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Nonparametric threshold estimation of spot volatility based on high-frequency data for time-dependent diffusion models with jumps

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Abstract

We construct a spot volatility kernel estimator of time-dependent diffusion models with jumps. Instead of idiomatic intraday return over an observation interval, in the proposed estimator, we use intraday range. Since the range represents the maximum difference among all observations within an interval, all data are used, and no information is lost. By setting a reasonable threshold and making the range not greater than it we effectively eliminate the negative effect of jump on volatility estimation. In this paper, we also prove the consistency and asymptotic normality of the estimator and testify its higher accuracy.

MSC: 60F05; 62G05; 62M99

Keywords: Spot volatility; Threshold; Range-based estimator; Time-dependent; High-frequency data

1 Introduction

In the analysis of financial markets, a correct description of the underlying variables is crucial. As a matter of fact, the dynamic rules of these underlying variables can often be evolved by the diffusion class models. Volatility is undoubtedly one of the most important indicators in diffusion model research. Its correct estimation and forecast play important roles in risk management, hedging, portfolio, and derivative pricing.

As we all know, the macro- and microeconomic environment is not static. Therefore it is necessary to assume that the spot volatility is not only related to a specific state variable, but also to time. In other words, the function of the underlying variables should satisfy a time-dependent diffusion process. Fan and Wang [1] used a kernel smoothing technique to consider spot volatility estimation for high-dimensional time-dependent diffusion models and proved the consistency and asymptotic normality of their proposed estimator. Zu and Boswijk [2] constructed a spot volatility estimator for high-frequency data contaminated by market noise and presented a data-driven method to select the scale and bandwidth parameters. For a continuous diffusion process

$$dX_t = \alpha_t\,dt + \beta_t\,dW_t,$$

Kristensen [3] constructed a weighted integral volatility estimator by kernel and proposed the following filtered spot volatility:

$$\hat{\beta}_t^2 = \sum_{i=1}^{n} K_h(t_{i-1} - t)(X_{t_i} - X_{t_{i-1}})^2, \tag{1}$$

where h is the bandwidth, K is the kernel function, and $K_h(\cdot) = K(\cdot/h)/h$. Under some weak conditions, he proved the following asymptotic normality of the estimator:

$$\sqrt{nh}\big(\hat{\beta}_t^2 - \beta_t^2\big) \xrightarrow{d} N\bigg(0, 2\beta_t^4 \int_R K^2(z)\,dz\bigg). \tag{2}$$

With the development of electronic trading technology and means, it becomes possible to conduct frequent trading in a shorter period of time, and financial activities can be carried out in scenario of higher frequency. It is unconvincing to use the continuous diffusion models to describe high-frequency and even ultrahigh-frequency data. Both theoretical and practical studies show that there are jumps in financial variables, which have important impacts in financial analysis (see Lee and Mykland [4] and Aït-Sahalia and Jacod [5]). In recent years, more and more scholars have begun to consider jump diffusion models to describe financial variables and studied their impacts on financial activities (see Zhu et al. [6] and Matenda and Chikodza [7]).

We consider the following jump diffusion model:

$$X_t = X_0 + \int_0^t \alpha_u\,du + \int_0^t \beta_u\,dW_u + \sum_{i=1}^{N_t} Y_i, \quad t \in (0, T], \tag{3}$$

where α_t, β_t are càdlàg, W_t is a standard Wiener process, $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process independent of W_t, N_t is a Poisson process with intensity λ, $\{Y_i, i = 1, 2, \ldots, N_t\}$ is the jump size at the ith jump point and independent of N_t.

To estimate the volatility better for jump diffusion models, eliminating the impacts of jumps is one of the good choices, among which the threshold technique is a common and effective method to remove jumps (Mancini [8], Mancini and Renò [9], Wang and Zhou [10], Song and Wang [11], and Sun and Yu [12]). Intuitively, it is plausible that we can remove the effects of jumps and estimate the spot volatility in equation (3) by modifying $\hat{\beta}_t^2$ in equation (1) as

$$\hat{\beta}_t^2 = \sum_{i=1}^{n} K_h(t_{i-1} - t)(X_{t_i} - X_{t_{i-1}})^2 \cdot I_{\{(X_{t_i} - X_{t_{i-1}})^2 \le \phi(\delta)\}}, \tag{4}$$

where $I_{\{\cdot\}}$ is the indicator function, $\phi(\delta)$ is a deterministic function of the time interval δ. As a matter of fact, it is inappropriate more or less to apply this return-based method directly to high-frequency data for the well-known negative effects of microscopic noise.

