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On a coupled system of fractional sum-difference equations with p-Laplacian operator

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Abstract

In this paper, we propose a nonlocal fractional sum-difference boundary value problem for a coupled system of fractional sum-difference equations with p-Laplacian operator. The problem contains both Riemann–Liouville and Caputo fractional difference with five fractional differences and four fractional sums. The existence and uniqueness result of the problem is studied by using the Banach fixed point theorem.

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Keywords: p-Laplacian operator; Existence and uniqueness; Coupled system of fractional difference equations; Boundary value problem

1 Introduction

Discrete fractional calculus and fractional difference equations have been widely studied. Goodrich and Peterson gave some useful basic definitions and properties of fractional difference calculus in the book [1]. Discrete fractional calculus can be applied in queuing problems, economics, logistic map, and electrical networks, see [2–4]. The extension of discrete fractional calculus has helped to build up some of the basic theory in this area, see [5–32] and the references cited therein.

The boundary value problem for fractional differential equations and the system of equations with p-Laplacian operator were presented in [33–39] and [40–44], respectively. Particularly, the boundary value problem for fractional difference equations with p-Laplacian operator was presented in [45–47]. In addition, the existence results of systems of fractional boundary value problems were presented in [48–55].

We observe that the boundary value problem of a coupled system of nonlinear fractional difference equations with p-Laplacian operator has not been studied. This result is the motivation for this research. In this paper, we aim to study the coupled system of nonlinear fractional sum-difference equations with p-Laplacian operator

$$\begin{aligned} \Delta_C^{\alpha_1}\phi_p\big[\Delta_C^{\beta_1}u_1(t)\big] = F_1\big[&t+\alpha_1+\beta_1-1, t+\alpha_2+\beta_2-1, \Delta^{\gamma_1}u_1(t+\alpha_1+\beta_1-\gamma_1),\\ &\Psi^{\omega_2}u_2(t+\alpha_2+\beta_2+\omega_2-1), u_2(t+\alpha_2+\beta_2+\omega_2-1)\big], \end{aligned} \tag{1.1}$$

$$\begin{aligned}\Delta_C^{\alpha_2}\phi_p\big[\Delta_C^{\beta_2}u_2(t)\big] = F_2\big[&t+\alpha_2+\beta_2-1, t+\alpha_1+\beta_1-1, \Delta^{\gamma_2}u_2(t+\alpha_2+\beta_2-\gamma_2),\\ &\Psi^{\omega_1}u_1(t+\alpha_1+\beta_1+\omega_1-1), u_1(t+\alpha_1+\beta_1+\omega_1-1)\big]\end{aligned}$$

with the nonlocal fractional sum and fractional difference boundary conditions

$$\begin{aligned}&\Delta_C^{\beta_1}u_1(\alpha_1-1)=0, \qquad u_1(T+\alpha_1+\beta_1)=\lambda_2\Delta^{-\theta_2}g_2(\eta_2+\theta_2)u_2(\eta_2+\theta_2),\\ &\Delta_C^{\beta_2}u_2(\alpha_2-1)=0, \qquad u_2(T+\alpha_2+\beta_2)=\lambda_1\Delta^{-\theta_1}g_1(\eta_1+\theta_1)u_1(\eta_1+\theta_1),\end{aligned} \tag{1.2}$$

where $t \in \mathbb{N}_{0,T} := \{0,1,\dots,T\}$, $\alpha_i,\beta_i,\gamma_i,\omega_i,\theta_i \in (0,1)$, $\alpha_i+\beta_i \in (1,2]$, $\lambda_i > 0$, $\eta_i \in \mathbb{N}_{\alpha_i+\beta_i-1,T+\alpha_i+\beta_i-1}$, $F_i \in C(\mathbb{N}_{\alpha_1+\beta_1-2,T+\alpha_1+\beta_1}\times\mathbb{N}_{\alpha_2+\beta_2-2,T+\alpha_2+\beta_2}\times\mathbb{R}^3,\mathbb{R})$, $g_i \in C(\mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}, \mathbb{R}^+)$, and for $\varphi_i : \mathbb{N}_{\alpha_1+\beta_1-2,T+\alpha_1+\beta_1}\times\mathbb{N}_{\alpha_2+\beta_2-2,T+\alpha_2+\beta_2}\to[0,\infty)$,

$$\begin{aligned}\Psi^{\omega_i}u_i(t+\omega_i) &:= \big[\Delta^{-\omega_i}\varphi_i u_i\big](t+\omega_i)\\ &= \frac{1}{\Gamma(\omega_i)}\sum_{s=\alpha_i+\beta_i-\omega_i-2}^{t-\omega_i}\big(t-\sigma(s)\big)^{\underline{\omega_i-1}}\varphi_i(t,s+\omega_i)u_i(s+\omega_i)\end{aligned}$$

for $i \in \{1,2\}$. For $p>1$, the p-Laplacian operator is defined as $\phi_p(x)=|x|^{p-2}x$, where ϕ_p is invertible and its inverse operator is ϕ_q, where $q>1$ is a constant such that $\frac{1}{p}+\frac{1}{q}=1$.

Our plan is as follows. In Sect. 2, we recall some basic knowledge and convert (1.1)–(1.2) to an equivalent summation equation and find its solution. In Sect. 3, we prove existence and uniqueness of the solution of boundary value problem (1.1)–(1.2) by using the Banach fixed point theorem. Some examples to illustrate our result are presented in the last section.

2 Preliminaries

Notations, definitions, and lemmas which are used in the main results are given as follows.

Definition 2.1 The generalized falling function is defined by $t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$ for any t and α for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}}=0$.

Lemma 2.1 ([5]) *Assume that the following factorial functions are well defined*:

(i) $(t-\mu)t^{\underline{\mu}} = t^{\underline{\mu+1}}$, *where* $\mu\in\mathbb{R}$.

(ii) *If* $t \le r$, *then* $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ *for any* $\alpha>0$.

Definition 2.2 Let $\alpha>0$ and f be defined on $\mathbb{N}_a$, the α-order fractional sum of f is defined by

$$\Delta^{-\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\big(t-\sigma(s)\big)^{\underline{\alpha-1}}f(s),$$

where $t\in\mathbb{N}_{a+\alpha}$ and $\sigma(s)=s+1$.

Definition 2.3 For $\alpha>0$ and f defined on $\mathbb{N}_a$, the α-order Riemann–Liouville fractional difference of f is defined by

$$\Delta^{\alpha}f(t) := \Delta^N\Delta^{-(N-\alpha)}f(t) = \frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{t+\alpha}\big(t-\sigma(s)\big)^{\underline{-\alpha-1}}f(s).$$

The α-order Caputo fractional difference of f is defined by

$$\Delta_C^\alpha f(t) := \Delta^{-(N-\alpha)}\Delta^N f(t) = \frac{1}{\Gamma(N-\alpha)}\sum_{s=a}^{t-(N-\alpha)}\left(t-\sigma(s)\right)^{\underline{N-\alpha-1}}\Delta^N f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha < N$. If $\alpha = N$, then $\Delta^\alpha f(t) = \Delta_C^\alpha f(t) = \Delta^N f(t)$.

Lemma 2.2 ([7]) *Let* $0 \le N-1 < \alpha \le N$. *Then*

$$\Delta^{-\alpha}\Delta_C^\alpha y(t) = y(t) + C_0 + C_1 t + \cdots + C_{N-1}t^{\underline{N-1}}$$

for some $C_i \in \mathbb{R}$, *with* $1 \le i \le N$.

We provide some properties of the p-Laplacian operator as follows.

(A1) If $1 < p < 2$, $xy > 0$ and $|x|, |y| \ge m > 0$, then

$$\left|\phi_p(x) - \phi_p(y)\right| \le (p-1)m^{p-2}|x-y|;$$

(A2) If $p > 2$, $xy > 0$ and $|x|, |y| \le M$, then

$$\left|\phi_p(x) - \phi_p(y)\right| \le (p-1)M^{p-2}|x-y|.$$

Next, we find a solution of the linear variant of boundary value problem (1.1)–(1.2) as shown in the following lemma.

Lemma 2.3 *For* $i,j \in \{1,2\}$ *and* $i \ne j$, *let* $\Lambda \ne 0$, $\alpha_i, \beta_i, \theta_i \in (0,1)$, $\alpha_i + \beta_i \in (1,2]$, $\lambda_i > 0$ *be given constants*, $h_i \in C(\mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}, \mathbb{R})$ *and* $g_i \in C(\mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}, \mathbb{R}^+)$ *be given functions. Then the linear variant problem given by*

$$\Delta_C^{\alpha_i}\phi_p\left[\Delta_C^{\beta_i}u_i(t)\right] = h_i(t+\alpha_i-1), \quad t \in \mathbb{N}_{0,T}, \tag{2.1}$$

$$\Delta_C^{\beta_i}u_i(\alpha_i-1) = 0, \tag{2.2}$$

$$u_i(T+\alpha_i+\beta_i) = \lambda_j\Delta^{-\theta_j}g_j(\eta_j+\theta_j)u_j(\eta_j+\theta_j), \quad \eta_j \in \mathbb{N}_{\alpha_j+\beta_j-1,T+\alpha_j+\beta_j-1} \tag{2.3}$$

has the unique solution (u_1, u_2), *where*

$$\begin{aligned} u_1(t_1) = {} & \frac{1}{\Gamma(\beta_1)}\sum_{s=\alpha_1-1}^{t_1-\beta_1}\left(t_1-\sigma(s)\right)^{\underline{\beta_1-1}}\phi_q\left[\frac{1}{\Gamma(\alpha_1)}\sum_{\xi=0}^{s-\alpha_1}\left(s-\sigma(\xi)\right)^{\underline{\alpha_1-1}}h_1(\xi+\alpha_1+\beta_1-1)\right] \\ & + \frac{1}{\Lambda}\left\{\left(\frac{\lambda_1}{\Gamma(\theta_1)}\sum_{s=\alpha_1+\beta_1-2}^{\eta_1}\left(\eta_1+\theta_1-\sigma(s)\right)^{\underline{\theta_1-1}}g_1(s)\right)\mathcal{P}[h_1,h_2] + \mathcal{Q}[h_1,h_2]\right\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} u_2(t_2) = {} & \frac{1}{\Gamma(\beta_2)}\sum_{s=\alpha_2-1}^{t_2-\beta_2}\left(t_2-\sigma(s)\right)^{\underline{\beta_2-1}}\phi_q\left[\frac{1}{\Gamma(\alpha_2)}\sum_{\xi=0}^{s-\alpha_2}\left(s-\sigma(\xi)\right)^{\underline{\alpha_2-1}}h_2(\xi+\alpha_2+\beta_2-1)\right] \\ & + \frac{1}{\Lambda}\left\{\left(\frac{\lambda_2}{\Gamma(\theta_2)}\sum_{s=\alpha_2+\beta_2-2}^{\eta_2}\left(\eta_2+\theta_2-\sigma(s)\right)^{\underline{\theta_1-1}}g_2(s)\right)\mathcal{Q}[h_1,h_2] + \mathcal{P}[h_1,h_2]\right\}, \end{aligned} \tag{2.5}$$

where $t_i \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}$, the constant Λ is defined by

$$\Lambda = \frac{\lambda_1\lambda_2}{\Gamma(\theta_1)\Gamma(\theta_2)} \sum_{s=\alpha_1+\beta_1-1}^{\eta_1} \big(\eta_1+\theta_1-\sigma(s)\big)^{\underline{\theta_1-1}} g_1(s) \times \sum_{s=\alpha_2+\beta_2-1}^{\eta_2} \big(\eta_2+\theta_2-\sigma(s)\big)^{\underline{\theta_2-1}} g_2(s) - 1, \tag{2.6}$$

and the functionals $\mathcal{P}[h_1,h_2]$, $\mathcal{Q}[h_1,h_2]$ are defined by

