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New quantum estimates in the setting of fractional calculus theory

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Abstract

In this article, the investigation is centered around the quantum estimates by utilizing quantum Hahn integral operator via the quantum shift operator ${}_\eta\psi_{\mathfrak{q}}(\zeta) = \mathfrak{q}\zeta + (1-\mathfrak{q})\eta$, $\zeta \in [\mu,\nu]$, $\eta = \mu + \frac{\omega}{(1-\mathfrak{q})}$, $0 < \mathfrak{q} < 1$, $\omega \geq 0$. Our strategy includes fractional calculus, Jackson's $\mathfrak{q}$-integral, the main ideas of quantum calculus, and a generalization used in the frame of convex functions. We presented, in general, three types of fractional quantum integral inequalities that can be utilized to explain orthogonal polynomials, and exploring some estimation problems with shifting estimations of fractional order ϱ_1 and the $\mathfrak{q}$-numbers have yielded fascinating outcomes. As an application viewpoint, an illustrative example shows the effectiveness of $\mathfrak{q}, \omega$-derivative for boundary value problem.

MSC: 26D15; 26D10; 90C23

Keywords: Hahn integral operator; Reverse Minkowski quantum Hahn integral inequality; Reverse Hölder quantum Hahn integral inequality

1 Introduction

Quantum difference operators have played a crucial role in the development of quantum calculus due to their fertile application, see [1–5]. Roughly speaking, a quantum calculus substitutes the classical derivative by a difference operator, which allows dealing with sets of nondifferentiable functions. In [6], Hahn a famous mathematician presented the Hahn difference operator $\mathcal{D}_{\mathfrak{q},\omega}$.

Suppose that h_1 defined on an interval $I \subseteq \mathbb{R}$ containing $\omega_0 := \frac{\omega}{1-\mathfrak{q}}$ is defined as

$$\mathcal{D}_{\mathfrak{q},\omega}h_1(\zeta) = \begin{cases} \frac{h_1(\mathfrak{q}\zeta+\omega)-h_1(\zeta)}{\zeta(\mathfrak{q}-1)+\omega}, & \zeta \neq \omega_0, \\ h_1'(\omega_0), & \zeta = \omega_0, \end{cases} \tag{1.1}$$

provided that h_1 is differentiable at ω_0, where $\mathfrak{q} \in (0,1)$ for some fixed $\omega \geq 0$.

The Hahn difference operator unifies (in the limit) the two most distinguished and extensively used quantum difference operators: the Jackson q-difference derivative $\mathcal{D}_{\mathfrak{q}}$ [7], where $\mathfrak{q} \in (0,1)$, defined by

$$\mathcal{D}_{\mathfrak{q}} h_1(\zeta) = \begin{cases} \frac{h_1(\zeta)-h_1(\mathfrak{q}\zeta)}{\zeta(1-\mathfrak{q})}, & \zeta \neq 0, \\ h_1'(0), & \zeta = 0, \end{cases} \tag{1.2}$$

provided that $h_1'(0)$ exists for $\omega = 0$, and the forward difference $\mathcal{D}_\omega$ for $\mathfrak{q} \to 1$ defined by

$$\mathcal{D}_\omega h_1(\zeta) = \frac{h_1(\zeta+\omega) - h_1(\zeta)}{\omega} \tag{1.3}$$

for some fixed $\omega > 0$. Hahn's difference operator has been applied successfully in the construction of families of orthogonal polynomials as well as in approximation problems [8–13].

In [14], some concepts of fractional quantum calculus were introduced in terms of a $\mathfrak{q}$-shifting operator ${}_\eta\psi_{\mathfrak{q}}(\zeta) = \mathfrak{q}\zeta + (1-\mathfrak{q})\eta$.

Consider an interval $I = [\mu,\nu] \subseteq \mathbb{R}$. The two quantum numbers $0 < \mathfrak{q} < 1$, $\omega \geq 0$ can generate a point η of Hahn calculus on an interval $[\mu,\nu]$ by

$$\eta = \mu + \frac{\omega}{1-\mathfrak{q}},$$

which states that $\eta \in [\mu,\nu]$ for all consequences of our investigation. The quantum Hahn shifting operator is stated as

$${}_\eta\psi_{\mathfrak{q}}(\zeta) = \mathfrak{q}\zeta + (1-\mathfrak{q})\eta, \quad \zeta \in [\mu,\nu]. \tag{1.4}$$

It follows that the iterated κ-times quantum shifting is introduced by

$${}_\eta\psi_{\mathfrak{q}}^{\kappa}(\zeta) = {}_\eta\psi_{\mathfrak{q}}^{\kappa-1}\big({}_\eta\psi_{\mathfrak{q}}(\zeta)\big) = \mathfrak{q}^\kappa\zeta + \big(1-\mathfrak{q}^\kappa\big)\eta,$$

with ${}_\eta\psi_{\mathfrak{q}}^{0}(\zeta) = \zeta$ for $\zeta \in [\mu,\nu]$.

Let us demonstrate the preliminaries of quantum Hahn calculus on an interval $[\mu,\nu]$ which are the results in [15] modified according to (1.4).

Definition 1.1 Let h_1 be a function defined on $[\mu,\nu]$. The quantum Hahn difference operator is defined as

$${}_\mu\mathcal{D}_{\mathfrak{q},\omega} h_1(\zeta) = \begin{cases} \frac{h_1(\zeta)-h_1({}_\eta\psi_{\mathfrak{q}}(\zeta))}{\zeta(\mathfrak{q}-1)+\omega}, & \zeta \neq \eta, \\ h_1'(\eta), & \zeta \neq \eta, \end{cases} \tag{1.5}$$

provided that h_1 is differentiable at η.

Definition 1.2 Assume that $h_1 : [\mu,\nu] \to \mathbb{R}$ is a given function and two points $x, y \in [\mu,\nu]$. The $\mathfrak{q}$, ω-quantum Hahn integral of h_1 from x to y is defined by

$$\int_x^y h_1(s)_\mu \, d_{\mathfrak{q},\omega}s := \int_\eta^y h_1(s)_\mu \, d_{\mathfrak{q},\omega}s - \int_\eta^x h_1(s)_\mu \, d_{\mathfrak{q},\omega}s, \tag{1.6}$$

where

$$\int_{\eta}^{\zeta} h_1(s)_{\mu}\, d_{\mathrm{q},\omega}s = \left[\zeta -_{\eta} \psi_{\mathrm{q}}(\zeta)\right]\sum_{i=0}^{\infty}\mathrm{q}^i h_1\left({}_{\eta}\psi_{\mathrm{q}}^i(\zeta)\right) \tag{1.7}$$

for $\zeta \in [\mu,\nu]$, provided that the series converge at $\zeta = x$ and $\zeta = y$. The function h_1 is said to be q,ω-integrable on $[\mu,\nu]$ if (1.7) exists for all $\zeta \in [\mu,\nu]$.

