\begin{aligned} L_{30} &= L_{10}^2 - L_{20}^2, & L_{31} &= 2L_{10}L_{12} - 2L_{20}L_{22} - L_{21}^2, \\ L_{32} &= L_{12}^2 + 2L_{10}L_{14} - L_{22}^2 - 2L_{20}L_{24} - 2L_{21}L_{23}, \\ L_{33} &= 2L_{10} + 2L_{12}L_{14} + 2L_{22}L_{24} - L_{23}^2 - 2L_{21}L_{25}, \\ L_{34} &= L_{14}^2 + 2L_{12} - L_{24}^2 - 2L_{23}L_{25}, & L_{35} &= 2L_{14} - L_{25}^2. \end{aligned}$$

Let $\varpi = \omega^2$. Then Eq. (18) becomes

$$\varpi^6 + L_{35}\varpi^5 + L_{34}\varpi^4 + L_{33}\varpi^3 + L_{32}\varpi^2 + L_{31}\varpi + L_{30} = 0. \tag{19}$$

If all the coefficients of system (2) are given, then we can obtain all the coefficients of Eq. (19). Therefore, we assume that (H_{41}) : Eq. (19) has six positive roots denoted by $\varpi_1, \varpi_2, \dots, \varpi_6$, respectively. Then, Eq. (18) has positive roots $\omega_i = \sqrt{\varpi_i}$, $i = 1, 2, \dots, 6$. For ω_i ,

$$\tau_i^{(j)} = \frac{1}{\omega_i} \left\{ \arccos \left[\frac{L_{24}\omega_i^4 + L_{22}\omega_i^2 + L_{20}}{\omega_i^6 + L_{14}\omega_i^4 + L_{12}\omega_i^2 + L_{10}} \right] + 2j\pi \right\},$$

where $i = 1, 2, \dots, 6$ and $j = 0, 1, 2, \dots$. Define

$$\tau_0 = \tau_{i0}^{(0)} = \min \{ \tau_i^{(0)} \}, \quad \omega_0 = \omega_{i0}, \quad i = 1, 2, \dots, 6.$$

It follows from Eq. (16) that

$$\left[\frac{d\lambda(\tau)}{d\tau} \right]^{-1} = - \frac{2Q_{42}\lambda + Q_{41} + (3\lambda^2 + 2P_{42}\lambda + P_{41})e^{\lambda\tau} + T_{41}e^{-\lambda\tau}}{(\lambda^4 + P_{42}\lambda^3 + P_{41}\lambda^2 + P_{40}\lambda)e^{\lambda\tau} - (T_{41}\lambda^2 + T_{40}\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda},$$

which leads to

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_0}^{-1} = - \frac{L_{41}L_{43} + L_{42}L_{44}}{L_{43}^2 + L_{44}^2},$$

where

$$\begin{aligned} L_{41} &= (P_{41} + T_{41} - 3\omega_0^2) \cos \tau_0\omega_0 - 2P_{42}\omega_0 \cos \tau_0\omega_0 + Q_{41}, \\ L_{42} &= (P_{41} - T_{41} - 3\omega_0^2) \sin \tau_0\omega_0 + 2P_{42}\omega_0 \sin \tau_0\omega_0 + 2Q_{42}\omega_0, \\ L_{43} &= (\omega_0^4 - P_{41}\omega_0^2 + T_{41}\omega_0^2) \cos \tau_0\omega_0 - (P_{40}\omega_0 + T_{40}\omega_0 - P_{42}\omega_0^3) \sin \tau_0\omega_0, \\ L_{44} &= (\omega_0^4 - P_{41}\omega_0^2 - T_{41}\omega_0^2) \sin \tau_0\omega_0 + (P_{40}\omega_0 - T_{40}\omega_0 - P_{42}\omega_0^3) \cos \tau_0\omega_0. \end{aligned}$$

Obviously, if $(H_{42}): L_{41}L_{43} + L_{42}L_{44} \neq 0$ holds,

$$\text{sign} \left\{ \frac{d(\text{Re}(\lambda))}{d\tau} \right\}_{\tau=\tau_0} = \text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau=\tau_0} \neq 0.$$

Theorem 3 *If $R_0 > 1$ and conditions (H_1) , (H_{41}) , and (H_{42}) hold, then the disease-present equilibrium $E^*(S^*, I^*, R^*)$ of system (2) is locally asymptotically stable when $\tau \in [0, \tau_0)$; system (2) undergoes a Hopf bifurcation when $\tau = \tau_0$ and a family of periodic solutions bifurcate from the disease-present equilibrium $E^*(S^*, I^*, R^*)$ near $\tau = \tau_0$.*

Case 5. $\tau_1 > 0, \tau_2 \in (0, \tau_{20})$. Let $\lambda = i\omega_{1*}$ be the root of Eq. (5), then

$$\begin{cases} L_{51}(\omega_{1*}) \sin \tau_1 \omega_{1*} + L_{52}(\omega_{1*}) \cos \tau_1 \omega_{1*} = L_{53}(\omega_{1*}), \\ L_{51}(\omega_{1*}) \cos \tau_1 \omega_{1*} - L_{52}(\omega_{1*}) \sin \tau_1 \omega_{1*} = L_{54}(\omega_{1*}), \end{cases} \tag{20}$$

where

$$\begin{aligned} L_{51}(\omega_{1*}) &= Q_1 \omega_{1*} + T_1 \omega_{1*} \cos \tau_2 \omega_{1*} - T_0 \sin \tau_2 \omega_{1*}, \\ L_{52}(\omega_{1*}) &= Q_0 - Q_2 \omega_{1*}^2 + T_1 \omega_{1*} \sin \tau_2 \omega_{1*} + T_0 \cos \tau_2 \omega_{1*}, \\ L_{53}(\omega_{1*}) &= P_2 \omega_{1*}^2 - P_0 - S_1 \omega_{1*} \sin \tau_2 \omega_{1*} - (S_0 - S_2 \omega_{1*}^2) \cos \tau_2 \omega_{1*}, \\ L_{54}(\omega_{1*}) &= \omega_{1*}^3 - P_1 \omega_{1*} - S_1 \omega_{1*} \cos \tau_2 \omega_{1*} + (S_0 - S_2 \omega_{1*}^2) \sin \tau_2 \omega_{1*}. \end{aligned} \tag{21}$$

Thus, one can obtain the following equation with respect to ω_{1*} :

$$L_{60}(\omega_{1*}) + L_{61}(\omega_{1*}) \cos \tau_2 \omega_{1*} + L_{62}(\omega_{1*}) \sin \tau_2 \omega_{1*} = 0 \tag{22}$$

with

$$\begin{aligned} L_{60}(\omega_{1*}) &= \omega_{1*}^6 + (P_2^2 - 2P_1 - S_2^2 - Q_2^2) \omega_{1*}^4 \\ &\quad + (P_1^2 - 2P - 0P_2 + S_1^2 - 2S_0S_2 - Q_1^2 + 2Q_0Q_2 - T_1^2) \omega_{1*}^2 \\ &\quad + P_0^2 + S_0^2 - T_0^2, \\ L_{61}(\omega_{1*}) &= 2[(P_2S_2 - S_1) \omega_{1*}^4 + (P_1S_1 - P_0S_2 - P_2S_0 - Q_1T_1 + Q_2T_0) \omega_{1*}^2 \\ &\quad + P_0S_0 - Q_0T_0], \\ L_{62}(\omega_{1*}) &= 2[-S_2 \omega_{1*}^5 + (P_1S_2 - P_2S_1 + S_0 + Q_2T_1) \omega_{1*}^3 \\ &\quad + (P_0S_1 - S_0P_1 - Q_0T_1 + Q_1T_0) \omega_{1*}]. \end{aligned}$$

Similar as in Case 4, we suppose that Eq. (22) has at least one positive root. Without loss of generality, we denote the positive roots of Eq. (22) as $\omega_{11}^*, \omega_{12}^*, \dots, \omega_{1k}^*$, respectively. For $\omega_{1i}^*, i = 0, 1, \dots, k$,

$$\tau_{1i}^{*(j)} = \frac{1}{\omega_{1i}^*} \left\{ \arccos \left[\frac{l_{51}(\omega_{1i}^*) \times L_{54}(\omega_{1i}^*) + L_{52}(\omega_{1i}^*) \times L_{53}(\omega_{1i}^*)}{L_{51}^2(\omega_{1i}^*) + L_{52}^2(\omega_{1i}^*)} \right] + 2j\pi \right\},$$

where $i = 1, 2, \dots, k$ and $j = 0, 1, 2, \dots$. Define

$$\tau_{10}^* = \tau_{1i0}^{*(0)} = \min\{\tau_{1i}^{*(0)}\}, \quad \omega_{10}^* = \omega_{1i0}, \quad i = 1, 2, \dots, k.$$

Then

$$\left[\frac{d\lambda(\tau_1)}{d\tau_1} \right]^{-1} = \frac{L_{71}(\lambda)}{L_{72}(\lambda)} - \frac{\tau_1}{\lambda},$$

with

$$\begin{aligned} L_{71}(\lambda) &= 3\lambda^2 + 2P_2\lambda + P_1 + (2Q_2\lambda + Q_1)e^{-\lambda\tau_1} \\ &\quad + (-\tau_2S_2\lambda^2 + (2S_2 - \tau_2S_1)\lambda + S_1 - \tau_2S_0)e^{-\lambda\tau_2} \\ &\quad + (-\tau_2T_1\lambda + T_1 - \tau_2T_0)e^{-\lambda(\tau_1+\tau_2)}, \\ L_{72}(\lambda) &= (Q_2\lambda^3 + Q_1\lambda^2 + Q_0\lambda)e^{-\lambda\tau_1} + (T_1\lambda^2 + T_0\lambda)e^{-\lambda(\tau_1+\tau_2)}. \end{aligned}$$