$$\begin{aligned}
&\mathcal{P}[h_1,h_2] \\
&= \frac{1}{\Gamma(\beta_1)} \sum_{s=\alpha_1-1}^{T+\alpha_1} \big(T+\alpha_1+\beta_1-\sigma(s)\big)^{\underline{\beta_1-1}} \\
&\quad\times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_1)}\sum_{\xi=0}^{s-\alpha_1}\big(s-\sigma(\xi)\big)^{\underline{\alpha_1-1}} h_1(\xi+\alpha_1+\beta_1-1)\Bigg] \\
&\quad- \frac{\lambda_2}{\Gamma(\beta_2)\Gamma(\theta_2)} \sum_{r=\alpha_2+\beta_2-1}^{\eta_2}\sum_{s=\alpha_2-1}^{r-\beta_2} \big(\eta_2+\theta_2-\sigma(r)\big)^{\underline{\theta_2-1}} \big(r-\sigma(s)\big)^{\underline{\beta_2-1}} g_2(r) \\
&\quad\times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_2)}\sum_{\xi=0}^{s-\alpha_2}\big(s-\sigma(\xi)\big)^{\underline{\alpha_2-1}} h_2(\xi+\alpha_2+\beta_2-1)\Bigg],
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\mathcal{Q}[h_1,h_2] \\
&= \frac{1}{\Gamma(\beta_2)} \sum_{s=\alpha_2-1}^{T+\alpha_2} \big(T+\alpha_2+\beta_2-\sigma(s)\big)^{\underline{\beta_2-1}} \\
&\quad\times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_2)}\sum_{\xi=0}^{s-\alpha_2}\big(s-\sigma(\xi)\big)^{\underline{\alpha_2-1}} h_2(\xi+\alpha_2+\beta_2-1)\Bigg] \\
&\quad- \frac{\lambda_1}{\Gamma(\beta_1)\Gamma(\theta_1)} \sum_{r=\alpha_1+\beta_1-1}^{\eta_1}\sum_{s=\alpha_1-1}^{r-\beta_1} \big(\eta_1+\theta_1-\sigma(r)\big)^{\underline{\theta_1-1}} \big(r-\sigma(s)\big)^{\underline{\beta_1-1}} g_1(r) \\
&\quad\times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_1)}\sum_{\xi=0}^{s-\alpha_1}\big(s-\sigma(\xi)\big)^{\underline{\alpha_1-1}} h_1(\xi+\alpha_1+\beta_1-1)\Bigg].
\end{aligned} \tag{2.8}$$

Proof For $i,j \in \{1,2\}$ and $i \neq j$, taking the fractional sum of order α_i for (2.1), we have

$$\phi_p\big[\Delta_C^{\beta_i} u_i(t)\big] = C_{0i} + \frac{1}{\Gamma(\alpha_i)} \sum_{s=0}^{t-\alpha_i} \big(t-\sigma(s)\big)^{\underline{\alpha_i-1}} h_i(s+\alpha_i+\beta_i-1) \tag{2.9}$$

for $t \in \mathbb{N}_{\alpha_i-1,T+\alpha_i}$.

From boundary condition (2.2), it implies that

$$C_{0i} = 0.$$

Then from (2.9) we have

$$\Delta_C^{\beta_i} u_i(t) = \phi_q\left[\frac{1}{\Gamma(\alpha_i)}\sum_{s=0}^{t-\alpha_i}\bigl(t-\sigma(s)\bigr)^{\underline{\alpha_i-1}} h_i(s+\alpha_i+\beta_i-1)\right]. \tag{2.10}$$

Next, taking the fractional sum of order β_i for (2.10), we have

$$\begin{aligned} u_i(t) &= C_{1i} + \frac{1}{\Gamma(\beta_i)}\sum_{s=\alpha_i-1}^{t-\beta_i}\bigl(t-\sigma(s)\bigr)^{\underline{\beta_i-1}} \\ &\quad\times \phi_q\left[\frac{1}{\Gamma(\alpha_i)}\sum_{\xi=0}^{s-\alpha_i}\bigl(s-\sigma(\xi)\bigr)^{\underline{\alpha_i-1}} h_i(\xi+\alpha_i+\beta_i-1)\right] \end{aligned} \tag{2.11}$$

for $t \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}$.

Using the fractional sum of order θ_i for (2.11), we get

$$\begin{aligned} \Delta^{-\theta_i} u(t) &= \frac{C_{1i}}{\Gamma(\theta_i)}\sum_{s=\alpha_i+\beta_i-2}^{t-\theta_i}\bigl(t-\sigma(s)\bigr)^{\underline{\theta_i-1}} \\ &\quad+ \frac{1}{\Gamma(\theta_i)\Gamma(\beta_i)}\sum_{r=\alpha_i+\beta_i-1}^{t-\theta_i}\sum_{s=\alpha_i-1}^{t-\beta_i}\bigl(t-\sigma(r)\bigr)^{\underline{\theta_i-1}}\bigl(r-\sigma(s)\bigr)^{\underline{\beta_i-1}} \\ &\quad\times \phi_q\left[\frac{1}{\Gamma(\alpha_i)}\sum_{\xi=0}^{s-\alpha_i}\bigl(s-\sigma(\xi)\bigr)^{\underline{\alpha_i-1}} h_i(\xi+\alpha_i+\beta_i-1)\right] \end{aligned} \tag{2.12}$$

for $t \in \mathbb{N}_{\alpha_i+\beta_i+\theta_i-3,T+\alpha_i+\beta_i+\theta_i}$.

Using boundary condition (2.3) implies

$$\begin{aligned} &C_{11} - C_{12}\frac{\lambda_2}{\Gamma(\theta_2)}\sum_{s=\alpha_2+\beta_2-2}^{\eta_2}\bigl(\eta_2+\theta_2-\sigma(s)\bigr)^{\underline{\theta_2-1}} g_2(s) \\ &\quad= \frac{\lambda_2}{\Gamma(\theta_2)\Gamma(\beta_2)}\sum_{r=\alpha_2+\beta_2-1}^{\eta_2}\sum_{s=\alpha_2-1}^{r-\beta_2}\bigl(\eta_2+\theta_2-\sigma(r)\bigr)^{\underline{\theta_2-1}}\bigl(r-\sigma(s)\bigr)^{\underline{\beta_2-1}} g_2(r) \\ &\qquad\times \phi_q\left[\frac{1}{\Gamma(\alpha_2)}\sum_{\xi=0}^{s-\alpha_2}\bigl(s-\sigma(\xi)\bigr)^{\underline{\alpha_2-1}} h_2(\xi+\alpha_2+\beta_2-1)\right] \\ &\qquad- \frac{1}{\Gamma(\beta_1)}\sum_{s=\alpha_1-1}^{T+\alpha_1}\bigl(T+\alpha_1+\beta_1-\sigma(s)\bigr)^{\underline{\beta_1-1}} \\ &\qquad\times \phi_q\left[\frac{1}{\Gamma(\alpha_1)}\sum_{\xi=0}^{s-\alpha_1}\bigl(s-\sigma(\xi)\bigr)^{\underline{\alpha_1-1}} h_1(\xi+\alpha_1+\beta_1-1)\right] \end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
&C_{21} - C_{22}\frac{\lambda_1}{\Gamma(\theta_1)}\sum_{s=\alpha_1+\beta_1-2}^{\eta_1}\big(\eta_1+\theta_1-\sigma(s)\big)^{\underline{\theta_1-1}}g_1(s) \\
&\quad = \frac{\lambda_1}{\Gamma(\theta_1)\Gamma(\beta_1)}\sum_{r=\alpha_1+\beta_1-1}^{\eta_1}\sum_{s=\alpha_1-1}^{r-\beta_1}\big(\eta_1+\theta_1-\sigma(r)\big)^{\underline{\theta_1-1}}\big(r-\sigma(s)\big)^{\underline{\beta_1-1}}g_1(r) \\
&\qquad \times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_1)}\sum_{\xi=0}^{s-\alpha_1}\big(s-\sigma(\xi)\big)^{\underline{\alpha_1-1}}h_1(\xi+\alpha_1+\beta_1-1)\Bigg] \\
&\qquad - \frac{1}{\Gamma(\beta_2)}\sum_{s=\alpha_2-1}^{T+\alpha_2}\big(T+\alpha_2+\beta_2-\sigma(s)\big)^{\underline{\beta_2-1}} \\
&\qquad \times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_2)}\sum_{\xi=0}^{s-\alpha_2}\big(s-\sigma(\xi)\big)^{\underline{\alpha_2-1}}h_2(\xi+\alpha_2+\beta_2-1)\Bigg]. \qquad (2.14)
\end{aligned}$$

C_{11}, C_{12} can be represented by solving equations (2.13) and (2.14) as

$$C_{11} = \frac{1}{\Lambda}\Bigg\{\Bigg(\frac{\lambda_1}{\Gamma(\theta_1)}\sum_{s=\alpha_1+\beta_1-2}^{\eta_1}\big(\eta_1+\theta_1-\sigma(s)\big)^{\underline{\theta_1-1}}g_1(s)\Bigg)\mathcal{P}[h_1,h_2] + \mathcal{Q}[h_1,h_2]\Bigg\} \qquad (2.15)$$

and

$$C_{12} = \frac{1}{\Lambda}\Bigg\{\Bigg(\frac{\lambda_2}{\Gamma(\theta_2)}\sum_{s=\alpha_2+\beta_2-2}^{\eta_2}\big(\eta_2+\theta_2-\sigma(s)\big)^{\underline{\theta_1-1}}g_2(s)\Bigg)\mathcal{Q}[h_1,h_2] + \mathcal{P}[h_1,h_2]\Bigg\}, \qquad (2.16)$$

where Λ, $\mathcal{P}(h_1,h_2)$ and $\mathcal{Q}(h_1,h_2)$ are defined as (2.6)–(2.8), respectively.

After substituting C_{11} and C_{12} into (2.11), we obtain (2.4) and (2.5). □

3 Existence and uniqueness result

In this section, we study the existence and uniqueness result for problem (1.1)–(1.2). For each $i,j \in \{1,2\}$ and $i \neq j$, we let $E_i : C(\mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i},\mathbb{R})$ be the Banach space for all functions on $\mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}$. Clearly, the product space $\mathcal{C} = E_1 \times E_2$ is the Banach space. Define the spaces

$$\mathcal{C}_i = \big\{(u_1,u_2) \in \mathcal{C} : \Delta^{\gamma_i}u_i(t_i-\gamma_i+1) \in E_i\big\}, \quad t_i \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i},$$

with the norm

$$\big\|(u_1,u_2)\big\|_{\mathcal{C}_i} = \max\big\{\big\|\Delta^{\gamma_i}u_i\big\|, \|u_j\|\big\},$$

where

$$\big\|\Delta^{\gamma_i}u_i\big\| = \max_{t_i\in\mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}}\big|\Delta^{\gamma_i}u_i(t_i-\gamma_i+1)\big| \quad \text{and} \quad \|u_j\| = \max_{t_j\in\mathbb{N}_{\alpha_j+\beta_j-2,T+\alpha_j+\beta_j}}\big|u_j(t_j)\big|.$$

Obviously, the space $(\mathcal{C}_1 \cap \mathcal{C}_2, \|(u_1,u_2)\|_{\mathcal{C}_1\cap\mathcal{C}_2})$ is also the Banach space with the norm

$$\big\|(u_1,u_2)\big\|_{\mathcal{C}_1\cap\mathcal{C}_2} = \max\big\{\big\|(u_1,u_2)\big\|_{\mathcal{C}_1}, \big\|(u_1,u_2)\big\|_{\mathcal{C}_2}\big\}.$$

