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# Adaptive control realization for canonic Caputo fractional-order systems with actuator nonlinearity: application to mechatronic devices

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## Abstract

Nonlinearities, such as dead-zone, backlash, hysteresis, and saturation, are common in the mechanical and mechatronic systems' components and actuators. Hence, an effective control strategy should take into account such nonlinearities which, if unaccounted for, may cause serious response problems and might even result in system failure. Input saturation is one of the most common nonlinearities in practical control systems. So, this article introduces a novel adaptive variable structure control strategy for nonlinear Caputo fractional-order systems despite the saturating inputs. Owing to the complex nature of the fractional-order systems and lack of proper identification strategies for such systems, this research focuses on the canonic systems with complete unknown dynamics and even those with model uncertainties and external noise. Using mathematical stability theory and adaptive control strategy, a simple stable integral sliding mode control is proposed. The controller will be shown to be effective against actuator saturation as well as unknown characteristics and system uncertainties. Finally, two case studies, including a mechatronic device, are considered to illustrate the effectiveness and practicality of the proposed controller in the applications.

**Keywords:** Variable structure control; Unknown dynamical system; Fractional differential equation; Adaptive approach; Mechanical system

## 1 Introduction

Leibniz and L'Hopital introduced the notion of fractional-order differential equations [1]. Despite its long history, fractional calculus did not see engineering applications until many centuries later. In the last three decades, however, fractional calculus has found a number of applications in science and engineering [1]. Nowadays, fractional-order derivatives and integrals are extensively adopted for precise description and modeling of a wide range of physical phenomena which are observed in practical systems and applications. As an example, application of fractional calculus in accurately describing the behavior of oscillators [2], medicine [3], mechanical devices [4], electrical systems [5], granular soils [6], circuits [7], and financial systems [8] has been reported in the recent literature. As a result,

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the dynamical motions of real world physical systems have been more fittingly investigated by fractional-order differential and/or difference equations rather than the integer-order ones. Given the fact that high fidelity model of physical systems can be described by fractional-order systems, the area has received a great deal of interest in the control community which has focused its attention to stability and control problems in systems represented by non-integer-order differential equations [9, 10].

Several control design strategies have been proposed for control and stabilization of nonlinear systems of fractional-order systems. Owing to the intrinsic desirable characteristics of sliding mode control (SMC), such as robustness against parameter variations and external noise, easy realization, quick response to the external input, and suitable transient action, the most suggested control methodology for stabilizing nonlinear fractional-order systems is of variable structure type. For instance, in the works [9] and [10], the author designed several finite-time SMC approaches for the canonical form of nonlinear fractional systems while considering the influences of external disturbances and system uncertainties. In [11], a cable-driven manipulator with system uncertainties was stabilized via a composite adaptive fractional-order sliding mode controller. Adaptive controllers are the other class of controllers applied for fractional systems. A fractional-order PID control scheme with an exact stability condition was proposed in [12] for a bilateral teleoperation. The work in [13] presented a fractional-order adaptive control approach for robotic manipulators subject to system uncertainties and external noise. Liu et al. [14] derived an intelligent adaptive dynamic surface feedback control method for nonlinear fractional-order systems.

However, all of the aforementioned studies have assumed that the dynamics of nonlinear fractional systems is fully (or partially) known in advance. Nevertheless, most physical systems inherently possess nonlinear dynamics with no exact and/or straightforward information about their structure. Undertaking such fractional systems with limited knowledge about their dynamics has rarely been studied in the literature. Moreover, in real world applications, applied fractional systems unavoidably involve additive model uncertainties and external noise. The effects of plant uncertainties and external disturbances may destabilize the system and degrade the prescribed performance of the system. Therefore, a control engineer should pay considerable attention to such issues in order to successfully design a robust stable system with desired performance.

Components used in an engineered system, including those used in actuators in a control system, have physical limits in their operation. Actuator components may include electronic circuitry with operational amplifiers, etc.; electromechanical apparatus; mechatronic machines, pneumatic devices, and hydraulic machines; and others, most of which are limited in their range of operation and involve nonlinearities. Many scholars have indicated that actuator saturation becomes a serious nonlinearity in control devices. The appearance of actuator saturation in industrial processes will introduce a lower efficiency, bias harmful fluctuations on the output signals along with unwanted responses, or could even result in system failure as well as unstable processes [15–18]. Hence, a special attention should be paid to the effects of such nonlinearities in the procedure of synthesis, design, and application of fractional-order processes.

The problem of control of uncertain fractional-order systems with actuator saturation was investigated in [19], and an adaptive fuzzy approximation approach was presented to deal with the uncertain parts of the system dynamics. The research [20] investigated

the existence of actuator saturation in linear fractional processes, and some applicable hypotheses were obtained using the conventional Lyapunov theorems. In [21], an adaptive fractional-order control algorithm was developed for control of robotic manipulators with actuator saturation. However, the convergence to zero was not guaranteed in mathematical synthesis of that research, and the outputs of the system might involve steady state errors. The work [22] adopted a useful mathematical lemma along with a special function to compensate the influences of situation nonlinearities in fractional-order processes using some state feedback robust control algorithms. The paper [23] applied the same lemma in [22] to asymptotically stabilize linear non-integer-order processes in spite of sector actuation nonlinearities through a linear matrix inequality approach. In [24], a disturbance estimator control algorithm was proposed to synchronize to chaotic fractional devices in the presence of actuator saturation and measurement noises. Nevertheless, that work supposed a fully known structure and dynamics for the fractional processes, which makes the control design to be extensive and far from practical standpoint. Soorki and Tavazoei [25] developed an asymptotic state feedback control strategy for convergence analysis of linear non-integer-order swarm groups with actuator saturation. The research [26] provided some results on the stability region and the disturbance rejection properties of the linear fractional processes working with saturating actuators. Wang et al. [27] proposed a backstepping-based neural network control algorithm for stabilization of a class of fractional-order plants in the presence of dead-zone input nonlinearity. However, most of the above-mentioned research works have been developed either for linear fractional systems where the outputs of the controlled systems involve steady state errors or for the systems with known structures and dynamics.

