Nasiruzzaman and Mursaleen *Advances in Difference Equations* (2020) 2020:393
https://doi.org/10.1186/s13662-020-02848-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Approximation by Jakimovski–Leviatan-beta operators in weighted space

M. Nasiruzzaman[1] and M. Mursaleen[2,3,4*]

*Correspondence: mursaleenm@gmail.com
[2]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan
[3]Department of Mathematics, Aligarh Muslim University, Aligarh, 202002, India
Full list of author information is available at the end of the article

Abstract
The main purpose of this article is to introduce a more generalized version of Jakimovski–Leviatan-beta operators through the Appell polynomials. We present some uniform convergence results of these operators via Korovkin's theorem and obtain the rate of convergence by using the modulus of continuity and Lipschitz class. Moreover, we obtain the approximation with the help of Peetre's K-functional and give some direct theorems.

MSC: 41A10; 41A25; 41A36

Keywords: Appell polynomials; Jakimovski–Leviatan operators; Korovkin's theorem; Modulus of continuity; Lipschitz functions; Peetre's *K*-functional

1 Introduction

Approximation theory basically deals with problems to find approximation of functions by simpler functions like polynomial. Bernstein [8] was first to construct a sequence of positive linear operators to provide a constructive proof of the well-known Weierstrass approximation theorem. Since then several operators have been defined to study approximation properties in different spaces.

In [13], Jakimovski and Leviatan introduced the following operators and obtained some of their approximation properties:

$$R_m(h;y) = \frac{e^{-my}}{P(1)} \sum_{k=0}^{\infty} Q_k(my) h\left(\frac{k}{m}\right). \tag{1.1}$$

For all $h \in E[0,\infty)$, the set of functions of exponential type on $[0,\infty)$ with $|h(x)| \leq \beta e^{\alpha x}$, $\alpha, \beta > 0$, where $Q_k(y) = \sum_{j=0}^{k} b_j \frac{y^{k-j}}{(k-j)!}$ $(k \in \mathbb{N})$, are Appell polynomials [5] defined by the identity

$$P(w)e^{wy} = \sum_{m=0}^{\infty} Q_m(y) w^m, \tag{1.2}$$

such that $P(w) = \sum_{m=0}^{\infty} b_m w^m$, $P(1) \neq 0$ is an analytic function in the disk $|w| < r$ $(r > 1)$. Note that, for $P(1) = 1$, $Q_m(y) = \frac{y}{m!}$ and operators (1.1) are reduced to Favard–Szász operators:

$$S_m(h;y) = e^{-my}\sum_{k=0}^{\infty}\frac{(my)^k}{k!}h\left(\frac{k}{m}\right). \tag{1.3}$$

Recently, Büyükyazıcı *et al.* [9] studied the following operators:

$$S_m^*(h;y) = \frac{e^{-\frac{m}{b_m}y}}{P(1)}\sum_{k=0}^{\infty}Q_k\left(\frac{m}{b_m}y\right)h\left(\frac{k}{m}b_m\right). \tag{1.4}$$

In this paper, we generalize the above operators and study their several approximation properties. We investigate a Korovkin-type theorem and obtain the order of convergence by using the modulus of continuity. Furthermore, we obtain the approximation with the help of Lipschitz continuous functions and give some direct theorems.

For more details on the related work, we refer to [1–4, 6, 7, 11, 14, 15, 17–19, 21, 23–26].

We define an integral type modification of Jakimovski–Leviatan operators by introducing the sequences of unbounded and increasing functions $\{u_m\}$, $\{v_m\}$ such that

$$\lim_{m\to\infty}\frac{u_m}{v_m} = 1 + O\left(\frac{1}{v_m}\right) \quad \text{and} \quad \lim_{m\to\infty}\frac{1}{v_m}\to 0. \tag{1.5}$$

Let $m\in\mathbb{N}$, $\phi > m$ and

$$h\in C_\phi[0,\infty) = \left\{h\in C[0,\infty) : h(t) = O\left(t^\phi\right)\right\}. \tag{1.6}$$