In view of the advantages of the range-based method, such as estimating accuracy, data integrity, and power of antinoise (see Christensen and Podolskij [13, 14], Liu et al. [15], Vortelinos [16], and Xu et al. [17]), in this paper, we use the range to replace the return $(X_{t_i} - X_{t_{i-1}})$ in equation (4). Meanwhile, since the range is defined as the maximum difference between two state variables within a given timing spacing, dominating its square no more than a specific threshold, we can propose a range-based threshold spot volatility estimator.

The rest part is fixed up as follows. In Sect. 2, we give some necessary technical conditions and construct the estimator of interest. Section 3 proves the consistency and asymptotic normality of the estimator. Conclusions are presented in Sect. 4.

2 Construction of estimator

For discussion, we decompose the process X_t as $X = X^{(C)} + X^{(J)}$, where $X^{(C)}$ and $X^{(J)}$ are the continuous and jump parts of the process, respectively. Some necessary constraints are given further.

T1 The process α_t is a second-order differentiable measurable process and satisfies

$$\left|\alpha_t^{(i)}\right| = O_P(1), \quad t \in [0,T], i = 0,1,2.$$

T2 The process β_t is differentiable and satisfies

$$\sup\left\{|\beta_s - \beta_t|, s,t \in [0,T], |s-t| \le \xi\right\} = O_P\left(\xi^{1/2}|\log \xi|^{1/2}\right)$$

and

$$\sup_{0\le t\le T} \beta_t^2 = O_P(1).$$

T3 Kernel function K is differentiable with support $[-1,1]$ and satisfies

$$\int_{-1}^{1} K(c)\,dc = 1$$

and

$$\int_{-1}^{1} K^2(c)\,dc, \int_{-1}^{1} K^3(c)\,dc, \int_{-1}^{1} K'(c)\,dc = O_P(1).$$

Without loss of generality, we assume that the samples are selected at the equal time spacing. Given n observations in the interval $[0,T]$, the time spacing between two adjacent observations is $\delta = T/n$. For sampling of nonequal time spacing, it suffices to define $\delta = \max_i(t_i - t_{i-1})$ $(i = 1,2,\dots,n)$.

We define the range of a process X_t in $[t_{i-1}, t_i]$ as

$$r_{X_{t_i,\delta}} = \sup_{t_{i-1}\le\varsigma,\tau\le t_i} \{X_\varsigma - X_\tau\}.$$

For a scaled Wiener process $X_t = \beta W_t$, Parkinson [18] proposed the pth moment generating function of its range in $[t_{i-1}, t_i]$ as

$$E\left[r^p_{X_{t_i,\delta}}\right] = \lambda_p \delta_i^{p/2} \beta^p \quad (p \ge 1), \tag{5}$$

where $\lambda_p = E[r^p_{W_{1,1}}]$.

Now we present the nonparametric spot volatility estimator

$$\hat{\beta}_t^2 = \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right) r_{X_{t_i,\delta}}^2 I_{\{r_{X_{t_i,\delta}}^2 \le \phi(\delta)\}}, \tag{6}$$

where $I_{\{\cdot\}}$ is the indicator function, and $\phi(\delta)$ is a deterministic function of the time spacing δ and satisfies

$$\lim_{\delta\to 0}\phi(\delta) = 0 \tag{7}$$

and

$$\lim_{\delta\to 0}\bigl(\delta \log\delta/\phi(\delta)\bigr) = 0. \tag{8}$$

Remark 1 In the estimator $\hat{\beta}_t^2$ of equation (6), we replace the term $(X_i - X_{t_{i-1}})^2$ in return-based estimators (see equation (1)) with range term $r_{X_{t_i,\delta}}^2/\mu_2$. As mentioned by Christensen and Podolskij [13], when the sampling frequency is not very high (say, 2- or 3-h returns), the return-based method is simple and efficient, but with the increase of frequency (1-s returns and even tick-by-tick returns), it will be seriously affected by microscopic noise. The main advantages of the range-based method are reflected in two aspects. On one hand, it requires no sparse sampling, uses the entire data without being affected by the noise, and thus ensures the integrity of information. On the other hand, it has a higher efficiency.

