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Existence results of fractional delta–nabla difference equations via mixed boundary conditions

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Abstract

In this article, we purpose existence results for a fractional delta–nabla difference equations with mixed boundary conditions by using Banach contraction principle and Schauder's fixed point theorem. Our problem contains a nonlinear function involving fractional delta and nabla differences. Moreover, our problem contains different orders in four fractional delta differences, four fractional nabla differences, one fractional delta sum, and one fractional nabla sum. Finally, we present some illustrative examples.

MSC: 39A05; 39A12

Keywords: Fractional delta–nabla difference equation; Mixed boundary value problem; Existence; Uniqueness

1 Introduction

Simultaneously with the development of the theory and application of differential calculus, difference calculus has also received more intense attention. In this article, we study the evolution of fractional difference calculus. Recently, fractional difference calculus became an attractive field to researchers since it can be used in ecology, biology, and other applied sciences [1–4].

In general, difference calculus is divided into two types, namely delta and nabla difference calculus. The fractional delta and nabla difference calculus has been studied in many research works such as [5–25] and [26–37], respectively. However, there are a few papers studying delta–nabla calculus, such as the delta–nabla calculus of variations [38–40], systems of delta–nabla fractional difference inclusions [41], and the discrete delta–nabla fractional boundary value problems with p-Laplacian [42].

The results mentioned above are the motivation for this research. In this paper, we study the existence of solutions of a fractional delta–nabla difference equation with mixed fractional delta–nabla difference–sum boundary conditions given by

$$\Delta^{\alpha} u(t) = F\big[t+\alpha-1, u(t+\alpha-1), \Delta^{\theta} u(t+\alpha-\theta+1), \nabla^{\gamma} u(t+\alpha+1)\big],$$

$$\begin{aligned} &\Delta^{\beta-k}u(\alpha-\beta-1)=\eta_k\nabla^{\omega-k}u(\alpha-1-k), \quad k=0,1, \\ &\Delta^{-\beta}u(T+\alpha+\beta)=\lambda\nabla^{-\omega}u(T+\alpha), \end{aligned} \tag{1.1}$$

where $t\in\mathbb{N}_{0,T}:=\{0,1,\ldots,T\}$; $\alpha\in(2,3]$; $\theta,\gamma,\beta,\omega\in(1,2]$; $T\in\mathbb{N}$; η_0, η_1, λ are given constants; and $F\in C(\mathbb{N}_{\alpha-3,T+\alpha}\times\mathbb{R}^3,\mathbb{R})$.

In Sect. 2, we provide some basic knowledge about delta and nabla difference calculus and investigate results for a linear variant of the boundary value problem (1.1). In Sect. 3, we present the existence results of (1.1) by using Banach contraction principle and Schauder's theorem. Then, we give some examples to illustrate our results.

2 Preliminaries

This section is divided into two parts. The first contains the notations, definitions, and lemmas which are used in the main results. In the second part, we provide a lemma presenting a linear variant of problem (1.1).

The forward jump operator is defined by $\sigma(t):=t+1$, and the backward jump operator is defined by $\rho(t):=t-1$.

For $t,\alpha\in\mathbb{R}$, the generalized falling function is defined by

$$t^{\underline{\alpha}}:=\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)},$$

where $t+1-\alpha$ is not a pole of the Gamma function. If $t+1-\alpha$ is a pole and $t+1$ is not a pole, then $t^{\underline{\alpha}}=0$.

The generalized rising function is defined by

$$t^{\overline{\alpha}}:=\frac{\Gamma(t+\alpha)}{\Gamma(t)},$$

where t and $t+\alpha$ are not poles of the Gamma function. If t is a pole and $t+\alpha$ is not a pole, then $t^{\overline{\alpha}}=0$.

Definition 2.1 ([10]) For $\alpha>0$ and f defined on $\mathbb{N}_a:=\{a,a+1,\ldots\}$, the α-order fractional delta sum of f is defined by

$$\Delta_a^{-\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{\underline{\alpha-1}}f(s), \quad t\in\mathbb{N}_{a+\alpha},$$

and the α-order Riemann–Liouville fractional delta difference of f is defined by

$$\Delta_a^{\alpha}f(t):=\Delta^N\Delta^{-(N-\alpha)}f(t)=\frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{t+\alpha}\bigl(t-\sigma(s)\bigr)^{\underline{-\alpha-1}}f(s), \quad t\in\mathbb{N}_{a+N-\alpha},$$

where $N\in\mathbb{N}$ is such that $0\le N-1<\alpha<N$.

For convenience, the notations $\Delta^{-\alpha}f(t)$ and $\Delta^{\alpha}f(t)$ are used instead of $\Delta_a^{-\alpha}f(t)$ and $\Delta_a^{\alpha}f(t)$, respectively.

Definition 2.2 ([29]) For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order fractional nabla sum of f is defined by

$$\nabla^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} \bigl(t-\rho(s)\bigr)^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_a,$$

and the α-order Riemann–Liouville fractional nabla difference of f is defined by

$$\nabla^{\alpha} f(t) := \nabla^N \nabla^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t} \bigl(t-\rho(s)\bigr)^{\overline{-\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+N},$$

where $N \in \mathbb{N}$ is such that $0 \le N-1 < \alpha < N$.

Lemma 2.1 ([11]) *Let $0 \le N-1 < \alpha \le N$, $N \in \mathbb{N}$ and $y : \mathbb{N}_a \to \mathbb{R}$. Then,*

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = y(t) + C_1(t-a)^{\underline{\alpha-1}} + C_2(t-a)^{\underline{\alpha-2}} + \cdots + C_N(t-a)^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.