Further,

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau_1} \right]_{\tau_1=\tau_{10}^*}^{-1} = \frac{L_{81}L_{83} + L_{82}L_{84}}{L_{83}^2 + L_{84}^2},$$

where

$$\begin{aligned} L_{81} &= 3(\omega_{10}^*)^2 + P_1 + 2Q_2\omega_{10}^* \sin \tau_{10}^* \omega_{10}^* + Q_1 \cos \tau_{10}^* \omega_{10}^* \\ &\quad + (2S_2 - \tau_2S_1) \sin \tau_2 \omega_{10}^* + (\tau_2S_2(\omega_{10}^*)^2 + S_1 - \tau_2S_0) \cos \tau_2 \omega_{10}^* \\ &\quad - \tau_2T_2\omega_{10}^* \sin(\tau_{10}^* + \tau_2) + (T_1 - \tau_2T_0) \cos(\tau_{10}^* + \tau_2), \\ L_{82} &= 2P_2L_{81}\omega_{10}^* + 2Q_2\omega_{10}^* \cos \tau_{10}^* \omega_{10}^* - Q_1 \sin \tau_{10}^* \omega_{10}^* \\ &\quad + (2S_2 - \tau_2S_1) \cos \tau_2 \omega_{10}^* - (\tau_2S_2(\omega_{10}^*)^2 + S_1 - \tau_2S_0) \sin \tau_2 \omega_{10}^* \\ &\quad - \tau_2T_2\omega_{10}^* \cos(\tau_{10}^* + \tau_2) - (T_1 - \tau_2T_0) \sin(\tau_{10}^* + \tau_2), \\ L_{83} &= (Q_0\omega_{10}^* - Q_2(\omega_{10}^*)^3) \sin \tau_{10}^* \omega_{10}^* - Q_1(\omega_{10}^*)^2 \cos \tau_{10}^* \omega_{10}^* \\ &\quad + T_0\omega_{10}^* \sin(\tau_{10}^* + \tau_2) - T_1(\omega_{10}^*)^2 \cos(\tau_{10}^* + \tau_2), \\ L_{84} &= (Q_0\omega_{10}^* - Q_2(\omega_{10}^*)^3) \cos \tau_{10}^* \omega_{10}^* + Q_1(\omega_{10}^*)^2 \sin \tau_{10}^* \omega_{10}^* \\ &\quad + T_0\omega_{10}^* \cos(\tau_{10}^* + \tau_2) + T_1(\omega_{10}^*)^2 \sin(\tau_{10}^* + \tau_2). \end{aligned}$$

Obviously, if (H_{52}) : $L_{81}L_{83} + L_{82}L_{84} \neq 0$ holds,

$$\operatorname{sign} \left\{ \frac{d(\operatorname{Re}(\lambda))}{d\tau_1} \right\}_{\tau_1=\tau_{10}^*} = \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau_1} \right) \right\}_{\tau_1=\tau_{10}^*} \neq 0.$$

Theorem 4 *If $R_0 > 1$ and conditions (H_1) , (H_{51}) , and (H_{52}) hold, then the disease-present equilibrium $E^*(S^*, I^*, R^*)$ of system (2) is locally asymptotically stable when $\tau_1 \in [0, \tau_{10}^*]$; system (2) undergoes a Hopf bifurcation when $\tau_1 = \tau_{10}^*$ and a family of periodic solutions bifurcate from the disease-present equilibrium $E^*(S^*, I^*, R^*)$ near $\tau_1 = \tau_{10}^*$.*

3 Global stability of the disease-present equilibrium

Theorem 5 *If $\max\{l_1, l_2, l_3\} < 0$, with*

$$\begin{aligned}
 l_1 &= \left[\frac{p_{11} + q_{11}}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} + \frac{q_{21}M_1\tau_1}{m_2} \right) + \frac{q_{21}}{m_2} \left(1 + \frac{M_2\tau_1}{m_2}(q_{22} + s_{22}) + \frac{s_{32}M_2\tau_1}{m_3} \right) \right. \\
 &\quad \left. + \frac{q_{12}q_{21}M_2\tau_1}{m_1m_2} + \frac{p_{31}}{m_3} \right] < 0, \\
 l_2 &= \left[\frac{p_{12} + q_{12}}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} + \frac{q_{21}M_1\tau_1}{m_2} \right) \right. \\
 &\quad \left. + \frac{p_{22} + q_{22} + s_{22}}{m_2} \left(\frac{q_{22}M_2\tau_1}{m_2} + \frac{s_{22}M_2\tau_2}{m_2} + 1 \right) + \frac{q_{12}M_2}{m_1m_2} (p_{22}\tau_1 + q_{22}\tau_1 + s_{22}\tau_2) \right. \\
 &\quad \left. + \frac{s_{32}M_2}{m_2m_3} (p_{22}\tau_2 + q_{22}\tau_1 + s_{22}\tau_2) + \frac{p_{32} + s_{32}}{m_3} \right] < 0, \\
 l_3 &= \left[\frac{p_{13}}{m_1} + \frac{p_{33}}{m_3} + \frac{p_{13}M_1\tau_1}{m_1} \left(\frac{q_{11}}{m_1} + \frac{q_{21}}{m_2} \right) \right] < 0,
 \end{aligned}$$

where $m_1 < S(t) < M_1$, $m_2 < I(t) < M_2$, and $m_3 < R(t) < M_3$ for $t > 0$, the disease-present equilibrium E^* of system (2) is globally asymptotically stable.

Proof Let $S(t) = S^*e^{x(t)}$, $I(t) = I^*e^{y(t)}$, $R(t) = R^*e^{z(t)}$. Then $E^*(S^*, I^*, R^*)$ becomes the trivial equilibrium for $x(t) = y(t) = z(t) = 0$ for all $t > 0$, and system (4) can be reduced in the following form:

$$\begin{aligned}
 \frac{dx}{dt} &= \frac{p_{11}S^*}{S} [e^{x(t)} - 1] + \frac{p_{12}I^*}{S} [e^{y(t)} - 1] + \frac{p_{13}R^*}{S} [e^{z(t)} - 1] \\
 &\quad + \frac{q_{11}S^*}{S} [e^{x(t-\tau_1)} - 1] + \frac{q_{12}I^*}{S} [e^{y(t-\tau_1)} - 1], \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{p_{22}I^*}{I} [e^{y(t)} - 1] + \frac{q_{21}S^*}{I} [e^{x(t-\tau_1)} - 1] + \frac{q_{22}I^*}{I} [e^{y(t-\tau_1)} - 1] \\
 &\quad + \frac{s_{22}I^*}{I} [e^{y(t-\tau_2)} - 1], \tag{24}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dz}{dt} &= \frac{p_{31}S^*}{R} [e^{x(t)} - 1] + \frac{p_{32}I^*}{R} [e^{y(t)} - 1] + \frac{p_{33}R^*}{R} [e^{z(t)} - 1] \\
 &\quad + \frac{s_{32}I^*}{R} [e^{y(t-\tau_2)} - 1]. \tag{25}
 \end{aligned}$$

Now, we have

$$e^{x(t-\tau_1)} = e^{x(t)} - \int_{t-\tau_1}^t e^{x(\omega)} \frac{dx}{d\omega} d\omega, \tag{26}$$

$$e^{y(t-\tau_1)} = e^{y(t)} - \int_{t-\tau_1}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega. \tag{27}$$

Now, by using Eq. (23), Eq. (26), and Eq. (27), we have

$$\begin{aligned}
 \frac{dx}{dt} &= \frac{p_{11}S^*}{S} [e^{x(t)} - 1] + \frac{p_{12}I^*}{S} [e^{y(t)} - 1] + \frac{p_{13}R^*}{S} [e^{z(t)} - 1] \\
 &\quad + \frac{q_{11}S^*}{S} [e^{x(t)} - 1] - \frac{q_{11}S^*}{S} \int_{t-\tau_1}^t e^{x(\omega)} \frac{dx}{d\omega} d\omega
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{q_{12}I^*}{S} [e^{y(t)} - 1] - \frac{q_{12}I^*}{S} \int_{t-\tau_1}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega \\
 = & \frac{p_{11}S^*}{S} [e^{x(t)} - 1] + \frac{p_{12}I^*}{S} [e^{y(t)} - 1] + \frac{p_{13}R^*}{S} [e^{z(t)} - 1] \\
 & + \frac{q_{11}S^*}{S} [e^{x(t)} - 1] + \frac{q_{12}I^*}{S} [e^{y(t)} - 1] \\
 & - \frac{q_{12}I^*}{S} \int_{t-\tau_1}^t e^{y(\omega)} \left[\frac{p_{22}I^*}{I} [e^{y(\omega)} - 1] + \frac{q_{21}S^*}{I} [e^{x(\omega-\tau_1)} - 1] \right. \\
 & \left. + \frac{q_{22}I^*}{I} [e^{y(\omega-\tau_1)} - 1] + \frac{s_{22}I^*}{I} [e^{y(\omega-\tau_2)} - 1] \right] d\omega \\
 & - \frac{q_{11}S^*}{S} \int_{t-\tau_1}^t e^{x(\omega)} \left[\frac{p_{11}S^*}{S} [e^{x(\omega)} - 1] + \frac{p_{12}I^*}{S} [e^{y(\omega)} - 1] \right. \\
 & \left. + \frac{p_{13}R^*}{S} [e^{z(\omega)} - 1] + \frac{q_{11}S^*}{S} [e^{x(\omega-\tau_1)} - 1] + \frac{q_{12}I^*}{S} [e^{y(\omega-\tau_1)} - 1] \right] d\omega.
 \end{aligned}$$

Let $V_1(t) = |x(t)|$,

$$\begin{aligned}
 D^+ V_1 \leq & \frac{p_{11}S^*}{m_1} |e^{x(t)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(t)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(t)} - 1| \\
 & + \frac{q_{11}S^*}{m_1} |e^{x(t)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(t)} - 1| \\
 & + \frac{q_{11}S^*}{m_1} \int_{t-\tau_1}^t e^{x(\omega)} \left[\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(\omega)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(\omega)} - 1| \right. \\
 & \left. + \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right] d\omega \\
 & + \frac{q_{12}I^*}{m_1} \int_{t-\tau_1}^t e^{y(\omega)} \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\
 & \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega.
 \end{aligned}$$

We find that there exists $t_1 > 0$ such that $S^* e^{x(t)} < M_1$, $I^* e^{y(t)} < M_2$, $\forall t > t_1$, and for $t > t_1 + \tau$, where $\tau = \max\{\tau_1, \tau_2\}$, we have

$$\begin{aligned}
 D^+ V_1 \leq & \frac{(p_{11} + q_{11})S^*}{m_1} |e^{x(t)} - 1| + \frac{(p_{12} + q_{12})I^*}{m_1} |e^{y(t)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(t)} - 1| \\
 & + \frac{q_{11}M_1}{m_1} \int_{t-\tau_1}^t \left[\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(\omega)} - 1| + \frac{p_{13}R^*}{m_1} |z(\omega) - 1| \right. \\
 & \left. + \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right] d\omega \\
 & + \frac{q_{12}M_2}{m_1} \int_{t-\tau_1}^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\
 & \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega. \tag{28}
 \end{aligned}$$