Let $\mathcal{U} = \mathcal{C}_1 \cap \mathcal{C}_2$. The operator $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ is defined by

$$\big(\mathcal{T}(u_1,u_2)\big)(t_1,t_2) = \Big(\big(\mathcal{T}_1(u_1,u_2)\big)(t_1,t_2), \big(\mathcal{T}_2(u_1,u_2)\big)(t_1,t_2)\Big) \tag{3.1}$$

and

$$\begin{aligned}
&\big(\mathcal{T}_1(u_1,u_2)\big)(t_1,t_2) \\
&\quad = \frac{1}{\Gamma(\beta_1)} \sum_{s=\alpha_1-1}^{t_1-\beta_1} \big(t_1 - \sigma(s)\big)^{\underline{\beta_1-1}} \\
&\qquad \times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_1)} \sum_{\xi=\alpha_1+\beta_1-1}^{s+\beta_1-1} \big(s+\alpha_1+\beta_1-1-\sigma(\xi)\big)^{\underline{\alpha_1-1}} F_1^*\big[u(t_2,\xi)\big]\Bigg] \\
&\qquad + \frac{1}{\Lambda}\Bigg\{\Bigg(\frac{\lambda_1}{\Gamma(\theta_1)} \sum_{s=\alpha_1+\beta_1-2}^{\eta_1} \big(\eta_1+\theta_1-\sigma(s)\big)^{\underline{\theta_1-1}} g_1(s)\Bigg) \mathcal{P}[F_1,F_2](u_1,u_2) \\
&\qquad + \mathcal{Q}[F_1,F_2](u_1,u_2)\Bigg\},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\big(\mathcal{T}_2(u_1,u_2)\big)(t_1,t_2) \\
&\quad = \frac{1}{\Gamma(\beta_2)} \sum_{s=\alpha_2-1}^{t_2-\beta_2} \big(t_2 - \sigma(s)\big)^{\underline{\beta_2-1}} \\
&\qquad \times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_2)} \sum_{\xi=\alpha_2+\beta_2-1}^{s+\beta_2-1} \big(s+\alpha_2+\beta_2-1-\sigma(\xi)\big)^{\underline{\alpha_2-1}} F_2^*\big[u(t_1,\xi)\big]\Bigg] \\
&\qquad + \frac{1}{\Lambda}\Bigg\{\Bigg(\frac{\lambda_2}{\Gamma(\theta_2)} \sum_{s=\alpha_2+\beta_2-2}^{\eta_2} \big(\eta_2+\theta_2-\sigma(s)\big)^{\underline{\theta_1-1}} g_2(s)\Bigg) \mathcal{Q}[F_1,F_2](u_1,u_2) \\
&\qquad + \mathcal{P}[F_1,F_2](u_1,u_2)\Bigg\},
\end{aligned} \tag{3.3}$$

where $t_i \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}$, Λ is defined as (2.6), and the functionals $\mathcal{P}[F_1,F_2](u_1,u_2)$, $\mathcal{Q}[F_1,F_2](u_1,u_2)$ are defined by

$$\begin{aligned}
&\mathcal{P}[F_1,F_2](u_1,u_2) \\
&\quad = \frac{1}{\Gamma(\beta_1)} \sum_{s=\alpha_1-1}^{T+\alpha_1} \big(T+\alpha_1+\beta_1-\sigma(s)\big)^{\underline{\beta_1-1}} \\
&\qquad \times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_1)} \sum_{\xi=\alpha_1+\beta_1-1}^{s+\beta_1-1} \big(s+\alpha_1+\beta_1-1-\sigma(\xi)\big)^{\underline{\alpha_1-1}} F_1^*\big[u(t_2,\xi)\big]\Bigg] \\
&\qquad - \frac{\lambda_2}{\Gamma(\beta_2)\Gamma(\theta_2)} \sum_{r=\alpha_2+\beta_2-1}^{\eta_2} \sum_{s=\alpha_2-1}^{r-\beta_2} \big(\eta_2+\theta_2-\sigma(r)\big)^{\underline{\theta_2-1}} \big(r-\sigma(s)\big)^{\underline{\beta_2-1}} g_2(r) \\
&\qquad \times \phi_q\Bigg[\frac{1}{\Gamma(\alpha_2)} \sum_{\xi=\alpha_2+\beta_2-1}^{s+\beta_2-1} \big(s+\alpha_2+\beta_2-1-\sigma(\xi)\big)^{\underline{\alpha_2-1}} F_2^*\big[u(t_1,\xi)\big]\Bigg],
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\mathcal{Q}[F_1,F_2](u_1,u_2)\\
&\quad= \frac{1}{\Gamma(\beta_2)}\sum_{s=\alpha_2-1}^{T+\alpha_2}\bigl(T+\alpha_2+\beta_2-\sigma(s)\bigr)^{\underline{\beta_2-1}}\\
&\qquad\times\phi_q\Biggl[\frac{1}{\Gamma(\alpha_2)}\sum_{\xi=\alpha_2+\beta_2-1}^{s+\beta_2-1}\bigl(s+\alpha_2+\beta_2-1-\sigma(\xi)\bigr)^{\underline{\alpha_2-1}}F_2^*\bigl[u(t_1,\xi)\bigr]\Biggr]\\
&\qquad-\frac{\lambda_1}{\Gamma(\beta_1)\Gamma(\theta_1)}\sum_{r=\alpha_1+\beta_1-1}^{\eta_1}\sum_{s=\alpha_1-1}^{r-\beta_1}\bigl(\eta_1+\theta_1-\sigma(r)\bigr)^{\underline{\theta_1-1}}\bigl(r-\sigma(s)\bigr)^{\underline{\beta_1-1}}g_1(r)\\
&\qquad\times\phi_q\Biggl[\frac{1}{\Gamma(\alpha_1)}\sum_{\xi=\alpha_1+\beta_1-1}^{s+\beta_1-1}\bigl(s+\alpha_1+\beta_1-1-\sigma(\xi)\bigr)^{\underline{\alpha_1-1}}F_1^*\bigl[u(t_2,\xi)\bigr]\Biggr],
\end{aligned}\tag{3.5}$$

with

$$F_i^*\bigl[u(t_j,\xi)\bigr]=F_i\bigl[t_j,\xi,\Delta^{\gamma_i}u_i(\xi-\gamma_i+1),\Psi^{\omega_j}u_j(t_j+\omega_j),u_j(t_j)\bigr].\tag{3.6}$$

For each $i,j\in\{1,2\}$ and $i\neq j$, we define the operators $(\mathcal{T}_i^0(u_1,u_2))(t_1,t_2)$ and $(\mathcal{T}_i^*(u_1,u_2))(t_1,t_2)$ by

$$\bigl(\mathcal{T}_1^0(u_1,u_2)\bigr)(t_1,t_2)=\phi_q\Biggl[\sum_{\xi=\alpha_1+\beta_1-1}^{t_1+\beta_1-1}\frac{(t_1+\alpha_1+\beta_1-1-\sigma(\xi))^{\underline{\alpha_1-1}}}{\Gamma(\alpha_1)}F_1^*\bigl[u(t_2,\xi)\bigr]\Biggr],\tag{3.7}$$

$$\bigl(\mathcal{T}_2^0(u_1,u_2)\bigr)(t_1,t_2)=\phi_q\Biggl[\sum_{\xi=\alpha_2+\beta_2-1}^{t_2+\beta_2-1}\frac{(t_2+\alpha_2+\beta_2-1-\sigma(\xi))^{\underline{\alpha_2-1}}}{\Gamma(\alpha_2)}F_2^*\bigl[u(t_1,\xi)\bigr]\Biggr],\tag{3.8}$$

and

$$\begin{aligned}
&\bigl(\mathcal{T}_1^*(u_1,u_2)\bigr)(t_1,t_2)\\
&\quad=\frac{1}{\Gamma(\beta_1)}\sum_{s=\alpha_1-1}^{t_1-\beta_1}\bigl(t_1-\sigma(s)\bigr)^{\underline{\beta_1-1}}(u_1,u_2)(t_2,s)+\frac{1}{\Lambda}\Biggl\{\mathcal{Q}^*[F_1,F_2](u_1,u_2)\\
&\qquad+\Biggl(\frac{\lambda_1}{\Gamma(\theta_1)}\sum_{s=\alpha_1+\beta_1-2}^{\eta_1}\bigl(\eta_1+\theta_1-\sigma(s)\bigr)^{\underline{\theta_1-1}}g_1(s)\Biggr)\mathcal{P}^*[F_1,F_2](u_1,u_2)\Biggr\},
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
&\bigl(\mathcal{T}_2^*(u_1,u_2)\bigr)(t_1,t_2)\\
&\quad=\frac{1}{\Gamma(\beta_2)}\sum_{s=\alpha_2-1}^{t_2-\beta_2}\bigl(t_2-\sigma(s)\bigr)^{\underline{\beta_2-1}}(u_1,u_2)(t_1,s)+\frac{1}{\Lambda}\Biggl\{\mathcal{P}^*[F_1,F_2](u_1,u_2)\\
&\qquad+\Biggl(\frac{\lambda_2}{\Gamma(\theta_2)}\sum_{s=\alpha_2+\beta_2-2}^{\eta_2}\bigl(\eta_2+\theta_2-\sigma(s)\bigr)^{\underline{\theta_1-1}}g_2(s)\Biggr)\mathcal{Q}^*[F_1,F_2](u_1,u_2)\Biggr\},
\end{aligned}\tag{3.10}$$

where $t_i \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}$, and the functionals $\mathcal{P}^*[F_1,F_2](u_1,u_2)$, $\mathcal{Q}^*[F_1,F_2](u_1,u_2)$ are defined by

$$\begin{aligned}
&\mathcal{P}^*[F_1,F_2](u_1,u_2)\\
&\quad= \frac{1}{\Gamma(\beta_1)}\sum_{s=\alpha_1-1}^{T+\alpha_1}\bigl(T+\alpha_1+\beta_1-\sigma(s)\bigr)^{\underline{\beta_1-1}}(u_1,u_2)(t_2,s)\\
&\qquad- \frac{\lambda_2}{\Gamma(\beta_2)\Gamma(\theta_2)}\sum_{r=\alpha_2+\beta_2-1}^{\eta_2}\sum_{s=\alpha_2-1}^{r-\beta_2}\bigl(\eta_2+\theta_2-\sigma(r)\bigr)^{\underline{\theta_2-1}}\bigl(r-\sigma(s)\bigr)^{\underline{\beta_2-1}}\\
&\qquad\times g_2(r)(u_1,u_2)(t_1,s),
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\mathcal{Q}^*[F_1,F_2](u_1,u_2)\\
&\quad= \frac{1}{\Gamma(\beta_2)}\sum_{s=\alpha_2-1}^{T+\alpha_2}\bigl(T+\alpha_2+\beta_2-\sigma(s)\bigr)^{\underline{\beta_2-1}}(u_1,u_2)(t_1,s)\\
&\qquad- \frac{\lambda_1}{\Gamma(\beta_1)\Gamma(\theta_1)}\sum_{r=\alpha_1+\beta_1-1}^{\eta_1}\sum_{s=\alpha_1-1}^{r-\beta_1}\bigl(\eta_1+\theta_1-\sigma(r)\bigr)^{\underline{\theta_1-1}}\bigl(r-\sigma(s)\bigr)^{\underline{\beta_1-1}}\\
&\qquad\times g_1(r)(u_1,u_2)(t_2,s).
\end{aligned} \tag{3.12}$$

Let $\mathcal{T}_i = \mathcal{T}_i^* \circ \mathcal{T}_i^0$, then $\mathcal{T}_i$ and $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ are continuous and compact operators. Note that problem (1.1)–(1.2) has solutions if and only if the operator $\mathcal{T}$ has fixed points.

In the case $p > 2$, we have $1 < q < 2$ due to $\frac{1}{p}+\frac{1}{q} = 1$ and the following theorem is obtained.

Theorem 3.1 *Let $p > 2$ for each $i,j \in \{1,2\}$, $i \neq j$, $F_i \in C(\mathbb{N}_{\alpha_1+\beta_1-2,T+\alpha_1+\beta_1} \times \mathbb{N}_{\alpha_2+\beta_2-2,T+\alpha_2+\beta_2} \times \mathbb{R}^3,\mathbb{R})$, $\varphi_i \in C(\mathbb{N}_{\alpha_1+\beta_1-2,T+\alpha_1+\beta_1} \times \mathbb{N}_{\alpha_2+\beta_2-2,T+\alpha_2+\beta_2},[0,\infty))$ with $\varphi_i^o = \max\{\varphi(t_i-1,s)\}$. In addition, suppose that:*

(H1) *There exist constants $\chi_i > 0$ and $0 < \delta < \frac{1}{2-q}$ such that*

$$\chi_i \Delta_C^{\alpha}\bigl(t_i^{\underline{\alpha_i}}\bigr)^{\delta} \le F_i[t_i,t_j,x,y,z]$$

for any $(t_i,t_j,x,y,z) \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i} \times \mathbb{N}_{\alpha_j+\beta_j-2,T+\alpha_j+\beta_j} \times \mathbb{R}^3$.