Lemma 2.2 ([29]) *Let $0 \le N-1 < \alpha \le N$, $N \in \mathbb{N}$ and $y : \mathbb{N}_{a+1} \to \mathbb{R}$. Then,*

$$\nabla^{-\alpha}\nabla^{\alpha} y(t) = \begin{cases} y(t), & \alpha \notin \mathbb{N}, \\ y(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a), & \alpha = N, \end{cases}$$

for all $t \in \mathbb{N}_{a+N}$.

The solution of a linear variant of the boundary value problem (1.1) is given in the following lemma.

Lemma 2.3 *Let $\Lambda \neq 0$, $\alpha \in (2,3]$; $\beta, \omega \in (1,2]$; $T \in \mathbb{N}$; η_0, η_1, λ be given constants; and $h \in C(\mathbb{N}_{\alpha-3,T+\alpha}, \mathbb{R})$. Then,*

$$\Delta^{\alpha} u(t) = h(t+\alpha-1), \quad t \in \mathbb{N}_{0,T}, \tag{2.1}$$

$$\Delta^{\beta-k} u(\alpha-\beta-1) = \eta_k \nabla^{\omega-k} u(\alpha-1-k), \quad k = 0,1, \tag{2.2}$$

$$\Delta^{-\beta} u(T+\alpha+\beta) = \lambda \nabla^{-\omega} u(T+\alpha) \tag{2.3}$$

has the unique solution given by

$$\begin{aligned} u(t) &= \frac{\Phi[h]}{\Lambda}\bigl[\mathcal{A}_1 t^{\underline{\alpha-1}} + \mathcal{A}_2 t^{\underline{\alpha-2}} + \mathcal{A}_3 t^{\underline{\alpha-3}}\bigr] \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{\underline{\alpha-1}} h(s+\alpha-1), \end{aligned} \tag{2.4}$$

where the functional $\Phi[h]$ and the constants Λ, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are defined as

$$
\begin{aligned}
\Phi[h] = {}& \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} \left[\lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} - \frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} \right] \\
& \times \left(s-\sigma(r)\right)^{\underline{\alpha-1}} h(r+\alpha-1), \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda = {}& \left[\Gamma(\alpha-2)\left((1-\beta)-\eta_1(1-\omega)\right) + \Gamma(\alpha-1)(1-\eta_1)\right] \\
& \times \Bigg\{ \left[\Gamma(\alpha-1)(\eta_0\omega-\beta) + \Gamma(\alpha)(\eta_0-1)\right] \sum_{s=\alpha-1}^{T+\alpha} \left[\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} \right. \\
& \left. - \lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} \right] s^{\underline{\alpha-1}} + \Gamma(\alpha)(\eta_0-1) \sum_{s=\alpha-2}^{T+\alpha} \left[\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} \right. \\
& \left. - \lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} \right] s^{\underline{\alpha-2}} \Bigg\} + \Gamma(\alpha-1)(\eta_1-1) \Bigg\{ \left[\frac{\Gamma(\alpha-2)}{2} \left(\eta_0(1-\omega)\omega \right. \right. \\
& \left. \left. - (1-\beta)\beta\right) + \Gamma(\alpha-1)(\eta_0\omega-\beta) + \Gamma(\alpha)(1-\eta_0) \right] \\
& \times \sum_{s=\alpha-1}^{T+\alpha} \left[\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} - \lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} \right] s^{\underline{\alpha-1}} + \Gamma(\alpha) \\
& \times (\eta_0-1) \sum_{s=\alpha-3}^{T+\alpha} \left[\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} - \lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} \right] s^{\underline{\alpha-3}} \Bigg\}, \qquad (2.6)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_1 = {}& \left[\Gamma(\alpha-1)(\eta_0\omega-\beta) + \Gamma(\alpha)(\eta_0-1)\right]\left[\Gamma(\alpha-2)\left((1-\beta)-\eta_1(1-\omega)\right) \right. \\
& \left. + \Gamma(\alpha-1)(1-\eta_1)\right] - \left[\frac{\Gamma(\alpha-2)}{2} \left(\eta_0(1-\omega)\omega - (1-\beta)\beta\right) \right. \\
& \left. + \Gamma(\alpha-1)(\eta_0\omega-\beta) + \Gamma(\alpha)(1-\eta_0) \right] \Gamma(\alpha-1)(1-\eta_1), \qquad (2.7)
\end{aligned}
$$

$$
\mathcal{A}_2 = \Gamma(\alpha)(\eta_0-1)\left[\Gamma(\alpha-2)\left((1-\beta)-\eta_1(1-\omega)\right) + \Gamma(\alpha-1)(1-\eta_1)\right], \qquad (2.8)
$$

$$
\mathcal{A}_3 = \Gamma(\alpha)\Gamma(\alpha-1)(\eta_0-1)(1-\eta_1). \qquad (2.9)
$$

Proof By using the fractional delta sum of order α in (2.1), we have

$$
u(t) = C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + C_3 t^{\underline{\alpha-3}} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \left(t-\sigma(s)\right)^{\underline{\alpha-1}} h(s+\alpha-1), \qquad (2.10)
$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$. Then taking the fractional delta difference of order $\beta - k$ in (2.10) where $k = 0, 1$, we get

$$
\begin{aligned}
\Delta^{\beta-k} u(t) = {}& \frac{1}{\Gamma(k-\beta)} \sum_{s=\alpha-3}^{t+\beta-k} \left(t-\sigma(s)\right)^{\underline{-\beta+k-1}} \left[C_1 s^{\underline{\alpha-1}} + C_2 s^{\underline{\alpha-2}} + C_3 s^{\underline{\alpha-3}}\right] \\
& + \frac{1}{\Gamma(k-\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{t+\beta-k} \sum_{r=0}^{s-\alpha} \left(t-\sigma(s)\right)^{\underline{-\beta+k-1}} \left(s-\sigma(r)\right)^{\underline{\alpha-1}} h(r+\alpha-1), \qquad (2.11)
\end{aligned}
$$

for $t \in \mathbb{N}_{\alpha-\beta-1,T+\alpha-\beta+k}$.