Before approaching the main definitions of fractional quantum Hahn calculus on $[\mu,\nu]$, we present the η-power function which is stated as follows:

$$(n-m)_{\eta}^{(0)} = 1, \qquad (n-m)_{\eta}^{(\kappa)} = \prod_{i=0}^{\kappa-1}\left(n - {}_{\eta}\psi_{\mathrm{q}}^i(m)\right), \quad \kappa \in \mathbb{N}\cup\{\infty\}.$$

Precisely, if $\alpha \in \mathbb{R}$, then

$$(n-m)_{\eta}^{(\alpha)} = \prod_{i=0}^{\infty}\frac{(n - {}_{\eta}\psi_{\mathrm{q}}^i(m))}{(n - {}_{\eta}\psi_{\mathrm{q}}^{\alpha+i}(m))}, \tag{1.8}$$

with ${}_{\eta}\psi_{\mathrm{q}}^{\varsigma}(m) = \mathrm{q}^{\varsigma}m + (1-\mathrm{q}^{\varsigma})\eta$, $\varsigma \in \mathbb{R}$.

The q-gamma function can be defined as

$$\Gamma_{\mathrm{q}}(\alpha) = \frac{(1-\mathrm{q})_0^{(\alpha-1)}}{(1-\mathrm{q})^{\alpha-1}}, \quad \alpha \in \mathbb{R}\setminus\{0,-1,-2,\ldots\}. \tag{1.9}$$

Obviously, $\Gamma_{\mathrm{q}}(\alpha+1) = [\alpha]_{\mathrm{q}}\Gamma_{\mathrm{q}}(\alpha)$, where $[c]_{\mathrm{q}} = \frac{(1-\mathrm{q}^c)}{1-\mathrm{q}}$, $c \in \mathbb{R}$, where q is the quantum number.

Now we consider the concept of Riemann–Liouville type of fractional derivative and integral of quantum Hahn calculus on an interval $[\mu,\nu]$ which is proposed by [16].

Definition 1.3 ([16]) Suppose that a function $h_1 : [\mu,\nu] \to \mathbb{R}$ is said to be fractional quantum Hahn derivative of Riemann–Liouville type of order $\varrho_1 \geq 0$ if

$$\left({}_{\mu}\mathcal{D}_{\mathrm{q},\omega}^{\varrho_1}h_1\right)(\zeta) = \frac{1}{\Gamma_{\mathrm{q}}(n-\varrho_1)}{}_{\mu}\mathcal{D}_{\mathrm{q},\omega}^{n}\int_{\mu}^{\zeta}\left(\zeta - {}_{\eta}\psi_{\mathrm{q}}(s)\right)_{\eta}^{(n-\varrho_1-1)}h_1(s)_{\mu}\, d_{\mathrm{q},\omega}, \tag{1.10}$$

where n is the smallest integer greater than or equal to ϱ_1.

Definition 1.4 ([16]) Suppose that a function $h_1 : [\mu,\nu] \to \mathbb{R}$ is said to be fractional quantum Hahn integral of Riemann–Liouville type of order $\varrho_1 \geq 0$ if

$$\left({}_{\mu}\mathcal{I}_{\mathrm{q},\omega}^{\varrho_1}h_1\right)(\zeta) = \frac{1}{\Gamma_{\mathrm{q}}(\varrho_1)}\int_{\mu}^{\zeta}\left(\zeta - {}_{\eta}\psi_{\mathrm{q}}(s)\right)_{\eta}^{(\varrho_1-1)}h_1(s)_{\mu}\, d_{\mathrm{q},\omega}, \quad \varrho_1 > 0, \zeta \in [\mu,\nu]. \tag{1.11}$$

Also, $({}_{\mu}\mathcal{I}_{\mathrm{q},\omega}^{0}h_1)(\zeta) = h_1(\zeta)$ and provided the right-hand side exists.

Theorem 1.5 ([16]) *Let* $\varrho_1, \varrho_2 \in \mathbb{R}^+$, $\vartheta \in (-1,\infty)$, *and* $\eta \in [\mu,\nu.]$ *The following formula hold*:

$$\left({}_{\mu}\mathcal{I}_{\mathrm{q},\omega}^{\varrho_1}(z-\mu)_{\eta}^{(\vartheta)}\right)(\zeta) = \frac{\Gamma_{\mathrm{q}}(\vartheta+1)}{\Gamma_{\mathrm{q}}(\varrho_1+\vartheta+1)}(\zeta-\mu)_{\eta}^{(\varrho_1+\vartheta)},$$

$$\big({}_{\mu}\mathcal{D}^{\varrho_1}_{\mathfrak{q},\omega}(z-\mu)^{(\vartheta)}_{\eta}\big)(\zeta)=\frac{\Gamma_{\mathfrak{q}}(\vartheta+1)}{\Gamma_{\mathfrak{q}}(\varrho_1-\vartheta+1)}(\zeta-\mu)^{(\varrho_1-\vartheta)}_{\eta}.$$

Fractional calculus has increased a great deal of consideration from analysts because of their expertise to increase certifiable issues utilized in different fields of science and engineering. Various physical phenomena in signal processing, control theory, electrochemistry of erosion, potential theory, acoustics, and electromagnetic are absolutely displayed by the utilization of its applications [17–19]. The utilities of variants have gained considerable importance among researchers for fixed point theorems and existence and uniqueness of solutions for differential equations. Numerous numerical and analytical methods have been recommended for the advancement of integral inequalities [20–39].

Tariboon et al. [40] expounded the concept of $\mathfrak{q}$-derivatives over the definite interval $[\mu,\nu]\subset\mathbb{R}$ and contemplated several versions on quantum analogues, for example, $\mathfrak{q}$-Cauchy–Schwarz inequality, the $\mathfrak{q}$-Grüss–Chebyshev integral inequality, the $\mathfrak{q}$-Grüss inequality, and other integral inequalities by classical convexity.

In this study we derive several new variants via quantum Hahn fractional integral operator concerning convex functions, reverse Minkowski and reverse Hölder inequalities via quantum Hahn integral operator. The concept is relatively new and appears to have opened new doors of research towards different areas of research including meteorology, quantum mechanics, biosciences, chaos, image processing, power-law, biochemistry, physics, and several others. The integral inequalities by quantum Hahn integral operator can focus on a predetermined number of complex problems on one hand, and on the other hand their applications can likewise catch various sorts of complexities. In this manner assembling these three speculations can help us to comprehend the complexities of existing nature in a vastly improved manner. Quantum fractional integrals have fascinated the attention of practically all scientists from various fields of science. It is noted that the quantum fractional estimate is able to appreciate some kind of self-similarities.

2 Some new generalizations by fractional quantum Hahn integral operator

Assume that for quantum numbers $0<\mathfrak{q}_i<1$, $\omega_i\geq 0$, $i=1,2$, and the points $\eta_i\in[\mu,\nu]$, $i=1,2$, can be defined as

$$\eta_i=\frac{\omega_i+(1-\mathfrak{q}_i)\mu}{1-\mathfrak{q}_i}.$$

Theorem 2.1 *Let f_1 and h_1 be two $\mathfrak{q}_1,\omega_1$-integrable functions defined on $[\mu,\nu]$, and $f_1\leq h_1$ on $[\mu,\nu]$. If $\frac{f_1}{h_1}$ is decreasing, f_1 is increasing on $[\mu,\nu]$, then, for any convex function Θ having $\Theta(0)=0$, the inequality*

$$\frac{{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}[f_1(\xi)]}{{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}[h_1(\xi)]}\geq\frac{{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}[\Theta(f_1(\xi))]}{{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}[\Theta(h_1(\xi))]}\tag{2.1}$$

holds for $\xi\in[\mu,\nu]$, $\varrho_1,>0$, $0<\mathfrak{q}_1<1$, and $\omega_1\geq 0$.