(H2) *There exist constants $L_i, M_i, N_i > 0$ such that, for each $t_i \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}$ and $u_1,u_2,u_3,v_1,v_2,v_3 \in \mathbb{R}$,*

$$\begin{aligned}
&\bigl|F_i[t_i,t_j,u_1,u_2,u_3]-F_i[t_i,t_j,v_1,v_2,v_3]\bigr|\\
&\quad\le L_i|u_1-v_1| + M_j|u_2-v_2| + N_j|u_3-v_3|.
\end{aligned}$$

(H3) *$g_i < g_i(t_i) < G_i$ for each $t_i \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}$.*

Then problem (1.1)–(1.2) has a unique solution provided that

$$\begin{aligned}
\Phi := \max\biggl\{&\mathcal{K}_1\chi_1^{q-2}\Theta_1 + \mathcal{K}_2\chi_2^{q-2}\Omega_2, \mathcal{K}_1\chi_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Omega_1 + \mathcal{K}_2\chi_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Theta_2,\\
&\mathcal{K}_1\chi_1^{q-2}\Omega_1 + \mathcal{K}_2\chi_2^{q-2}\Theta_2, \mathcal{K}_1\chi_1^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\Theta_1 + \mathcal{K}_2\chi_2^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\Omega_2\biggr\}\\
&< 1,
\end{aligned} \tag{3.13}$$

where

$$\mathcal{K}_i = \left[L_i + N_j + M_j\varphi_j^o \frac{(T+\omega_j+2)^{\underline{\omega_j}}}{\Gamma(\omega_j+1)}\right]\frac{(q-1)}{\Gamma(\alpha_i+1)}, \tag{3.14}$$

$$\begin{aligned}\Omega_i &= \left[1 + \frac{\lambda_i G_i(\eta_i-\alpha_i-\beta_i+\theta_i+2)^{\underline{\theta_i}}}{|\Lambda|\Gamma(\theta_i+1)}\right]\frac{1}{\Gamma(\beta_i)}\sum_{s=\alpha_i-1}^{T+\alpha_i}\bigl(T+\alpha_i+\beta_i-\sigma(s)\bigr)^{\underline{\beta_i-1}}\bigl(s^{\underline{\alpha_i}}\bigr)^{\delta(q-2)+1} \\ &\quad + \frac{\lambda_i G_i(\eta_i-\alpha_i-\beta_i+\theta_i+1)^{\underline{\theta_i}}}{|\Lambda|\Gamma(\theta_i+1)}\cdot\frac{1}{\Gamma(\beta_i)}\sum_{s=\alpha_i-1}^{\eta_i-\beta_i}\bigl(\eta_i-\sigma(s)\bigr)^{\underline{\beta_i-1}}\bigl(s^{\underline{\alpha_i}}\bigr)^{\delta(q-2)+1},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\Theta_i &= \frac{1}{|\Lambda|\Gamma(\beta_i)}\sum_{s=\alpha_i-1}^{T+\alpha_i}\bigl(T+\alpha_i+\beta_i-\sigma(s)\bigr)^{\underline{\beta_i-1}}\bigl(s^{\underline{\alpha_i}}\bigr)^{\delta(q-2)+1} \\ &\quad + \frac{\lambda_1\lambda_2 G_1 G_2}{|\Lambda|}\cdot\frac{(\eta_j-\alpha_j-\beta_j+\theta_j+2)^{\underline{\theta_j}}}{\Gamma(\theta_j+1)}\cdot\frac{(\eta_i-\alpha_i-\beta_i+\theta_i+1)^{\underline{\theta_i}}}{\Gamma(\theta_i+1)} \\ &\quad \times \frac{1}{\Gamma(\beta_i)}\sum_{s=\alpha_i-1}^{\eta_i-\beta_i}\bigl(\eta_i-\sigma(s)\bigr)^{\underline{\beta_i-1}}\bigl(s^{\underline{\alpha_i}}\bigr)^{\delta(q-2)+1}.\end{aligned} \tag{3.16}$$

Proof For each $i,j \in \{1,2\}$, $i \neq j$, by (H1) we have

$$\chi_i\bigl(t_i^{\underline{\alpha_i}}\bigr)^{\delta} \leq \frac{1}{\Gamma(\alpha_i)}\sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1}\bigl(t_i+\alpha_i+\beta_i-1-\sigma(\xi)\bigr)^{\underline{\alpha_i-1}}F_i^*(t_j,\xi). \tag{3.17}$$

By (A1), (H2), and the definition of operator $\mathcal{T}_i^0$, for any $(u_1,u_2),(v_1,v_2) \in \mathcal{C}$, we have

$$\begin{aligned}&\bigl|\bigl(\mathcal{T}_i^0(u_1,u_2)\bigr)(t_1,t_2) - \bigl(\mathcal{T}_i^0(v_1,v_2)\bigr)(t_1,t_2)\bigr| \\ &\quad = \Biggl|\phi_q\Biggl[\sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1}\frac{(t_i+\alpha_i+\beta_i-1-\sigma(\xi))^{\underline{\alpha_i-1}}}{\Gamma(\alpha_i)}F_i^*\bigl[u(t_j,\xi)\bigr]\Biggr] \\ &\qquad - \phi_q\Biggl[\sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1}\frac{(t_i+\alpha_i+\beta_i-1-\sigma(\xi))^{\underline{\alpha_i-1}}}{\Gamma(\alpha_i)}F_i^*\bigl[v(t_j,\xi)\bigr]\Biggr]\Biggr| \\ &\quad \leq (q-1)\bigl(\chi_i\bigl(t_i^{\underline{\alpha_i}}\bigr)^{\delta}\bigr)^{q-2}\sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1}\frac{(t_i+\alpha_i+\beta_i-1-\sigma(\xi))^{\underline{\alpha_i-1}}}{\Gamma(\alpha_i)}\bigl|F_i^*\bigl[u(t_j,\xi)\bigr]-F_i^*\bigl[v(t_j,\xi)\bigr]\bigr| \\ &\quad \leq (q-1)\chi_i^{q-2}\frac{(t_i^{\underline{\alpha_i}})^{\delta(q-2)+1}}{\Gamma(\alpha_i+1)}\bigl[L_i\bigl\|\Delta^{\gamma_i}u_i-\Delta^{\gamma_i}v_i\bigr\| + M_j\bigl\|\Psi^{\omega_i}u_i-\Psi^{\omega_j}v_i\bigr\| + N_j\|u_j-v_j\|\bigr] \\ &\quad \leq (q-1)\chi_i^{q-2}\frac{(t_i^{\underline{\alpha_i}})^{\delta(q-2)+1}}{\Gamma(\alpha_i+1)}\Biggl[L_i\bigl\|\Delta^{\gamma_i}u_i-\Delta^{\gamma_i}v_i\bigr\| \\ &\qquad + \biggl(N_j + M_j\varphi_j^o\frac{(T+\omega_j+2)^{\underline{\omega_j}}}{\Gamma(\omega_j+1)}\biggr)\|u_j-v_j\|\Biggr] \\ &\quad \leq (q-1)\chi_i^{q-2}\frac{(t_i^{\underline{\alpha_i}})^{\delta(q-2)+1}}{\Gamma(\alpha_i+1)}\biggl[L_i+N_j+M_j\varphi_j^o\frac{(T+\omega_j+2)^{\underline{\omega_j}}}{\Gamma(\omega_j+1)}\biggr] \\ &\qquad \times \bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{C}_i}.\end{aligned} \tag{3.18}$$

Using (3.18) and (H3), we have

$$
\begin{aligned}
&\left|\mathcal{P}^*[F_1,F_2](u_1,u_2)-\mathcal{P}^*[F_1,F_2](v_1,v_2)\right| \\
&\quad\le \frac{1}{\Gamma(\beta_1)}\sum_{s=\alpha_1-1}^{T+\alpha_1}\left(T+\alpha_1+\beta_1-\sigma(s)\right)^{\underline{\beta_1-1}}\left|\left(\mathcal{T}_i^0(u_1,u_2)\right)(s,t_2)-\left(\mathcal{T}_i^0(v_1,v_2)\right)(s,t_2)\right| \\
&\qquad-\frac{\lambda_2}{\Gamma(\beta_2)\Gamma(\theta_2)}\sum_{r=\alpha_2+\beta_2-1}^{\eta_2}\sum_{s=\alpha_2-1}^{r-\beta_2}\left(\eta_2+\theta_2-\sigma(r)\right)^{\underline{\theta_2-1}}\left(r-\sigma(s)\right)^{\underline{\beta_2-1}}g_2(r) \\
&\qquad\times\left|\left(\mathcal{T}_i^0(u_1,u_2)\right)(t_1,s)-\left(\mathcal{T}_i^0(v_1,v_2)\right)(t_1,s)\right| \\
&\quad\le \frac{\chi_1^{q-2}\mathcal{K}_1}{\Gamma(\beta_1)}\left\|(u_1-v_1,u_2-v_2)\right\|_{\mathcal{C}_1}\sum_{s=\alpha_1-1}^{T+\alpha_1}\left(T+\alpha_1+\beta_1-\sigma(s)\right)^{\underline{\beta_1-1}}\left(s^{\underline{\alpha_1}}\right)^{\delta(q-2)+1} \\
&\qquad-\frac{\lambda_2G_2\chi_2^{q-2}\mathcal{K}_2}{\Gamma(\beta_2)\Gamma(\theta_2)}\left\|(u_1-v_1,u_2-v_2)\right\|_{\mathcal{C}_2}\sum_{r=\alpha_2+\beta_2-1}^{\eta_2}\sum_{s=\alpha_2-1}^{r-\beta_2}\left(\eta_2+\theta_2-\sigma(r)\right)^{\underline{\theta_2-1}} \\
&\qquad\times\left(r-\sigma(s)\right)^{\underline{\beta_2-1}}\left(s^{\underline{\alpha_2}}\right)^{\delta(q-2)+1},
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
&\left|\mathcal{Q}^*[F_1,F_2](u_1,u_2)-\mathcal{Q}^*[F_1,F_2](v_1,v_2)\right| \\
&\quad= \frac{\chi_2^{q-2}\mathcal{K}_2}{\Gamma(\beta_2)}\left\|(u_1-v_1,u_2-v_2)\right\|_{\mathcal{C}_2}\sum_{s=\alpha_2-1}^{T+\alpha_2}\left(T+\alpha_2+\beta_2-\sigma(s)\right)^{\underline{\beta_2-1}}\left(s^{\underline{\alpha_2}}\right)^{\delta(q-2)+1} \\
&\qquad-\frac{\lambda_1G_1\chi_1^{q-2}\mathcal{K}_1}{\Gamma(\beta_1)\Gamma(\theta_1)}\left\|(u_1-v_1,u_2-v_2)\right\|_{\mathcal{C}_1}\sum_{r=\alpha_1+\beta_1-1}^{\eta_1}\sum_{s=\alpha_1-1}^{r-\beta_1}\left(\eta_1+\theta_1-\sigma(r)\right)^{\underline{\theta_1-1}} \\
&\qquad\times\left(r-\sigma(s)\right)^{\underline{\beta_1-1}}\left(s^{\underline{\alpha_1}}\right)^{\delta(q-2)+1}.
\end{aligned}
\tag{3.20}
$$