Taking the fractional nabla difference of order $\omega - k$ of (2.10) where $k = 0, 1$, we obtain

$$\begin{aligned}\nabla^{\omega-k}u(t) &= \frac{1}{\Gamma(k-\omega)}\sum_{s=\alpha-3}^{t}\big(t-\rho(s)\big)^{\overline{-\omega+k-1}}\big[C_1 s^{\underline{\alpha-1}} + C_2 s^{\underline{\alpha-2}} + C_3 s^{\underline{\alpha-3}}\big] \\ &\quad + \frac{1}{\Gamma(k-\omega)\Gamma(\alpha)}\sum_{s=\alpha}^{t}\sum_{r=0}^{s-\alpha}\big(t-\rho(s)\big)^{\overline{-\omega+k-1}}\big(s-\sigma(r)\big)^{\underline{\alpha-1}}h(r+\alpha-1), \qquad (2.12)\end{aligned}$$

for $t \in \mathbb{N}_{\alpha-k-1,T+\alpha}$.

We now substitute $t = \alpha - \beta - 1$ into (2.11) and $t = \alpha - 1 - k$ into (2.12), then apply condition (2.2). So, we have

$$\begin{aligned}&\frac{1}{\Gamma(k-\beta)}\sum_{s=\alpha-3}^{\alpha-k-1}\big(\alpha-\beta-1-\sigma(s)\big)^{\underline{-\beta+k-1}}\big[C_1 s^{\underline{\alpha-1}} + C_2 s^{\underline{\alpha-2}} + C_3 s^{\underline{\alpha-3}}\big] \\ &= \frac{\eta_k}{\Gamma(k-\omega)}\sum_{s=\alpha-3}^{\alpha-k-1}\big(\alpha-k-1-\rho(s)\big)^{\overline{-\omega+k-1}}\big[C_1 s^{\underline{\alpha-1}} + C_2 s^{\underline{\alpha-2}} + C_3 s^{\underline{\alpha-3}}\big], \\ &k = 0, 1. \qquad (2.13)\end{aligned}$$

With $k = 0$ and $k = 1$ in (2.13), we obtain the equations

$$\begin{aligned}(E_1)\quad & C_1\Gamma(\alpha)(1-\eta_0) + C_2\big[\Gamma(\alpha-1)(\eta_0\omega-\beta) + \Gamma(\alpha)(\eta_0-1)\big] + C_3\bigg[\frac{\Gamma(\alpha-2)}{2} \\ &\times \big(\eta_0(1-\omega)\omega - (1-\beta)\beta\big) + \Gamma(\alpha-1)(\eta_0\omega-\beta) + \Gamma(\alpha)(1-\eta_0)\bigg] = 0, \\ (E_2)\quad & C_2\Gamma(\alpha-1)(1-\eta_1) + C_3\big[\Gamma(\alpha-2)\big((1-\beta)-\eta_1(1-\omega)\big) + \Gamma(\alpha-1)(1-\eta_1)\big] = 0.\end{aligned}$$

After taking the fractional delta sum of order β for (2.10), we get

$$\begin{aligned}\Delta^{-\beta}u(t) &= \frac{1}{\Gamma(\beta)}\sum_{s=\alpha-3}^{t-\beta}\big(t-\sigma(s)\big)^{\underline{\beta-1}}\big[C_1 s^{\underline{\alpha-1}} + C_2 s^{\underline{\alpha-2}} + C_3 s^{\underline{\alpha-3}}\big] \\ &\quad + \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\beta}\sum_{r=0}^{s-\alpha}\big(t-\sigma(s)\big)^{\underline{\beta-1}}\big(s-\sigma(r)\big)^{\underline{\alpha-1}}h(r+\alpha-1), \qquad (2.14)\end{aligned}$$

for $t \in \mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta}$.

Using the fractional nabla sum of order ω for (2.10), we obtain

$$\begin{aligned}\nabla^{-\omega}u(t) &= \frac{1}{\Gamma(\omega)}\sum_{s=\alpha-3}^{t}\big(t-\rho(s)\big)^{\overline{\omega-1}}\big[C_1 s^{\underline{\alpha-1}} + C_2 s^{\underline{\alpha-2}} + C_3 s^{\underline{\alpha-3}}\big] \\ &\quad + \frac{1}{\Gamma(\omega)\Gamma(\alpha)}\sum_{s=\alpha}^{t}\sum_{r=0}^{s-\alpha}\big(t-\rho(s)\big)^{\overline{\omega-1}}\big(s-\sigma(r)\big)^{\underline{\alpha-1}}h(r+\alpha-1), \qquad (2.15)\end{aligned}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

We now substitute $t = T + \alpha + \beta$ into (2.14) and $t = T + \alpha$ into (2.15), then apply condition (2.3). So, we have

$$\begin{aligned}
(E_3)\quad & C_1 \sum_{s=\alpha-1}^{T+\alpha} \left[\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} - \lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} \right] s^{\underline{\alpha-1}} \\
& + C_2 \sum_{s=\alpha-2}^{T+\alpha} \left[\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} - \lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} \right] s^{\underline{\alpha-2}} \\
& + C_3 \sum_{s=\alpha-3}^{T+\alpha} \left[\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} - \lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} \right] s^{\underline{\alpha-3}} \\
& = \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} \left[\lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} - \frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} \right] \\
& \times \big(s-\sigma(r)\big)^{\underline{\alpha-1}} h(r+\alpha-1).
\end{aligned}$$