For $Q_r(y)\geq 0$ and $P(1)\neq 0$, we define

$$L_m^{u_m,v_m}(h;y) = \frac{e^{-u_m y}}{P(1)}\sum_{r=0}^{\infty}Q_r(u_m y)\frac{1}{B(r+1,m)}\int_0^\infty\frac{t^r}{(1+t)^{r+m+1}}h\left(\frac{mt}{v_m}\right)\,\mathrm{d}t. \tag{1.7}$$

2 Moments

Lemma 2.1 *Suppose $e_i = t^{i-1}$ $(i=1,2,3)$. Then the following hold true for operators* (1.7):

(1) $L_m^{u_m,v_m}(e_1;y) = 1$;

(2) $L_m^{u_m,v_m}(e_2;y) = \frac{u_m}{v_m}\frac{m}{(m-1)}y + \frac{1}{v_m}\frac{m}{(m-1)}\left(\frac{P'(1)}{P(1)}+1\right)$;

(3) $L_m^{u_m,v_m}(e_3;y) =$
$\left(\frac{u_m}{v_m}\right)^2\frac{m^2}{(m-2)(m-1)}y^2 + 2\frac{u_m}{v_m^2}\frac{m^2}{(m-2)(m-1)}\left(\frac{P'(1)}{P(1)}+2\right)y + \frac{1}{v_m^2}\frac{m^2}{(m-2)(m-1)}\left(\frac{P''(1)}{P(1)}+4\frac{P'(1)}{P(1)}+2\right)$.

Proof We can easily see that

$$\sum_{r=0}^{\infty}Q_r(u_m y) = P(1)e^{u_m y}, \tag{2.1}$$

$$\sum_{r=0}^{\infty}rQ_r(u_m y) = \left[P'(1) + (u_m y)P(1)\right]e^{u_m y}, \tag{2.2}$$

$$\sum_{r=0}^{\infty}r^2Q_r(u_m y) = \left[P''(1) + 2(u_m y)P'(1) + P'(1) + (u_m y)P(1) + (u_m y)^2P(1)\right]e^{u_m y}. \tag{2.3}$$

(1) Take $h = e_1$

$$\begin{aligned}
L_m^{u_m,v_m}(e_1;x) &= \frac{e^{-u_m y}}{P(1)} \sum_{r=0}^{\infty} Q_r(u_m y) \frac{1}{B(r+1,m)} \int_0^{\infty} \frac{t^r}{(1+t)^{r+m+1}}\,\mathrm{d}t \\
&= \frac{e^{-u_m y}}{P(1)} \sum_{r=0}^{\infty} Q_r(u_m y) \frac{B(r+1,m)}{B(r+1,m)} \\
&= 1.
\end{aligned}$$

(2) Take $h = e_2$

$$\begin{aligned}
L_m^{u_m,v_m}(e_2;y) &= \frac{m e^{-u_m y}}{v_m P(1)} \sum_{r=0}^{\infty} Q_r(u_m y) \frac{1}{B(r+1,m)} \int_0^{\infty} \frac{t^{r+1}}{(1+t)^{r+m+1}}\,\mathrm{d}t \\
&= \frac{m e^{-u_m y}}{v_m P(1)} \sum_{r=0}^{\infty} Q_r(u_m y) \frac{B(r+2,m-1)}{B(r+1,m)} \\
&= \frac{m}{v_m(m-1)} \frac{e^{-u_m y}}{P(1)} \sum_{r=0}^{\infty} (r+1) Q_r(u_m y) \frac{B(r+1,m)}{B(r+1,m)} \\
&= \frac{m}{v_m(m-1)} + \frac{m}{v_m(m-1)} \frac{e^{-u_m y}}{P(1)} \sum_{r=0}^{\infty} r Q_r(u_m y) \\
&= \frac{m}{v_m(m-1)} + \frac{m u_m}{v_m(m-1)} \left(y + \frac{1}{u_m} \frac{P'(1)}{P(1)} \right).
\end{aligned}$$