Due to the form of Eq. (28), we consider the following functional:

$$\begin{aligned} \tilde{V}_1(t) = & V_1(t) + \frac{q_{11}M_1}{m_1} \int_{t-\tau_1}^t \int_h^t \left[\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(\omega)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(\omega)} - 1| \right. \\ & \left. + \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right] d\omega dh \\ & + \frac{q_{11}^2 M_1 S^* \tau_1}{m_1^2} \int_{t-\tau_1}^t |e^{x(\omega)} - 1| d\omega + \frac{q_{11}q_{12}M_1 I^* \tau_1}{m_1^2} \int_{t-\tau_1}^t |e^{y(\omega)} - 1| d\omega \\ & + \frac{q_{12}M_2}{m_1} \int_{t-\tau_1}^t \int_h^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\ & \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega dh \\ & + \frac{q_{12}q_{21}M_2 S^* \tau_1}{m_1 m_2} \int_{t-\tau_1}^t |e^{x(\omega)} - 1| d\omega \\ & + \frac{q_{12}q_{22}M_2 I^* \tau_1}{m_1 m_2} \int_{t-\tau_1}^t |e^{y(\omega)} - 1| d\omega + \frac{q_{12}s_{22}M_2 I^* \tau_2}{m_1 m_2} \int_{t-\tau_2}^t |e^{y(\omega)} - 1| d\omega. \end{aligned}$$

Thus,

$$\begin{aligned} D^+ \tilde{V}_1(t) \leq & D^+ V_1(t) + \frac{q_{11}M_1 \tau_1}{m_1} \left[\frac{p_{11}S^*}{m_1} |e^{x(t)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(t)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(t)} - 1| \right] \\ & + \frac{q_{11}^2 M_1 S^* \tau_1}{m_1^2} |e^{x(t)} - 1| + \frac{q_{11}q_{12}M_1 I^* \tau_1}{m_1^2} |e^{y(t)} - 1| \\ & - \left[\frac{q_{11}M_1}{m_1} \int_{t-\tau_1}^t \left(\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(\omega)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(\omega)} - 1| \right. \right. \\ & \left. \left. + \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right) d\omega \right] \\ & + \frac{q_{12}p_{22}M_2 I^* \tau_1}{m_1 m_2} |e^{y(t)} - 1| + \frac{q_{12}q_{21}M_2 S^* \tau_1}{m_1 m_2} |e^{x(t)} - 1| \\ & + \frac{q_{12}q_{22}M_2 I^* \tau_1}{m_1 m_2} |e^{y(t)} - 1| + \frac{q_{12}s_{22}M_2 I^* \tau_2}{m_1 m_2} |e^{y(t)} - 1| \\ & - \left[\frac{q_{12}M_2}{m_1} \int_{t-\tau_1}^t \left(\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \right. \\ & \left. \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right) d\omega \right]. \end{aligned}$$

Further,

$$\begin{aligned} D^+ \tilde{V}_1(t) \leq & \left[\frac{(p_{11} + q_{11})S^*}{m_1} + \frac{q_{11}(p_{11} + q_{11})M_1 \tau_1 S^*}{m_1^2} + \frac{q_{12}q_{21}M_2 \tau_1 S^*}{m_1 m_2} \right] |e^{x(t)} - 1| \\ & + \left[\frac{(p_{12} + q_{12})I^*}{m_1} + \frac{q_{11}(p_{12} + q_{12})M_1 \tau_1 I^*}{m_1^2} + \frac{q_{12}(p_{22} \tau_1 + q_{22} \tau_1 + s_{22} \tau_2)M_2 I^*}{m_1 m_2} \right] \\ & \times |e^{y(t)} - 1| \\ & + \left[\frac{p_{13}R^*}{m_1} + \frac{q_{11}p_{13}R^* M_1 \tau_1}{m_1^2} \right] |e^{z(t)} - 1|. \end{aligned}$$

Then

$$\begin{aligned}
 D^+ \tilde{V}_1(t) \leq & \left[\frac{(p_{11} + q_{11})S^*}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} \right) + \frac{q_{12}q_{21}M_2S^*\tau_1}{m_1m_2} \right] |e^{x(t)} - 1| \\
 & + \left[\frac{(p_{12} + q_{12})I^*}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} \right) + \frac{q_{12}(p_{22}\tau_1 + q_{22}\tau_1 + s_{22}\tau_2)M_2I^*}{m_1m_2} \right] \\
 & \times |e^{y(t)} - 1| \\
 & + \left[\frac{p_{13}R^*}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} \right) \right] |e^{z(t)} - 1|. \tag{29}
 \end{aligned}$$

Next,

$$e^{y(t-\tau_2)} = e^{y(t)} - \int_{t-\tau_2}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega. \tag{30}$$

Then Eq. (24) can be rewritten as follows:

$$\begin{aligned}
 \frac{dy}{dt} = & \frac{p_{22}I^*}{I} [e^{y(t)} - 1] + \frac{q_{21}S^*}{I} \left[(e^{x(t)} - 1) - \int_{t-\tau_1}^t e^{x(\omega)} \frac{dx}{d\omega} d\omega \right] \\
 & + \frac{q_{22}I^*}{I} \left[(e^{y(t)} - 1) - \int_{t-\tau_1}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega \right] + \frac{s_{22}I^*}{I} \left[(e^{y(t)} - 1) - \int_{t-\tau_2}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega \right] \\
 = & \frac{p_{22}I^*}{I} [e^{y(t)} - 1] + \frac{q_{21}S^*}{I} [e^{x(t)} - 1] + \frac{q_{22}I^*}{I} [e^{y(t)} - 1] + \frac{s_{22}I^*}{I} [e^{y(t)} - 1] \\
 & - \frac{q_{21}S^*}{I} \int_{t-\tau_1}^t e^{x(\omega)} \frac{dx}{d\omega} d\omega - \frac{q_{22}I^*}{I} \int_{t-\tau_1}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega - \frac{s_{22}I^*}{I} \int_{t-\tau_2}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega \\
 = & \frac{(p_{22} + q_{22} + s_{22})I^*}{I} [e^{y(t)} - 1] + \frac{q_{21}S^*}{I} [e^{x(t)} - 1] \\
 & - \frac{q_{21}S^*}{I} \int_{t-\tau_1}^t e^{x(\omega)} \left[\frac{p_{11}S^*}{S} [e^{x(\omega)} - 1] + \frac{p_{12}I^*}{S} [e^{y(\omega)} - 1] + \frac{p_{13}R^*}{S} [e^{z(\omega)} - 1] \right. \\
 & \left. + \frac{q_{11}S^*}{S} [e^{x(\omega-\tau_1)} - 1] + \frac{q_{12}I^*}{S} [e^{y(\omega-\tau_1)} - 1] \right] d\omega \\
 & - \frac{q_{22}I^*}{I} \int_{t-\tau_1}^t e^{y(\omega)} \left[\frac{p_{22}I^*}{I} [e^{y(\omega)} - 1] + \frac{q_{21}S^*}{I} [e^{x(\omega-\tau_1)} - 1] + \frac{q_{22}I^*}{I} [e^{y(\omega-\tau_1)} - 1] \right. \\
 & \left. + \frac{s_{22}I^*}{I} [e^{y(\omega-\tau_2)} - 1] \right] d\omega - \frac{s_{22}I^*}{I} \int_{t-\tau_2}^t e^{y(\omega)} \left[\frac{p_{22}I^*}{I} [e^{y(\omega)} - 1] \right. \\
 & \left. + \frac{q_{21}S^*}{I} [e^{x(\omega-\tau_1)} - 1] + \frac{q_{22}I^*}{I} [e^{y(\omega-\tau_1)} - 1] + \frac{s_{22}I^*}{I} [e^{y(\omega-\tau_2)} - 1] \right] d\omega.
 \end{aligned}$$

Let $V_2(t) = |y(t)|$,

$$\begin{aligned}
 D^+ V_2 \leq & \frac{(p_{22} + q_{22} + s_{22})I^*}{m_2} |e^{y(t)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(t)} - 1| \\
 & + \frac{q_{21}S^*}{m_2} \int_{t-\tau_1}^t e^{x(\omega)} \left[\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(\omega)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(\omega)} - 1| \right. \\
 & \left. + \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right] d\omega
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{q_{22}I^*}{m_2} \int_{t-\tau_1}^t e^{y(\omega)} \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\
 & + \left. \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{\omega-\tau_2} - 1| \right] d\omega \\
 & + \frac{s_{22}I^*}{m_2} \int_{t-\tau_2}^t e^{y(\omega)} \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| \right. \\
 & + \left. \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega.
 \end{aligned}$$

Then, we find that there exists $t_1 > 0$ such that $S^*e^{x(t)} < M_1, I^*e^{y(t)} < M_2, \forall t > t_1$, and for $t > t_1 + \tau$, where $\tau = \max\{\tau_1, \tau_2\}$, we have

$$\begin{aligned}
 D^+V_2 \leq & \frac{(p_{22} + q_{22} + s_{22})I^*}{m_2} |e^{y(t)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(t)} - 1| \\
 & + \frac{q_{21}M_1}{m_2} \int_{t-\tau_1}^t \left[\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(\omega)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(\omega)} - 1| \right. \\
 & + \left. \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right] d\omega + \frac{q_{22}M_2}{m_2} \int_{t-\tau_1}^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| \right. \\
 & + \left. \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{x(\omega-\tau_2)} - 1| \right] d\omega \\
 & + \frac{s_{22}M_2}{m_2} \int_{t-\tau_2}^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| \right. \\
 & + \left. \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega. \tag{31}
 \end{aligned}$$

Again due to the form of Eq. (31), we consider the following Lyapunov functional:

$$\begin{aligned}
 \tilde{V}_2(t) = & V_2(t) + \frac{q_{21}M_1}{m_2} \int_{t-\tau_1}^t \int_k^t \left[\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(\omega)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(\omega)} - 1| \right. \\
 & + \left. \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right] d\omega dk \\
 & + \frac{q_{11}q_{21}S^*M_1\tau_1}{m_1m_2} \int_{t-\tau_1}^t |e^{x(\omega)} - 1| d\omega + \frac{q_{12}q_{21}M_1I^*\tau_1}{m_1m_2} \int_{t-\tau_1}^t |e^{y(\omega)} - 1| d\omega \\
 & + \frac{q_{22}M_2}{m_2} \int_{t-\tau_1}^t \int_k^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| \right. \\
 & + \left. \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega dk + \frac{q_{21}q_{22}S^*M_2\tau_1}{m_2^2} \int_{t-\tau_1}^t |e^{x(\omega)} - 1| d\omega \\
 & + \frac{q_{22}^2I^*M_2\tau_1}{m_2^2} \int_{t-\tau_1}^t |e^{y(\omega)} - 1| d\omega + \frac{q_{22}s_{22}M_2I^*\tau_2}{m_2^2} \int_{t-\tau_2}^t |e^{y(\omega)} - 1| d\omega \\
 & + \frac{s_{22}M_2}{m_2} \int_{t-\tau_2}^t \int_k^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| \right. \\
 & + \left. \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega dk + \frac{q_{21}s_{22}M_2S^*\tau_1}{m_2^2} \int_{t-\tau_1}^t |e^{x(\omega)} - 1| d\omega \\
 & + \frac{s_{22}q_{22}M_2I^*\tau_1}{m_2^2} \int_{t-\tau_1}^t |e^{y(\omega)} - 1| d\omega + \frac{s_{22}^2M_2I^*\tau_2}{m_2^2} \int_{t-\tau_2}^t |e^{y(\omega)} - 1| d\omega. \tag{32}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 D^+ \tilde{V}_2(t) \leq & D^+ V_2(t) + \left[\frac{q_{21}M_1\tau_1}{m_2} \left(\frac{p_{11}S^*}{m_1} |e^{x(t)} - 1| + \frac{p_{12}I^*}{m_1} |e^{y(t)} - 1| + \frac{p_{13}R^*}{m_1} |e^{z(t)} - 1| \right) \right. \\
 & \left. + \frac{q_{11}q_{12}S^*M_1\tau_1}{m_1m_2} |e^{x(t)} - 1| + \frac{q_{12}q_{21}M_1I^*\tau_1}{m_1m_2} |e^{y(t)} - 1| \right] \\
 & - \left[\frac{q_{21}M_1}{m_2} \int_{t-\tau_1}^t \left(\frac{p_{11}S^*}{m_1} |e^{x(\omega)} - 1| + \frac{p_{12}I^*}{m_1} |e_1^{y(\omega)}| + \frac{p_{13}R^*}{m_1} |e^{z(\omega)} - 1| \right) \right. \\
 & \left. + \frac{q_{11}S^*}{m_1} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{12}I^*}{m_1} |e^{y(\omega-\tau_1)} - 1| \right) d\omega \Big] + \frac{p_{22}q_{22}M_2I^*\tau_1}{m_2^2} |e^{y(t)} - 1| \\
 & + \frac{q_{21}q_{22}S^*M_2\tau_1}{m_2^2} |e^{x(t)} - 1| + \frac{q_{22}^2I^*M_2\tau_1}{m_2^2} |e^{y(t)} - 1| + \frac{q_{22}s_{22}M_2I^*\tau_2}{m_2^2} |e^{y(t)} - 1| \\
 & - \left[\frac{q_{22}M_2}{M_2} \int_{t-\tau_1}^t \left(\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| \right) \right. \\
 & \left. + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right) d\omega \Big] + \frac{s_{22}p_{22}M_2I^*\tau_2}{m_2^2} |e^{y(t)} - 1| \\
 & + \frac{q_{21}s_{22}M_2S^*\tau_1}{m_2^2} |e^{x(t)} - 1| + \frac{s_{22}q_{22}M_2I^*\tau_1}{m_2^2} |e^{y(t)} - 1| \\
 & + \frac{s_{22}^2M_2I^*\tau_2}{m_2^2} |e^{y(t)} - 1| - \left[\frac{s_{22}M_2}{m_2} \int_{t-\tau_2}^t \left(\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| \right) \right. \\
 & \left. + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right) d\omega \Big]. \quad (33)
 \end{aligned}$$

Then

$$\begin{aligned}
 D^+ \tilde{V}_2(t) \leq & \left[\frac{q_{21}S^*}{m_2} + \frac{q_{21}(p_{11} + q_{11})S^*M_1\tau_1}{m_1m_2} + \frac{q_{21}(q_{22} + s_{22})M_2S^*\tau_1}{m_2^2} \right] |e^{x(t)} - 1| \\
 & + \left[\frac{(p_{22} + q_{22} + s_{22})I^*}{m_2} + \frac{q_{21}(p_{12} + q_{12})M_1I^*\tau_1}{m_1m_2} + \frac{q_{22}(p_{22} + q_{22})M_2I^*\tau_1}{m_2^2} \right. \\
 & \left. + \frac{q_{22}s_{22}M_2I^*\tau_2}{m_2^2} + \frac{s_{22}(p_{22} + s_{22})M_2I^*\tau_2}{m_2^2} + \frac{s_{22}q_{22}M_2I^*\tau_1}{m_2^2} \right] |e^{y(t)} - 1| \\
 & + \frac{p_{13}q_{21}R^*M_1\tau_1}{m_1m_2} |e^{z(t)} - 1| \\
 = & \frac{q_{21}S^*}{m_2} \left[1 + \frac{M_1\tau_1}{m_1} (p_{11} + q_{11}) + \frac{M_2\tau_2}{m_2} (q_{22} + s_{22}) \right] |e^{x(t)} - 1| \\
 & + \frac{I^*}{m_2} \left[(p_{22} + q_{22} + s_{22}) + \frac{q_{21}M_1\tau_1(p_{12} + q_{12})}{m_1} + \frac{q_{22}(p_{22} + q_{22} + s_{22})M_2\tau_2}{m_2} \right. \\
 & \left. + \frac{s_{22}(p_{22} + q_{22} + s_{22})M_2\tau_2}{m_2} \right] |e^{y(t)} - 1| + \frac{p_{13}q_{21}R^*M_1\tau_1}{m_1m_2} |e^{z(t)} - 1|. \quad (34)
 \end{aligned}$$

Next, Eq. (25) can be rewritten as the following form:

$$\begin{aligned}
 \frac{dz}{dt} = & \frac{p_{31}S^*}{R} [e^{x(t)} - 1] + \frac{p_{32}I^*}{R} [e^{y(t)} - 1] + \frac{p_{33}R^*}{R} [e^{z(t)} - 1] \\
 & + \frac{s_{32}I^*}{R} [e^{y(t)} - 1] - \frac{s_{32}I^*}{R} \int_{t-\tau_2}^t e^{y(\omega)} \frac{dy}{d\omega} d\omega
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{p_{31}S^*}{R} [e^{x(t)} - 1] + \frac{(p_{32} + s_{32})I^*}{R} [e^{y(t)} - 1] + \frac{p_{33}R^*}{R} [e^{z(t)} - 1] \\
 &\quad - \frac{s_{32}I^*}{R} \left[\int_{t-\tau_2}^t e^{y(\omega)} \left(\frac{p_{22}I^*}{I} (e^{y(\omega)} - 1) + \frac{q_{21}S^*}{I} (e^{x(\omega-\tau_1)} - 1) \right. \right. \\
 &\quad \left. \left. + \frac{q_{22}I^*}{I} (e^{y(\omega-\tau_1)} - 1) + \frac{s_{22}I^*}{I} (e^{y(\omega-\tau_2)} - 1) \right) d\omega \right].
 \end{aligned}$$

Let $V_3(t) = |z(t)|$.

$$\begin{aligned}
 D^+ V_3 \leq & \frac{p_{31}S^*}{m_3} |e^{x(t)} - 1| + \frac{(p_{32} + s_{32})I^*}{m_3} |e^{y(t)} - 1| + \frac{p_{33}R^*}{m_3} |e^{z(t)} - 1| \\
 & + \frac{s_{32}I^*}{m_3} \int_{t-\tau_2}^t e^{y(\omega)} \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\
 & \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega.
 \end{aligned}$$

Then we find that there exists $t_1 > 0$ such that $I^*e^{y(\omega)} < M_2, \forall t > t_1$, and for $t > t + \tau$, $\tau = \max\{\tau_1, \tau_2\}$. We have

$$\begin{aligned}
 D^+ V_3 \leq & \frac{p_{31}S^*}{m_3} |e^{x(t)} - 1| + \frac{(p_{32} + s_{32})I^*}{m_3} |e^{y(t)} - 1| + \frac{p_{33}R^*}{m_3} |e^{z(t)} - 1| \\
 & + \frac{s_{32}M_2}{m_3} \int_{t-\tau_2}^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\
 & \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega. \tag{35}
 \end{aligned}$$

Again due to the form of Eq. (35), we consider the following Lyapunov functional, and we have

$$\begin{aligned}
 \tilde{V}_3(t) = & V_3(t) + \frac{s_{32}M_2}{m_3} \int_{t-\tau_2}^t \int_l^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\
 & \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega dl \\
 & + \frac{s_{32}q_{21}M_2S^*\tau_1}{m_2m_3} \int_{t-\tau_1}^t |e^{x(\omega)} - 1| d\omega + \frac{s_{32}q_{22}M_2I^*\tau_1}{m_2m_3} \int_{t-\tau_1}^t |e^{y(\omega)} - 1| d\omega \\
 & + \frac{s_{22}s_{32}M_2I^*\tau_2}{m_2m_3} \int_{t-\tau_2}^t |e^{y(\omega)} - 1| d\omega.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \tilde{V}_3(t) \leq & D^+ V_3(t) + \frac{s_{32}p_{22}M_2I^*\tau_2}{m_2m_3} |e^{y(t)} - 1| + \frac{s_{32}q_{21}M_2S^*\tau_1}{m_2m_3} |e^{x(t)} - 1| \\
 & + \frac{s_{32}q_{22}M_2I^*\tau_1}{m_2m_3} |e^{y(t)} - 1| + \frac{s_{22}s_{32}M_2I^*\tau_2}{m_2m_3} |e^{y(t)} - 1| \\
 & - \frac{s_{32}M_2}{m_3} \int_{t-\tau_2}^t \left[\frac{p_{22}I^*}{m_2} |e^{y(\omega)} - 1| + \frac{q_{21}S^*}{m_2} |e^{x(\omega-\tau_1)} - 1| \right. \\
 & \left. + \frac{q_{22}I^*}{m_2} |e^{y(\omega-\tau_1)} - 1| + \frac{s_{22}I^*}{m_2} |e^{y(\omega-\tau_2)} - 1| \right] d\omega.
 \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{V}_3(t) \leq & \left[\frac{p_{31}S^*}{m_3} + \frac{s_{32}q_{21}M_2S^*\tau_1}{m_2m_3} \right] |e^{x(t)} - 1| \\ & + \left[\frac{(p_{32} + s_{32})I^*}{m_3} + \frac{s_{32}(p_{22}\tau_2 + q_{22}\tau_1 + s_{22}\tau_2)M_2I^*}{m_2m_3} \right] |e^{y(t)} - 1| \\ & + \frac{p_{33}R^*}{m_3} |e^{z(t)} - 1|. \end{aligned} \tag{36}$$