Finding the solution of equations (E_1)–(E_3), we have

$$C_i = \frac{\Phi[h]}{\Lambda} \mathcal{A}_i, \quad i = 1, 2, 3,$$

where $\Phi[h]$, Λ, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are defined by (2.5)–(2.9), respectively. Substituting the constants C_1 through C_3 into (2.10), we get the unique solution as (2.4). □

3 Main results

In this section, we show existence results of problem (1.1). Let $\mathcal{C} = C(\mathbb{N}_{\alpha-3,T+\alpha}, \mathbb{R})$ be the Banach space of functions u with the norm defined by

$$\|u\|_{\mathcal{C}} = \max\big\{ \|u\|, \big\|\Delta^{\theta} u\big\|, \big\|\nabla^{\gamma} u\big\| \big\},$$

where $\|u\| = \max_{t\in\mathbb{N}_{\alpha-3,T+\alpha}} |u(t)|$, $\|\Delta^{\theta} u\| = \max_{t\in\mathbb{N}_{\alpha-3,T+\alpha}} |\Delta^{\theta} u(t-\theta+2)|$ and $\|\nabla^{\gamma} u\| = \max_{t\in\mathbb{N}_{\alpha-3,T+\alpha}} |\nabla^{\gamma} u(t+2)|$. We define the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}
(\mathcal{F}u)(t) = {} & \frac{\Phi[F(u)]}{\Lambda} \big[\mathcal{A}_1 t^{\underline{\alpha-1}} + \mathcal{A}_2 t^{\underline{\alpha-2}} + \mathcal{A}_3 t^{\underline{\alpha-3}} \big] + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \big(t-\sigma(s)\big)^{\underline{\alpha-1}} \\
& \times F\big[s+\alpha-1, u(s+\alpha-1), \Delta^{\theta} u(s+\alpha-\theta+1), \nabla^{\gamma} u(s+\alpha+1)\big],
\end{aligned} \tag{3.1}$$

where $\Lambda \neq 0$, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are given in Lemma 2.3 and the functional $\Phi[F(u)]$ is given by

$$\begin{aligned}
& \Phi\big[F(u)\big] \\
& \quad = \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} \left[\lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} - \frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)} \right] \big(s-\sigma(r)\big)^{\underline{\alpha-1}} \\
& \qquad \times F\big[r+\alpha-1, u(r+\alpha-1), \Delta^{\theta} u(r+\alpha-\theta+1), \nabla^{\gamma} u(r+\alpha+1)\big].
\end{aligned} \tag{3.2}$$

The boundary value problem (1.1) has solutions if and only if operator $\mathcal{F}$ has fixed points.

Theorem 3.1 *Let $F : \mathbb{N}_{\alpha-3,T+\alpha} \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and suppose that the following conditions hold*:

(H_1) *There exist constants $L_1, L_2, L_3 > 0$ such that for each $t \in \mathbb{N}_{\alpha-3,T+\alpha}$ and $u_i, v_i \in \mathbb{R}$, $i = 1, 2, 3$,*

$$\bigl|F(t,u_1,u_2,u_3) - F(t,v_1,v_2,v_3)\bigr| \le L_1|u_1 - v_1| + L_2|u_2 - v_2| + L_3|u_3 - v_3|,$$

(H_2) $[L_1 + L_2 + L_3] \max\{\Omega_1, \Omega_2, \Omega_3\} < 1$,

then problem (1.1) *has a unique solution on $\mathbb{N}_{\alpha-3,T+\alpha}$, where*

$$\Theta = \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\biggl[\lambda\frac{(T+\omega)^{\underline{\omega}}}{\Gamma(\omega+1)} - \frac{(T+\beta)^{\underline{\beta}}}{\Gamma(\beta+1)}\biggr], \tag{3.3}$$

$$\Omega_1 = \frac{\Theta}{|\Lambda|}\bigl[|\mathcal{A}_1|(T+\alpha)^{\underline{\alpha-1}} + |\mathcal{A}_2|(T+\alpha)^{\underline{\alpha-2}} + |\mathcal{A}_3|(T+\alpha)^{\underline{\alpha-3}}\bigr] + \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}, \tag{3.4}$$

$$\begin{aligned}\Omega_2 = \frac{(T-\theta+5)^{\underline{\theta-1}}}{|\Gamma(1-\theta)|}\biggl\{&\frac{\Theta}{|\Lambda|}\bigl[|\mathcal{A}_1|(T+\alpha+2)^{\underline{\alpha-1}} + |\mathcal{A}_2|(T+\alpha+2)^{\underline{\alpha-2}} \\ &+ |\mathcal{A}_3|(T+\alpha+2)^{\underline{\alpha-3}}\bigr] + \frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\biggr\},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\Omega_3 = \frac{(T-\gamma+5)^{\underline{\gamma-1}}}{|\Gamma(1-\gamma)|}\biggl\{&\frac{\Theta}{|\Lambda|}\bigl[|\mathcal{A}_1|(T+\alpha+2)^{\underline{\alpha-1}} + |\mathcal{A}_2|(T+\alpha+2)^{\underline{\alpha-2}} \\ &+ |\mathcal{A}_3|(T+\alpha+2)^{\underline{\alpha-3}}\bigr] + \frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\biggr\}.\end{aligned} \tag{3.6}$$