According to the above discussion, few studies in the literature deal with stabilization of fractional-order systems with input saturation. However, here we propose a switching type adaptive SMC method for a widely used class of nonlinear fractional systems i.e. the canonical (normal) systems. Many of the existing real world physical systems are represented by the said canonical form, and many others can be easily transformed to this type of systems using a proper change of coordinates and mapping. After formulating the problem, a suitable fractional integral type sliding surface with desired dynamics is designed. We assume that the bounds and the rate of the input saturation function are fully unknown. For tackling such a situation, an adaptive rule is suggested and implemented. To consider the effect of uncertain terms and external noise, an unknown time-varying bounded uncertain term is included in the system model. It is also assumed that there is no information about the bounds of this term. On the basis of the parameter separation principle [28], we parameterize the nonlinear dynamics of the system to some bounded unknown terms. In this case, several other appropriate adaptive rules are proposed to undertake the parameterized unknown dynamics of the fractional system. Afterward, using the derived update laws and the fractional Lyapunov stability theory, a switching control signal is proposed to guarantee the occurrence of the sliding mode. At last, for evaluating the effectiveness of the proposed adaptive variable structure controller, two illustrative examples, including the control of a chaotic system and a mechatronic device, are involved in the paper.

It is worth mentioning the main contributions and motivations of this research, which are as follows: (i) Owing to the more precise ability of the fractional-order systems along with more interesting stability properties rather than the conventional integer-order sys-

tems, this research aims to investigate the control problem of canonical fractional-order nonlinear systems whose dynamics is applicable for most chaotic and mechatronic systems; (ii) Since input saturation nonlinearity does exist in practical realizations of the controllers, this work is inspired to propose an adaptive scheme to undertake the saturating control signals; (iii) We assume that there is no information available for the nonlinear dynamics of the system and, therefore, the introduced control technology would be able to stabilize the system without requiring exact modeling data which would lessen the burden of complexity of modeling procedure; (iv) The robustness of the closed-loop controlled system against lumped uncertainties and external disturbances is guaranteed using a simple adaptive mechanism; and (v) We suppose that the uncertain parts are fully unknown and there is no prior knowledge about the bounds in which this feature facilitates development of a simple controller with less measurement requirements.

To the best of our knowledge, the problem of input saturation with unknown bounds for nonlinear fractional-order plants with fully unknown dynamics, model uncertainties, and the external noise has not been considered before, and it is addressed for the first time in this research. The remainder of this paper is structured as follows. In Sect. 2, some preliminaries on fractional calculus as well as fractional stability theory are given. Section 3 deals with problem formulation. In Sect. 4, the proposed adaptive variable structure controller design is presented. Numerical computer simulations are presented in Sect. 5. Finally, concluding remarks are made in Sect. 6.

## 2 Fractional calculus preliminaries

**Definition 1** ([1]) The Caputo fractional derivative of a function  $f(t)$  is defined as follows:

$${}_C D_{t_0}^\alpha f(t) = I_{t_0}^{m-\alpha} \frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \tag{1}$$

where  $m - 1 < \alpha < m \in \mathbb{N}$ .

**Property 1** ([1]) The following equality holds for  $m = 1$ :

$${}_C D_{t_0}^\alpha I_{t_0}^\alpha f(t) = f(t). \tag{2}$$

**Property 2** ([1]) For the Caputo definition, the following equality is satisfied for  $m = 1$ :

$$I_{t_0}^\alpha {}_C D_{t_0}^\alpha f(t) = f(t) - f(t_0). \tag{3}$$

**Note 1** In this paper, we use the Caputo definition of the fractional derivatives and for convenience denote it by  $D^\alpha$ .

**Lemma 1** ([29]) Assume  $f(t) \in R$  to be a continuous and derivable function. Then, for any time instant  $t \geq t_0$ , the following inequality holds:

$$\frac{1}{2} {}_C D_{t_0}^\alpha f^2(t) \leq f(t) {}_C D_{t_0}^\alpha f(t). \tag{4}$$

**Lemma 2** ([28]) For any real-valued continuous function  $f(x, y)$ , there are smooth scalar-valued functions  $a(x) \geq 1$  and  $b(y) \geq 1$  such that the following inequality holds for all values

of  $x$  and  $y$ :

$$|f(x, y)| \leq a(x)b(y). \tag{5}$$

*Remark 1* It is easy to check that inequality (5) can be rewritten as follows:

$$|f(x, y)| \leq A(x)B(y) + c, \tag{6}$$

where  $A(x) \geq 0$  and  $B(y) \geq 0$  are smooth scalar-valued functions and  $c > 0$  is a regulating constant.

### 3 System dynamics and problem formulation

It is well known that many real world systems, such as robot manipulators, mass-spring-damper systems, structure dynamics, most of mechatronic and mechanical systems, and many of chaotic models belong to a special class of nonlinear systems called canonical (or normal) systems. Furthermore, a wide range of other classes of nonlinear systems can be transformed into the canonic forms using some mappings [27]. Therefore, in this paper a class of uncertain  $n$ -dimensional fractional-order systems in the canonic form is considered and is described as follows:

$$\begin{cases} D^\alpha x_i = x_{i+1}, & i \leq 1 \leq n - 1, \\ D^\alpha x_n = f(X, \rho) + \Delta f(X, t) + sat(u(t)), & i = n, \end{cases} \tag{7}$$

where  $\alpha \in (0, 1)$  is the order of the system,  $X(t) = [x_1, x_2, \dots, x_n]^T \in R^n$  is the state vector,  $f(X, \rho) \in R$  is a nonlinear smooth function of  $X$ , and  $\rho$  in which  $\rho \in R^m$  is the unknown parameter vector,  $\Delta f(X, t) \in R$  represents an unknown uncertainty and external disturbance term,  $u(t) \in R$  is the control signal and  $sat(\cdot)$  represents the saturation function as follows:

$$sat(u(t)) = \begin{cases} u_H & \text{if } u(t) \geq u^h, \\ \theta u(t) & \text{if } u^l \leq u(t) \leq u^h, \\ u_L & \text{if } u(t) \leq u^l, \end{cases} \tag{8}$$

where  $u_H, u^h \in R^+$  and  $u_L, u^l \in R^-$  are the bounds of the saturation function and  $\theta \in R$  is the saturation slope.

**Assumption 1** All parameters of the saturation function are bounded yet unknown.

Now, one can rewrite the saturation function (8) in the following form:

$$sat(u(t)) = u(t) + \Delta u(u(t)), \tag{9}$$

where  $\Delta u(u(t))$  is given as follows:

$$\Delta(u(t)) = \begin{cases} u_H - u(t) & \text{if } u(t) \geq u^h, \\ (\theta - 1)u(t) & \text{if } u^l \leq u(t) \leq u^h, \\ u_L - u(t) & \text{if } u(t) \leq u^l. \end{cases} \tag{10}$$

**Assumption 2** Based on Assumption 1 and the boundedness property of practical control signals, one can conclude that the term  $\Delta u(u(t))$  should be always bounded. Therefore, the following inequity should be held in practice:

$$|\Delta u(u(t))| \leq M < \infty, \tag{11}$$

where  $M$  is a positive constant.