Let us define the functional $V(t) = \tilde{V}_1(t) + \tilde{V}_2(t) + \tilde{V}_3(t)$. Then

$$\begin{aligned} D^+V(t) &= D^+\tilde{V}_1(t) + D^+\tilde{V}_2(t) + D^+\tilde{V}_3(t) \\ &\leq \left[\frac{(p_{11} + s_{11})S^*}{m_1} + \frac{q_{11}(p_{11} + q_{11})M_1S^*\tau_1}{m_1^2} + \frac{q_{12}q_{21}M_2S^*\tau_1}{m_1m_2} + \frac{q_{21}S^*}{m_2} \right. \\ &\quad \left. + \frac{q_{21}(p_{11} + q_{11})S^*M_1\tau_1}{m_1m_2} + \frac{q_{21}(q_{22} + s_{22})M_2\tau_1}{m_2^2} + \frac{p_{31}S^*}{m_3} + \frac{s_{32}q_{21}M_2S^*\tau_1}{m_2m_3} \right] \\ &\quad \times |e^{x(t)} - 1| \\ &\quad + \left[\frac{(p_{12} + q_{12})I^*}{m_1} + \frac{q_{11}(p_{12} + q_{12})M_1I^*\tau_1}{m_1^2} + \frac{q_{12}(p_{22}\tau_1 + q_{22}\tau_1 + s_{22}\tau_2)M_2I^*}{m_1m_2} \right. \\ &\quad + \frac{(p_{22} + q_{22} + s_{22})I^*}{m_2} + \frac{q_{21}(p_{12} + q_{12})M_1I^*\tau_1}{m_1m_2} + \frac{q_{22}(p_{22} + q_{22})M_2I^*\tau_1}{m_2^2} \\ &\quad + \frac{q_{22}s_{22}M_2I^*\tau_2}{m_2^2} + \frac{s_{22}(p_{22} + s_{22})M_2I^*\tau_2}{m_2^2} + \frac{s_{22}q_{22}M_2I^*\tau_1}{m_2^2} + \frac{(p_{32} + s_{32})I^*}{m_3} \\ &\quad \left. + \frac{s_{32}(p_{22}\tau_2 + q_{22}\tau_1 + s_{22}\tau_2)M_2I^*}{m_2m_3} \right] |e^{y(t)} - 1| \\ &\quad + \left[\frac{p_{13}R^*}{m_1} + \frac{q_{11}p_{13}R^*M_1\tau_1}{m_1^2} + \frac{p_{13}q_{21}R^*M_1\tau_1}{m_1m_2} + \frac{p_{33}R^*}{m_3} \right] |e^{z(t)} - 1| \\ &= \left[\frac{(p_{11} + q_{11})S^*}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} + \frac{q_{21}M_1\tau_1}{m_2} \right) \right. \\ &\quad + \frac{q_{21}S^*}{m_2} \left(1 + \frac{M_2\tau_1}{m_2}(q_{22} + s_{22}) + \frac{s_{32}M_2\tau_1}{m_3} \right) \\ &\quad \left. + \frac{q_{12}q_{21}M_2S^*\tau_1}{m_1m_2} + \frac{p_{31}S^*}{m_3} \right] |e^{x(t)} - 1| \\ &\quad + \left[\frac{(p_{12} + q_{12})I^*}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} + \frac{q_{21}M_1\tau_1}{m_2} \right) \right. \\ &\quad + (p_{22} + q_{22} + s_{22}) \left(\frac{q_{22}M_2I^*\tau_1}{m_2^2} + \frac{s_{22}M_2I^*\tau_2}{m_2^2} + \frac{I^*}{m_2} \right) \\ &\quad + \frac{q_{12}M_2I^*}{m_1m_2}(p_{22}\tau_1 + q_{22}\tau_1 + s_{22}\tau_2) \\ &\quad \left. + \frac{s_{32}M_2I^*}{m_2m_3}(p_{22}\tau_2 + q_{22}\tau_1 + s_{22}\tau_2) + \frac{(p_{32} + s_{32})I^*}{m_3} \right] |e^{y(t)} - 1| \\ &\quad + \left[\frac{p_{13}R^*}{m_1} + \frac{p_{33}R^*}{m_3} + \frac{p_{13}R^*M_1\tau_1}{m_1} \left(\frac{q_{11}}{m_1} + \frac{q_{21}}{m_2} \right) \right] |e^{z(t)} - 1|. \end{aligned}$$

Thus,

$$\begin{aligned}
 D^+ V(t) \leq & S^* \left[\frac{(p_{11} + q_{11})}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} + \frac{q_{21}M_1\tau_1}{m_2} \right) \right. \\
 & + \frac{q_{21}}{m_2} \left(1 + \frac{M_2\tau_1}{m_2}(q_{22} + s_{22}) + \frac{s_{32}M_2\tau_1}{m_3} \right) + \frac{q_{12}q_{21}M_2\tau_1}{m_1m_3} + \frac{p_{31}}{m_3} \left. \right] |e^{x(t)} - 1| \\
 & + I^* \left[\frac{(p_{12} + q_{12})}{m_1} \left(1 + \frac{q_{11}M_1\tau_1}{m_1} + \frac{q_{21}M_1\tau_1}{m_2} \right) \right. \\
 & + \frac{(p_{22} + q_{22} + s_{22})}{m_2} \left(\frac{q_{22}M_2\tau_1}{m_2} + \frac{s_{22}M_2\tau_2}{m_2} + 1 \right) \\
 & + \frac{q_{12}M_2}{m_1m_2}(p_{22}\tau_1 + q_{22}\tau_1 + s_{22}\tau_2) + \frac{s_{32}M_2}{m_2m_3}(p_{22}\tau_2 + q_{22}\tau_1 + s_{22}\tau_2) + \frac{p_{32} + s_{32}}{m_3} \left. \right] \\
 & \times |e^{y(t)} - 1| \\
 & + R^* \left[\frac{p_{13}}{m_1} + \frac{p_{33}}{m_3} + \frac{p_{13}M_1\tau_1}{m_1} \left(\frac{q_{11}}{m_1} + \frac{q_{21}}{m_2} \right) \right] |e^{z(t)} - 1|.
 \end{aligned}$$

Then

$$D^+ V(t) \leq l_1 S^* |e^{x(t)} - 1| + l_2 I^* |e^{y(t)} - 1| + l_3 R^* |e^{z(t)} - 1|.$$

Since model (4) is positive invariant, for all $t > t_1^*$, we have

$$\bar{S} > S(t) = S^* e^{x(t)} > \underline{S},$$

$$\bar{I} > I(t) = I^* e^{y(t)} > \underline{I},$$

$$\bar{R} > R(t) = R^* e^{z(t)} > \underline{R}.$$

According to the mean value theorem, we have

$$S^* |e^{x(t)} - 1| = S^* e^{\mu_1(t)} |x(t)| > m_1 |x(t)|,$$

$$I^* |e^{y(t)} - 1| = I^* e^{\mu_2(t)} |y(t)| > m_2 |y(t)|,$$

$$R^* |e^{z(t)} - 1| = R^* e^{\mu_3(t)} |z(t)| > m_3 |z(t)|.$$

We know that $S^* e^{\mu_1(t)}$ lies between S^* and $S(t)$, $I^* e^{\mu_2(t)}$ lies between I^* and $I(t)$, and $R^* e^{\mu_3(t)}$ lies between R^* and $R(t)$. Therefore

$$D^+ V(t) \leq -l [|x(t)|, |y(t)|, |z(t)|],$$

where

$$-l = \max\{l_1 \bar{S}, l_2 \bar{I}, l_3 \bar{R}\} < 0.$$

Hence, by using Lyapunov stability theory, the disease-present equilibrium E^* of system (2) is globally asymptotically stable. Hence, the proof is completed. \square

4 Direction and stability of Hopf bifurcation

In this section, we shall derive the explicit formulae that determine the direction and stability of the Hopf bifurcation at the critical value of τ_{10}^* when $\tau_1 > 0$ and $\tau_2 \in [0, \tau_{20})$. We assume that $\tau_{20}^* < \tau_{10}^*$, where $\tau_{20}^* \in [0, \tau_{20})$. For convenience, let $\tau_1 = \tau_{10}^* + \varrho$ ($\varrho \in \mathbb{R}$), $u_1 = S(\tau_1 t)$, $u_2 = I(\tau_1 t)$, $u_3 = R(\tau_1 t)$. System (2) becomes

$$\dot{u}(t) = L_\varrho(u_t) + F(\varrho, u_t), \tag{37}$$

where $u(t) = (u_1, u_2, u_3)^T \in C = C([-1, 0], \mathbb{R}^3)$ and $L_\varrho: C \rightarrow \mathbb{R}^3$ and $F: \mathbb{R} \times C \rightarrow \mathbb{R}^3$ are defined as

$$L_\varrho \phi = (\tau_{10}^* + \varrho) \left(L_{1\max} \phi(0) + L_{2\max} \phi \left(-\frac{\tau_{20}^*}{\tau_{10}^*} \right) + L_{3\max} \phi(-1) \right), \tag{38}$$

and

$$F(\varrho, \phi) = (\tau_{10}^* + \varrho) [F_1, F_2, F_3]^T,$$

$$L_{1\max} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \quad L_{2\max} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & s_{32} & 0 \end{pmatrix}, \tag{39}$$

$$L_{3\max} = \begin{pmatrix} q_{11} & q_{12} & 0 \\ q_{21} & q_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$F_1 = p_{14}\phi_1^2(0) + p_{15}\phi_1(0)\phi_2(0) + p_{16}\phi_1(0)\phi_3(0) + p_{17}\phi_2(0)\phi_3(0) \\ + p_{18}\phi_1(-1)\phi_2(-1) + p_{19}\phi_2^2(0) + p_{110}\phi_3^2(0),$$

$$F_2 = p_{23}\phi_1(-1)\phi_2(-1),$$

$$F_3 = p_{34}\phi_1^2(0) + p_{35}\phi_1(0)\phi_2(0) + p_{36}\phi_1(0)\phi_3(0) + p_{37}\phi_2(0)\phi_3(0) \\ + p_{38}\phi_2^2(0) + p_{39}\phi_3^2(0),$$

with

$$p_{14} = -\frac{b(1-q)}{K}, \quad p_{15} = -\frac{(1-q)(1+r)b}{K}, \quad p_{16} = -\frac{b(1-q)}{K},$$

$$p_{17} = -\frac{(1-q)(1+r)b}{K}, \quad p_{18} = -\beta, \quad p_{19} = \frac{(1-q)br}{K},$$

$$p_{110} = \frac{b(1-q)}{K}, \quad p_{23} = \beta, \quad p_{34} = -\frac{qb}{K}, \quad p_{35} = -\frac{qb(1+r)}{K},$$

$$p_{36} = -\frac{qb}{K}, \quad p_{37} = -\frac{qb(1+r)}{K}, \quad p_{38} = -\frac{qbr}{K}, \quad p_{39} = -\frac{qb}{K}.$$

Accordingly, there exists $\eta(\theta, \varrho)$ such that

$$L_\varrho \phi = \int_{-1}^0 d\eta(\theta, \varrho) \phi(\theta). \tag{40}$$

Choosing

$$\eta(\theta, \varrho) = \begin{cases} (\tau_{10}^* + \varrho)(L_{1\max} + L_{2\max} + L_{3\max}), & \theta = 0, \\ (\tau_{10}^* + \varrho)(L_{2\max} + L_{3\max}), & \theta \in [-\frac{\tau_{20}^*}{\tau_{10}^*}, 0), \\ (\tau_{10}^* + \varrho)L_{3\max}, & \theta \in (-1, -\frac{\tau_{20}^*}{\tau_{10}^*}), \\ 0, & \theta = -1. \end{cases} \tag{41}$$

For $\phi \in C([-1, 0], \mathbb{R}^3)$, define

$$A(\varrho)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 d\eta(\theta, \varrho)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\varrho)\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\varrho, \phi), & \theta = 0. \end{cases}$$

System (37) becomes

$$\dot{u}(t) = A(\varrho)u_t + R(\varrho)u_t. \tag{42}$$

For $\varphi \in C^1([0, 1], (\mathbb{R}^3)^*)$, define

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^0 d\eta^T(s, 0)\varphi(-s), & s = 0, \end{cases}$$

with the bilinear inner form for $A(0)$ and A^*

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^\theta \bar{\varphi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{43}$$

where $\eta(\theta) = \eta(\theta, 0)$.