Proof Letting $u, v \in \mathcal{C}$, for each $t \in \mathbb{N}_{\alpha-3,T+\alpha}$, we have

$$\begin{aligned}&\bigl|\Phi\bigl[F(u)\bigr] - \Phi\bigl[F(v)\bigr]\bigr| \\ &\le \frac{1}{\Gamma(\alpha)}\sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\biggl[\lambda\frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} - \frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)}\biggr]\bigl(s-\sigma(r)\bigr)^{\underline{\alpha-1}} \\ &\quad\times\bigl[L_1\bigl|u(r+\alpha-1) - v(r+\alpha-1)\bigr| + L_2\bigl|\Delta^\theta u(r+\alpha-\theta+1) - \Delta^\theta v(r+\alpha-\theta+1)\bigr| \\ &\quad + L_3\bigl|\nabla^\gamma u(r+\alpha+1) - \nabla^\gamma v(r+\alpha+1)\bigr|\bigr] \\ &\le \bigl[L_1\|u-v\| + L_2\bigl\|\Delta^\theta u - \Delta^\theta v\bigr\| + L_3\bigl\|\nabla^\gamma u - \nabla^\gamma v\bigr\|\bigr]\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} \\ &\quad\times\Biggl|\lambda\sum_{s=\alpha}^{T+\alpha}\frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} - \sum_{s=\alpha}^{T+\alpha}\frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)}\Biggr| \\ &= \|u-v\|_{\mathcal{C}}[L_1 + L_2 + L_3]\Theta\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\bigl|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\bigr| \\ &\le \biggl|\frac{\Phi[F(u)] - \Phi[F(v)]}{\Lambda}\biggr|\bigl[|\mathcal{A}_1|(T+\alpha)^{\underline{\alpha-1}} + |\mathcal{A}_2|(T+\alpha)^{\underline{\alpha-2}} + |\mathcal{A}_3|(T+\alpha)^{\underline{\alpha-3}}\bigr] \\ &\quad + \sum_{s=0}^{T}\frac{(T+\alpha-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)}\bigl[L_1\bigl|u(s+\alpha-1) - v(s+\alpha-1)\bigr| + L_2\bigl|\Delta^\theta u(s+\alpha-\theta+1)\end{aligned}$$

$$
\begin{aligned}
&- \Delta^{\theta} v(s+\alpha-\theta+1)\big| + L_3\big|\nabla^{\gamma} u(s+\alpha+1) - \nabla^{\gamma} v(s+\alpha+1)\big|\big] \\
&\leq \big[L_1\|u-v\| + L_2\big\|\Delta^{\theta} u - \Delta^{\theta} v\big\| + L_3\big\|\nabla^{\gamma} u - \nabla^{\gamma} v\big\|\big]\bigg\{\frac{\Theta}{|\Lambda|}\big[|\mathcal{A}_1|(T+\alpha)^{\underline{\alpha-1}} \\
&+ |\mathcal{A}_2|(T+\alpha)^{\underline{\alpha-2}} + |\mathcal{A}_3|(T+\alpha)^{\underline{\alpha-3}}\big] + \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\bigg\} \\
&\leq \|u-v\|_{\mathcal{C}}[L_1+L_2+L_3]\Omega_1.
\end{aligned} \tag{3.8}
$$

We find that

$$
\begin{aligned}
&\big(\Delta^{\theta}\mathcal{F}u\big)(t-\theta+2) \\
&= \frac{\Phi[F(u)]}{\Lambda\Gamma(-\theta)}\sum_{s=\alpha-3}^{t+2}\big(t-\theta+2-\sigma(s)\big)^{\underline{-\theta-1}}\big[|\mathcal{A}_1|s^{\underline{\alpha-1}} + |\mathcal{A}_2|s^{\underline{\alpha-2}} + |\mathcal{A}_3|s^{\underline{\alpha-3}}\big] \\
&+ \frac{1}{\Gamma(-\theta)\Gamma(\alpha)}\sum_{s=\alpha}^{t+2}\sum_{r=0}^{s-\alpha}\big(t-\theta+2-\sigma(s)\big)^{\underline{-\theta-1}}\big(s-\sigma(r)\big)^{\underline{\alpha-1}} \\
&\times F\big[r+\alpha-1, u(r+\alpha-1), \Delta^{\theta} u(r+\alpha-\theta+1), \nabla^{\gamma} u(r+\alpha+1)\big]
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&\big(\nabla^{\gamma}\mathcal{F}u\big)(t+2) \\
&= \frac{\Phi[F(u)]}{\Lambda\Gamma(-\gamma)}\sum_{s=\alpha-3}^{t+2}\big(t+2-\rho(s)\big)^{\overline{-\gamma-1}}\big[|\mathcal{A}_1|s^{\underline{\alpha-1}} + |\mathcal{A}_2|s^{\underline{\alpha-2}} + |\mathcal{A}_3|s^{\underline{\alpha-3}}\big] \\
&+ \frac{1}{\Gamma(-\gamma)\Gamma(\alpha)}\sum_{s=\alpha}^{t+2}\sum_{r=0}^{s-\alpha}\big(t+2-\rho(s)\big)^{\overline{-\gamma-1}}\big(s-\sigma(r)\big)^{\underline{\alpha-1}} \\
&\times F\big[r+\alpha-1, u(r+\alpha-1), \Delta^{\theta} u(r+\alpha-\theta+1), \nabla^{\gamma} u(r+\alpha+1)\big].
\end{aligned} \tag{3.10}
$$

Since

$$
\big|\big(\Delta^{\theta}\mathcal{F}u\big)(t-\theta+2) - \big(\Delta^{\theta}\mathcal{F}v\big)(t-\theta+2)\big| \leq \|u-v\|_{\mathcal{C}}[L_1+L_2+L_3]\Omega_2, \tag{3.11}
$$

$$
\big|\big(\nabla^{\gamma}\mathcal{F}u\big)(t+2) - \big(\nabla^{\gamma}\mathcal{F}v\big)(t+2)\big| \leq \|u-v\|_{\mathcal{C}}[L_1+L_2+L_3]\Omega_3, \tag{3.12}
$$

we get

$$
\big\|(\mathcal{F}u) - (\mathcal{F}v)\big\|_{\mathcal{C}} \leq [L_1+L_2+L_3]\max\{\Omega_1,\Omega_2,\Omega_3\}\|u-v\|_{\mathcal{C}}. \tag{3.13}
$$

By (H_2), we get $\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_{\mathcal{C}} < \|u-v\|_{\mathcal{C}}$.