**Assumption 3** Since the uncertain terms and external noises are always bounded in practical situations, the term  $\Delta f(X, t)$  is assumed to be bounded by

$$|\Delta f(X, t)| \leq \gamma < \infty, \tag{12}$$

where  $\gamma$  is a positive constant.

**Assumption 4** It is assumed that both the constants  $M$  and  $\gamma$  are unknown.

*Remark 2* The nonlinear smooth function  $f(X, \rho)$  satisfies the following inequality:

$$|f(X, \rho)| \leq \Theta \|X\| + c, \tag{13}$$

where  $\Theta$  is a positive constant.

*Proof* According to Lemma 2 and Remark 1, one has

$$|f(X, \rho)| \leq A(X)B(\rho) + c. \tag{14}$$

Since  $\rho$  is a constant vector, we have

$$B(\rho) \leq \|\rho\| < \mathcal{E} < \infty, \tag{15}$$

where  $\mathcal{E}$  is a positive unknown constant.

Now, based on Lemma 2 since  $A(X)$  is a smooth scalar-valued function, it should be Lipschitz in  $X$ . In other words, one has

$$|A(X) - A(Y)| \leq L \|X - Y\|, \tag{16}$$

where  $L > 0$  is the unknown Lipschitz constant.

For the case  $Y = 0$ , and since  $Y = 0$  is the equilibrium point of the system (i.e.  $A(0) = 0$  for  $\rho \neq 0$ ), one obtains

$$|A(X)| \leq L \|X\|. \tag{17}$$

So, according to inequalities (14), (15), and (17), letting set  $\mathcal{E}L < \Theta < \infty$ , we get

$$|f(X, \rho)| \leq \mathcal{E}L \|X\| + c < \Theta \|X\| + c. \tag{18}$$

This completes the proof. □

**Assumption 5** The nonlinear part of the system  $f(X, \rho)$  is fully unknown and, therefore, the parameter  $\Theta$  is also unknown.

*Remark 3* On the basis of inequalities (12) and (13), one can get the following inequality:

$$|f(X, \rho)| + |\Delta f(X, t)| \leq \Theta \|X\| + \Pi, \tag{19}$$

where  $c + \gamma < \Pi < \infty$  is an unknown constant.

*Remark 4* The control objective pursued in this paper is to propose a robust fractional-order variable structure SMC to stabilize the origin of system (7) in the presence of unknown nonlinear dynamics, unknown uncertainties and external noises, and unknown input saturation with satisfying Assumptions 1–5.

#### 4 Design of adaptive switching variable structure controller

In this section, first an integral type fractional sliding manifold is given and the stability of the resulting sliding motion is analyzed. Next, some adaptive rules as well as sliding mode control laws are proposed to make the occurrence of the sliding motion satisfying all limitations faced in the system and actuator.

As step one in the design procedure of the variable structure SMC, the following fractional-order integral type sliding manifold is adopted in this paper:

$$s(t) = x_n + D^{-\alpha} \sum_{i=1}^n c_i x_i, \tag{20}$$

where  $x_i, i = 1, 2, \dots, n$ , are the system states and  $c_i, i = 1, 2, \dots, n$ , are sliding mode parameters to be set later.

According to the sliding mode control theory, once the system reaches the sliding manifold (i.e. once the sliding motion takes place), one has the following equalities [30]:

$$s(t) = 0, \quad \dot{s}(t) = 0. \tag{21}$$

Consequently, noting to (1) for  $0 < \alpha \leq 1$ ,  $D^\alpha s(t)$  is related to  $\dot{s}(t)$  as  $D^\alpha s(t) = \frac{1}{\Gamma(1-\alpha)} \times \int_{t_0}^t \frac{\dot{s}(\tau)}{(t-\tau)^\alpha} d\tau$ . In other words,  $\dot{s}(t) = 0$  implies  $D^\alpha s(t) = 0$ . Therefore, using (21) one can conclude that in the sliding motion we should have

$$s(t) = 0, \quad D^\alpha s(t) = 0. \tag{22}$$

So, obtaining the fractional derivative of the sliding manifold (20) and noting Property 1, one gets

$$D^\alpha s(t) = D^\alpha x_n + \sum_{i=1}^n c_i x_i = 0 \Rightarrow D^\alpha x_n = - \sum_{i=1}^n c_i x_i. \tag{23}$$

Thus, the following dynamic equations represent the closed-loop dynamics of the system in the sliding motion:

$$\begin{cases} D^\alpha x_i = x_{i+1}, & i \leq 1 \leq n - 1, \\ D^\alpha x_n = -\sum_{i=1}^n c_i x_i, & i = n. \end{cases} \tag{24}$$

One can choose the parameters  $c_i$  in which the eigenvalues of linear system (24) satisfy the stability condition  $|\arg(\text{eig } A)| > \frac{\alpha\pi}{2}$  mentioned in [31].

In step two of the control design for the system, we propose suitable update rules to account for the unknown dynamics and parameters as well as uncertainties and external noises of the system. Towards that, the following fractional-order update rules are proposed:

$$\begin{aligned} D^\alpha \hat{\Theta} &= p|s|\|X\|, \\ D^\alpha \hat{\Pi} &= q|s|, \\ D^\alpha \hat{M} &= r|s|, \end{aligned} \tag{25}$$

in which  $\hat{\Theta}$ ,  $\hat{\Pi}$ , and  $\hat{M}$  are adaptive parameters to be estimations of the unknown parameters  $\Theta$ ,  $\Pi$ , and  $M$ , respectively, and  $p$ ,  $q$ , and  $r$  are positive constants called adaptation gains.

The last step in our controller design is to propose the final switching control signal to force the system states to reach the sliding manifold and remain there for the subsequent times. In this regard, the following simple switching control law is proposed:

$$u(t) = -\left[ \sum_{i=1}^n (c_i x_i) + (\hat{\Theta} \|X\| + \hat{\Pi} + \hat{M} + k) \text{sgn}(s) \right], \tag{26}$$

where  $k > 0$  is the switch gain and  $\text{sgn}(\cdot)$  is the standard sign function.