Remark 2 The deterministic function $\phi(\delta)$ is a threshold used to determine whether a jump occurs. Theoretically, any function that satisfies equations (7) and (8) can be selected as the threshold function. The power function δ^a (for any $a \in (0,1)$) is a possible option for the function $\phi(\delta)$, since it satisfies equations (7) and (8). In diffusion models with finite activity jumps, Yu et al. [19] chose $\phi(\delta) = \delta^{0.49}$, but in diffusion models with finite and infinite activity jumps, Mancini [8] chose $\phi(\delta) = \delta^{0.99}$. Mancini and Renò [9] even thought that a time-varying function could be selected.

Remark 3 By setting a reasonable threshold $\phi(\delta)$ and controlling the range in a proper interval no more than the threshold, as $\delta \to 0$, the estimator $\hat{\beta}_t^2$ can effectively eliminate the influence of jumps. Therefore it is an ideal estimator of spot volatility in the jump diffusion models, which are more in line with the realities of financial markets.

3 Consistency and asymptotic normality

Lemma 1 (Cai et al. [20]) *Suppose that* $\sum_{i=1}^{N_t} Y_i$ *is the jump process in equation* (3) *and* $\phi(\delta)$ *satisfies equations* (7) *and* (8). *Suppose that for all* $t \in (0,T]$,

$$P(Y_{N_t} = 0, \Delta N_t \neq 0) = 0.$$

Then

$$I_{\{r_{X_{t_i,\delta}}^2 \le \phi(\delta)\}} = I_{\{\Delta_i N = 0\}} \quad (\forall i = 1,2,\ldots,n). \tag{9}$$

Remark 4 Lemma 1 is an extension of Theorem 1 in [8]. In equation (9), the return is replaced by the range. For small δ, if the squared range $r^2_{X_{t_i,\delta}}$ in the interval $[t_{i-1}, t_i]$ is not larger than the threshold $\phi(\delta)$, then no jumps will appear in the interval. Since the range represents the maximum difference among all observations within an interval, more high-frequency data are used, and the ability to detect jumps is better.

Remark 5 The condition $P(Y_{N_t} = 0, \Delta N_t \neq 0) = 0$ was also used in [8]; it means that the probability is 0 when $\Delta N_t \neq 0$ and the jump size at the N_tth jump point equals 0.

Theorem 2 *Suppose that X_t satisfies equation (3) and* T1–T3 *hold. If $\delta \to 0$ so that*

$$\delta/h \to 0 \tag{10}$$

and $\phi(\delta)$ satisfies equations (7) and (8), then

$$\hat{\beta}^2_t \xrightarrow{P} \beta^2_t,$$

where the symbol "$\xrightarrow{P}$" denotes the convergence in probability.

Proof We first prove that $\hat{\beta}^2_t \xrightarrow{P} \frac{1}{h\mu_2}\sum_{i=1}^n K(\frac{t_i-t}{h})\beta^2_{t_i} r^2_{W_{t_i,\delta}}$. We can see from the definition of $\hat{\beta}^2_t$ and Lemma 1 that

$$\hat{\beta}^2_t = \frac{1}{h\mu_2}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right) r^2_{X_{t_i,\delta}} I_{\{\Delta_i N=0\}} = A - B,$$

where

$$\begin{aligned} A &= \frac{1}{h\mu_2}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right) r^2_{X^C_{t_i,\delta}} \\ &= \frac{1}{h\mu_2}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left(\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau \beta_s\,dW_s\right)\right)^2, \\ B &= \frac{1}{h\mu_2}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left(\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau \beta_s\,dW_s\right)\right)^2 I_{\{\Delta_i N\neq 0\}}. \end{aligned}$$

Using the triangle inequality, we obtain

$$\begin{aligned} &\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left(\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau \beta_s\,dW_s\right)\right)^2 \\ &\quad\le 2\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\int_\varsigma^\tau \alpha_s\,ds\right)^2 + 2\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\int_\varsigma^\tau \beta_s\,dW_s\right)^2 \\ &\quad= D + E. \end{aligned}$$

Obviously,

$$D = 2\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\int_\varsigma^\tau \alpha_s\,ds\right)^2 = O_P\left(\delta^2\right).$$