Hence, $\mathcal{F}$ is a contraction. By the Banach contraction principle, we conclude that $\mathcal{F}$ has a unique fixed point which is a unique solution of the problem (1.1) for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$. □

We next show that our problem (1.1) has at least one solution as follows.

Lemma 3.1 (Arzelá–Ascoli theorem, [43]) *A set of functions in $C[a,b]$ with the sup-norm is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 3.2 ([43]) *If a set is closed and relatively compact, then it is compact.*

Lemma 3.3 (Schauder's fixed point theorem, [44]) *Let (D,d) be a complete metric space, U a closed convex subset of D, and $T : D \to D$ a map such that the set $Tu : u \in U$ is relatively compact in D. Then, the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Theorem 3.2 *Suppose that (H_1) and (H_2) hold. Then problem (1.1) has at least one solution on $\mathbb{N}_{\alpha-3,T+\alpha}$.*

Proof Step I. We verify that $\mathcal{F}$ maps bounded sets into bounded sets in B_R, where we consider $B_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq R\}$.

Let $\max_{t\in\mathbb{N}_{\alpha-3,T+\alpha}} |F(t,0,0,0)| = M$ and choose a constant

$$R \geq \frac{M \max\{\Omega_1, \Omega_2, \Omega_3\}}{1 - [L_1 + L_2 + L_3] \max\{\Omega_1, \Omega_2, \Omega_3\}}. \tag{3.14}$$

For each $u \in B_R$, we obtain

$$\begin{aligned}
&\bigl|\Phi\bigl[F(u)\bigr]\bigr| \\
&\quad\leq \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} \biggl[\lambda \frac{(T+\alpha-\rho(s))^{\overline{\omega-1}}}{\Gamma(\omega)} - \frac{(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}}{\Gamma(\beta)}\biggr] \bigl(s-\sigma(r)\bigr)^{\underline{\alpha-1}} \\
&\qquad \times \bigl[\bigl|F\bigl[r+\alpha-1, u(r+\alpha-1), \Delta^{\theta} u(r+\alpha-\theta+1), \nabla^{\gamma} u(r+\alpha+1)\bigr] \\
&\qquad - F(r+\alpha-1,0,0,0)\bigr| + \bigl|F(r+\alpha-1,0,0,0)\bigr|\bigr] \\
&\quad\leq \bigl[\bigl[L_1\|u\| + L_2\bigl\|\Delta^{\theta} u\bigr\| + L_3\bigl\|\nabla^{\gamma} u\bigr\|\bigr] + M\bigr] \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)\Gamma(T+1)} \\
&\qquad \times \biggl[\lambda \frac{\Gamma(T+\omega+1)}{\Gamma(\omega+1)} - \frac{\Gamma(T-\beta+1)}{\Gamma(\beta+1)}\biggr] \\
&\quad\leq \bigl[(L_1+L_2+L_3)\|u\|_{\mathcal{C}} + M\bigr]\Theta
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\bigl|(\mathcal{F}u)(t)\bigr| \\
&\quad\leq \biggl|\frac{\Phi[F(u)]}{\Lambda}\biggr| \bigl[|\mathcal{A}_1|(T+\alpha)^{\underline{\alpha-1}} + |\mathcal{A}_2|(T+\alpha)^{\underline{\alpha-2}} + |\mathcal{A}_3|(T+\alpha)^{\underline{\alpha-3}}\bigr] \\
&\qquad + \sum_{s=0}^{T} \frac{(T+\alpha-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)} \bigl[\bigl|F\bigl[s+\alpha-1, u(s+\alpha-1), \Delta^{\theta} u(s+\alpha-\theta+1), \\
&\qquad \nabla^{\gamma} u(s+\alpha+1)\bigr] - F(s+\alpha-1,0,0,0)\bigr| + \bigl|F(s+\alpha-1,0,0,0)\bigr|\bigr] \\
&\quad\leq \bigl[(L_1+L_2+L_3)\|u\|_{\mathcal{C}} + M\bigr] \biggl\{\frac{\Theta}{|\Lambda|}\bigl[|\mathcal{A}_1|(T+\alpha)^{\underline{\alpha-1}} + |\mathcal{A}_2|(T+\alpha)^{\underline{\alpha-2}} \\
&\qquad + |\mathcal{A}_3|(T+\alpha)^{\underline{\alpha-3}}\bigr] + \frac{\Gamma(T+\alpha+1)}{\Gamma(\alpha+1)\Gamma(T+1)}\biggr\} \\
&\quad\leq \bigl[(L_1+L_2+L_3)\|u\|_{\mathcal{C}} + M\bigr]\Omega_1.
\end{aligned} \tag{3.16}$$

Since

$$\left|\left(\Delta^{\theta}\mathcal{F}u\right)(t-\theta+2)\right| \leq \left[(L_1+L_2+L_3)\|u\|_{\mathcal{C}}+M\right]\Omega_2, \tag{3.17}$$

$$\left|\left(\nabla^{\gamma}\mathcal{F}u\right)(t+2)\right| \leq \left[(L_1+L_2+L_3)\|u\|_{\mathcal{C}}+M\right]\Omega_3, \tag{3.18}$$

this implies that

$$\begin{aligned}\left\|(\mathcal{F}u)(t)\right\|_{\mathcal{C}} &\leq \left[(L_1+L_2+L_3)\|u\|_{\mathcal{C}}+M\right]\max\{\Omega_1,\Omega_2,\Omega_3\}\\ &\leq R.\end{aligned} \tag{3.19}$$

We find that $\|\mathcal{F}u\|_{\mathcal{C}} \leq R$. Hence, $\mathcal{F}$ is uniformly bounded.