(3) Take $h = e_3$

$$\begin{aligned}
L_m^{u_m,v_m}(e_3;y) &= \frac{m^2 e^{-u_m y}}{v_m^2 P(1)} \sum_{r=0}^{\infty} Q_r(u_m y) \frac{1}{B(r+1,m)} \int_0^{\infty} \frac{t^{r+2}}{(1+t)^{r+m+1}}\,\mathrm{d}t \\
&= \frac{m^2}{v_m^2(m-2)(m-1)} \frac{e^{-u_m y}}{P(1)} \sum_{r=0}^{\infty} Q_r(u_m y) \left(r^2+3r+2\right) \\
&= \frac{2m^2}{v_m^2(m-2)(m-1)} + \frac{3m^2}{v_m^2(m-2)(m-1)} \left(\frac{P'(1)}{P(1)} + u_m y \right) \\
&\quad + \frac{m^2}{v_m^2(m-2)(m-1)} \left(\frac{P''(1)}{P(1)} + 2u_m y \frac{P'(1)}{P(1)} + \frac{P'(1)}{P(1)} + u_m y + u_m^2 y^2 \right). \qquad \square
\end{aligned}$$

Lemma 2.2 *If $\Upsilon_j = (e_2 - y)^j$ for $j = 1, 2$, then we have*

1° $L_m^{u_m,v_m}(\Upsilon_1;y) = \left(\frac{u_m}{v_m}\frac{m}{(m-1)} - 1\right)y + \frac{1}{v_m}\frac{m}{(m-1)}\left(\frac{P'(1)}{P(1)} + 1\right)$;

2° $L_m^{u_m,v_m}(\Upsilon_2;y) = \left[\left(\frac{u_m}{v_m}\right)^2 \frac{m^2}{(m-2)(m-1)} - 2\frac{u_m}{v_m}\frac{m}{(m-1)} + 1\right]y^2 + \left[2\frac{u_m}{v_m^2}\frac{m^2}{(m-2)(m-1)}\left(\frac{P'(1)}{P(1)} + 2\right) - 2\frac{u_m}{v_m}\frac{m}{(m-1)}\left(\frac{P'(1)}{P(1)} + 1\right)\right]y + \frac{1}{v_m^2}\frac{m^2}{(m-2)(m-1)}\left(\frac{P''(1)}{P(1)} + \frac{P'(1)}{P(1)} + 2\right)$.

3 Approximation in a weighted space

In this section we present approximation results in the space $C_B[0,\infty)$ of all bounded and continuous functions on $[0,\infty)$, which is a linear normed space with the norm

$$\|h\|_{C_B[0,\infty)} = \sup_{y \geq 0} \left| h(y) \right|.$$

For our convenience, we rewrite

$$C_\gamma[0,\infty) = \{h \in C[0,\infty) : |h(t)| \le K(1+t)^\gamma \text{ for some } K > 0\}.$$

Let

$$E = \left\{h \in C[0,\infty) : \lim_{y\to\infty} \frac{h(y)}{1+y^2} \text{ exists}\right\}.$$

Theorem 3.1 *Let $h \in C_\gamma[0,\infty) \cap E$ and $\gamma \ge 2$. Then*

$$\lim_{m\to\infty} L_m^{u_m,v_m}(h;y) = h(y) \quad \textit{uniformly.}$$

Proof In the lighting of Korovkin's theorem and Lemma 2.1, it is obvious that

$$\lim_{m\to\infty} L_m^{u_m,v_m}(e_j;y) = y^{j-1}, \quad j = 1,2,3. \qquad \square$$