*Remark 5* It is noted that the constant coefficients  $p$ ,  $q$ , and  $r$  in update rules (25) tune the convergence rate of the adaptive parameters  $\hat{\Theta}$ ,  $\hat{\Pi}$ , and  $\hat{M}$ . Although one can choose any positive large values for these constants, according to the control signal (26), such high values may result in a big control effort which can restrict the physical realization of the controller. On the other hand, very small coefficient parameters would not be able to undertake the effects of uncertainties and input saturation. So, a trade-off between the rate of lumped uncertainty compensation and control energy can be performed by the designer according to the application requirements.

The following theorem uses the fractional Lyapunov stability theory to prove the existence of the sliding mode.

**Theorem 1** *Assume that the uncertain dynamical fractional-order canonical system (7) with saturating input is forced by the proposed adaptive variable structure controller composed of (20), (25), and (26). If Assumptions 1–5 are held, then the states of this system will attain the sliding surface  $s(t) = 0$ .*



*Proof* On the basis of the fractional stability theorem, we adopt a Lyapunov function candidate for the system as follows:

$$V(t) = \frac{1}{2} \left( s^2 + \frac{1}{p}(\hat{\Theta} - \Theta)^2 + \frac{1}{q}(\hat{\Pi} - \Pi)^2 + \frac{1}{r}(\hat{M} - M)^2 \right). \tag{27}$$

Taking fractional time derivative of the Lyapunov function and using Lemma 1, one obtains

$$\begin{aligned} D^\alpha V(t) &= \frac{1}{2} D^\alpha \left( s^2 + \frac{1}{p}(\hat{\Theta} - \Theta)^2 + \frac{1}{q}(\hat{\Pi} - \Pi)^2 + \frac{1}{r}(\hat{M} - M)^2 \right) \\ &\leq s D^\alpha s + \frac{1}{p}(\hat{\Theta} - \Theta) D^\alpha (\hat{\Theta} - \Theta) + \frac{1}{q}(\hat{\Pi} - \Pi) D^\alpha (\hat{\Pi} - \Pi) \\ &\quad + \frac{1}{r}(\hat{M} - M) D^\alpha (\hat{M} - M). \end{aligned} \tag{28}$$

Owing to the Caputo derivative property for the constants (i.e.  $D^\alpha k = 0$ ), we have

$$D^\alpha V(t) \leq s D^\alpha s + \frac{1}{p}(\hat{\Theta} - \Theta) D^\alpha \hat{\Theta} + \frac{1}{q}(\hat{\Pi} - \Pi) D^\alpha \hat{\Pi} + \frac{1}{r}(\hat{M} - M) D^\alpha \hat{M}. \tag{29}$$

Inserting  $D^\alpha s$  from (23) into the above inequality, one gets

$$D^\alpha V(t) \leq s \left( D^\alpha x_n + \sum_{i=1}^n c_i x_i \right) + \frac{1}{p}(\hat{\Theta} - \Theta) D^\alpha \hat{\Theta} + \frac{1}{q}(\hat{\Pi} - \Pi) D^\alpha \hat{\Pi} + \frac{1}{r}(\hat{M} - M) D^\alpha \hat{M}. \tag{30}$$

Using the system state equations (7) and Eq. (9), one has

$$\begin{aligned} D^\alpha V(t) &\leq s \left( f(X, \rho) + \Delta f(X, t) + u(t) + \Delta u(u(t)) + \sum_{i=1}^n c_i x_i \right) \\ &\quad + \frac{1}{p}(\hat{\Theta} - \Theta) D^\alpha \hat{\Theta} + \frac{1}{q}(\hat{\Pi} - \Pi) D^\alpha \hat{\Pi} + \frac{1}{r}(\hat{M} - M) D^\alpha \hat{M}. \end{aligned} \tag{31}$$

Based on the absolute value operator characteristic, inequality (31) can be rewritten as follows:

$$\begin{aligned} D^\alpha V(t) &\leq |s| (|f(X, \rho)| + |\Delta f(X, t)| + |\Delta u(u(t))|) + s \left( u(t) + \sum_{i=1}^n c_i x_i \right) \\ &\quad + \frac{1}{p}(\hat{\Theta} - \Theta) D^\alpha \hat{\Theta} + \frac{1}{q}(\hat{\Pi} - \Pi) D^\alpha \hat{\Pi} + \frac{1}{r}(\hat{M} - M) D^\alpha \hat{M}. \end{aligned} \tag{32}$$

The usage of Assumptions 1–5 and adaptation laws (25) results in the following inequality:

$$\begin{aligned} D^\alpha V(t) &\leq |s| (\Theta \|X\| + \Pi + M) + s \left( u(t) + \sum_{i=1}^n c_i x_i \right) + \frac{1}{p}(\hat{\Theta} - \Theta) D^\alpha \hat{\Theta} \\ &\quad + \frac{1}{q}(\hat{\Pi} - \Pi) D^\alpha \hat{\Pi} + \frac{1}{r}(\hat{M} - M) D^\alpha \hat{M} \end{aligned} \tag{33}$$

After some simplifications and introduction of the control input (26) to the above inequality, one has

$$D^\alpha V(t) \leq s(\hat{\Theta} \|X\| + \hat{\Gamma} + \hat{M} + k) \operatorname{sgn}(s) + \hat{\Theta} |s| \|X\| + \hat{\Gamma} |s| + \hat{M} |s|. \tag{34}$$

Noting the definition  $\operatorname{sgn}(s) = \frac{|s|}{s}$ , we obtain

$$D^\alpha V(t) \leq -|s|(\hat{\Theta} \|X\| + \hat{\Gamma} + \hat{M} + k) + \hat{\Theta} |s| \|X\| + \hat{\Gamma} |s| + \hat{M} |s|. \tag{35}$$

It is obvious that

$$D^\alpha V(t) \leq -k|s|. \tag{36}$$

So, referencing to [32], one can conclude that the states of fractional-order system (7) reach  $s(t) = 0$ . Thus, the proof is complete.  $\square$

*Remark 6* It is known that owing to the appearance of the discontinuous sign function in the control signals of the conventional sliding mode controllers, chattering phenomenon might appear on the outputs. One alternative to reduce the effects of chattering is known as boundary layer method. In this method, the sign function is approximated by a saturation function. However, in this article, the continuous smooth function tanh is utilized to substitute the sign function in the control input. This alternative eliminates the discontinuity of the control signal and, therefore, chattering phenomenon is evaded.