Proof Since Θ is convex having $\Theta(0)=0$, then $\frac{\Theta(\xi)}{\xi}$ is increasing. This and the fact that $f_1(\xi)\leq h_1(\xi)$ gives

$$\frac{\Theta(f_1(\xi))}{f_1(\xi)}\leq\frac{\Theta(h_1(\xi))}{h_1(\xi)}.$$

Also, f_1 and $\frac{\Theta(\xi)}{\xi}$ are increasing, then the function $\frac{\Theta(f_1(\xi))}{f_1(\xi)}$ is also increasing. Clearly $\frac{f_1(\xi)}{h_1(\xi)}$ is decreasing. Thus, for all $\lambda, \rho \in [\mu, \xi)$, we have

$$\left(\frac{\Theta(f_1(\lambda))}{f_1(\lambda)} - \frac{\Theta(f_1(\rho))}{f_1(\rho)}\right)\left(\frac{f_1(\rho)}{h_1(\rho)} - \frac{f_1(\lambda)}{h_1(\lambda)}\right) \geq 0. \tag{2.2}$$

It follows that

$$\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}\frac{f_1(\rho)}{h_1(\rho)} + \frac{\Theta(f_1(\rho))}{f_1(\rho)}\frac{f_1(\lambda)}{h_1(\lambda)} - \frac{\Theta(f_1(\rho))}{f_1(\rho)}\frac{f_1(\rho)}{h_1(\rho)} - \frac{\Theta(f_1(\lambda))}{f_1(\lambda)}\frac{f_1(\lambda)}{h_1(\lambda)} \geq 0. \tag{2.3}$$

Multiplying (2.3) by $h_1(\lambda)h_1(\rho)$, we have

$$\begin{aligned}&\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}f_1(\rho)h_1(\lambda) + \frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\lambda)h_1(\rho)\\&\quad - \frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\rho)h_1(\lambda) - \frac{\Theta(f_1(\lambda))}{f_1(\lambda)}f_1(\lambda)h_1(\rho) \geq 0.\end{aligned} \tag{2.4}$$

Multiplying (2.4) by $\frac{(\xi - {}_{\lambda_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu, \xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration with respect to λ on $[\mu, \xi)$, one obtains

$$\begin{aligned}&\int_\mu^\xi \frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}f_1(\rho)h_1(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\&\quad + \int_\mu^\xi \frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\lambda)h_1(\rho)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\&\quad - \int_\mu^\xi \frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\rho)h_1(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\&\quad - \int_\mu^\xi \frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}f_1(\lambda)h_1(\rho)\,d_{\mathfrak{q}_1,\omega_1}\lambda \geq 0.\end{aligned} \tag{2.5}$$

From this it follows that

$$\begin{aligned}&f_1(\rho)\,{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\right) + \left(\frac{\Theta(f_1(\rho))}{f_1(\rho)}h_1(\rho)\right){}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(f_1(\xi)\big)\\&\quad - \left(\frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\rho)\right){}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(h_1(\xi)\big) - h_1(\rho)\,{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}f_1(\xi)\right) \geq 0.\end{aligned} \tag{2.6}$$

Again, multiplying both sides of (2.6) by $\frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\rho))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\rho \in [\mu, \xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration with respect to ρ on $[\mu, \xi)$, one obtains

$$\begin{aligned}&{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(f_1(\xi)\big)\,{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\right) + {}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\right){}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(f_1(\xi)\big)\\&\quad \geq {}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(h_1(\xi)\big)\,{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(\Theta\big(f_1(\xi)\big)\big) + {}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(\Theta\big(f_1(\xi)\big)\big)^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\big(h_1(\xi)\big).\end{aligned} \tag{2.7}$$

It follows that

$$\frac{{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(f_1(\xi))}{{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1(\xi))} \geq \frac{{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(\Theta(f_1(\xi)))}{{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(\frac{\Theta(f_1(\xi))}{f_1(\xi)}(h_1(\xi)))}. \tag{2.8}$$

Now, since $f_1 \leq h_1$ on $[\mu,\nu]$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda,\rho \in [\mu,\xi)$, we have

$$\frac{\Theta(f_1(\lambda))}{f_1(\lambda)} \leq \frac{\Theta(h_1(\lambda))}{h_1(\lambda)}. \tag{2.9}$$

Multiplying both sides of (2.9) by $\frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu,\xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration with respect to λ on $[\mu,\xi)$, one obtains

$$\begin{aligned}
&\int_\mu^\xi \frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)} \frac{\Theta(f_1(\lambda))}{f_1(\lambda)} h_1(\lambda)\, d_{\mathfrak{q}_1,\omega_1}\lambda \\
&\quad \leq \int_\mu^\xi \frac{(\xi - {}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)} \frac{\Theta(h_1(\lambda))}{h_1(\lambda)} h_1(\lambda)\, d_{\mathfrak{q}_1,\omega_1}\lambda,
\end{aligned} \tag{2.10}$$

which, in view of (2.7), can be written as

$${}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)} h_1(\xi)\right) \leq {}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(\Theta\big(h_1(\lambda)\big)\big). \tag{2.11}$$

Hence, from (2.8) and (2.11), we get (2.1). □

Theorem 2.2 *Let f_1 and h_1 be two $\mathfrak{q}_i$, ω_i-integrable functions defined on $[\mu,\nu]$, $i = 1,2$ and $f_1 \leq h_1$ on $[\mu,\nu]$. If $\frac{f_1}{h_1}$ is decreasing, f_1 is increasing on $[\mu,\nu]$, then for any convex function Θ having $\Theta(0) = 0$, the inequality*

$$\frac{{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[f_1(\xi)]\,{}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}[\Theta(h_1(\xi))] + {}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}[f_1(\xi)]\,{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[\Theta(h_1(\xi))]}{{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[h_1(\xi)]\,{}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}[\Theta(f_1(\xi))] + {}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}[h_1(\xi)]\,{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[\Theta(f_1(\xi))]} \geq 1 \tag{2.12}$$

holds for $\xi \in [\mu,\nu]$, $\varrho_1,\varrho_2 > 0$, $0 < \mathfrak{q}_i < 1$, and $\omega_i \geq 0$, $i = 1,2$.

Proof Since f_1 is increasing along with the function $\frac{\Theta(f_1(\xi))}{f_1(\xi)}$, using the convexity of Θ having $\Theta(0) = 0$, the function $\frac{\Theta(\xi)}{\xi}$ is increasing.

Thus, for all $\lambda,\rho \in [\mu,\xi)$.