Following Gadžiev [12], we recall the weighted spaces for which the analogues of Korovkin's theorem hold [20]. Let ϕ be a continuous and strictly increasing function and $\varrho(y) = 1 + \phi^2(y)$, $\lim_{y\to\infty} \varrho(y) = \infty$. Moreover,

$$B_\varrho[0,\infty) = \{h : |h(y)| \le K_h \varrho(y)\} \quad \text{and} \quad C_\varrho[0,\infty) = B_\varrho[0,\infty) \cap C[0,\infty)$$

with $\|h\|_\varrho = \sup_{y\ge 0} \frac{|h(y)|}{\varrho(y)}$, where the constant K_h depends only on h. The sequence of positive linear operators $\{L_m^{u_m,v_m}\}_{m\ge 1}$ maps $C_\varrho[0,\infty)$ into $B_\varrho[0,\infty)$ if and only if

$$|L_m^{u_m,v_m}(\varrho;y)| \le K\varrho(y),$$

where $y \in [0,\infty)$ and $K > 0$ is a constant. Finally, let $C_\varrho^0[0,\infty) \subset C_\varrho[0,\infty)$ satisfying

$$\lim_{y\to\infty} \frac{h(y)}{\varrho(y)} = K_h.$$

Theorem 3.2 (cf. [16]) *Let the sequence of positive linear operators K_m, acting from $C_\varrho[0,\infty)$to $B_\varrho[0,\infty)$, satisfy the conditions*

$$\lim_{m\to\infty} \|K_m(t^k;y) - y^k\|_\varrho = 0 \quad (k = 0,1,2).$$

Then, for each $h \in C_\varrho^0[0,\infty)$,

$$\lim_{m\to\infty} \|K_m h - h\|_\varrho = 0.$$

Theorem 3.3 *For every $h \in C_\varrho^0[0,\infty)$, operators $L_m^{u_m,v_m}$ satisfy*

$$\lim_{m\to\infty} \|L_m^{u_m,v_m}(h;y) - h\|_\varrho = 0.$$

Proof In view of Theorem 3.2, suppose $\phi(y) = y$, then $\varrho(y) = 1 + y^2$. Since

$$\left\| L_m^{u_m,v_m}(e_i;y) - y^{i-1} \right\|_\varrho = \sup_{y\geq 0} \frac{|L_m^{u_m,v_m}(e_i;y) - y^{i-1}|}{1+y^2}.$$

For i = 1, from Lemma 2.1 we get

$$\lim_{m\to\infty} \left\| L_m^{u_m,v_m}(e_1;y) - 1 \right\|_\varrho = 0.$$

For $i = 2$,

$$\begin{aligned} \sup_{y\geq 0} \frac{|L_m^{u_m,v_m}(e_2;y) - y|}{1+y^2} &\leq \left| \frac{u_m}{v_m} \frac{m}{(m-1)} - 1 \right| \sup_{y\geq 0} \frac{y}{1+y^2} \\ &\quad + \left| \frac{m}{v_m(m-1)} \left(1 + \frac{P'(1)}{P(1)}\right) \right| \sup_{y\geq 0} \frac{1}{1+y^2}. \end{aligned}$$

Therefore

$$\lim_{m\to\infty} \left\| L_m^{u_m,v_m}(e_2;y) - y \right\|_\varrho = 0.$$

For $i = 3$,

$$\begin{aligned} \sup_{y\geq 0} \frac{|L_m^{u_m,v_m}(e_2;y) - y^2|}{1+y^2} &\leq \left| \left(\frac{u_m}{v_m}\right)^2 \frac{m^2}{(m-2)(m-1)} - 1 \right| \sup_{y\geq 0} \frac{y^2}{1+y^2} \\ &\quad + \left| \frac{u_m}{v_m^2} \frac{m^2}{(m-2)(m-1)} \left(2\frac{P'(1)}{P(1)} + 4\right) \right| \sup_{y\geq 0} \frac{y}{1+y^2} \\ &\quad + \left| \frac{m^2}{v_m^2(m-2)(m-1)} \left(\frac{P''(1)}{P(1)} + 4\frac{P'(1)}{P(1)} + 2 \right) \right| \sup_{y\geq 0} \frac{1}{1+y^2}. \end{aligned}$$

Therefore,

$$\lim_{m\to\infty} \left\| L_m^{u_m,v_m}(e_3;y) - y^2 \right\|_\varrho = 0.$$

Lemma 2.1 implies that

$$\lim_{m\to\infty} \left\| L_m^{u_m,v_m}(h;y) - h \right\|_\varrho = 0. \qquad \square$$