### 5 Numerical examples

This section presents two case studies to illustrate the usefulness and efficacy of the proposed fractional adaptive variable structure control in dealing with the unknown dynamical systems subject to saturating inputs. The numerical algorithm introduced in [33] with a step time of 0.001 is adopted for solving the fractional-order equations.

#### 5.1 Case study 1

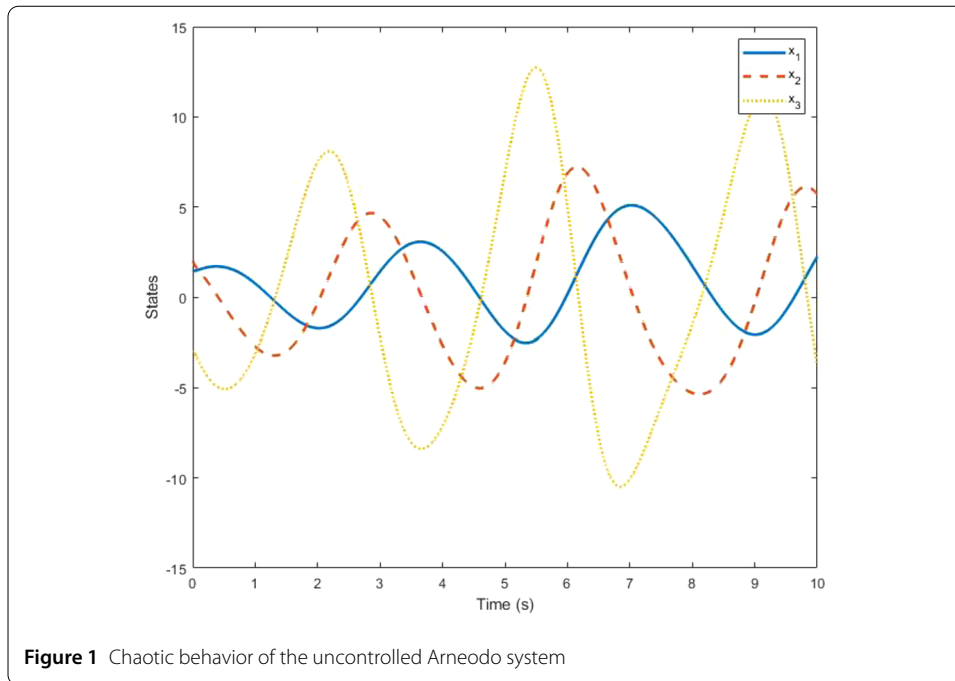
In this case study, the fractional-order Arneodo system (37) is stabilized using the proposed adaptive SMC strategy:

$$\begin{cases} D^\alpha x_1 = x_2, \\ D^\alpha x_2 = x_3, \\ D^\alpha x_3 = 5.5x_1 - 3.5x_2 - x_3 + x_1^3 + \Delta f(X, t) + \operatorname{sat}(u(t)), \end{cases} \tag{37}$$

It is shown that this system can exhibit chaotic behavior when the fractional order  $\alpha$  is set to 0.97 [34]. The chaotic behavior of this system without any control input is depicted in Fig. 1.

As mentioned before, we assume that the dynamics of the system is fully unknown and bounded by inequality (13). To show the robustness of the designed adaptive controller against uncertainties and external noises, the following term is added to the dynamics of the system:

$$\Delta f(X, t) = 0.45 \sin(t)x_2 + 0.5 \cos(5t)x_3 + 0.3 \tanh(2t)x_1 - 0.45 \cos(3t). \tag{38}$$



The vector  $X(0) = [1.5, 2, -3]^T$  is taken as the initial conditions of the system. And, according to Eq. (20), the following sliding manifold is established:

$$s(t) = x_3 + D^{-0.97}(10x_1 + 7x_2 + 3x_3). \tag{39}$$

The saturation control is defined as follows:

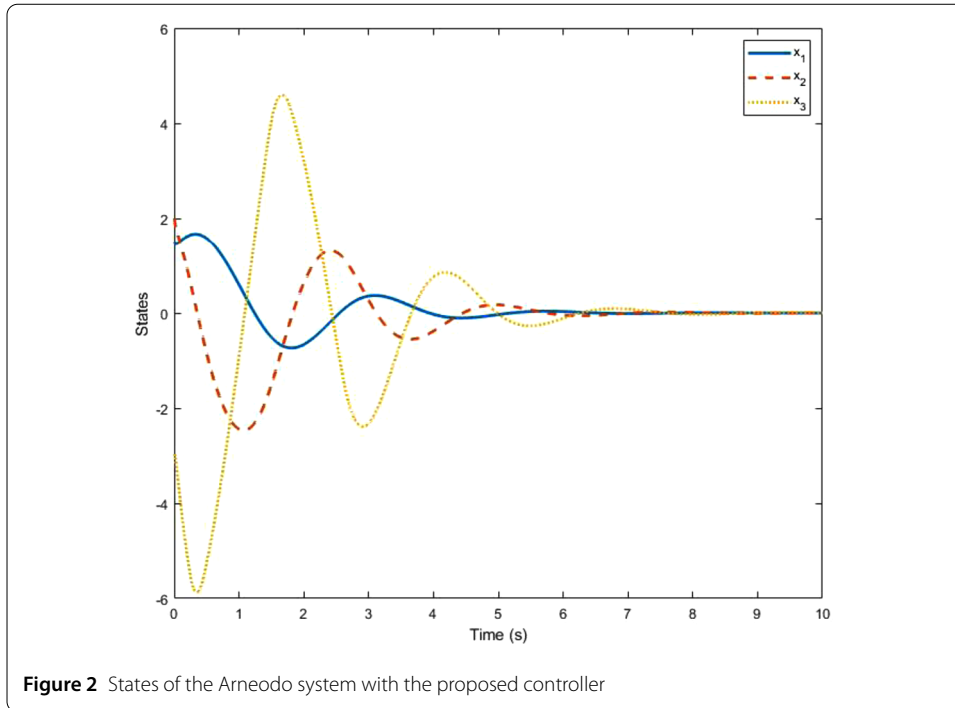
$$\text{sat}(u(t)) = \begin{cases} 5 & \text{if } u(t) \geq 5, \\ u(t) & \text{if } -5 \leq u(t) \leq 5, \\ -5 & \text{if } u(t) \leq -5. \end{cases} \tag{40}$$

Subsequently, with attention to the proposed adaptive controller in Eqs. (25) and (26), we implement the controller as follows:

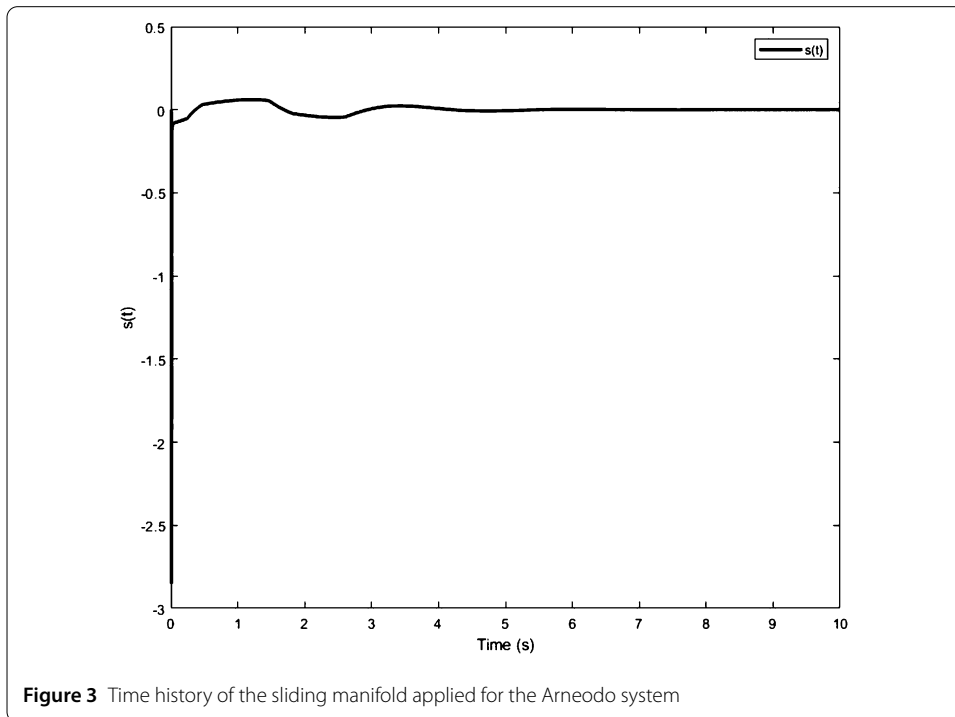
$$\begin{aligned} D^\alpha \hat{\Theta} &= |s| \|X\|, & \hat{\Theta}(0) &= 0, \\ D^\alpha \hat{\Pi} &= 0.1 |s|, & \hat{\Pi}(0) &= 0, \end{aligned} \tag{41}$$

$$\begin{aligned} D^\alpha \hat{M} &= 0.1 |s|, & \hat{M}(0) &= 0, \\ u(t) &= -(15x_1 + 12x_2 + 9x_3 + (\hat{\Theta} \|X\| + \hat{\Pi} + \hat{M} + 0.1) \tanh(50s)). \end{aligned} \tag{42}$$

Figure 2 depicts the state trajectories of the Arneodo system. One sees that the system states approach to the origin in less than eight seconds, which indicates that the fractional-order Arneodo system is indeed stabilized. Also, it is seen that the initial oscillations are soft and there are no steady state errors in the system response. The time histories of the sliding manifold and the saturated control input are given in Figs. 3 and 4, respectively. It can be seen that the saturation is undertaken using the adaptive method. Thus, the proposed control signal can be implemented using real physical actuators.



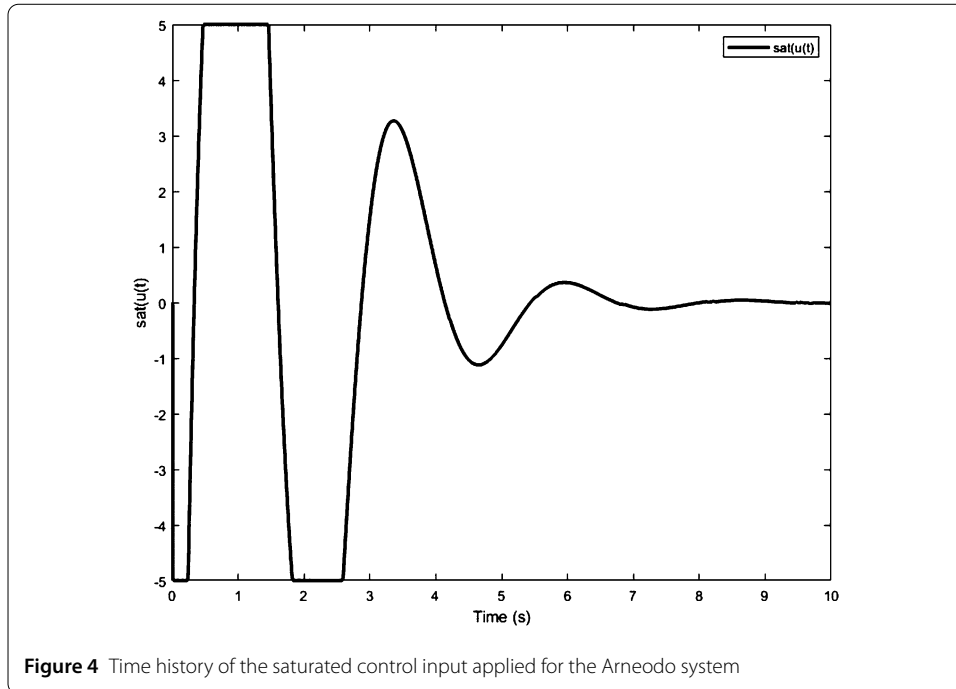
**Figure 2** States of the Arneodo system with the proposed controller



**Figure 3** Time history of the sliding manifold applied for the Arneodo system

### 5.2 Case study 2

This case study examines the efficient performance of the suggested simple adaptive variable structure controller in stabilization of a mechatronic system called fractional-order horizontal platform system (FHPS). In fact, a FHPS is a mechatronic device that can freely rotate around the horizontal axis. The horizontal platform devices are widely used in off-



shore and earthquake engineering. It has been shown that the FHPS possesses chaotic and oscillatory motions [35]. The dynamic behavior of FHPS is governed by the following equations and with  $\alpha = 0.1$  is illustrated in Fig. 5.

$$\begin{cases} D^\alpha x_1 = x_2, \\ D^\alpha x_2 = -ax_2 - b \sin(x_1) + l \sin(x_1) \cos(x_1) + F \cos(\omega t) + sat(u(t)), \end{cases} \tag{43}$$

where  $x_1$  is the rotation of the platform relative to the earth and  $F \cos(\omega t) = \cos 3.4 \cos 1.8$  is the harmonic torque and  $a = 4/3$ ,  $b = 3.776$ , and  $l = 4.6 \times 10^{-6}$  are system constant parameters.