For the term E, by the Burkholder–Davis–Gundy (BDG) inequality we obtain that

$$E = O_P\left(\int_{t_{i-1}}^{t_i} \beta_s^2\,ds\right) = O_P(\delta).$$

So

$$B = O_P\left(\frac{N_T\delta}{h\mu_2}\right) = O_P\left(\frac{\delta}{h}\right) \to 0.$$

Next, we prove that

$$A \xrightarrow{P} \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2. \tag{11}$$

We easily get

$$\begin{aligned} A - \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2 &= \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\left(r_{X_{t_i,\delta}^C}^2 - \beta_{t_i}^2 r_{W_{t_i,\delta}}^2\right) \\ &= F + G, \end{aligned}$$

where

$$F = \frac{2}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i} r_{W_{t_i,\delta}}\left(r_{X_{t_i,\delta}^C} - \beta_{t_i} r_{W_{t_i,\delta}}\right),$$

$$G = \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\left(r_{X_{t_i,\delta}^C} - \beta_{t_i} r_{W_{t_i,\delta}}\right)^2.$$

For the term G, we have

$$\begin{aligned} G &\le \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau (\beta_s - \beta_{t_i})\,dW_s\right|\right)^2 \\ &\le \frac{2}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds\right|\right)^2 \\ &\quad + \frac{2}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau (\beta_s - \beta_{t_i})\,dW_s\right|\right)^2 \\ &= G_1 + G_2. \end{aligned}$$

By condition T1 we have

$$\max_i \sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds\right| = O_P(\delta).$$

Therefore

$$G_1 = \frac{2}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds\right|\right)^2 = O_P(\delta).$$

By the BDG inequality there exists a constant $C(>0)$ such that

$$\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau (\beta_s-\beta_{t_i})\,dW_s\right|\right)^2 \le C\int_{t_{i-1}}^{t_i}(\beta_s-\beta_{t_i})^2\,ds.$$

Combining with condition T2, we obtain

$$\begin{aligned} G_2 &\le \frac{2C}{h\mu_2}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\int_{t_{i-1}}^{t_i}(\beta_s-\beta_{t_i})^2\,ds \\ &= O_P\big(\delta|\log\delta|\big). \end{aligned}$$

Hence

$$G = o_P(1).$$

Using the decomposition similar to G, we get

$$F \le \frac{2}{h\mu_2}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\beta_{t_i} r_{W_{t_i,\delta}} \sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau (\beta_s-\beta_{t_i})\,dW_s\right|.$$

Using Hölder's inequality, we have

$$\begin{aligned} F \le{}& \frac{2}{\mu_2}\left(\frac{1}{h}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2\right)^{1/2} \\ &\cdot\left(\frac{1}{h}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau (\beta_s-\beta_{t_i})\,dW_s\right|\right)^2\right)^{1/2}. \end{aligned}$$

Using Hölder's inequality again, we have

$$\begin{aligned} E[F] \le{}& \frac{2}{\mu_2}\left(E\left[\frac{1}{h}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2\right]\right)^{1/2} \\ &\cdot\left(E\left[\frac{1}{h}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau (\beta_s-\beta_{t_i})\,dW_s\right|\right)^2\right]\right)^{1/2}. \end{aligned}$$

From equation (5) we have $E[r_{W_{t_i,\delta}}^2]=\mu_2\delta$, and therefore

$$\frac{2}{\mu_2}\left(E\left[\frac{1}{h}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2\right]\right)^{1/2} = O_P(1).$$

By the discussion of G we have obtained

$$\begin{aligned} &\left(E\left[\frac{1}{h}\sum_{i=1}^n K\left(\frac{t_i-t}{h}\right)\left(\sup_{t_{i-1}\le\varsigma,\tau\le t_i}\left|\int_\varsigma^\tau \alpha_s\,ds + \int_\varsigma^\tau (\beta_s-\beta_{t_i})\,dW_s\right|\right)^2\right]\right)^{1/2} \\ &\quad= O_P\big(\delta^{1/2}|\log\delta|^{1/2}\big). \end{aligned}$$