Multiplying (2.6) by $\frac{(\xi - {}_{\eta_2}\psi_{\mathfrak{q}_2}(\rho))_{\eta_2}^{(\varrho_2-1)}}{\Gamma_{\mathfrak{q}_2}(\varrho_2)}$, $\rho \in [\mu,\xi)$ and taking the $\mathfrak{q}_2$, ω_2-integration with respect to ρ on $[\mu,\xi)$, one obtains

$$\begin{aligned}
&{}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}\big(f_1(\xi)\big)\,{}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)} h_1(\xi)\right) \\
&\qquad + {}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)} h_1(\xi)\right){}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(f_1(\xi)\big) \\
&\quad \geq {}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(h_1(\xi)\big)\,{}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)} f_1(\xi)\right) \\
&\qquad + {}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)} f_1(\xi)\right){}_\mu\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\varrho_2}\big(h_1(\xi)\big).
\end{aligned} \tag{2.13}$$

Now, since $f_1 \leq h_1$ on $[\mu,\nu]$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda,\rho \in [\mu,\xi)$, we have

$$\frac{\Theta(f_1(\lambda))}{f_1(\lambda)} \leq \frac{\Theta(h_1(\lambda))}{h_1(\lambda)}. \tag{2.14}$$

Multiplying both sides of (2.14) by $\frac{(\xi-{}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu,\xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration with respect to λ on $[\mu,\xi)$, one obtains

$$\begin{aligned}&\int_{\mu}^{\xi}\frac{(\xi-{}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}h_1(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\&\quad\leq\int_{\mu}^{\xi}\frac{(\xi-{}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(h_1(\lambda))}{h_1(\lambda)}h_1(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda,\end{aligned}\tag{2.15}$$

which, in view of (2.7), can be written as

$${}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\right)\leq{}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(h_1(\xi))}{h_1(\xi)}h_1(\xi)\right).\tag{2.16}$$

Hence from (2.11), (2.13), and (2.16), we get our desired result. □

We further have the subsequent main result.

Theorem 2.3 *Let f_1, h_1, and $\mathcal{W}$ be three $\mathfrak{q}_1$, ω_1-integrable functions defined on $[\mu,\nu]$, $i = 1,2$, and $f_1\leq h_1$ on $[\mu,\nu]$. If $\frac{f_1}{h_1}$ is decreasing, f_1 is increasing on $[\mu,\nu]$, then, for any convex function Θ having $\Theta(0) = 0$, the inequality*

$$\frac{{}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[f_1(\xi)]}{{}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[h_1(\xi)]}\geq\frac{{}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[\Theta(f_1(\xi))\mathcal{W}(\xi)]}{{}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}[\Theta(h_1(\xi))\mathcal{W}(\xi)]}\tag{2.17}$$

holds for $\xi\in[\mu,\nu]$, $\varrho_1>0$, $0<\mathfrak{q}_1<1$, and $\omega_1\geq 0$.

Proof Since $f_1\leq h_1$ on $[\mu,\nu]$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda,\rho\in[\mu,\xi)$, we have

$$\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}\leq\frac{\Theta(h_1(\lambda))}{h_1(\lambda)}.\tag{2.18}$$

Multiplying both sides of (2.18) by $\frac{(\xi-{}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}h_1(\lambda)\mathcal{W}(\lambda)$, $\lambda\in[\mu,\xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration with respect to λ on $[\mu,\xi)$, we get

$$\begin{aligned}&\int_{\mu}^{\xi}\frac{(\xi-{}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}h_1(\lambda)\mathcal{W}(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\&\quad\leq\int_{\mu}^{\xi}\frac{(\xi-{}_{\eta_1}\psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(h_1(\lambda))}{h_1(\lambda)}h_1(\lambda)\mathcal{W}(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda,\end{aligned}\tag{2.19}$$

which, in view of (2.7), can be written as

$${}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\right)\leq{}_{\mu}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\bigl(\Theta\bigl(h_1(\xi)\bigr)\mathcal{W}(\xi)\bigr).\tag{2.20}$$

Also, since Θ is convex with $\Theta(0) = 0$, the function $\frac{\Theta(\xi)}{\xi}$ is increasing. As f_1 is increasing, so is the function $\frac{\Theta(f_1(\xi))}{f_1(\xi)}$. Obviously, the function $\frac{f_1(\xi)}{h_1(\xi)}$ is decreasing for all $\lambda,\rho\in[\mu,\xi)$, we have

$$\left(\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}\mathcal{W}(\lambda)-\frac{\Theta(f_1(\rho))}{f_1(\rho)}\mathcal{W}(\rho)\right)\bigl(f_1(\rho)h_1(\lambda)-f_1(\lambda)h_1(\rho)\bigr)\geq 0.\tag{2.21}$$

It follows that

$$\frac{\Theta(f_1(\lambda))\mathcal{W}(\lambda)}{f_1(\lambda)}f_1(\rho)h_1(\lambda)+\frac{\Theta(f_1(\rho))\mathcal{W}(\rho)}{f_1(\rho)}f_1(\lambda)h_1(\rho) - \frac{\Theta(f_1(\rho))\mathcal{W}(\rho)}{f_1(\rho)}f_1(\rho)h_1(\lambda)-\frac{\Theta(f_1(\lambda))\mathcal{W}(\lambda)}{f_1(\lambda)}f_1(\lambda)h_1(\rho)\geq 0. \tag{2.22}$$

Multiplying (2.22) by $\frac{(\xi-_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda\in[\mu,\xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu,\xi)$, one obtains

$$\begin{aligned}&\int_\mu^\xi \frac{(\xi-_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}f_1(\rho)h_1(\lambda)\mathcal{W}(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\ &\quad+\int_\mu^\xi \frac{(\xi-_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\lambda)h_1(\rho)\mathcal{W}(\rho)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\ &\quad-\int_\mu^\xi \frac{(\xi-_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\rho)h_1(\lambda)\mathcal{W}(\rho)\,d_{\mathfrak{q}_1,\omega_1}\lambda\\ &\quad-\int_\mu^\xi \frac{(\xi-_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\frac{\Theta(f_1(\lambda))}{f_1(\lambda)}f_1(\lambda)h_1(\rho)\mathcal{W}(\lambda)\,d_{\mathfrak{q}_1,\omega_1}\lambda\geq 0.\end{aligned} \tag{2.23}$$

From this it follows that

$$\begin{aligned}&f_1(\rho)\,{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\right)\\ &\quad+\left(\frac{\Theta(f_1(\rho))}{f_1(\rho)}h_1(\rho)\mathcal{W}(\rho)\right){}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(f_1(\xi)\big)\\ &\quad-\left(\frac{\Theta(f_1(\rho))}{f_1(\rho)}f_1(\rho)\mathcal{W}(\rho)\right){}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(h_1(\xi)\big)\\ &\quad-h_1(\rho)\,{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}f_1(\xi)\mathcal{W}(\xi)\right)\geq 0.\end{aligned} \tag{2.24}$$

Again, multiplying both sides of (2.24) by $\frac{(\xi-_{\eta_1}\Psi_{\mathfrak{q}_1}(\rho))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\rho\in[\mu,\xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to ρ on $[\mu,\xi)$, one obtains

$$\begin{aligned}&{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(f_1(\xi)\big){}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\right)\\ &\qquad+{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\left(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\right){}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(f_1(\xi)\big)\\ &\quad\geq {}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(h_1(\xi)\big){}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(\Theta f_1(\xi)\mathcal{W}(\xi)\big)\\ &\qquad+{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(\Theta f_1(\xi)\mathcal{W}(\xi)\big){}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(h_1(\xi)\big).\end{aligned} \tag{2.25}$$

It follows that

$$\frac{{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(f_1(\xi))}{{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(f_1(\xi))}\geq\frac{{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(\Theta(f_1(\xi))\mathcal{W}(\xi))}{{}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(\frac{\Theta(f_1(\xi))}{f_1(\xi)}(h_1(\xi))\mathcal{W}(\xi))}. \tag{2.26}$$