The aim of this case study is to stabilize the FHPS with fully unknown dynamics, uncertain parameters, and external noises subjected to the input saturation nonlinearity. Therefore, the dynamics of the system is disturbed by the uncertain terms as follows:

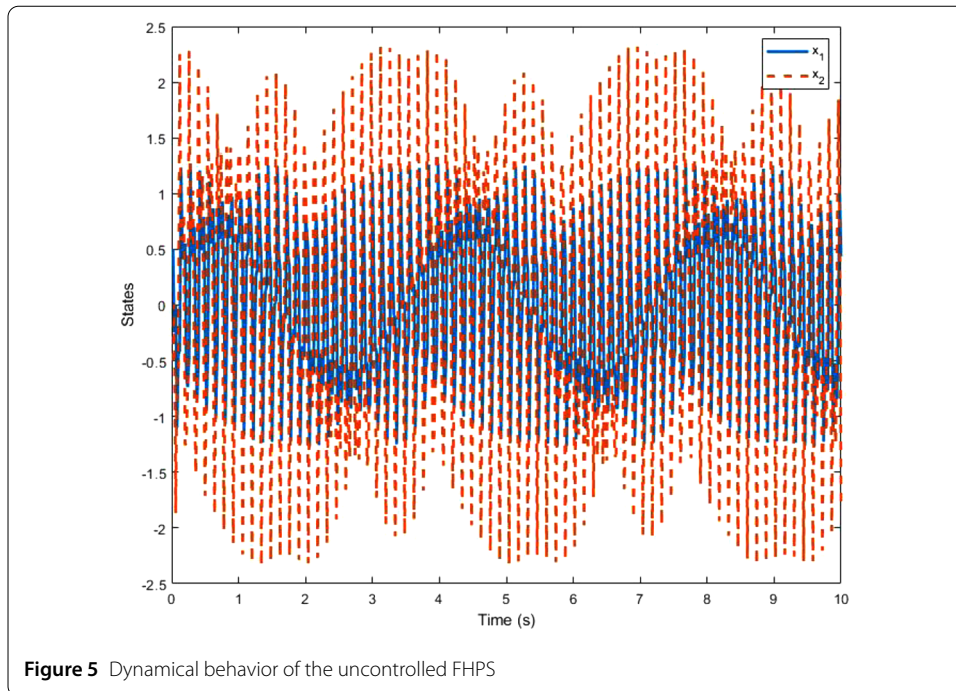
$$\Delta f(X, t) = 0.4 \sin(2t)x_2 + 0.4 \tanh(3t)x_1 + 0.5 \sin(t). \tag{44}$$

The vector  $X(0) = [0.5, -0.5]^T$  is taken as the initial conditions of the system. According to Eq. (20), the following sliding manifold is established:

$$s(t) = x_2 + D^{-0.1}(15x_1 + 10x_2). \tag{45}$$

The saturation control is also defined as follows:

$$sat(u(t)) = \begin{cases} 1 & \text{if } u(t) \geq 1, \\ 0.5u(t) & \text{if } -1 \leq u(t) \leq 1, \\ -1 & \text{if } u(t) \leq -1. \end{cases} \tag{46}$$



Subsequently, with attention to the proposed adaptive controller in Eqs. (25) and (26), we implement the controller as follows:

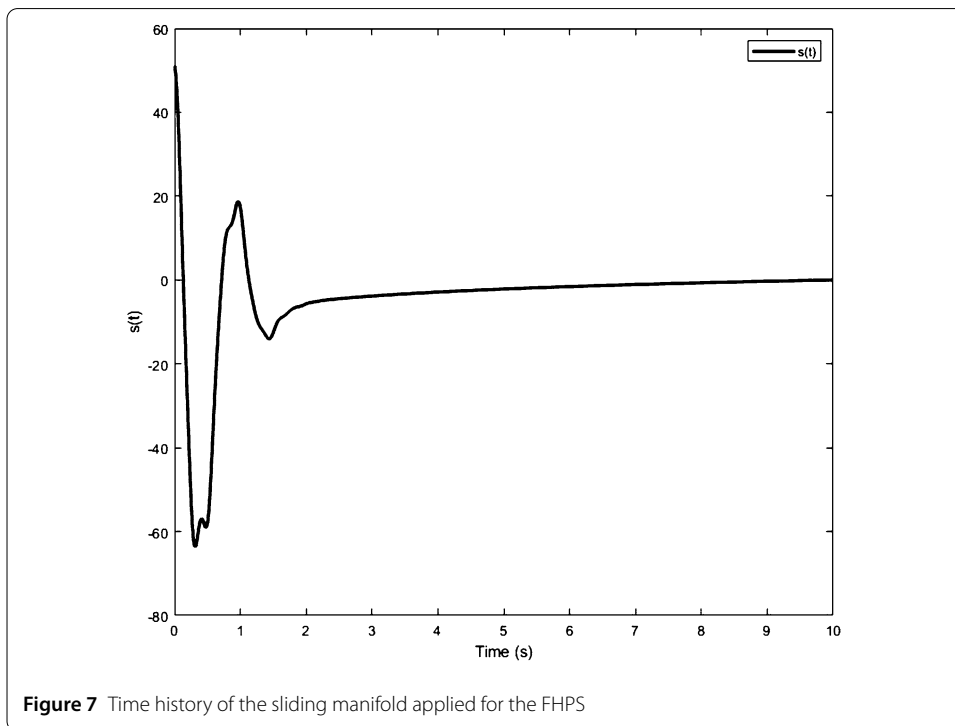
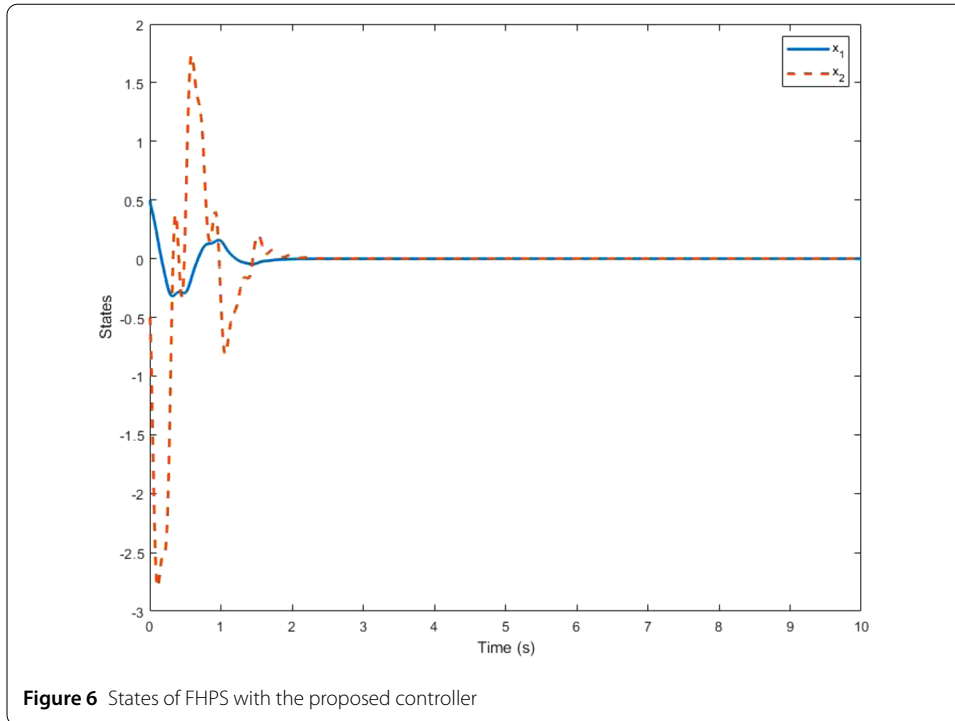
$$\begin{aligned}
 D^\alpha \hat{\Theta} &= |s| \|X\|, & \hat{\Theta}(0) &= 0, \\
 D^\alpha \hat{\Gamma} &= 0.1|s|, & \hat{\Gamma}(0) &= 0,
 \end{aligned} \tag{47}$$