Hence

$$A - \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2 = o_P(1).$$

So equation (11) is proved. In the rest of the proof, it suffices to prove

$$\begin{aligned}
&\frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2 \xrightarrow{P} \beta_t^2, \\
&E\left[\frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2\right] \\
&\quad = \frac{\delta}{h}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 \\
&\quad = \frac{1}{h}\sum_{i=1}^{n}\beta_{t_i}^2\int_{t_{i-1}}^{t_i}\left(K\left(\frac{t_i - t}{h}\right) - K\left(\frac{u - t}{h}\right)\right) du \\
&\qquad + \frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K\left(\frac{u - t}{h}\right)\left(\beta_{t_i}^2 - \beta_u^2\right) du \\
&\qquad + \frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K\left(\frac{u - t}{h}\right)\beta_u^2\, du \\
&\quad = H_1 + H_2 + H_3.
\end{aligned} \tag{12}$$

For H_1, using Taylor's formula, we have

$$\begin{aligned}
H_1 &= \frac{1}{h}\sum_{i=1}^{n}\beta_{t_i}^2\int_{t_{i-1}}^{t_i}\left(K'\left(\frac{u - t}{h}\right)\cdot\frac{t_i - u}{h} + o\left(\frac{t_i - u}{h}\right)\right) du \\
&= O_P\left(\frac{\delta}{h^2}\int_0^T K'\left(\frac{u - t}{h}\right) du\right) \\
&= O_P\left(\frac{\delta}{h}\int_{-1}^{1} K'(s)\, ds\right).
\end{aligned}$$

By condition T3 and equation (10) we have $H_1 = O_P(\delta/h) = o_P(1)$. Using condition T2, we easily get

$$H_2 = \delta^{1/2}|\log\delta|^{1/2} = o_P(1).$$

For the term H_3, let

$$\frac{u - t}{h} = s.$$

Then

$$\begin{aligned} H_3 &= \int_{-1}^{1} K(s)\beta_{sh+t}^2\,ds \\ &= \int_{-1}^{1} K(s)\bigl(\beta_t^2 + O_P\bigl(h^{1/2}|\log h|^{1/2}\bigr)\bigr)\,ds \\ &= \beta_t^2 + O_P\bigl(h^{1/2}|\log h|^{1/2}\bigr). \end{aligned}$$

Combining with the discussions of the terms H_1, H_2, and H_3, we get

$$E\left[\frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2\right] \xrightarrow{P} \beta_t^2.$$

Let

$$\rho_i = \frac{1}{h\mu_2} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2\bigl(r_{W_{t_i,\delta}}^2 - E\bigl[r_{W_{t_i,\delta}}^2\bigr]\bigr).$$

Then

$$E\bigl[\rho_i^2\bigr] = \frac{\delta^2}{h^2} K^2\left(\frac{t_i - t}{h}\right)\beta_{t_i}^4 \cdot \mathrm{M}_2,$$

where $\mathrm{M}_2 = (\mu_4 - \mu_2^2)/\mu_2^2$. By using a decomposition similar to equation (12) we can obtain

$$\begin{aligned} \sum_{i=1}^{n} E\bigl[\rho_i^2\bigr] &= \frac{\delta \mathrm{M}_2 \beta_t^4}{h}\int_{-1}^{1} K^2(s)\,ds + O_P\left(\frac{\delta|\log h|^{1/2}}{h^{1/2}}\right) \\ &= o_P(1). \end{aligned}$$

Therefore

$$\frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2 \xrightarrow{P} \beta_t^2.$$

These complete the proof of Theorem 2. □

Theorem 3 *Suppose the process X_t satisfies the conditions of Theorem 2. If $\delta \to 0$ so that*

$$\frac{h^2|\log h|}{\delta} = o_P(1), \tag{13}$$

then

$$\sqrt{\frac{h}{\delta}}\bigl(\hat{\beta}_t^2 - \beta_t^2\bigr) \xrightarrow{d} N\left(0, \mathrm{M}_2\beta_t^4 \int_{-1}^{1} K^2(s)\,ds\right), \tag{14}$$

where $\mathrm{M}_2 = (\mu_4 - \mu_2^2)/\mu_2^2$, and the symbol "$\xrightarrow{d}$" denotes the convergence in distribution.