Hence from (2.20) and (2.26), we get our required result. □

Theorem 2.4 *Let f_1, h_1, and $\mathcal{W}$ be three $\mathfrak{q}_i$, ω_i-integrable functions defined on $[\mu,\nu]$, $i = 1,2$, and $f_1 \leq h_1$ on $[\mu,\nu]$. If $\frac{f_1}{h_1}$ is decreasing, f_1 is increasing on $[\mu,\nu]$, then, for any convex function Θ having $\Theta(0) = 0$, the inequality*

$$\begin{aligned}&\bigl({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl[f_1(\xi)\bigr]{}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl[\Theta\bigl(f_1(\xi)\bigr)\mathcal{W}(\xi)\bigr]+{}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl[f_1(\xi)\bigr]{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl[\Theta\bigl(f_1(\xi)\bigr)\mathcal{W}(\xi)\bigr]\bigr)\\&\quad\Big/\bigl({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl[h_1(\xi)\bigr]{}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl[\Theta\bigl(f_1(\xi)\bigr)\mathcal{W}(\xi)\bigr]\\&\quad+{}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl[h_1(\xi)\bigr]{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl[\Theta\bigl(f_1(\xi)\bigr)\mathcal{W}(\xi)\bigr]\bigr)\geq 1\end{aligned}\tag{2.27}$$

holds for $\xi \in [\mu,\nu]$, $\varrho_1,\varrho_2 > 0$, $0 < \mathfrak{q}_i < 1$, and $\omega_i \geq 0$, $i = 1,2$.

Proof Multiplying both sides of (2.24) by $\frac{(\xi-\eta_2\Psi_{\mathfrak{q}_2}(\rho))^{(\varrho_2-1)}_{\eta_2}}{\Gamma_{\mathfrak{q}_2}(\varrho_2)}$, $\rho \in [\mu,\xi)$ and taking the $\mathfrak{q}_2$, ω_2-integration of the resulting inequality with respect to ρ on $[\mu,\xi)$, one obtains

$$\begin{aligned}&{}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl(f_1(\xi)\bigr){}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\biggl(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\biggr)\\&\qquad+{}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\biggl(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\biggr){}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl(f_1(\xi)\bigr)\\&\quad\geq{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl(h_1(\xi)\bigr){}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl(\Theta\bigl(f_1(\xi)\bigr)\mathcal{W}(\xi)\bigr)\\&\qquad+{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl(\Theta\bigl(f_1(\xi)\bigr)\mathcal{W}(\xi)\bigr){}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl(h_1(\xi)\bigr).\end{aligned}\tag{2.28}$$

Now, since $f_1 \leq h_1$ on $[\mu,\nu]$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda,\rho \in [\mu,\xi)$, we have

$$\frac{\Theta(f_1(\lambda))}{f_1(\lambda)} \leq \frac{\Theta(h_1(\lambda))}{h_1(\lambda)}.\tag{2.29}$$

Multiplying both sides of (2.29) by $\frac{(\xi-\eta_1\Psi_{\mathfrak{q}_1}(\lambda))^{(\varrho_1-1)}_{\eta_1}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}h_1(\lambda)\mathcal{W}(\lambda)$, $\lambda \in [\mu,\xi)$ and taking the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu,\xi)$, one obtains

$${}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\biggl(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\biggr)\leq{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\bigl(\Theta\bigl(h_1(\xi)\bigr)\mathcal{W}(\xi)\bigr).\tag{2.30}$$

Similarly, multiplying both sides of (2.29) by $\frac{(\xi-\eta_2\Psi_{\mathfrak{q}_2}(\rho))^{(\varrho_2-1)}_{\eta_2}}{\Gamma_{\mathfrak{q}_2}(\varrho_2)}f_1(\rho)\mathcal{W}(\lambda)$, $\rho \in [\mu,\xi)$ and taking the $\mathfrak{q}_2$, ω_2-integration of the resulting inequality with respect to ρ on $[\mu,\xi)$, one obtains

$${}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\biggl(\frac{\Theta(f_1(\xi))}{f_1(\xi)}h_1(\xi)\mathcal{W}(\xi)\biggr)\leq{}_{\mu}\mathcal{I}^{\varrho_2}_{\mathfrak{q}_2,\omega_2}\bigl(\Theta\bigl(h_1(\xi)\bigr)\mathcal{W}(\xi)\bigr).\tag{2.31}$$

Hence from (2.28), (2.30), and (2.31), we get our desired result. □

3 New estimates for reverse Minkowski inequality by fractional quantum Hahn integral operator

This segment comprises our principal involvement of establishing the proof of the reverse Minkowski inequalities via fractional quantum Hahn integral operator defined in (1.11).

Theorem 3.1 *For $p \geq 1$ and $\omega_1 \geq 0$, let f_1 and h_1 be two $\mathfrak{q}_1, \omega_1$-integrable functions defined on $[0,\infty)$ such that, for all $\xi > \mu$, ${}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^p(\xi) < \infty$ and ${}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} f_1^p(\xi) < \infty$. If $0 < \theta_1 \leq \frac{h_1(\lambda)}{f_1(\lambda)} \leq \theta_2$ for $\theta_1, \theta_2 \in \mathbb{R}^+$ and for all $\lambda \in [\mu,\xi]$, then*

$$\begin{aligned}&\big({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^p(\xi)\big)^{\frac{1}{p}} + \big({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} f_1^p(\xi)\big)^{\frac{1}{p}} \\ &\quad\leq \frac{1+\theta_2(\theta_1+2)}{(\theta_1+1)(\theta_2+1)}\big({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(h_1+f_1)^p(\xi)\big)^{\frac{1}{p}}\end{aligned} \tag{3.1}$$

holds for $\xi \in [\mu,\nu]$, $\varrho_1 > 0$, and $0 < \mathfrak{q}_1 < 1$.

Proof By the suppositions mentioned in Theorem 3.1, $\frac{h_1(\lambda)}{f_1(\lambda)} \leq \theta_2$, $\mu \leq \lambda \leq \xi$, it tends to be composed

$$(M+1)^p h_1^p(\lambda) \leq M^p\big(h_1(\lambda)+f_1(\lambda)\big)^p. \tag{3.2}$$

If we multiply both sides of (3.2) with $\frac{(\xi - {}_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))^{(\varrho_1-1)}_{\eta_1}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu,\xi)$ and take the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu,\xi)$, we obtain

$$\begin{aligned}&\frac{(M+1)^p}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\int_{\mu}^{\xi}\big(\xi - {}_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda)\big)^{(\varrho_1-1)}_{\eta_1} h_1^p(\lambda)\,d\lambda_{\mathfrak{q}_1,\omega_1} \\ &\quad\leq \frac{M^p}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\int_{\mu}^{\xi}\big(\xi - {}_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda)\big)^{(\varrho_1-1)}_{\eta_1}\big(h_1(\lambda)+f_1(\lambda)\big)^p\,d\lambda_{\mathfrak{q}_1,\omega_1}.\end{aligned} \tag{3.3}$$

Similarly, it can be written as

$$\big({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^p(\xi)\big)^{\frac{1}{p}} \leq \frac{\theta_2}{\theta_2+1}\big({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(h_1+f_1)^p(\xi)\big)^{\frac{1}{p}}. \tag{3.4}$$

In contrast, as $mf_1(\lambda) \leq h_1(\lambda)$, it follows

$$\left(1+\frac{1}{\theta_1}\right)^p f_1^p(\lambda) \leq \left(\frac{1}{\theta_1}\right)^p\big(h_1(\lambda)+f_1(\lambda)\big)^p. \tag{3.5}$$

Again, if we multiply both sides of (3.5) with $\frac{(\xi - {}_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))^{(\varrho_1-1)}_{\eta_1}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu,\xi)$ and take the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu,\xi)$, we obtain

$$\big({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} f_1^p(\xi)\big)^{\frac{1}{p}} \leq \frac{1}{\theta_1+1}\big({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(h_1+f_1)^p(\xi)\big)^{\frac{1}{p}}. \tag{3.6}$$

Thus adding (3.4) and (3.6) yields the desired inequality. □

Inequality (3.1) is referred to as the reverse Minkowski inequality by fractional quantum Hahn integral operator.