From (3.19)–(3.20), it implies that

$$
\begin{aligned}
&\left|\left(\mathcal{T}_1(u_1,u_2)\right)(t_1,t_2)-\left(\mathcal{T}_1(v_1,v_2)\right)(t_1,t_2)\right| \\
&\quad=\left|\left(\mathcal{T}_1^*\left(\mathcal{T}_1^o(u_1,u_2)\right)\right)(t_1,t_2)-\left(\mathcal{T}_1^*\left(\mathcal{T}_1^o(v_1,v_2)\right)\right)(t_1,t_2)\right| \\
&\quad\le\frac{1}{\Gamma(\beta_1)}\sum_{s=\alpha_1-1}^{t_1-\beta_1}\left(t_1-\sigma(s)\right)^{\underline{\beta_1-1}}\left|\left(\mathcal{T}_1^0(u_1,u_2)\right)(t_1,t_2)-\left(\mathcal{T}_1^0(v_1,v_2)\right)(t_1,t_2)\right| \\
&\qquad+\frac{1}{|\Lambda|}\left\{\left(\frac{\lambda_1}{\Gamma(\theta_1)}\sum_{s=\alpha_1+\beta_1-2}^{\eta_1}\left(\eta_1+\theta_1-\sigma(s)\right)^{\underline{\theta_1-1}}g_1(s)\right)\right. \\
&\qquad\times\left|\mathcal{P}^*[F_1,F_2](u_1,u_2)-\mathcal{P}^*[F_1,F_2](v_1,v_2)\right| \\
&\qquad\left.+\left|\mathcal{Q}^*[F_1,F_2](u_1,u_2)-\mathcal{Q}^*[F_1,F_2](v_1,v_2)\right|\right\} \\
&\quad\le\frac{\chi_1^{q-2}\mathcal{K}_1\|(u_1-v_1,u_2-v_2)\|_{\mathcal{C}_1}}{\Gamma(\beta_1)}\sum_{s=\alpha_1-1}^{t_1-\beta_1}\left(t_1-\sigma(s)\right)^{\underline{\beta_1-1}}\left(s^{\underline{\alpha_1}}\right)^{\delta(q-2)+1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}\Bigg\{\frac{\lambda_1 G_1(\eta_1-\alpha_1-\beta_1+\theta_1+2)^{\underline{\theta_1}}}{|\Lambda|\Gamma(\theta_1+1)} \\
&\times\Bigg[\frac{\chi_1^{q-2}\mathcal{K}_1}{\Gamma(\beta_1)}\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{C}_1}\sum_{s=\alpha_1-1}^{T+\alpha_1}\big(T+\alpha_1+\beta_1-\sigma(s)\big)^{\underline{\beta_1-1}}\big(s^{\underline{\alpha_1}}\big)^{\delta(q-2)+1} \\
&+\frac{\lambda_2 G_2\chi_2^{q-2}\mathcal{K}_2}{\Gamma(\beta_2)\Gamma(\theta_2)}\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{C}_2}\sum_{r=\alpha_2+\beta_2-1}^{\eta_2}\sum_{s=\alpha_2-1}^{r-\beta_2}\big(\eta_2+\theta_2-\sigma(r)\big)^{\underline{\theta_2-1}} \\
&\times\big(r-\sigma(s)\big)^{\underline{\beta_2-1}}\big(s^{\underline{\alpha_2}}\big)^{\delta(q-2)+1}\Bigg] \\
&+\Bigg[\frac{\chi_2^{q-2}\mathcal{K}_2}{\Gamma(\beta_2)}\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{C}_2}\sum_{s=\alpha_2-1}^{T+\alpha_2}\big(T+\alpha_2+\beta_2-\sigma(s)\big)^{\underline{\beta_2-1}}\big(s^{\underline{\alpha_2}}\big)^{\delta(q-2)+1} \\
&+\frac{\lambda_1 G_1\chi_1^{q-2}\mathcal{K}_1}{\Gamma(\beta_1)\Gamma(\theta_1)}\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{C}_1}\sum_{r=\alpha_1+\beta_1-1}^{\eta_1}\sum_{s=\alpha_1-1}^{r-\beta_1}\big(\eta_1+\theta_1-\sigma(r)\big)^{\underline{\theta_1-1}} \\
&\times\big(r-\sigma(s)\big)^{\underline{\beta_1-1}}\big(s^{\underline{\alpha_1}}\big)^{\delta(q-2)+1}\Bigg]\Bigg\} \\
\le{}&\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{U}}\big\{\mathcal{K}_1\chi_1^{q-2}\Omega_1+\mathcal{K}_2\chi_2^{q-2}\Theta_2\big\}. \qquad (3.21)
\end{aligned}
$$

Similarly, we can find that

$$
\begin{aligned}
&\big|\big(\mathcal{T}_2(u_1,u_2)\big)(t_1,t_2)-\big(\mathcal{T}_2(v_1,v_2)\big)(t_1,t_2)\big| \\
&\quad\le\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{U}}\big\{\mathcal{K}_1\chi_1^{q-2}\Theta_1+\mathcal{K}_2\chi_2^{q-2}\Omega_2\big\}. \qquad (3.22)
\end{aligned}
$$

Next, taking the fractional difference of order γ_1, γ_2 for (3.2) and (3.3), respectively, we obtain

$$
\begin{aligned}
&\Delta^{\gamma_1}\big(\mathcal{T}_1(u_1,u_2)\big)(t_1,t_2) \\
&\quad=\Delta^{\gamma_1}\big(\mathcal{T}_1^*\big(\mathcal{T}_1^o(u_1,u_2)\big)\big)(t_1,t_2) \\
&\quad=\frac{1}{\Gamma(-\gamma_1)\Gamma(\beta_1)}\sum_{r=\alpha_1+\beta_1-1}^{t_1+\gamma_1}\sum_{s=\alpha_1-1}^{r-\beta_1}\big(t_1-\sigma(r)\big)^{\underline{-\gamma_1-1}}\big(r-\sigma(s)\big)^{\underline{\beta_1-1}}\big(\mathcal{T}_1^o(u_1,u_2)\big)(t_2,s) \\
&\qquad+\frac{1}{\Lambda}\Bigg\{\frac{\lambda_1}{\Gamma(\theta_1)\Gamma(-\gamma_1)}\sum_{r=\alpha_1+\beta_1-1}^{t_1+\gamma_1}\sum_{s=\alpha_1+\beta_1-2}^{\eta_1}\big(t_1-\sigma(r)\big)^{\underline{-\gamma_1-1}}\big(\eta_1+\theta_1-\sigma(s)\big)^{\underline{\theta_1-1}}g_1(s) \\
&\qquad\times\mathcal{P}^*[F_1,F_2](u_1,u_2)+\frac{1}{\Gamma(-\gamma_1)}\sum_{s=\alpha_1+\beta_1-1}^{t_1+\gamma_1}\big(t_1-\sigma(s)\big)^{\underline{-\gamma_1-1}}\mathcal{Q}^*[F_1,F_2](u_1,u_2)\Bigg\} \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta^{\gamma_2}\big(\mathcal{T}_2(u_1,u_2)\big)(t_1,t_2) \\
&\quad=\Delta^{\gamma_2}\big(\mathcal{T}_2^*\big(\mathcal{T}_2^o(u_1,u_2)\big)\big)(t_1,t_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(-\gamma_2)\Gamma(\beta_2)} \sum_{r=\alpha_2+\beta_2-1}^{t_2+\gamma_2} \sum_{s=\alpha_2-1}^{r-\beta_2} \bigl(t_2-\sigma(r)\bigr)^{\underline{-\gamma_2-1}} \bigl(r-\sigma(s)\bigr)^{\underline{\beta_2-1}} \bigl(\mathcal{T}_2^o(u_1,u_2)\bigr)(t_1,s) \\
&\quad + \frac{1}{\Lambda}\Biggl\{ \frac{\lambda_2}{\Gamma(-\gamma_2)\Gamma(\theta_2)} \sum_{r=\alpha_2+\beta_2-1}^{t_2+\gamma_2} \sum_{s=\alpha_2+\beta_2-2}^{\eta_2} \bigl(t_2-\sigma(r)\bigr)^{\underline{-\gamma_2-1}} \bigl(\eta_2+\theta_2-\sigma(s)\bigr)^{\underline{\theta_1-1}} g_2(s) \\
&\quad \times \mathcal{Q}^*[F_1,F_2](u_1,u_2) + \frac{1}{\Gamma(-\gamma_2)} \sum_{s=\alpha_2+\beta_2-1}^{t_2+\gamma_2} \bigl(t_2-\sigma(s)\bigr)^{\underline{-\gamma_2-1}} \mathcal{P}^*[F_1,F_2](u_1,u_2) \Biggr\}, \qquad (3.24)
\end{aligned}
$$

where $t_i \in \mathbb{N}_{\alpha_i+\beta_i-\gamma_i+1,T+\alpha_i+\beta_i-\gamma_i}$. Therefore,

$$
\begin{aligned}
&\bigl|\Delta^{\gamma_1}\bigl(\mathcal{T}_1(u_1,u_2)\bigr)(t_1,t_2) - \Delta^{\gamma_1}\bigl(\mathcal{T}_1(v_1,v_2)\bigr)(t_1,t_2)\bigr| \\
&\quad \leq \frac{1}{\Gamma(-\gamma_1)\Gamma(\beta_1)} \sum_{r=\alpha_1+\beta_1-1}^{t_1+\gamma_1} \sum_{s=\alpha_1-1}^{r-\beta_1} \bigl(t_1-\sigma(r)\bigr)^{\underline{-\gamma_1-1}} \bigl(r-\sigma(s)\bigr)^{\underline{\beta_1-1}} \\
&\quad\quad \times \bigl|\bigl(\mathcal{T}_1^*\bigl(\mathcal{T}_1^o(u_1,u_2)\bigr)\bigr)(t_1,t_2) - \bigl(\mathcal{T}_1^*\bigl(\mathcal{T}_1^o(v_1,v_2)\bigr)\bigr)(t_1,t_2)\bigr| \\
&\quad\quad + \frac{1}{\Lambda}\Biggl\{ \frac{\lambda_1 G_1}{\Gamma(\theta_1)\Gamma(-\gamma_1)} \sum_{x=\alpha_1+\beta_1-1}^{t_1+\gamma_1} \sum_{y=\alpha_1+\beta_1-2}^{\eta_1} \bigl(t_1-\sigma(x)\bigr)^{\underline{-\gamma_1-1}} \bigl(\eta_1+\theta_1-\sigma(y)\bigr)^{\underline{\theta_1-1}} \\
&\quad\quad \times \bigl|\mathcal{P}^*[F_1,F_2](u_1,u_2) - \mathcal{P}^*[F_1,F_2](v_1,v_2)\bigr| \\
&\quad\quad + \frac{1}{\Gamma(-\gamma_1)} \sum_{x=\alpha_1+\beta_1-1}^{t_1+\gamma_1} \bigl(t_1-\sigma(x)\bigr)^{\underline{-\gamma_1-1}} \bigl|\mathcal{Q}^*[F_1,F_2](u_1,u_2) - \mathcal{Q}^*[F_1,F_2](v_1,v_2)\bigr| \Biggr\} \\
&\quad \leq \frac{\chi_1^{q-2}\mathcal{K}_1 \|(u_1-v_1,u_2-v_2)\|_{\mathcal{C}_1}}{\Gamma(-\gamma_1)\Gamma(\beta_1)} \cdot \frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)} \sum_{s=\alpha_1-1}^{T+\alpha_1} \bigl(T+\alpha_1+\beta_1-\sigma(s)\bigr)^{\underline{\beta_1-1}} \bigl(s^{\underline{\alpha_1}}\bigr)^{\delta(q-2)+1} \\
&\quad\quad + \frac{1}{\Lambda}\Biggl\{ \lambda_1 G_1 \cdot \frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)} \cdot \frac{(\eta_1-\alpha_1-\beta_1+\theta_1+2)^{\underline{\theta_1}}}{\Gamma(\theta_1+1)} \\
&\quad\quad \times \Biggl[\frac{\chi_1^{q-2}\mathcal{K}_1}{\Gamma(\beta_1)} \bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{C}_1} \sum_{s=\alpha_1-1}^{T+\alpha_1} \bigl(T+\alpha_1+\beta_1-\sigma(s)\bigr)^{\underline{\beta_1-1}} \bigl(s^{\underline{\alpha_1}}\bigr)^{\delta(q-2)+1} \\
&\quad\quad + \frac{\lambda_2 G_2 \chi_2^{q-2}\mathcal{K}_2}{\Gamma(\beta_2)\Gamma(\theta_2)} \bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{C}_2} \sum_{r=\alpha_2+\beta_2-1}^{\eta_2} \sum_{s=\alpha_2-1}^{r-\beta_2} \bigl(\eta_2+\theta_2-\sigma(r)\bigr)^{\underline{\theta_2-1}} \\
&\quad\quad \times \bigl(r-\sigma(s)\bigr)^{\underline{\beta_2-1}} \bigl(s^{\underline{\alpha_2}}\bigr)^{\delta(q-2)+1} \Biggr] + \frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)} \\
&\quad\quad \times \Biggl[\frac{\chi_2^{q-2}\mathcal{K}_2}{\Gamma(\beta_2)} \bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{C}_2} \sum_{s=\alpha_2-1}^{T+\alpha_2} \bigl(T+\alpha_2+\beta_2-\sigma(s)\bigr)^{\underline{\beta_2-1}} \bigl(s^{\underline{\alpha_2}}\bigr)^{\delta(q-2)+1} \\
&\quad\quad + \frac{\lambda_1 G_1 \chi_1^{q-2}\mathcal{K}_1}{\Gamma(\beta_1)\Gamma(\theta_1)} \bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{C}_1} \sum_{r=\alpha_1+\beta_1-1}^{\eta_1} \sum_{s=\alpha_1-1}^{r-\beta_1} \bigl(\eta_1+\theta_1-\sigma(r)\bigr)^{\underline{\theta_1-1}} \\
&\quad\quad \times \bigl(r-\sigma(s)\bigr)^{\underline{\beta_1-1}} \bigl(s^{\underline{\alpha_1}}\bigr)^{\delta(q-2)+1} \Biggr]\Biggr\} \\
&\quad \leq \bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}} \Biggl\{ \mathcal{K}_1\chi_1^{q-2} \frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)} \Omega_1 + \mathcal{K}_2\chi_2^{q-2} \frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)} \Theta_2 \Biggr\}. \qquad (3.25)
\end{aligned}
$$