Step II. Since F is a continuous function, the operator $\mathcal{F}$ is continuous on B_R.

Step III. We show that $\mathcal{F}$ is equicontinuous on B_R. For any $\epsilon > 0$, there exists a positive constant $\rho^* = \min\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6\}$ such that for $t_1,t_2 \in \mathbb{N}_{\alpha-3,T+\alpha}$,

$$\begin{aligned}
&\left|t_2^{\underline{\alpha-i}}-t_1^{\underline{\alpha-i}}\right| < \frac{\epsilon|\Lambda|}{4|\mathcal{A}_i|\Theta\|F\|}, \quad \text{if } |t_2-t_1|<\delta_i, i=1,2,3,\\
&\left|t_2^{\underline{\alpha}}-t_1^{\underline{\alpha}}\right| < \frac{\epsilon\Gamma(\alpha+1)}{4\|F\|}, \quad \text{if } |t_2-t_1|<\delta_4,\\
&\left|(t_2-\alpha-\theta+5)^{\underline{-\theta}}-(t_1-\alpha-\theta+5)^{\underline{-\theta}}\right|\\
&\quad < \left(\epsilon\left|\Gamma(1-\theta)\right|\right)\Bigg(\|F\|\bigg[\frac{\Theta}{|\Lambda|}\left(|\mathcal{A}_1|(T+\alpha+2)^{\underline{\alpha-1}}+|\mathcal{A}_2|(T+\alpha+2)^{\underline{\alpha-2}}\right.\\
&\qquad \left.+|\mathcal{A}_3|(T+\alpha+2)^{\underline{\alpha-3}}\right)+\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\bigg]\Bigg)^{-1},\\
&\quad \text{if } |t_2-t_1|<\delta_5,\\
&\left|(t_2-\alpha+6)^{\overline{-\gamma}}-(t_2-\alpha+6)^{\overline{-\gamma}}\right|\\
&\quad < \left(\epsilon\left|\Gamma(1-\gamma)\right|\right)\Bigg(\|F\|\bigg[\frac{\Theta}{|\Lambda|}\left(|\mathcal{A}_1|(T+\alpha+2)^{\underline{\alpha-1}}+|\mathcal{A}_2|(T+\alpha+2)^{\underline{\alpha-2}}\right.\\
&\qquad \left.+|\mathcal{A}_3|(T+\alpha+2)^{\underline{\alpha-3}}\right)+\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\bigg]\Bigg)^{-1},\\
&\quad \text{if } |t_2-t_1|<\delta_6.
\end{aligned}$$

Then, for $|t_2-t_1|<\rho^*$, we have

$$\begin{aligned}
&\left|(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1)\right|\\
&\quad \leq \left|\frac{\Phi[F(u)]}{\Lambda}\right|\Big[|\mathcal{A}_1|\left|t_2^{\underline{\alpha-1}}-t_2^{\underline{\alpha-1}}\right|+|\mathcal{A}_2|\left|t_2^{\underline{\alpha-1}}-t_2^{\underline{\alpha-2}}\right|+|\mathcal{A}_3|\\
&\qquad \times\left|t_2^{\underline{\alpha-1}}-t_2^{\underline{\alpha-3}}\right|\Big]+\Bigg|\sum_{s=0}^{t_2}\frac{(t_2-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)}F\big[s+\alpha-1,u(s+\alpha-1),\\
&\qquad \Delta^{\theta}u(s+\alpha-\theta+1),\nabla^{\gamma}u(s+\alpha+1)\big]-\sum_{s=0}^{t_1}\frac{(t_1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)}F\big[s+\alpha-1,
\end{aligned}$$

$$
\begin{aligned}
&u(s+\alpha-1), \Delta^{\theta} u(s+\alpha-\theta+1), \nabla^{\gamma} u(s+\alpha+1)\big]\Bigg| \\
&< \left|\frac{\mathcal{A}_1}{\Lambda}\right| \Theta \|F\| \left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right| + \left|\frac{\mathcal{A}_2}{\Lambda}\right| \Theta \|F\| \left|t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right| + \left|\frac{\mathcal{A}_3}{\Lambda}\right| \Theta \|F\| \left|t_2^{\underline{\alpha-3}} - t_1^{\underline{\alpha-3}}\right| \\
&\quad + \frac{\|F\|}{\Gamma(\alpha+1)} \left|t_2^{\underline{\alpha}} - t_1^{\underline{\alpha}}\right| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned} \tag{3.20}
$$

Similarly, we have

$$
\left|\left(\Delta^{\theta} \mathcal{F} u\right)(t_2-\theta+2) - \left(\Delta^{\theta} \mathcal{F} u\right)(t_1-\theta+2)\right| < \epsilon, \tag{3.21}
$$

$$
\left|\left(\nabla^{\gamma} \mathcal{F} u\right)(t_2+2) - \left(\nabla^{\gamma} \mathcal{F} u\right)(t_1+2)\right| < \epsilon. \tag{3.22}
$$