Theorem 3.2 *For $p \geq 1$ and $\omega_1 \geq 0$, let f_1 and h_1 be two $\mathfrak{q}_1, \omega_1$-integrable functions defined on $[0,\infty)$ such that, for all $\xi > \mu$, ${}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^p(\xi) < \infty$ and ${}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} f_1^p(\xi) < \infty$. If $0 < \theta_1 \leq \frac{h_1(\lambda)}{f_1(\lambda)} \leq$*

θ_2 for $\theta_1,\theta_2\in\mathbb{R}^+$ and for all $\lambda\in[\mu,\xi]$, then the inequality

$$\begin{aligned}&\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi)\right)^{\frac{2}{p}}+\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}f_1^p(\xi)\right)^{\frac{2}{p}}\\&\quad\le\left(\frac{(\theta_1+1)(\theta_2+1)}{\theta_2}-2\right)\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi)\right)^{\frac{1}{p}}\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi)\right)^{\frac{1}{p}}\end{aligned}\tag{3.7}$$

holds for $\xi\in[\mu,\nu]$, $\varrho_1>0$, and $0<\mathfrak{q}_1<1$.

Proof The product of inequalities (3.4) and (3.6) yields

$$\begin{aligned}&\left(\frac{(\theta_1+1)(\theta_2+1)}{\theta_2}-2\right)\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi)\right)^{\frac{1}{p}}\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}f_1^p(\xi)\right)^{\frac{1}{p}}\\&\quad\le\left[\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(h_1+f_1)^p(\xi)\right)^{\frac{1}{p}}\right]^2.\end{aligned}\tag{3.8}$$

Now, utilizing the Minkowski inequality to the right-hand side of (3.7), one obtains

$$\begin{aligned}&\left[\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}(h_1+f_1)^p(\xi)\right)^{\frac{1}{p}}\right]^2\\&\quad\le\left[{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi))^{\frac{1}{p}}+{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}f_1^p(\xi))^{\frac{1}{p}}\right]^2\\&\quad\le{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi))^{\frac{2}{p}}+{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi))^{\frac{2}{p}}+2\left[{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi))^{\frac{1}{p}}\right]\left[{}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi))^{\frac{1}{p}}\right].\end{aligned}\tag{3.9}$$

Thus, from inequalities (3.8) and (3.9), we attain inequality (3.7). □

4 New bounds for reverse Hölder inequality by fractional quantum Hahn integral operator

This section is dedicated to deriving bounds for reverse Hölder inequalities regarding fractional quantum Hahn integral operator.

Theorem 4.1 *For $p,q\ge1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\omega_1\ge0$, let f_1 and h_1 be two $\mathfrak{q}_1,\omega_1$-integrable functions defined on $[0,\infty)$ such that, for all $\xi>\mu$, ${}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi)<\infty$ and ${}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}f_1^p(\xi)<\infty$. If $0<\theta_1\le\frac{h_1(\lambda)}{f_1(\lambda)}\le\theta_2$ for $\theta_1,\theta_2\in\mathbb{R}^+$ and for all $\lambda\in[\mu,\xi]$, then the inequality*

$$\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^p(\xi)\right)^{\frac{1}{p}}\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}f_1^q(\xi)\right)^{\frac{1}{q}}\le\left(\frac{\theta_2}{\theta_1}\right)^{\frac{1}{pq}}\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}h_1^{\frac{1}{p}}(\xi)f_1^{\frac{1}{q}}(\xi)\right)\tag{4.1}$$

holds for $\xi\in[\mu,\nu]$, $\varrho_1>0$, and $0<\mathfrak{q}_1<1$.

Proof Under the given suppositions $\frac{h_1(\lambda)}{f_1(\lambda)}\le\theta_2$, $r_1\le\eta\le y$, we have

$$f_1^{\frac{1}{q}}(\lambda)\ge\theta_2^{\frac{-1}{q}}h_1^{\frac{1}{q}}(\lambda).\tag{4.2}$$

Taking product on both sides of (4.2) by $h_1^{\frac{1}{p}}(\lambda)$, it follows that

$$h_1^{\frac{1}{p}}(\lambda)f_1^{\frac{1}{q}}(\lambda)\ge\theta_2^{\frac{-1}{q}}h_1(\lambda).\tag{4.3}$$

If we multiply both sides of (4.3) with $\frac{(\xi -_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu,\xi)$ and take the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu,\xi)$, we obtain

$$\begin{aligned}&\frac{\theta_2^{\frac{-1}{q}}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\int_\mu^\xi \big(\xi -_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda)\big)_{\eta_1}^{(\varrho_1-1)} h_1(\lambda)\,d\lambda_{\mathfrak{q}_1,\omega_i}\\ &\quad\leq \frac{1}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}\int_\mu^\xi \big(\xi -_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda)\big)_{\eta_1}^{(\varrho_1-1)} h_1^{\frac{1}{p}}(\lambda) f_1^{\frac{1}{q}}(\lambda)\,d\lambda_{\mathfrak{q}_1,\omega_i}.\end{aligned}\tag{4.4}$$

Consequently, we have

$$\theta_2^{-\frac{1}{pq}}\big({}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^p(\xi)\big)^{\frac{1}{p}} \leq \big({}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^{\frac{1}{p}}(\xi) f_1^{\frac{1}{q}}(\xi)\big)^{\frac{1}{p}}.\tag{4.5}$$

In contrast, as $\theta_1 f_1(\lambda) \leq h_1(\lambda)$, therefore we have

$$\theta_1^{\frac{1}{p}} f_1^{\frac{1}{p}}(\lambda) \leq h_1^{\frac{1}{p}}(\lambda).\tag{4.6}$$

Again, if we multiply both sides of (4.6) by $f_1^{\frac{1}{q}}(\lambda)$ and invoke the relation $\frac{1}{p}+\frac{1}{q}=1$, it yields

$$\theta_1^{\frac{1}{p}} f_1(\lambda) \leq h_1^{\frac{1}{p}}(\lambda) f_1^{\frac{1}{q}}(\lambda).\tag{4.7}$$

If we multiply both sides of (4.7) with $\frac{(\xi -_{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu,\xi)$ and take the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu,\xi)$, we obtain

$$\theta_1^{\frac{1}{pq}}\big({}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} f_1(\xi)\big)^{\frac{1}{q}} \leq \big({}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^{\frac{1}{p}}(\xi) f_1^{\frac{1}{q}}(\xi)\big)^{\frac{1}{q}}.\tag{4.8}$$