4 Rate of convergence

We write $C_B^*[0,\infty)$ for the set of all uniformly continuous and bounded functions on $[0,\infty)$ with $\|f\|_{C_B[0,\infty)} = \sup_{y\in[0,\infty)} |f(y)|$. For

$$\omega^\circ(h,\delta^\circ) = \sup_{|t-y|\leq\delta} \left| h(t) - h(y) \right|, \quad \delta^\circ > 0, h \in C_B[0,\infty), \tag{4.1}$$

we have

$$\left| h(t) - h(y) \right| \leq \left(\frac{\delta^\circ + |t-y|}{\delta^\circ} \right) \omega^\circ(h,\delta^\circ) \tag{4.2}$$

and $\lim_{\delta^\circ\to 0+} \omega^\circ(h,\delta^\circ) = 0$.

For $C>0$ and $0<\zeta\leq 1$, the Lipschitz class is defined by

$$Lip_C(\zeta)=\bigl\{h:\bigl|h(\eta_1)-h(\eta_2)\bigr|\leq C|\eta_1-\eta_2|^{\zeta}\ \bigl(\eta_1,\eta_2\in[0,\infty)\bigr)\bigr\}. \tag{4.3}$$

Theorem 4.1 *Let $m\in\mathbb{N}$ and $m>2$, then for all $h\in C_B^*[0,\infty)$*

$$\bigl|L_m^{u_m,v_m}(h;y)-h(y)\bigr|\leq 2\omega^\circ(h;\Psi_m), \tag{4.4}$$

where $(\Psi_m)^2=\bigl[(\frac{u_m}{v_m})^2\frac{m^2}{(m-2)(m-1)}-2\frac{u_m}{v_m}\frac{m}{(m-1)}+1\bigr]y^2+\bigl[\frac{u_m}{v_m^2}\frac{m^2}{(m-2)(m-1)}\bigl(2\frac{P'(1)}{P(1)}+4\bigr)-\frac{1}{v_m}\frac{m}{(m-1)}\bigl(\frac{2P'(1)}{P(1)}+2\bigr)\bigr]y+\frac{1}{v_m^2}\frac{m^2}{(m-2)(m-1)}\bigl(\frac{P''(1)}{P(1)}+\frac{P'(1)}{P(1)}+2\bigr)$.

Proof We have

$$\begin{aligned}\bigl|L_m^{u_m,v_m}(h;y)-h(y)\bigr| &= \bigl|L_m^{u_m,v_m}(h;y)-h(y)L_m^{u_m,v_m}(e_1;y)\bigr|\\ &= \bigl|L_m^{u_m,v_m}\bigl(h(t)-h(y);y\bigr)\bigr|\\ &\leq L_m^{u_m,v_m}\bigl(\bigl|h(t)-h(y)\bigr|;y\bigr).\end{aligned}$$

In the light of (4.1) and (4.2), we get

$$\begin{aligned}\bigl|L_m^{u_m,v_m}(h;y)-h(y)\bigr| &\leq L_m^{u_m,v_m}\left(1+\frac{|t-y|}{\delta^\circ};y\right)\omega^\circ\bigl(h,\delta^\circ\bigr)\\ &= \left(1+\frac{1}{\delta^\circ}L_m^{u_m,v_m}\bigl(|t-y|;y\bigr)\right)\omega^\circ\bigl(h,\delta^\circ\bigr).\end{aligned}$$

Apply the Cauchy–Schwarz inequality

$$L_m^{u_m,v_m}\bigl(|t-y|;y\bigr)\leq\bigl[L_m^{u_m,v_m}(e_1;y)L_m^{u_m,v_m}\bigl((t-y)^2;y\bigr)\bigr]^{\frac{1}{2}}$$

so that

$$\bigl|L_m^{u_m,v_m}(h;y)-h(y)\bigr|\leq\left(1+\frac{1}{\delta^\circ}L_m^{u_m,v_m}(\Upsilon_2;y)^{\frac{1}{2}}\right)\omega^\circ\bigl(h,\delta^\circ\bigr). \tag{4.5}$$