So,

$$
\left\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\right\|_{\mathcal{C}} < \epsilon. \tag{3.23}
$$

Hence, the set $\mathcal{F}(B_R)$ is equicontinuous. Combining the results of Steps I to III with the Arzelá–Ascoli theorem, we get that $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous. By using Schauder fixed point theorem, we can conclude that boundary value problem (1.1) has at least one solution. □

4 Some examples

In this section, we provide a mixed boundary value problem for fractional delta–nabla difference equations and apply our results from the previous section as follows:

$$
\begin{aligned}
&\Delta^{\frac{5}{2}} u(t) = F\big[t+\alpha-1, u(t+\alpha-1), \Delta^{\theta} u(t+\alpha-\theta+1), \nabla^{\gamma} u(t+\alpha+1)\big], \\
&\Delta^{\frac{5}{4}-k} u\left(\frac{1}{4}\right) = \left(\frac{k+1}{3}\right) \nabla^{\frac{7}{6}-k} u\left(\frac{3}{2}-k\right), \quad k = 0, 1, \\
&\Delta^{-\frac{5}{4}} u\left(\frac{35}{4}\right) = 2\nabla^{-\frac{7}{6}} u\left(\frac{15}{2}\right).
\end{aligned} \tag{4.1}
$$

Here $\alpha = \frac{5}{2}$, $\theta = \frac{4}{3}$, $\gamma = \frac{8}{5}$, $\beta = \frac{5}{4}$, $\omega = \frac{7}{6}$, $\eta_0 = \frac{1}{3}$, $\eta_1 = \frac{2}{3}$, and $\lambda = 2$, $T = 5$. We find that

$$
|\Lambda| = 3.711, \qquad \Theta = 529.938, \qquad \Omega_1 = 5151.475, \qquad \Omega_2 = 87.307,
$$
$$
\Omega_3 = 53.338.
$$

(i) Let

$$
\begin{aligned}
&F\big[t, u(t), \Delta^{\theta} u(t-\theta+2), \nabla^{\gamma} u(t+2)\big] \\
&\quad = \frac{e^{-\cos^2 t}}{(t+100)^2} \cdot \frac{|u(t)| + 2|\Delta^{\frac{4}{3}} u(t+\frac{2}{3})| + 3|\nabla^{\frac{8}{5}} u(t+2)|}{[1+|u(t)|]}.
\end{aligned}
$$

Since (H_1) holds for each $t \in \mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$, we obtain

$$\begin{aligned}&\bigl|F\bigl[t,u(t),\Delta^{\theta}u(t-\theta+2),\nabla^{\gamma}u(t+2)\bigr]-F\bigl[t,v(t),\Delta^{\theta}v(t-\theta+2),\nabla^{\gamma}v(t+2)\bigr]\bigr|\\&\quad\le\frac{4}{46{,}225}\|u-v\|+\frac{8}{46{,}225}\bigl\|\Delta^{\theta}u-\Delta^{\theta}v\bigr\|+\frac{12}{46{,}225}\bigl\|\nabla^{\gamma}u-\nabla^{\gamma}v\bigr\|,\end{aligned}$$

so $L_1=\frac{4}{46{,}225}$, $L_2=\frac{8}{46{,}225}$, $L_3=\frac{12}{46{,}225}$.

Finally, we can show that $(H2)$ holds with

$$[L_1+L_2+L_3]\max\{\Omega_1,\Omega_2,\Omega_3\}=0.275<1.$$

Hence, by Theorem 3.1, Problem 4.1 has a unique solution on $\mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$. In addition, by Theorem 3.2, Problem 4.1 has at least one solution on $\mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$.

(ii) Let

$$\begin{aligned}&F\bigl[t,u(t),\Delta^{\theta}u(t-\theta+2),\nabla^{\gamma}u(t+2)\bigr]\\&\quad=\frac{e^{-\sin^2 t}}{2t+10}\cdot\frac{3|u(t)|+5|\Delta^{\frac{4}{3}}u(t+\frac{2}{3})|+2|\nabla^{\frac{8}{5}}u(t+2)|}{[1+|u(t)|]}.\end{aligned}$$

Since (H_1) holds for each $t \in \mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$, we obtain

$$\begin{aligned}&\bigl|F\bigl[t,u(t),\Delta^{\theta}u(t-\theta+2),\nabla^{\gamma}u(t+2)\bigr]-F\bigl[t,v(t),\Delta^{\theta}v(t-\theta+2),\nabla^{\gamma}v(t+2)\bigr]\bigr|\\&\quad\le\frac{3}{115}\|u-v\|+\frac{5}{115}\bigl\|\Delta^{\theta}u-\Delta^{\theta}v\bigr\|+\frac{2}{115}\bigl\|\nabla^{\gamma}u-\nabla^{\gamma}v\bigr\|,\end{aligned}$$

so $L_1=\frac{3}{115}$, $L_2=\frac{5}{115}$, $L_3=\frac{2}{115}$.

Finally, we show that $(H2)$ not holds with

$$[L_1+L_2+L_3]\max\{\Omega_1,\Omega_2,\Omega_3\}=41.473>1.$$

Therefore, Problem 4.1 is inconsistent with Theorem 3.1 and 3.2, which makes it impossible to conclude the existence results for this problem.

5 Conclusions

We consider a fractional delta–nabla difference equation with fractional delta–nabla sum-difference boundary value conditions. In our studies, we employ the Banach contraction principle to investigate the conditions for the existence and uniqueness of solution for our problem. In addition, the conditions for at least one solution is obtained by using the Schauder's fixed point theorem.

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