Multiplying (4.5) and (4.8), the required inequality (4.1) can be concluded. □

Theorem 4.2 *For $p,q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\omega_1 \geq 0$, let f_1 and h_1 be two $\mathfrak{q}_i$, ω_i-integrable functions defined on $[0,\infty)$ such that, for all $\xi > \mu$, ${}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1^p(\xi) < \infty$ and ${}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} f_1^p(\xi) < \infty$. If $0 < \theta_1 \leq \frac{h_1(\lambda)}{f_1(\lambda)} \leq \theta_2$ for $\theta_1,\theta_2 \in \mathbb{R}^+$ and for all $\lambda \in [\mu,\xi]$, then the inequality*

$$\begin{aligned}\big({}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1} h_1(\xi) f_1(\xi)\big) &\leq \frac{2^{p-1}\theta_2^p}{p(\theta_2+1)^p}\big({}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(h_1^p+f_1^p\big)(\xi)\big)\\ &\quad+\frac{2^{q-1}}{p(\theta_1+1)^p}\big({}_\mu\mathcal{I}^{\varrho_1}_{\mathfrak{q}_1,\omega_1}\big(h_1^q+f_1^q\big)(\xi)\big)\end{aligned}\tag{4.9}$$

holds for $\xi \in [\mu,\nu]$, $\varrho_1 > 0$, and $0 < \mathfrak{q}_1 < 1$.

Proof By the given assumption $\frac{h_1(\lambda)}{f_1(\lambda)} < \theta_2$, we have

$$(\theta_2+1)^p h_1^p(\lambda) \leq \theta_2^p (h_1+f_1)^p(\lambda).\tag{4.10}$$

If we multiply both sides of (4.10) with $\frac{(\xi - _{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu, \xi)$ and take the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu, \xi)$, we obtain

$$\begin{aligned} &\frac{(\theta_2+1)^p}{\Gamma_{\mathfrak{q}_1}(\varrho_1)} \int_\mu^\xi \big(\xi - _{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda)\big)_{\eta_1}^{(\varrho_1-1)} h_1^p(\lambda)\,d\lambda_{\mathfrak{q}_1,\omega_i} \\ &\quad \le \frac{\theta_2^p}{\Gamma_{\mathfrak{q}_1}(\varrho_1)} \int_\mu^\xi \big(\xi - _{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda)\big)_{\eta_1}^{(\varrho_1-1)} (h_1+f_1)^p(\lambda)\,d\lambda_{\mathfrak{q}_1,\omega_i}. \end{aligned} \tag{4.11}$$

It follows that

$$\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1} h_1^p(\xi)\big) \le \frac{\theta_2^p}{(\theta_2+1)^p}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1+f_1)^p(\xi)\big). \tag{4.12}$$

In contrast, using $0 < \theta_1 \le \frac{h_1(\lambda)}{f_1(\lambda)}$, $\mu < \lambda < \xi$, we have

$$(\theta_1+1)^q f_1^q(\lambda) \le (h_1+f_1)^q(\lambda). \tag{4.13}$$

Again, if we multiply both sides of (4.13) with $\frac{(\xi - _{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu, \xi)$ and take the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu, \xi)$, we obtain

$$\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1} f_1^q(\xi)\big) \le \frac{1}{(\theta_1+1)^q}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1+f_1)^q(\xi)\big). \tag{4.14}$$

Now, taking into account Young's inequality, we get

$$h_1(\lambda)f_1(\lambda) \le \frac{h_1^p(\lambda)}{p} + \frac{f_1^q(\lambda)}{q}. \tag{4.15}$$

Now, if we multiply both sides of (4.15) with $\frac{(\xi - _{\eta_1}\Psi_{\mathfrak{q}_1}(\lambda))_{\eta_1}^{(\varrho_1-1)}}{\Gamma_{\mathfrak{q}_1}(\varrho_1)}$, $\lambda \in [\mu, \xi)$ and take the $\mathfrak{q}_1$, ω_1-integration of the resulting inequality with respect to λ on $[\mu, \xi)$, we obtain

$$\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1f_1)(\xi)\big) \le \frac{1}{p}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1} h_1^p(\xi)\big) + \frac{1}{q}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1} f_1^q(\xi)\big). \tag{4.16}$$

With the aid of (4.12) and (4.14) into (4.16), one obtains

$$\begin{aligned} &\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1f_1)(\xi)\big) \\ &\quad \le \frac{1}{p}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1} h_1^p(\xi)\big) + \frac{1}{q}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1} f_1^q(\xi)\big) \\ &\quad \le \frac{\theta_2^p}{p(\theta_2+1)^p}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1+f_1)^p(\xi)\big) \\ &\qquad + \frac{1}{q(\theta_1+1)^q}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1+f_1)^q(\xi)\big). \end{aligned} \tag{4.17}$$

Using the inequality $(a_1+a_2)^s \le 2^{s-1}(a_1^s + a_2^s)$, $s > 1$, $a_1, a_2 > 0$, one can obtain

$$\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}(h_1+f_1)^p(\xi)\big) \le 2^{p-1}\big({}_\mu\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\varrho_1}\big(h_1^p+f_1^p\big)(\xi)\big) \tag{4.18}$$

and

$$\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathrm{q}_1,\omega_1}(h_1+f_1)^q(\xi)\right) \leq 2^{q-1}\left({}_{\mu}\mathcal{I}^{\varrho_1}_{\mathrm{q}_1,\omega_1}\left(h_1^q+f_1^q\right)(\xi)\right). \tag{4.19}$$

Hence the proof of (4.9) can be concluded from (4.17), (4.18), and (4.19) collectively. □

5 Application

This section is devoted to the application of q,ω-derivative equations concerning the nonlocal q,ω-integral boundary value problem of nonlinear fractional equations

$$\left({}_{\mu}\mathcal{D}^{\varrho_1}_{\mathrm{q},\omega}p\right)(\xi) + f_1\left(\mathrm{q}\xi+\omega, p(\mathrm{q}\xi+\omega)\right) = 0, \tag{5.1}$$

$$p(\mu) = 0, \qquad p(\nu) = \vartheta\left({}_{\mu}\mathcal{I}^{\varrho_2}_{\mathrm{q},\omega}p\right)(\sigma), \tag{5.2}$$

where $\vartheta > 0$ is a parameter with the assumption that f_1 is a real-valued continuous function, where $0 < \sigma < 1$, $\mathrm{q} \in (0,1)$, $1 \leq \varrho_1 \leq 3$, $0 < \varrho_2 \leq 3$. Also, ${}_{\mu}\mathcal{D}^{\varrho_1}_{\mathrm{q},\omega}$ denotes the quantum Hahn q,ω-derivative operator of order ϱ_1.