Suppose that $V(\theta) = (1, V_2, V_3)e^{i\tau_{10}^*\omega_{10}^*\theta}$ ($\theta \in [-1, 0]$) is the eigenvector of $A(0)$ associated with the eigenvalue $+i\omega_{10}\tau_{10}^*$ and $V^*(s) = D(1, V_2^*, V_3^*)e^{i\tau_{10}^*\omega_{10}^*s}$ ($s \in [0, 1]$) is the eigenvector of $A(0)$ associated with the eigenvalue $-i\omega_{10}\tau_{10}^*$. By the definitions of $A(0)$ and A^* , we can obtain

$$\begin{aligned} V_2 &= \frac{q_{21}e^{-i\tau_{10}^*\omega_{10}^*}}{i\omega_{10}^* - p_{22} - s_{22}e^{-i\tau_{20}^*\omega_{10}^*} - q_{22}e^{-i\tau_{10}^*\omega_{10}^*}}, \\ V_3 &= \frac{i\omega_{10}^* - p_{11} - q_{11}e^{-i\tau_{10}^*\omega_{10}^*} - (p_{12} + q_{12}e^{-i\tau_{10}^*\omega_{10}^*})V_2}{p_{13}}, \\ V_2^* &= -\frac{i\omega_{10}^* + p_{11} + q_{11}e^{i\tau_{10}^*\omega_{10}^*}}{q_{21}e^{i\tau_{10}^*\omega_{10}^*}}, \\ V_3^* &= -\frac{p_{13}}{i\omega_{10}^* + p_{33}}. \end{aligned}$$

From Eq. (43), we obtain

$$\begin{aligned} \bar{D} = & \left[1 + V_2 \bar{V}_2^* + V_3 \bar{V}_3^* + \tau_{20}^* e^{-i\tau_{10}^* \omega_{10}^*} (s_{22} \bar{V}_2^* + s_{32} \bar{V}_3^*) V_2 \right. \\ & \left. + \tau_{10}^* e^{-i\tau_{10}^* \omega_{10}^*} (q_{11} + q_{21} \bar{V}_2^*) + (q_{12} + q_{22} \bar{V}_2^*) V_2 \right]^{-1}, \end{aligned}$$

then $\langle V^*, V \rangle = 1$ and $\langle V^*, \bar{V} \rangle = 0$.

In what follows, we can obtain g_{20} , g_{11} , g_{02} , and g_{21} by using the algorithms in [41] and a similar process as that in [42–44]:

$$\begin{aligned} g_{20} = & 2\tau_{10}^* \bar{D} \left[p_{14} + p_{15} V_2 + p_{15} V_3 + p_{16} V_2 V_3 + p_{17} V_2 V_3 + p_{18} V_2 e^{-2i\tau_{10}^* \omega_{10}^*} \right. \\ & + p_{19} V_2^2 + P_{110} V_3^2 + p_{23} \bar{V}_2^* V_2 e^{-2i\tau_{10}^* \omega_{10}^*} + \bar{V}_3^* (p_{34} + p_{35} V_2 + p_{36} V_3 \\ & \left. + p_{37} V_2 V_3 + p_{38} V_2^2 + p_{39} V_3^2) \right], \end{aligned}$$

$$\begin{aligned} g_{11} = & \tau_{10}^* \bar{D} \left[2p_{14} + p_{15} (V_2 + \bar{V}_2) + p_{16} (V_3 + \bar{V}_3) + p_{17} (V_2 \bar{V}_3 + \bar{V}_2 V_3) \right. \\ & + p_{18} (V_2 + \bar{V}_2) + 2p_{19} V_2 \bar{V}_2 + 2p_{110} V_3 \bar{V}_3 + p_{23} \bar{V}_2^* (V_2 + \bar{V}_2) \\ & + \bar{V}_3^* (2p_{34} + p_{35} (V_2 + \bar{V}_2) + p_{36} (V_3 + \bar{V}_3) + p_{37} (V_2 \bar{V}_3 + \bar{V}_2 V_3) \\ & \left. + 2p_{38} V_2 \bar{V}_2 + 2p_{39} V_3 \bar{V}_3) \right], \end{aligned}$$

$$\begin{aligned} g_{02} = & 2\tau_{10}^* \bar{D} \left[p_{14} + p_{15} \bar{V}_2 + p_{15} \bar{V}_3 + p_{16} \bar{V}_2 \bar{V}_3 + p_{17} \bar{V}_2 \bar{V}_3 + p_{18} \bar{V}_2 e^{2i\tau_{10}^* \omega_{10}^*} \right. \\ & + p_{19} \bar{V}_2^2 + P_{110} \bar{V}_3^2 + p_{23} \bar{V}_2^* \bar{V}_2 e^{2i\tau_{10}^* \omega_{10}^*} + \bar{V}_3^* (p_{34} + p_{35} \bar{V}_2 + p_{36} \bar{V}_3 \\ & \left. + p_{37} \bar{V}_2 \bar{V}_3 + p_{38} \bar{V}_2^2 + p_{39} \bar{V}_3^2) \right], \end{aligned}$$

$$\begin{aligned} g_{21} = & 2\tau_{10}^* \bar{D} \left\{ p_{14} (2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + p_{15} \left(W_{11}^{(1)}(0) V_2 + \frac{1}{2} W_{20}^{(1)}(0) \bar{V}_2 \right. \right. \\ & + W_{11}^{(2)}(0) + \frac{1}{2} W_{20}^{(2)}(0) \left. \right) + p_{16} \left(W_{11}^{(1)}(0) V_3 + \frac{1}{2} W_{20}^{(1)}(0) \bar{V}_3 \right. \\ & + W_{11}^{(3)}(0) + \frac{1}{2} W_{20}^{(3)}(0) \left. \right) + p_{17} \left(W_{11}^{(2)}(0) V_3 + \frac{1}{2} W_{20}^{(2)}(0) \bar{V}_3 \right. \\ & + W_{11}^{(3)}(0) V_2 + \frac{1}{2} W_{20}^{(3)}(0) \bar{V}_2 \left. \right) + p_{18} \left(W_{11}^{(1)}(-1) V_2 e^{-i\tau_{10}^* \omega_{10}^*} \right. \\ & + \frac{1}{2} W_{20}^{(1)}(-1) \bar{V}_2 e^{i\tau_{10}^* \omega_{10}^*} + W_{11}^{(2)}(-1) e^{-i\tau_{10}^* \omega_{10}^*} + \frac{1}{2} W_{20}^{(2)}(-1) e^{i\tau_{10}^* \omega_{10}^*} \left. \right) \\ & + p_{19} (2W_{11}^{(2)}(0) + W_{20}^{(2)}(0)) + p_{110} (2W_{11}^{(3)}(0) + W_{20}^{(3)}(0)) \\ & + p_{23} \bar{V}_2^* \left(W_{11}^{(1)}(-1) V_2 e^{-i\tau_{10}^* \omega_{10}^*} + \frac{1}{2} W_{20}^{(1)}(-1) \bar{V}_2 e^{i\tau_{10}^* \omega_{10}^*} \right. \\ & + W_{11}^{(2)}(-1) e^{-i\tau_{10}^* \omega_{10}^*} + \frac{1}{2} W_{20}^{(2)}(-1) e^{i\tau_{10}^* \omega_{10}^*} \left. \right) + \bar{V}_3^* \left(p_{34} (2W_{11}^{(1)}(0) \right. \\ & + W_{20}^{(1)}(0)) + p_{35} \left(W_{11}^{(1)}(0) V_2 + \frac{1}{2} W_{20}^{(1)}(0) \bar{V}_2 + W_{11}^{(2)}(0) + \frac{1}{2} W_{20}^{(2)}(0) \right) \\ & + p_{36} \left(W_{11}^{(1)}(0) V_3 + \frac{1}{2} W_{20}^{(1)}(0) \bar{V}_3 + W_{11}^{(3)}(0) + \frac{1}{2} W_{20}^{(3)}(0) \right) \\ & + p_{37} \left(W_{11}^{(2)}(0) V_3 + \frac{1}{2} W_{20}^{(2)}(0) \bar{V}_3 + W_{11}^{(3)}(0) V_2 + \frac{1}{2} W_{20}^{(3)}(0) \bar{V}_2 \right) \\ & \left. + p_{38} (2W_{11}^{(2)}(0) + W_{20}^{(2)}(0)) + p_{39} (2W_{11}^{(3)}(0) + W_{20}^{(3)}(0)) \right\} \end{aligned}$$

with

$$W_{20}(\theta) = \frac{ig_{20}V(0)}{\tau_{10}^*\omega_{10}^*} e^{i\tau_{10}^*\omega_{10}^*\theta} + \frac{i\bar{g}_{02}\bar{V}(0)}{3\tau_{10}^*\omega_{10}^*} e^{-i\tau_{10}^*\omega_{10}^*\theta} + E_1 e^{2i\tau_{10}^*\omega_{10}^*\theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}V(0)}{\tau_{10}^*\omega_{10}^*} e^{i\tau_{10}^*\omega_{10}^*\theta} + \frac{i\bar{g}_{11}\bar{V}(0)}{\tau_{10}^*\omega_{10}^*} e^{-i\tau_{10}^*\omega_{10}^*\theta} + E_2$$

with

$$E_1 = 2 \begin{pmatrix} p'_{11} & -p_{12} - q_{12}e^{-2i\omega_{10}^*\tau_{10}^*} & -p_{13} \\ -q_{21}e^{-2i\omega_{10}^*\tau_{10}^*} & p'_{22} & 0 \\ -p_{31} & -p_{32} - s_{32}e^{-2i\omega_{10}^*\tau_{20}^*} & p'_{33} \end{pmatrix}^{-1} \times \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix},$$