Proof Decompose $(\hat{\beta}_t^2 - \beta_t^2)$ as

$$\begin{aligned}
&\left(\hat{\beta}_t^2 - \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 r_{W_{t_i,\delta}}^2\right) \\
&\quad + \frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 \left(r_{W_{t_i,\delta}}^2 - E\left[r_{W_{t_i,\delta}}^2\right]\right) \\
&\quad + \left(\frac{1}{h\mu_2}\sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right)\beta_{t_i}^2 E\left[r_{W_{t_i,\delta}}^2\right] - \beta_t^2\right) \\
&\quad = L_1 + L_2 + L_3.
\end{aligned}$$

By the proof of Theorem 2 we know that

$$L_1 = L_3 = O_P\left(h^{1/2}|\log h|^{1/2}\right),$$

and therefore we obtain from equation (13) that

$$\sqrt{\frac{h}{\delta}}L_1 = \sqrt{\frac{h}{\delta}}L_3 = O_P\left(\frac{h|\log h|^{1/2}}{\delta^{1/2}}\right) = o_P(1).$$

Now we discuss the term L_2. Similarly, from the proof of Theorem 2 we get that

$$\sqrt{\frac{h}{\delta}}L_2 = \sqrt{\frac{h}{\delta}}\sum_{i=1}^{n}\rho_i$$

and

$$\frac{h}{\delta}\sum_{i=1}^{n} E\left[\rho_i^2\right] = \mathrm{M}_2\beta_t^4\int_{-1}^{1} K^2(s)\,ds + O_P\left(h^{1/2}|\log h|^{1/2}\right).$$

As long as $(h^{3/2}/\delta^{3/2})\sum_{i=1}^{n} E[\rho_i^3] \to 0$ we further can conclude that $\sqrt{h/\delta}\rho_i$ $(i = 1, 2, \ldots, n)$ satisfies Lyapunov's condition:

$$\begin{aligned}
&\frac{h^{3/2}}{\delta^{3/2}}\sum_{i=1}^{n} E\left[\rho_i^3\right] \\
&\quad = \frac{h^{3/2}}{\delta^{3/2}} \cdot \frac{1}{h^3\mu_2^3}\sum_{i=1}^{n} K^3\left(\frac{t_i - t}{h}\right)\beta_{t_i}^6 \cdot E\left[\left(r_{W_{t_i,\delta}}^2 - E\left[r_{W_{t_i,\delta}}^2\right]\right)^3\right] \\
&\quad = \frac{\left(\frac{\mu_6}{\mu_2^3} - \frac{3\mu_4}{\mu_2^2} + 2\right)\delta^3}{h^{3/2}\delta^{3/2}}\sum_{i=1}^{n} K^3\left(\frac{t_i - t}{h}\right)\beta_{t_i}^6 \\
&\quad = O_P\left(\frac{\delta^{1/2}}{h^{1/2}}\right) \to 0.
\end{aligned}$$

By Lyapunov's central limit theorem we obtain

$$\sqrt{\frac{h}{\delta}}\sum_{i=1}^{n}\rho_i \xrightarrow{d} N\left(0, \mathrm{M}_2\beta_t^4\int_{-1}^{1} K^2(s)\,ds\right).$$

Further,

$$\sqrt{\frac{h}{\delta}}\bigl(\hat{\beta}_t^2 - \beta_t^2\bigr) \xrightarrow{d} N\biggl(0, \mathrm{M}_2\beta_t^4 \int_{-1}^{1} K^2(s)\,ds\biggr). \qquad \square$$

Remark 6 In spot volatility estimation for continuous diffusion models, Fan and Wang [1] selected the bandwidth $h \sim \delta^{1/2}/\log(1/\delta)$. In this case, equation (13) is also satisfied. Choosing $h = O(\delta^{1/2}/\log(1/\delta))$, we can obtain the convergence rate close to the optimal rate $n^{-1/4}$, which was in keeping with the rate in Mykland and Zhang [21] and Foster and Nelson [22]. Kristensen [3] chose a variable bandwidth $h = O(\delta^{1/(2\gamma+1)})$ by setting $0 < \gamma \leq 1$ (for models driven by a Wiener process, it is $0 < \gamma < 1/2$) and obtained the optimal attainable convergence rate $O_P(\delta^{\gamma/(2\gamma+1)})$.

Remark 7 It is worth mentioning that the constant M_2 in equation (14) is approximately equal to 0.4; however, the amount in equation (2) is 2 (the same amount in Theorem 1 in Fan and Wang [1]).

4 Conclusions

Combining with the range-based method and the threshold technique, we propose a nonparametric spot volatility estimation procedure for time-dependent diffusion models with jumps. Using the range instead of the return of the state variables, we employ the total data and improve the estimating precision. Meanwhile, restricting the squared range to be not greater than a specific threshold, the estimator is robust to the jumps.

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