Example 5.1 Suppose that a real-valued continuous function u is the unique solution for the boundary value problem

$$\left({}_{\mu}\mathcal{D}^{\varrho_1}_{1/2,\omega}p\right)(\zeta) + u(\zeta/2+\eta) = 0, \quad 1 \leq \varrho_1 \leq 3,$$

with the assumption that $z = \Gamma_{1/2}(\varrho_1+\varrho_2)(\nu-\mu)_\eta^{(\varrho_1-1)} - \eta\Gamma_{1/2}(\varrho_1)(\sigma-\mu)_\eta^{\varrho_1+\varrho_2-1} > 0$ subject to the boundary condition

$$p(\mu) = 0, \qquad p(\nu) = \vartheta\left({}_{\mu}\mathcal{I}^{\varrho_2}_{\mathrm{q},\omega}p\right)(\sigma), \quad 0 < \varrho_2 \leq 3, 0 < \sigma < 1,$$

is given by $p(\zeta) = \int_\mu^\nu \mathcal{Y}(\zeta, {}_\eta\psi_{1/2}(\lambda))u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda$, $\zeta \in [\mu,\nu]$, where

$$\mathcal{Y}\left(\zeta, {}_\eta\psi_{1/2}(\lambda)\right) = y\left(\zeta, {}_\eta\psi_{1/2}(\lambda)\right) + \frac{\vartheta(\zeta-\mu)_\eta^{\varrho_1-1}}{z}\mathcal{A}\left(\sigma, {}_\eta\psi_{1/2}(\lambda)\right),$$

$$y\left(\zeta, {}_\eta\psi_{1/2}(\lambda)\right) = \frac{1}{\Gamma_{1/2}(\varrho_1)}\begin{cases} \frac{(\zeta-\mu)_\eta^{(\varrho_1-1)}}{(\nu-\mu)_\eta^{(\varrho_1-1)}}(\nu - {}_\eta\psi_{1/2}(\lambda))_\eta^{(\varrho_1-1)} - (\zeta - {}_\eta\psi_{1/2}(\lambda))_\eta^{(\varrho_1-1)}, \\ \quad \mu \leq {}_\eta\psi_{1/2}(\lambda) \leq \zeta \leq \nu, \\ \frac{(\zeta-\mu)_\eta^{(\varrho_1-1)}}{(\nu-\mu)_\eta^{(\varrho_1-1)}}(\nu - {}_\eta\psi_{1/2}(\lambda))_\eta^{(\varrho_1-1)}, \quad \mu \leq {}_\eta\psi_{1/2}(\lambda) \leq \zeta \leq \nu, \end{cases}$$

$$\mathcal{A}\left(\sigma, {}_\eta\psi_{1/2}(\lambda)\right) = \begin{cases} \frac{1}{\Gamma_{1/2}(\varrho_1)}\frac{(\sigma-\mu)_\eta^{(\varrho_1-1)}}{(\nu-\mu)_\eta^{(\varrho_1-1)}}(\nu - {}_\eta\psi_{1/2}(\lambda))_\eta^{(\varrho_1-1)} - (\sigma - {}_\eta\psi_{1/2}(\lambda))_\eta^{(\varrho_1-1)}, \\ \quad \mu \leq {}_\eta\psi_{1/2}(\lambda) \leq \sigma \leq \nu, \\ \frac{(\sigma-\mu)_\eta^{(\varrho_1-1)}}{(\nu-\mu)_\eta^{(\varrho_1-1)}}(\nu - {}_\eta\psi_{1/2}(\lambda))_\eta^{(\varrho_1-1)}, \quad \mu \leq {}_\eta\psi_{1/2}(\lambda) \leq \sigma \leq \nu. \end{cases}$$

Proof Suppose that the given system has the following solution:

$$p(\zeta) = a_1(\zeta-\mu)_\eta^{(\varrho_1-1)} + a_2(\zeta-\mu)_\eta^{(\varrho_1-2)} - \frac{1}{\Gamma_{1/2}(\varrho_1)}\int_\mu^\zeta \left(\zeta - {}_\eta\psi_{1/2}(\lambda)\right)_\eta^{(\varrho_1-1)} u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda, \tag{5.3}$$

where a_1, a_2 are free of $\zeta, \zeta \in [\mu, \nu]$. Given that $p(\mu) = 0$, we have $a_2 = 0$. Under the boundary condition $p(\nu) = \vartheta({}_{\mu}\mathcal{I}^{\varrho_2}_{\mathrm{q},\omega}p)(\sigma)$, we obtain

$$\begin{aligned} a_1 = {}& \frac{\Gamma_{1/2}(\varrho_1+\varrho_2)}{z}\biggl\{\frac{1}{\Gamma_{1/2}(\varrho_1)}\int_{\mu}^{\nu}\bigl(\nu - {}_{\eta}\psi_{1/2}(\lambda)\bigr)_{\eta}^{(\varrho_1-1)}u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda \\ & - \frac{\vartheta}{\Gamma_{1/2}(\varrho_1+\varrho_2)}\int_{\mu}^{\sigma}\bigl(\sigma - {}_{\eta}\psi_{1/2}(\lambda)\bigr)_{\eta}^{(\varrho_1+\varrho_2-1)}u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda\biggr\}, \end{aligned}$$

which concludes

$$\begin{aligned} p(\zeta) = {}& \frac{\Gamma_{1/2}(\varrho_1+\varrho_2)(\zeta-\mu)_{\eta}^{(\varrho_1-1)}}{z}\biggl\{\frac{1}{\Gamma_{1/2}(\varrho_1)}\int_{\mu}^{\nu}\bigl(\nu - {}_{\eta}\psi_{1/2}(\lambda)\bigr)_{\eta}^{(\varrho_1-1)}u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda \\ & - \frac{\vartheta}{\Gamma_{1/2}(\varrho_1+\varrho_2)}\int_{\mu}^{\sigma}\bigl(\sigma - {}_{\eta}\psi_{1/2}(\lambda)\bigr)_{\eta}^{(\varrho_1+\varrho_2-1)}u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda\biggr\} \\ & - \frac{1}{\Gamma_{1/2}(\varrho_1)}\int_{\mu}^{\zeta}\bigl(\zeta - {}_{\eta}\psi_{1/2}(\lambda)\bigr)_{\eta}^{(\varrho_1-1)}u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda \\ = {}& \int_{\mu}^{\nu} y\bigl(\zeta, {}_{\eta}\psi_{1/2}(\lambda)\bigr)u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda \\ & + \frac{\vartheta(\zeta-\mu)_{\eta}^{(\varrho_1-1)}}{z}\int_{\mu}^{\nu}\mathcal{A}\bigl(\sigma, {}_{\eta}\psi_{1/2}(\lambda)\bigr)u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda \\ = {}& \int_{\mu}^{\nu}\mathcal{Y}\bigl(\zeta, {}_{\eta}\psi_{1/2}(\lambda)\bigr)u(\lambda/2+\eta)\,d_{1/2,\omega}\lambda. \end{aligned}$$

□

6 Conclusion

We suggested in this paper quantum calculus and fractional calculus conditions under which a Hahn integral operator with these new generalizations is reached using the quantum Hahn integral operator strategy. We supposed three various cases in the present research study. First is concerned with convex functions, the second one has identified reverse Minkowski inequalities, and the last one is the reverse Hölder inequalities. In each case, we presented several new generalizations that meet prerequisites and simultaneously are simpler to actualize. It is important that at the limit case, that is, when $\varrho_1 = 1$, $q \to 1$, and $\omega = 0$, we get classical integral inequalities from these quantum Hahn integral operators. It is hence evident that this new approach which we call fractional quantum calculus is better than both quantum and fractional calculus. We presented an example in Riemann–Liouville type q, ω-derivative in the boundary value problem to show the applicability of the proposed operator. Also, to catch more complexities of the attractors under scrutiny in the present research work, new examinations and applications can be investigated with some positive and new results in different fields of science, optics, and fractal theory. These new investigations will be displayed in future research work being prepared by the scientist of the present paper.

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