$$E_2 = - \begin{pmatrix} p_{11} + q_{11} & p_{12} + q_{12} & p_{13} \\ q_{21} & p_{22} + s_{22} + q_{22} & 0 \\ p_{31} & p_{32} + s_{32} & 0 \end{pmatrix}^{-1} \times \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix},$$

where

$$p'_{11} = 2i\omega_{10}^* - p_{11} - q_{11}e^{-2i\omega_{10}^*\tau_{10}^*},$$

$$p'_{22} = 2i\omega_{10}^* - p_{22} - s_{22}e^{-2i\omega_{10}^*\tau_{20}^*} - q_{22}e^{-2i\omega_{10}^*\tau_{10}^*},$$

$$p'_{33} = 2i\omega_{10}^* - p_{33},$$

and

$$E_1^{(1)} = p_{14} + p_{15}V_2 + p_{15}V_3 + p_{16}V_2V_3 + p_{17}V_2V_3 + p_{18}V_2e^{-2i\tau_{10}^*\omega_{10}^*} + p_{19}V_2^2 + p_{110}V_3^2,$$

$$E_1^{(2)} = p_{23}V_2e^{-2i\tau_{10}^*\omega_{10}^*},$$

$$E_1^{(3)} = p_{34} + p_{35}V_2 + p_{36}V_3 + p_{37}V_2V_3 + p_{38}V_2^2 + p_{39}V_3^2,$$

$$E_2^{(1)} = 2p_{14} + p_{15}(V_2 + \bar{V}_2) + p_{16}(V_3 + \bar{V}_3) + p_{17}(V_2\bar{V}_3 + \bar{V}_2V_3) + p_{18}(V_2 + \bar{V}_2) + 2p_{19}V_2\bar{V}_2 + 2p_{110}V_3\bar{V}_3,$$

$$E_2^{(2)} = p_{23}(V_2 + \bar{V}_2),$$

$$E_2^{(3)} = 2p_{34} + p_{35}(V_2 + \bar{V}_2) + p_{36}(V_3 + \bar{V}_3) + p_{37}(V_2\bar{V}_3 + \bar{V}_2V_3) + 2p_{38}V_2\bar{V}_2 + 2p_{39}V_3\bar{V}_3.$$

Thus,

$$C_1(0) = \frac{i}{2\tau_{10}^*\omega_{10}^*} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_{10}^*)\}},$$

$$\beta_2 = 2\operatorname{Re}\{C_1(0)\},$$

$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_{10}^*)\}}{\tau_{10}^*\omega_{10}^*},$$
(44)

in which μ_2 determines the direction of the Hopf bifurcation; β_2 determines the stability of bifurcating periodic solutions, and T_2 determines the period of bifurcating periodic solutions.

5 Numerical simulation

Choosing $q = 0.15, b = 34.45, K = 300, r = 0.2, \beta = 0.087, \mu = 2, \nu = 2, \gamma = 1.5$. Then system (2) takes the following form:

$$\begin{cases} \frac{dS(t)}{dt} = 29.2825(1 - \frac{S(t)+I(t)+R(t)}{300})(S(t) + 0.2I(t) + R(t)) \\ \quad - 0.087S(t - \tau_1)I(t - \tau_1) - 2S(t), \\ \frac{dI(t)}{dt} = 0.087S(t - \tau_1)I(t - \tau_1) - 2I(t) - 1.5I(t - \tau_2), \\ \frac{dR(t)}{dt} = 5.1675(1 - \frac{S(t)+I(t)+R(t)}{300})(S(t) + 0.2I(t) + R(t)) \\ \quad + 1.5I(t - \tau_2) + 2S(t) - 2R(t), \end{cases} \tag{45}$$

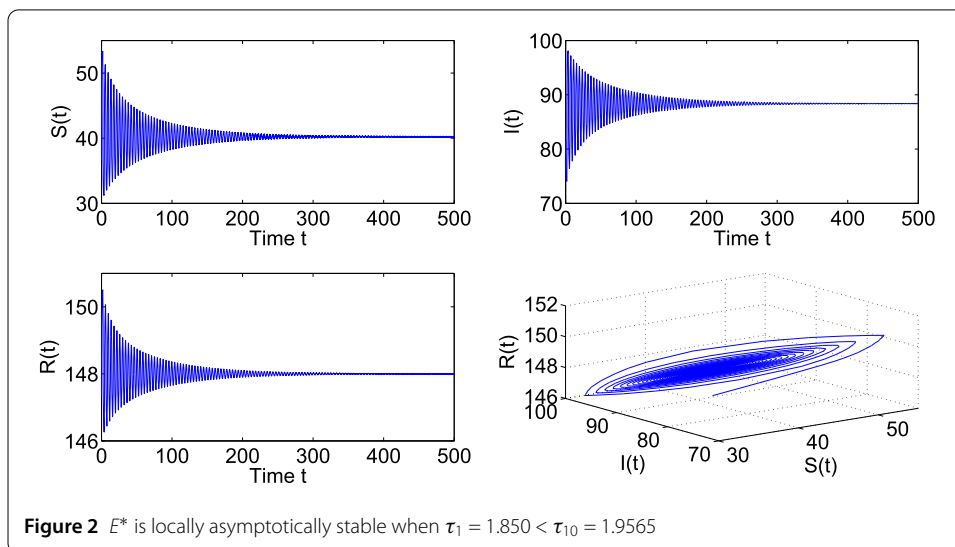
from which we obtain $R_0 = 2.9853 > 1$ and Eq. (45) becomes

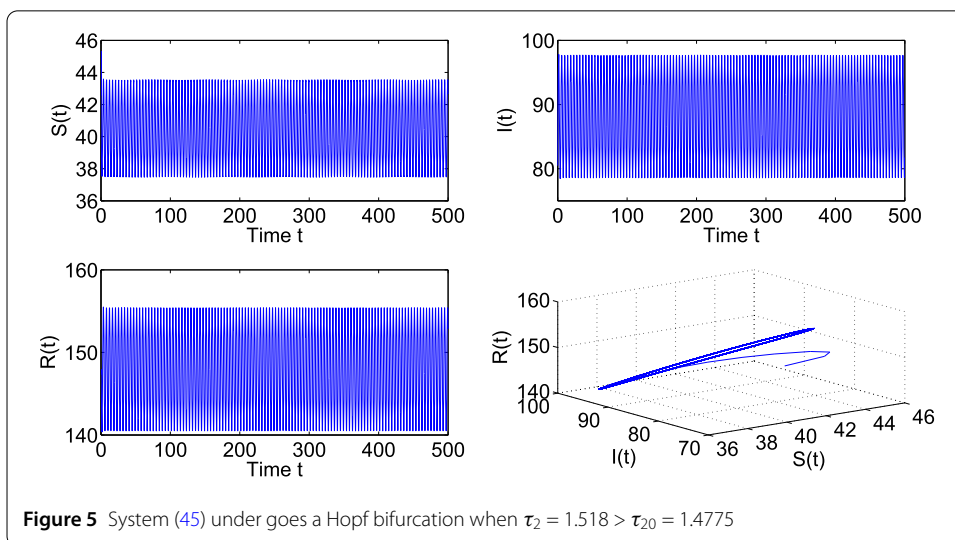
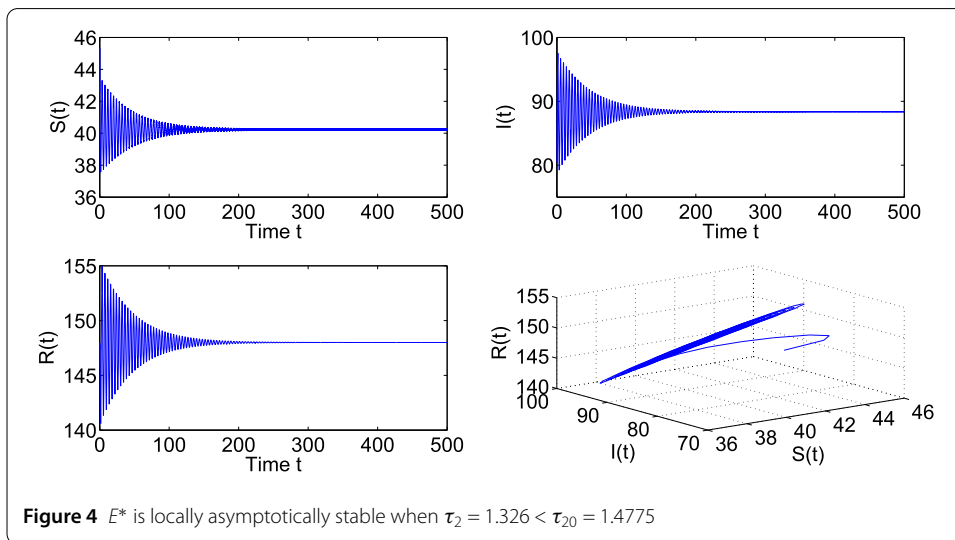
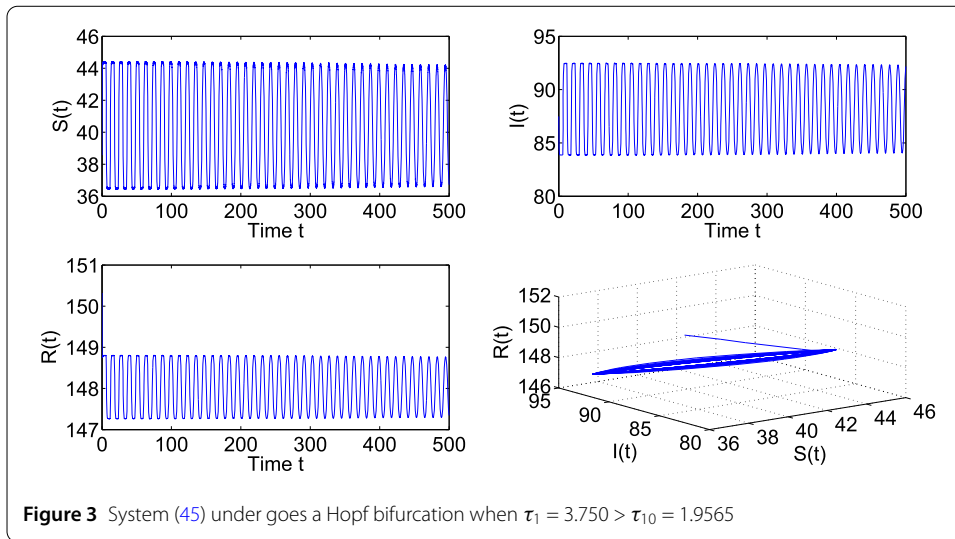
$$-0.2530(I^*)^2 + 2.7082I^* + 1736.3 = 0. \tag{46}$$