Similarly, we obtain

$$\begin{aligned}&\left|\Delta^{\gamma_2}\big(\mathcal{T}_2(u_1,u_2)\big)(t_1,t_2)-\Delta^{\gamma_2}\big(\mathcal{T}_2(v_1,v_2)\big)(t_1,t_2)\right|\\&\quad\le\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{U}}\left\{\mathcal{K}_1\chi_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Theta_1+\mathcal{K}_2\chi_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Omega_2\right\}.\end{aligned}\tag{3.26}$$

From (3.22) and (3.25), we find that

$$\begin{aligned}&\big\|\big(\mathcal{T}(u_1,u_2)\big)-\big(\mathcal{T}(v_1,v_2)\big)\big\|_{\mathcal{C}_1}\\&\quad<\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{U}}\\&\qquad\times\max\left\{\mathcal{K}_1\chi_1^{q-2}\Theta_1+\mathcal{K}_2\chi_2^{q-2}\Omega_2,\mathcal{K}_1\chi_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Omega_1+\mathcal{K}_2\chi_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Theta_2\right\}.\end{aligned}\tag{3.27}$$

In addition, by (3.21) and (3.26), we find that

$$\begin{aligned}&\big\|\big(\mathcal{T}(u_1,u_2)\big)-\big(\mathcal{T}(v_1,v_2)\big)\big\|_{\mathcal{C}_2}\\&\quad<\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{U}}\\&\qquad\times\max\left\{\mathcal{K}_1\chi_1^{q-2}\Omega_1+\mathcal{K}_2\chi_2^{q-2}\Theta_2,\mathcal{K}_1\chi_1^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\Theta_1+\mathcal{K}_2\chi_2^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\Omega_2\right\}.\end{aligned}\tag{3.28}$$

Hence, from (3.27) and (3.28), we can conclude that

$$\begin{aligned}&\big\|\big(\mathcal{T}(u_1,u_2)\big)-\big(\mathcal{T}(v_1,v_2)\big)\big\|_{\mathcal{U}}\\&\quad<\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{U}}\\&\qquad\times\max\bigg\{\mathcal{K}_1\chi_1^{q-2}\Theta_1+\mathcal{K}_2\chi_2^{q-2}\Omega_2,\mathcal{K}_1\chi_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Omega_1+\mathcal{K}_2\chi_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\Theta_2,\\&\qquad\mathcal{K}_1\chi_1^{q-2}\Omega_1+\mathcal{K}_2\chi_2^{q-2}\Theta_2,\mathcal{K}_1\chi_1^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\Theta_1+\mathcal{K}_2\chi_2^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\Omega_2\bigg\}\\&\quad=\big\|(u_1-v_1,u_2-v_2)\big\|_{\mathcal{U}}\Phi.\end{aligned}\tag{3.29}$$

By (3.13), $\mathcal{T}$ is a contraction mapping. Hence, by the Banach fixed point theorem, we get that $\mathcal{T}$ has a fixed point, which is a unique solution of problem (1.1)–(1.2). □

In the same manner as Theorem 3.1, we can obtain the following theorem.

Theorem 3.2 *Let* $p>2$, (H2)–(H3) *hold, and the following condition hold*:

(H4) *There exist constants* $\chi_i>0$ *and* $0<\delta<\frac{1}{2-q}$ *such that*

$$F_i[t_i,t_j,x,y,z]\le-\chi_i\Delta_C^{\alpha}\big(t_i^{\underline{\alpha_i}}\big)^{\delta}$$

for any $(t_i,t_j,x,y,z)\in\mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i}\times\mathbb{N}_{\alpha_j+\beta_j-2,T+\alpha_j+\beta_j}\times\mathbb{R}^3$.

Then problem (1.1)–(1.2) *has a unique solution.*

In the case $1 < p < 2$ and $q > 2$ since $\frac{1}{p} + \frac{1}{q} = 1$, we obtain the following theorem.

Theorem 3.3 *Let* $1 < p < 2$ *and* (H2)–(H3) *hold. For each* $i,j \in \{1,2\}$, $i \neq j$, $F_i \in C(\mathbb{N}_{\alpha_1+\beta_1-2,T+\alpha_1+\beta_1} \times \mathbb{N}_{\alpha_2+\beta_2-2,T+\alpha_2+\beta_2} \times \mathbb{R}^3, \mathbb{R})$, $\varphi_i \in C(\mathbb{N}_{\alpha_1+\beta_1-2,T+\alpha_1+\beta_1} \times \mathbb{N}_{\alpha_2+\beta_2-2,T+\alpha_2+\beta_2}, [0,\infty))$ *with* $\varphi_i^o = \max\{\varphi(t_i - 1, s)\}$. *Suppose that the following assumption holds*:

(H5) *There exists a nonnegative function* $k_i \in C(\mathbb{N}_{\alpha_1+\beta_1-2,T+\alpha_1+\beta_1} \times \mathbb{N}_{\alpha_2+\beta_2-2,T+\alpha_2+\beta_2}, [0,\infty))$ *and* $\mathcal{M}_i := \frac{1}{\Gamma(\alpha_i)} \sum_{\xi=\alpha_i+\beta_i-1}^{T+\alpha_i+\beta_i-1} (T + 2\alpha_i + \beta_i - 1 - \sigma(\xi))^{\underline{\alpha_i-1}} k_i(T + \alpha_j + \beta_j, \xi) > 0$ *such that*

$$F_i[t_i, t_j, x, y, z] \leq k_i(t_i, t_j)$$

for any $(t_i, t_j, x, y, z) \in \mathbb{N}_{\alpha_i+\beta_i-2,T+\alpha_i+\beta_i} \times \mathbb{N}_{\alpha_j+\beta_j-2,T+\alpha_j+\beta_j} \times \mathbb{R}^3$.

Then problem (1.1)–(1.2) *has a unique solution provided that*

$$\begin{aligned} \Upsilon := \max\Bigg\{ & \mathcal{K}_1 \mathcal{M}_1^{q-2} \bar{\Theta}_1 + \mathcal{K}_2 \mathcal{M}_2^{q-2} \bar{\Omega}_2, \mathcal{K}_1 \mathcal{M}_1^{q-2} \frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)} \bar{\Omega}_1 + \mathcal{K}_2 \mathcal{M}_2^{q-2} \frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)} \bar{\Theta}_2, \\ & \mathcal{K}_1 \mathcal{M}_1^{q-2} \bar{\Omega}_1 + \mathcal{K}_2 \mathcal{M}_2^{q-2} \bar{\Theta}_2, \mathcal{K}_1 \mathcal{M}_1^{q-2} \frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)} \bar{\Theta}_1 + \mathcal{K}_2 \mathcal{M}_2^{q-2} \frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)} \bar{\Omega}_2 \Bigg\} \\ < {} & 1, \end{aligned} \tag{3.30}$$

where $\mathcal{K}_i$ *is defined as* (3.14), *and*

$$\begin{aligned} \bar{\Omega}_i = {} & \left[1 + \frac{\lambda_i G_i (\eta_i - \alpha_i - \beta_i + \theta_i + 2)^{\underline{\theta_i}}}{|\Lambda| \Gamma(\theta_i + 1)}\right] \frac{1}{\Gamma(\beta_i)} \sum_{s=\alpha_i-1}^{T+\alpha_i} \big(T + \alpha_i + \beta_i - \sigma(s)\big)^{\underline{\beta_i-1}} s^{\underline{\alpha_i}} \\ & + \frac{\lambda_i G_i (\eta_i - \alpha_i - \beta_i + \theta_i + 1)^{\underline{\theta_i}}}{|\Lambda| \Gamma(\theta_i + 1)} \cdot \frac{1}{\Gamma(\beta_i)} \sum_{s=\alpha_i-1}^{\eta_i-\beta_i} \big(\eta_i - \sigma(s)\big)^{\underline{\beta_i-1}} s^{\underline{\alpha_i}}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} \bar{\Theta}_i = {} & \frac{1}{|\Lambda| \Gamma(\beta_i)} \sum_{s=\alpha_i-1}^{T+\alpha_i} \big(T + \alpha_i + \beta_i - \sigma(s)\big)^{\underline{\beta_i-1}} s^{\underline{\alpha_i}} \\ & + \frac{\lambda_1 \lambda_2 G_1 G_2}{|\Lambda|} \cdot \frac{(\eta_j - \alpha_j - \beta_j + \theta_j + 2)^{\underline{\theta_j}}}{\Gamma(\theta_j + 1)} \cdot \frac{(\eta_i - \alpha_i - \beta_i + \theta_i + 1)^{\underline{\theta_i}}}{\Gamma(\theta_i + 1)} \\ & \times \frac{1}{\Gamma(\beta_i)} \sum_{s=\alpha_i-1}^{\eta_i-\beta_i} \big(\eta_i - \sigma(s)\big)^{\underline{\beta_i-1}} s^{\underline{\alpha_i}}. \end{aligned} \tag{3.32}$$

Proof For each $i,j \in \{1,2\}$, $i \neq j$, by (H5) we have

$$\begin{aligned} & \left| \frac{1}{\Gamma(\alpha_i)} \sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1} \big(t_i + \alpha_i + \beta_i - 1 - \sigma(\xi)\big)^{\underline{\alpha_i-1}} F_i^*\big[u(t_j, \xi)\big] \right| \\ & \leq \frac{1}{\Gamma(\alpha_i)} \sum_{\xi=\alpha_i+\beta_i-1}^{T+\alpha_i+\beta_i-1} \big(T + 2\alpha_i + \beta_i - 1 - \sigma(\xi)\big)^{\underline{\alpha_i-1}} k_i(t_j, \xi) \\ & \leq \mathcal{M}_i. \end{aligned} \tag{3.33}$$