$$\begin{aligned}
 D^\alpha \hat{M} &= 0.1|s|, & \hat{M}(0) &= 0, \\
 u(t) &= -(15x_1 + 10x_2 + (\hat{\Theta} \|X\| + \hat{\Gamma} + \hat{M} + 0.1) \tanh(50s)).
 \end{aligned} \tag{48}$$

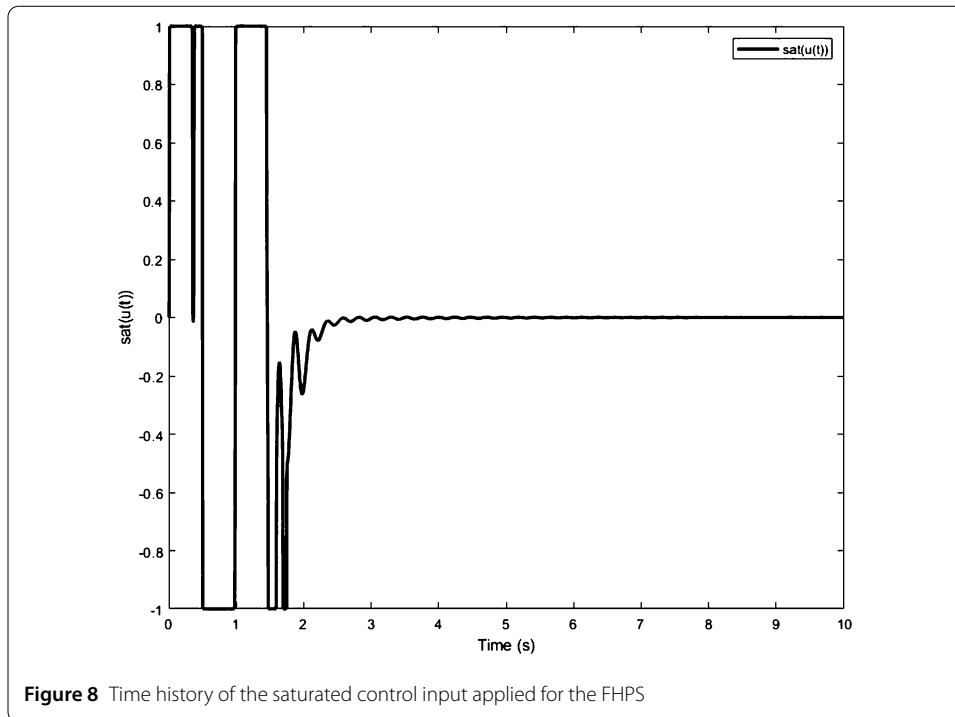
Figure 6 depicts the time responses of the system states controlled with the adaptive control algorithm. The results demonstrate that although the uncontrolled FHPS includes chaotic oscillations, the controller given in this paper can converge the system states to an equilibrium after passing some reasonable transient oscillations. This indicates that the developed adaptive sliding mode control strategy makes the closed-loop system cope with the fractional-order horizontal platform device’s unknown dynamics in spite of being some actuator saturation nonlinearities. The time responses of the active sliding surface and the final control signal applied to the chaotic system are shown in Figs. 7 and 8, respectively. The given curves in these figures point out that the sliding surface and the control signal converge to the origin indicating that the introduced controller possesses finite control effort and energy in a practical manner. Subsequently, it is concluded that the developed control signal can be implemented through the existing physical actuators for mechatronic devices.

### 6 Conclusion

Application and implementation of controllers designed for fractional nonlinear engineering systems has been an active area of research in recent years. A wide class of fractional



systems, in canonical forms, is considered in this paper. Owing to the existence of saturation phenomenon in practice, the influence of unknown saturating functions in the control signal is taken into account. On the other hand, due to the complex and uncertain dynamics of the recent applied fractional dynamical systems, it is assumed that the dynamics of the system is fully unknown and the system is subject to external noise and model un-



certainties. In this regard, a novel simple adaptive sliding mode controller is proposed to handle all the above-mentioned effects. The stable steady state behavior of the controlled system is ensured and is proved mathematically. Two case studies, which include the stabilization of a chaotic fractional Arneodo system and an oscillating dynamical horizontal platform mechatronic device, illustrate that not only there is no steady state error in the controlled output, but also satisfactory transient response is achieved. Also, boundedness of the saturated control inputs confirms that the findings of this research can be implemented in practice using physical devices and actuators.

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#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

MPA contributed to the design of the research. MS contributed to revising the paper. All authors read and approved the final manuscript.

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