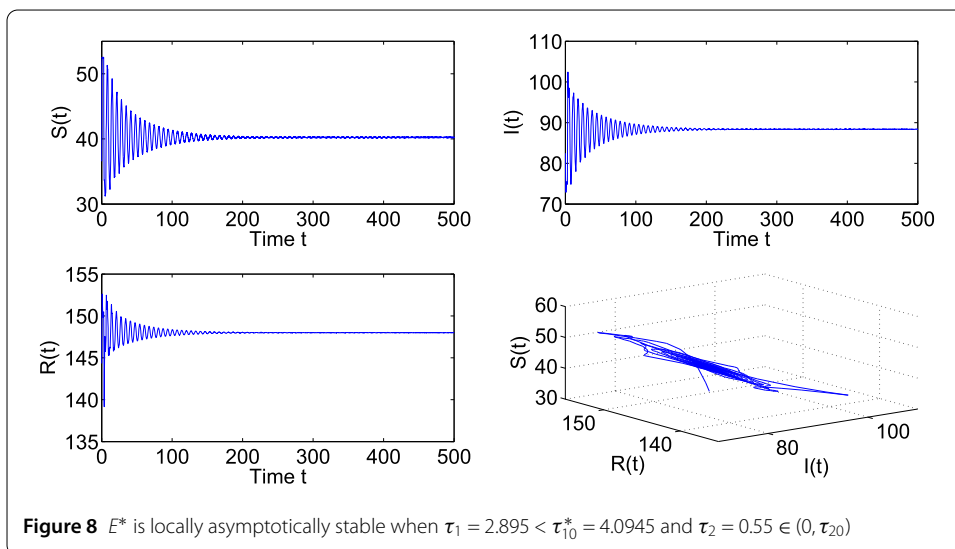
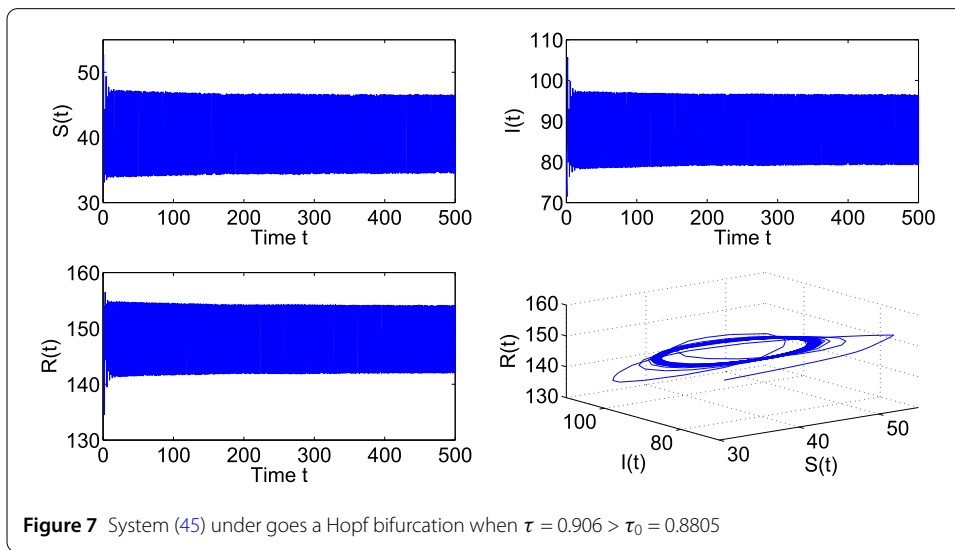
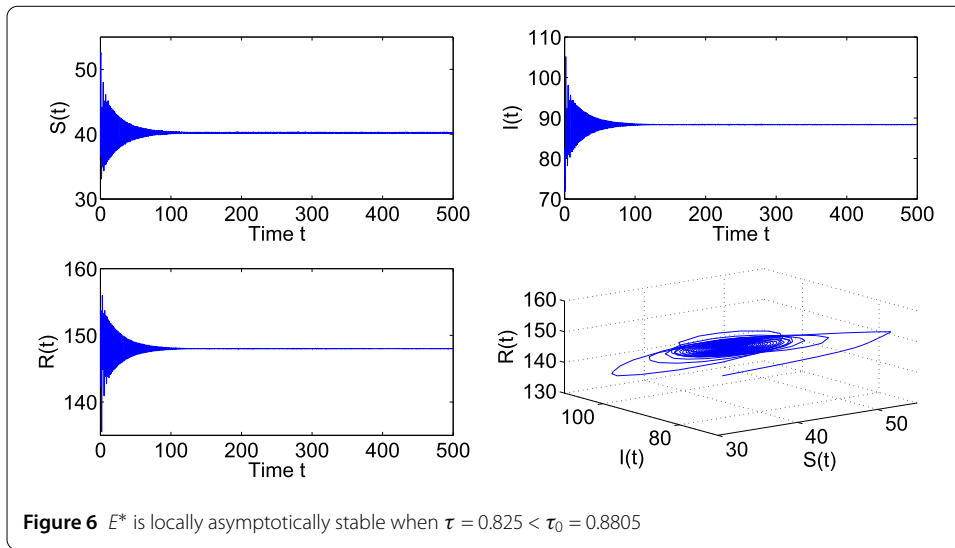
Then, the unique disease-present equilibrium $E^*(40.2299, 88.3672, 147.9940)$ can be obtained with the aid of Matlab software package, and assumption (H_1) is satisfied for system (45). Accordingly, system (45) is locally asymptotically stable for $\tau_1 = \tau_2 = 0$.

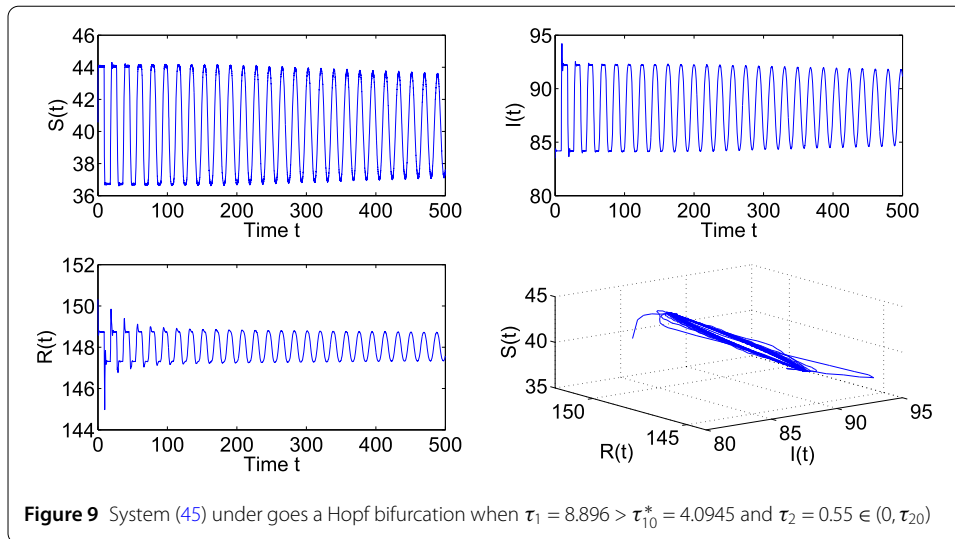
For Theorem 1, we can obtain $\omega_{10} = 0.3065$ and the critical value $\tau_{10} = 1.9565$ for the occurrence of the Hopf bifurcation. Therefore, conditions (H_{21}) and (H_{22}) of Theorem 1 hold. Figure 2 implies that the disease-present equilibrium $E^*(40.2299, 88.3672, 147.9940)$ is locally asymptotically stable when $\tau_1 \in [0, \tau_{10})$. Figure 3 reveals that system (45) loses its stability and a Hopf bifurcation appears once the value of τ_1 exceeds τ_{10} .

Similarly, we have $\omega_{20} = 1.0602$ and $\tau_{20} = 1.4775$ for Theorem 2, $\omega_0 = 2.0951$ and $\tau_0 = 0.8805$ for Theorem 3, $\omega_{10}^* = 1.0207$ and $\tau_{10}^* = 4.0945$ when $\tau_2 = 0.55 \in (0, \tau_{20}]$ for Theorem 4, respectively. The corresponding numerical simulations can be revealed by Figs. 4–5, Figs. 6–7, and Figs. 8–9, respectively.









Specially, we obtain $C_1(0) = -37.0955 + 9.0748i$, $\lambda(\tau_{10}^*) = 0.6902 - 1.3033i$ by some complex computation by means of Matlab package. Based on Eq. (44), we have $\mu_2 = 54.9913$, $\beta_2 = -74.19100$, and $T_2 = 14.9776$. Since $\mu_2 > 0$, $\beta_2 < 0$, and $T_2 > 0$, then the Hopf bifurcation is supercritical; the bifurcating periodic solutions are stable and increase.

6 Conclusions

A multitude of researchers have paid attention to porcine reproductive and respiratory syndrome due to its vast economic impact on pig production in many countries all over the world. Various strategies at farm level have been developed to combat porcine reproductive and respiratory syndrome. However, it is usually difficult to determine easily which strategy is most suitable in a given farm situation. Mathematical modeling has been extensively used to study and predict the propagation of infectious diseases in populations, and in view of this point, we propose a delayed SIR model for the transmission of PRRSV among a swine population by incorporating two delays into the model formulated in [26]. We derive sufficient conditions to ensure local stability of the disease-present equilibrium and the existence of Hopf bifurcation of the proposed model. Direction and stability of the model are investigated by applying the normal form theory and the center manifold theorem. The obtained findings are justified by computer numerical simulations.

Our findings demonstrate that the model is in an ideal stable state when the value of the two delays is below the corresponding critical value at which the Hopf bifurcation occurs. In this case, we can predict and control the transmission of porcine reproductive and respiratory syndrome virus in a swine population. However, the proposed model will lose its stability and a Hopf bifurcation occurs once the value of the two delays exceeds the corresponding critical value. In this case, the transmission of porcine reproductive and respiratory syndrome virus will be out of control. Thus, it can be concluded that the two delays have an important role on the stability of the model, and our findings in this paper have important theoretical significance and practical value for predicting and controlling the transmission of porcine reproductive and respiratory syndrome virus in a swine population.

The obtained theoretical results in the present paper have shown that the occurrence of the Hopf bifurcation of system (2) is harmful and should be controlled. Therefore, it is an

interesting issue to investigate Hopf bifurcation control by means of feedback methods or nonlinear time delay feedback methods [45, 46]. Especially, we are interested in the Hopf bifurcation control issue of fractional-order delayed system (2), which will be our next research work.

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Availability of data and materials

All of the authors declare that all the data can be accessed in our manuscript in the numerical simulation section.

Competing interests

The authors declare that there is no conflict of interests.

Authors' contributions

Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

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