By (A2), (H2), and the definition of operator $\mathcal{T}_i^0$, for any $(u_1,u_2),(v_1,v_2)\in\mathcal{C}$, we have

$$\begin{aligned}
&\bigl|\bigl(\mathcal{T}_i^0(u_1,u_2)\bigr)(t_1,t_2)-\bigl(\mathcal{T}_i^0(v_1,v_2)\bigr)(t_1,t_2)\bigr|\\
&=\Biggl|\phi_q\Biggl[\sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1}\frac{(t_i+\alpha_i+\beta_i-1-\sigma(\xi))^{\underline{\alpha_i-1}}}{\Gamma(\alpha_i)}F_i^*\bigl[u(t_j,\xi)\bigr]\Biggr]\\
&\quad-\phi_q\Biggl[\sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1}\frac{(t_i+\alpha_i+\beta_i-1-\sigma(\xi))^{\underline{\alpha_i-1}}}{\Gamma(\alpha_i)}F_i^*\bigl[v(t_j,\xi)\bigr]\Biggr]\Biggr|\\
&\leq(q-1)\mathcal{M}_i^{q-2}\sum_{\xi=\alpha_i+\beta_i-1}^{t_i+\beta_i-1}\frac{(t_i+\alpha_i+\beta_i-1-\sigma(\xi))^{\underline{\alpha_i-1}}}{\Gamma(\alpha_i)}\bigl|F_i^*\bigl[u(t_j,\xi)\bigr]-F_i^*\bigl[v(t_j,\xi)\bigr]\bigr|\\
&\leq(q-1)\mathcal{M}_i^{q-2}\frac{t_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\bigl[L_i\bigl\|\Delta^{\gamma_i}u_i-\Delta^{\gamma_i}v_i\bigr\|+M_j\bigl\|\Psi^{\omega_i}u_i-\Psi^{\omega_j}v_i\bigr\|+N_j\|u_j-v_j\|\bigr]\\
&\leq(q-1)\mathcal{M}_i^{q-2}\frac{t_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\Biggl[L_i\bigl\|\Delta^{\gamma_i}u_i-\Delta^{\gamma_i}v_i\bigr\|+\Biggl(N_j+M_j\varphi_j^o\frac{(T+\omega_j+2)^{\underline{\omega_j}}}{\Gamma(\omega_j+1)}\Biggr)\|u_j-v_j\|\Biggr]\\
&\leq(q-1)\mathcal{M}_i^{q-2}\frac{t_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\Biggl[L_i+N_j+M_j\varphi_j^o\frac{(T+\omega_j+2)^{\underline{\omega_j}}}{\Gamma(\omega_j+1)}\Biggr]\\
&\quad\times\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{C}_i}.
\end{aligned}\tag{3.34}$$

Then, by (3.19) and (3.20), we have

$$\begin{aligned}
&\bigl|\bigl(\mathcal{T}_1(u_1,u_2)\bigr)(t_1,t_2)-\bigl(\mathcal{T}_1(v_1,v_2)\bigr)(t_1,t_2)\bigr|\\
&\quad\leq\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}\max\bigl\{\mathcal{K}_1\mathcal{M}_1^{q-2}\bar{\Omega}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\bar{\Theta}_2\bigr\},
\end{aligned}\tag{3.35}$$

$$\begin{aligned}
&\bigl|\bigl(\mathcal{T}_2(u_1,u_2)\bigr)(t_1,t_2)-\bigl(\mathcal{T}_2(v_1,v_2)\bigr)(t_1,t_2)\bigr|\\
&\quad\leq\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}\max\bigl\{\mathcal{K}_1\mathcal{M}_1^{q-2}\bar{\Theta}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\bar{\Omega}_2\bigr\}.
\end{aligned}\tag{3.36}$$

Similarly as in Theorem 3.1, we obtain

$$\begin{aligned}
&\bigl|\Delta^{\gamma_1}\bigl(\mathcal{T}_1(u_1,u_2)\bigr)(t_1,t_2)-\Delta^{\gamma_21}\bigl(\mathcal{T}_1(v_1,v_2)\bigr)(t_1,t_2)\bigr|\\
&\quad\leq\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}\Biggl\{\mathcal{K}_1\mathcal{M}_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Omega}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Theta}_2\Biggr\},
\end{aligned}\tag{3.37}$$

$$\begin{aligned}
&\bigl|\Delta^{\gamma_2}\bigl(\mathcal{T}_2(u_1,u_2)\bigr)(t_1,t_2)-\Delta^{\gamma_2}\bigl(\mathcal{T}_2(v_1,v_2)\bigr)(t_1,t_2)\bigr|\\
&\quad\leq\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}\Biggl\{\mathcal{K}_1\mathcal{M}_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Theta}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Omega}_2\Biggr\}.
\end{aligned}\tag{3.38}$$

By (3.36) and (3.37), we have

$$\begin{aligned}
&\bigl\|\bigl(\mathcal{T}(u_1,u_2)\bigr)-\bigl(\mathcal{T}(v_1,v_2)\bigr)\bigr\|_{\mathcal{C}_1}\\
&\quad<\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}
\end{aligned}$$

$$\begin{aligned}&\times \max\Biggl\{\mathcal{K}_1\mathcal{M}_1^{q-2}\bar{\Theta}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\bar{\Omega}_2,\mathcal{K}_1\mathcal{M}_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Omega}_1\\&+\mathcal{K}_2\mathcal{M}_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Theta}_2\Biggr\}.\end{aligned}\tag{3.39}$$

By (3.35) and (3.38), we have

$$\begin{aligned}&\bigl\|\bigl(\mathcal{T}(u_1,u_2)\bigr)-\bigl(\mathcal{T}(v_1,v_2)\bigr)\bigr\|_{\mathcal{C}_2}\\&\quad<\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}\\&\qquad\times \max\Biggl\{\mathcal{K}_1\mathcal{M}_1^{q-2}\bar{\Omega}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\bar{\Theta}_2,\mathcal{K}_1\mathcal{M}_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Theta}_1\\&\qquad+\mathcal{K}_2\mathcal{M}_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Omega}_2\Biggr\}.\end{aligned}\tag{3.40}$$

Therefore, by (3.39) and (3.40), we can conclude that

$$\begin{aligned}&\bigl\|\bigl(\mathcal{T}(u_1,u_2)\bigr)-\bigl(\mathcal{T}(v_1,v_2)\bigr)\bigr\|_{\mathcal{U}}\\&\quad<\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}\\&\qquad\times \max\Biggl\{\mathcal{K}_1\mathcal{M}_1^{q-2}\bar{\Theta}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\bar{\Omega}_2,\\&\qquad\mathcal{K}_1\mathcal{M}_1^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Omega}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\frac{(T+2)^{\underline{-\gamma_1}}}{\Gamma(1-\gamma_1)}\bar{\Theta}_2,\\&\qquad\mathcal{K}_1\mathcal{M}_1^{q-2}\bar{\Omega}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\bar{\Theta}_2,\mathcal{K}_1\mathcal{M}_1^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\bar{\Theta}_1+\mathcal{K}_2\mathcal{M}_2^{q-2}\frac{(T+2)^{\underline{-\gamma_2}}}{\Gamma(1-\gamma_2)}\bar{\Omega}_2\Biggr\}\\&\quad=\bigl\|(u_1-v_1,u_2-v_2)\bigr\|_{\mathcal{U}}\Upsilon.\end{aligned}\tag{3.41}$$

By (3.30), we can conclude that $\mathcal{T}$ is a contraction mapping. Hence, by the Banach fixed point theorem, $\mathcal{T}$ has a fixed point, which is a unique solution of problem (1.1)–(1.2). □

4 Some examples

In this section, we consider some examples to illustrate our main result.

Example 4.1 Consider the following fractional sum boundary value problem:

$$\begin{aligned}&\Delta_C^{\frac{1}{2}}\bigl[\phi_{\frac{5}{2}}\bigl(\Delta_C^{\frac{2}{3}}u_1\bigr)\bigr](t)\\&\quad=F_1\biggl[t+\frac{1}{6},t+\frac{1}{8},\Delta^{\frac{1}{3}}u_1\biggl(t+\frac{5}{6}\biggr),\Psi^{\frac{1}{4}}u_2\biggl(t+\frac{3}{8}\biggr),u_2\biggl(t+\frac{1}{8}\biggr)\biggr],\\&\Delta_C^{\frac{3}{4}}\bigl[\phi_{\frac{5}{2}}\bigl(\Delta_C^{\frac{3}{8}}u_2\bigr)\bigr](t)\\&\quad=F_2\biggl[t+\frac{1}{6},t+\frac{1}{8},\Delta^{\frac{2}{3}}u_2\biggl(t+\frac{11}{24}\biggr),\Psi^{\frac{3}{4}}u_1\biggl(t+\frac{11}{12}\biggr),u_1\biggl(t+\frac{1}{6}\biggr)\biggr],\end{aligned}\tag{4.1}$$

subject to nonlocal fractional sum boundary conditions of the form

$$\begin{aligned} &\Delta_C^{\frac{2}{3}} u_1\left(-\frac{1}{2}\right) = 0, \qquad u_1\left(\frac{67}{6}\right) = 3\Delta^{-\frac{2}{5}} e^{2\sin(\frac{221}{40}\pi)} u_2\left(\frac{221}{40}\right), \\ &\Delta_C^{\frac{3}{4}} u_1\left(-\frac{1}{4}\right) = 0, \qquad u_1\left(\frac{89}{8}\right) = 2\Delta^{-\frac{1}{4}} e^{\cos(\frac{41}{12}\pi)} u_1\left(\frac{41}{12}\right), \end{aligned} \tag{4.2}$$

where $t \in \mathbb{N}_{0,10}$. Functions F_1, F_2 are determined by

$$\begin{aligned} &F_1\left[t_1, t_2, \Delta^{\frac{1}{3}} u_1\left(t_1 - \frac{2}{3}\right), \Psi^{\frac{1}{4}} u_2\left(t_2 + \frac{1}{4}\right), u_2(t_2)\right] \\ &\quad = 3t_1^2\left[1 + \frac{1}{50{,}000e^7}\sin^2\left(\Delta^{\frac{1}{3}} u_1\left(t_1 - \frac{2}{3}\right)\right)\right] \\ &\qquad + 2t_2^2\left[1 + \frac{1}{40{,}000e^8}\cos^2\left(\Psi^{\frac{1}{4}} u_2\left(t_2 + \frac{1}{4}\right)\right)\right] \\ &\qquad + t_2^2\left[1 + \frac{1}{60{,}000e^6}\sin^2\left(u_2(t_2)\right)\right], \\ &F_2\left[t_1, t_2, \Delta^{\frac{2}{3}} u_2\left(t_2 - \frac{1}{3}\right), \Psi^{\frac{3}{4}} u_1\left(t_1 + \frac{3}{4}\right), u_1(t_1)\right] \\ &\quad = 2t_2^2\left[1 + \frac{1}{60{,}000e^6}\sin^2\left(\Delta^{\frac{2}{3}} u_2\left(t_2 - \frac{1}{3}\right)\right)\right] \\ &\qquad + 2t_1^2\left[1 + \frac{1}{50{,}000e^7}\cos^2\left(\Psi^{\frac{3}{4}} u_1\left(t_1 + \frac{3}{4}\right)\right)\right] \\ &\qquad + 3t_1^2\left[1 + \frac{1}{40{,}000e^8}\sin^2\left(u_1(t_1)\right)\right], \end{aligned}$$

and

$$\Psi^{\frac{3}{4}} u_1\left(t_1 + \frac{3}{4}\right) = \frac{1}{\Gamma(\frac{3}{4})}\sum_{s=-\frac{19}{12}}^{t_1-\frac{3}{4}} \left(t_1 - \sigma(s)\right)^{\underline{\frac{3}{4}-1}} \frac{e^{-s}}{(t_1+10)^3} u_1\left(t_1 + \frac{3}{4}\right), \tag{4.3}$$

$$\Psi^{\frac{1}{4}} u_2\left(t_2 + \frac{1}{4}\right) = \frac{1}{\Gamma(\frac{1}{4})}\sum_{s=-\frac{9}{8}}^{t_1-\frac{1}{4}} \left(t_2 - \sigma(s)\right)^{\underline{\frac{1}{4}-1}} \frac{e^{-s}}{(t_2+20)^2} u_2\left(t_2 + \frac{1}{4}\right). \tag{4.4}$$

Here, $p = \frac{5}{2}$, $q = \frac{5}{3}$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{3}{4}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{3}{8}$, $\gamma_1 = \frac{1}{3}$, $\gamma_2 = \frac{2}{3}$, $\omega_1 = \frac{3}{4}$, $\omega_2 = \frac{1}{4}$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{2}{5}$, $\eta_1 = \frac{19}{6}$, $\eta_2 = \frac{41}{8}$, $\lambda_1 = 2$, $\lambda_2 = 3$, $T = 10$, $g_1(t_1) = e^{\cos t_1\pi}$, $g_2(t_2) = e^{2\sin t_2\pi}$, $\varphi_1(t_1, s) = \frac{e^{-s}}{(t_1+10)^3}$, $\varphi_2 = \frac{e^{-s}}{(t_2+20)^2}$, and $\varphi_1^o = \frac{216}{166{,}375e^{1/6}} \approx 0.0011$, $\varphi_2^o = \frac{64}{23{,}409e^{1/8}} \approx 0.0024$.

Let $t_1 \in \mathbb{N}_{-\frac{5}{6}, \frac{67}{6}}$ and $t_2 \in \mathbb{N}_{-\frac{7}{8}, \frac{89}{8}}$. Taking $\chi_1 = 3$, $\chi_2 = 2$ and $1 = \delta < \frac{1}{2-q} = 3$, we have

$$\begin{aligned} &\chi_1 \Delta_C^{\frac{1}{2}}\left(t_1^{1/2}\right) \le 3t_1^2 \le 3t_1^2 + 3t_2^2 \le F_1[t_1, t_2, x, y, z], \\ &\chi_2 \Delta_C^{\frac{3}{4}}\left(t_2^{3/4}\right) \le 2t_2^2 \le 2t_2^2 + 5t_1^2 \le F_2[t_1, t_2, x, y, z]. \end{aligned}$$

Thus, (H1) holds.

For $(u_1,u_2),(v_1,v_2)\in\mathcal{C}$, we have

$$\begin{aligned}
&\bigl|F_1\bigl[t_1,t_2,\Delta^{\frac{1}{3}}u_1,\Psi^{\frac{1}{4}}u_2,u_2\bigr]-F_1\bigl[t_1,t_2,\Delta^{\frac{1}{3}}v_1,\Psi^{\frac{1}{4}}v_2,v_2\bigr]\bigr|\\
&\quad\le\frac{3t_1^2}{50{,}000e^7}\bigl|\Delta^{\frac{1}{3}}u_1-\Delta^{\frac{1}{3}}v_1\bigr|+\frac{2t_2^2}{40{,}000e^8}\bigl|\Psi^{\frac{1}{4}}u_2-\Psi^{\frac{1}{4}}v_2\bigr|+\frac{t_2^2}{60{,}000e^6}|u_2-v_2|,\\
&\bigl|F_2\bigl[t_1,t_2,\Delta^{\frac{2}{3}}u_2,\Psi^{\frac{3}{4}}u_1,u_1\bigr]-F_2\bigl[t_1,t_2,\Delta^{\frac{2}{3}}v_2,\Psi^{\frac{3}{4}}v_1,v_1\bigr]\bigr|\\
&\quad\le\frac{2t_2^2}{60{,}000e^6}\bigl|\Delta^{\frac{2}{3}}u_2-\Delta^{\frac{2}{3}}v_2\bigr|+\frac{2t_1^2}{50{,}000e^7}\bigl|\Psi^{\frac{3}{4}}u_1-\Psi^{\frac{3}{4}}v_1\bigr|+\frac{3t_2^2}{40{,}000e^8}|u_1-v_1|.
\end{aligned}$$

Thus, (H2) holds with $L_1 = 6.211\times10^{-6}$, $L_2 = 9.307\times10^{-6}$, $M_1 = 4.141\times10^{-6}$, $M_2 = 1.889\times10^{-6}$, $N_1 = 2.856\times10^{-6}$, and $N_2 = 4.653\times10^{-6}$.

Since $\frac{1}{e}\le g_1(t_1)\le e$ and $\frac{1}{e^2}\le g_2(t_2)\le e^2$.

Thus, (H3) holds with $g_1=\frac{1}{e}$, $g_2=\frac{1}{e^2}$ and $G_1=e$, $G_2=e^2$.

Finally, we find that

$$\begin{aligned}
&\Lambda\ge0.029,\qquad \mathcal{K}_1=0.286,\qquad \mathcal{K}_2=0.574,\qquad \Omega_1=3783.803,\\
&\Omega_2=31{,}848.989,\qquad \Theta_1=39{,}305.323,\qquad\text{and}\qquad \Theta_2=55{,}288.515.
\end{aligned}$$

Therefore, we have

$$\Phi=\max\{0.446,0.410,0.130,0.030\}=0.446<1.$$

Hence, by Theorem 3.1, boundary value problem (4.1)–(4.2) has a unique solution.

Example 4.2 Consider the following fractional sum boundary value problem:

$$\begin{aligned}
&\Delta_C^{\frac{1}{2}}\bigl[\phi_{\frac{5}{2}}\bigl(\Delta_C^{\frac{2}{3}}u_1\bigr)\bigr](t)\\
&\quad=H_1\biggl[t+\frac{1}{6},t+\frac{1}{8},\Delta^{\frac{1}{3}}u_1\biggl(t+\frac{5}{6}\biggr),\Psi^{\frac{1}{4}}u_2\biggl(t+\frac{3}{8}\biggr),u_2\biggl(t+\frac{1}{8}\biggr)\biggr],\\
&\Delta_C^{\frac{3}{4}}\bigl[\phi_{\frac{5}{2}}\bigl(\Delta_C^{\frac{3}{8}}u_2\bigr)\bigr](t)\\
&\quad=H_2\biggl[t+\frac{1}{6},t+\frac{1}{8},\Delta^{\frac{2}{3}}u_2\biggl(t+\frac{11}{24}\biggr),\Psi^{\frac{3}{4}}u_1\biggl(t+\frac{11}{12}\biggr),u_1\biggl(t+\frac{1}{6}\biggr)\biggr],
\end{aligned}\tag{4.5}$$

where $t\in\mathbb{N}_{0,10}$, and the nonlocal fractional sum boundary conditions satisfy (4.2). Functions H_1, H_2 are determined by

$$\begin{aligned}
&H_1\biggl[t_1,t_2,\Delta^{\frac{1}{3}}u_1\biggl(t_1-\frac{2}{3}\biggr),\Psi^{\frac{1}{4}}u_2\biggl(t_2+\frac{1}{4}\biggr),u_2(t_2)\biggr]\\
&\quad=\frac{3t_1^2}{500{,}000e^7}\sin^2\biggl(\Delta^{\frac{1}{3}}u_1\biggl(t_1-\frac{2}{3}\biggr)\biggr)\\
&\qquad+\frac{2t_2^2}{400{,}000e^8}\biggl[\cos^2\biggl(\Psi^{\frac{1}{4}}u_2\biggl(t_2+\frac{1}{4}\biggr)\biggr)+\sin^2\bigl(u_2(t_2)\bigr)\biggr],\\
&H_2\biggl[t_1,t_2,\Delta^{\frac{2}{3}}u_2\biggl(t_2-\frac{1}{3}\biggr),\Psi^{\frac{3}{4}}u_1\biggl(t_1+\frac{3}{4}\biggr),u_1(t_1)\biggr]
\end{aligned}$$

$$= \frac{2t_2^2}{6{,}000{,}000e^6}\sin^2\left(\Delta^{\frac{2}{3}}u_2\left(t_2-\frac{1}{3}\right)\right) + \frac{t_1^2}{5{,}000{,}000e^7}\left[\cos^2\left(\Psi^{\frac{3}{4}}u_1\left(t_1+\frac{3}{4}\right)\right)+\sin^2\left(u_1(t_1)\right)\right],$$

where $\Psi^{\frac{3}{4}}u_1$, $\Psi^{\frac{1}{4}}u_2$ are defined as (4.3) and (4.4), respectively.

Let $t_1 \in \mathbb{N}_{-\frac{5}{6},\frac{67}{6}}$ and $t_2 \in \mathbb{N}_{-\frac{7}{8},\frac{89}{8}}$. Using $g_1(t_1,t_2) = \frac{3t_1^2}{500{,}000e^7} + \frac{2t_2^2}{400{,}000e^8}$ and $g_2(t_1,t_2) = \frac{2t_2^2}{6{,}000{,}000e^6} + \frac{t_1^2}{5{,}000{,}000e^7}$, we have

$$\mathcal{M}_1 = 0.000709 \quad\text{and}\quad \mathcal{M}_2 = 0.00272.$$

For $(u_1,u_2),(v_1,v_2) \in \mathcal{C}$, we have

$$\begin{aligned}
&\left|F_1\left[t_1,t_2,\Delta^{\frac{1}{3}}u_1,\Psi^{\frac{1}{4}}u_2,u_2\right]-F_1\left[t_1,t_2,\Delta^{\frac{1}{3}}v_1,\Psi^{\frac{1}{4}}v_2,v_2\right]\right| \\
&\quad\le \frac{3t_1^2}{500{,}000e^7}\left|\Delta^{\frac{1}{3}}u_1-\Delta^{\frac{1}{3}}v_1\right| + \frac{2t_2^2}{40000e^8}\left[\left|\Psi^{\frac{1}{4}}u_2-\Psi^{\frac{1}{4}}v_2\right|+|u_2-v_2|\right], \\
&\left|F_2\left[t_1,t_2,\Delta^{\frac{2}{3}}u_2,\Psi^{\frac{3}{4}}u_1,u_1\right]-F_2\left[t_1,t_2,\Delta^{\frac{2}{3}}v_2,\Psi^{\frac{3}{4}}v_1,v_1\right]\right| \\
&\quad\le \frac{2t_2^2}{600{,}000e^6}\left|\Delta^{\frac{2}{3}}u_2-\Delta^{\frac{2}{3}}v_2\right| + \frac{2t_1^2}{500{,}000e^7}\left[\left|\Psi^{\frac{3}{4}}u_1-\Psi^{\frac{3}{4}}v_1\right|+|u_1-v_1|\right].
\end{aligned}$$

Thus, (H2) holds with $L_1 = 6.211 \times 10^{-7}$, $L_2 = 9.307 \times 10^{-7}$, $M_1 = N_1 = 2.070 \times 10^{-7}$, and $M_2 = N_2 = 9.447 \times 10^{-8}$.

From Example 4.1, we get $\Lambda \ge 0.029$, $g_1 = \frac{1}{e}$, $g_2 = \frac{1}{e^2}$ and $G_1 = e$, $G_2 = e^2$.

Finally, we find that

$$\mathcal{K}_1 = 5.385 \times 10^{-7}, \qquad \mathcal{K}_2 = 8.261 \times 10^{-7}, \qquad \bar{\Omega}_1 = 4993.134,$$
$$\bar{\Omega}_2 = 33{,}202.614, \qquad \bar{\Theta}_1 = 44{,}000.064 \quad\text{and}\quad \bar{\Theta}_2 = 77{,}432.180.$$

Hence,

$$\Upsilon = \max\{0.285, 0.019, 0.489, 0.155\} = 0.489 < 1.$$

From Theorem 3.3, we can conclude that boundary value problem (4.5) and (4.2) has a unique solution.

5 Conclusions

We have proved existence and uniqueness results of the nonlocal fractional sum boundary value problem for a coupled system of fractional sum-difference equations with p-Laplacian operator (1.1)–(1.2) by using the Banach fixed point theorem. Our problem contains both Riemann–Liouville and Caputo fractional difference with five fractional differences and four fractional sums.

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