Choosing $\delta^\circ=\Psi_m=\sqrt{L_m^{u_m,v_m}(\Upsilon_2;y)}$ yields the result. □

Remark 4.2 For $u_m=v_m=1$, the above estimate is reduced to [22], i.e.,

$$\bigl|L_m(h;y)-h(y)\bigr|\leq 2\omega^\circ(h,\Phi_m), \tag{4.6}$$

where $(\Phi_m)^2=\bigl(\frac{m^2}{(m-2)(m-1)}-\frac{2m}{(m-1)}+1\bigr)y^2+\bigl[\frac{2m}{(m-2)(m-1)}\bigl(\frac{P'(1)}{P(1)}\bigr)+2\bigr)-\frac{2}{(m-1)}\bigl(\frac{P'(1)}{P(1)}+1\bigr)\bigr]y+\frac{1}{(m-2)(m-1)}\bigl(\frac{P''(1)}{P(1)}+\frac{P'(1)}{P(1)}+2\bigr)$.

Theorem 4.3 *For every $h\in\mathrm{Lip}_C(\zeta)$, we have*

$$\bigl|L_m^{u_m,v_m}(h;y)-h(y)\bigr|\leq C(\Psi_m)^{\zeta},$$

where $m>2$, $m\in\mathbb{N}$, and $\Psi_m=\sqrt{L_m^{u_m,v_m}(\Upsilon_2;y)}$ by Theorem 4.1.

Proof We use (4.3) and Hölder's inequality to get

$$
\begin{aligned}
&\left|L_m^{u_m,v_m}(h;y)-h(y)\right| \\
&\quad\leq \left|L_m^{u_m,v_m}\big(h(t)-h(y);y\big)\right| \\
&\quad\leq L_m^{u_m,v_m}\big(\left|h(t)-h(y)\right|;y\big) \\
&\quad\leq CL_m^{u_m,v_m}\big(|t-y|^{\zeta};y\big). \\
&\quad\leq C\frac{e^{-u_m y}}{v_m P(1)}\sum_{r=0}^{\infty}Q_r(u_m y)\frac{1}{B(r+1,m)}\int_0^{\infty}\frac{t^r}{(1+t)^{r+m+1}}|t-y|^{\zeta}\,\mathrm{d}t \\
&\quad= \frac{e^{-u_m y}}{v_m P(1)}\left(\sum_{r=0}^{\infty}Q_r(u_m y)\frac{1}{B(r+1,m)}\right)^{1-\frac{\zeta}{2}} \\
&\qquad\times\left(Q_r(u_m y)\frac{1}{B(r+1,m)}\right)^{\frac{\zeta}{2}}\int_0^{\infty}\frac{t^r}{(1+t)^{r+m+1}}|t-y|^{\zeta}\,\mathrm{d}t \\
&\quad\leq C\left(\frac{e^{-u_m y}}{v_m P(1)}\sum_{r=0}^{\infty}Q_r(u_m y)\frac{1}{B(r+1,m)}\int_0^{\infty}\frac{t^r}{(1+t)^{r+m+1}}\,\mathrm{d}t\right)^{1-\frac{\zeta}{2}} \\
&\qquad\times\left(\frac{e^{-u_m y}}{v_m P(1)}\sum_{r=0}^{\infty}Q_r(u_m y)\frac{1}{B(r+1,m)}\int_0^{\infty}\frac{t^r}{(1+t)^{r+m+1}}|t-y|^{2}\,\mathrm{d}t\right)^{\frac{\zeta}{2}} \\
&\quad= CL_m^{u_m,v_m}(\Upsilon_2;y)^{\frac{\zeta}{2}}.
\end{aligned}
$$

This completes the proof. □

5 Direct theorems

Let

$$C_B^{\kappa}[0,\infty)=\left\{f\in C_B[0,\infty):f',f''\in C_B[0,\infty)\right\}\tag{5.1}$$

with the norm

$$\|f\|_{C_B^{\kappa}[0,\infty)}=\|f\|_{C_B[0,\infty)}+\left\|f'\right\|_{C_B[0,\infty)}+\left\|f''\right\|_{C_B[0,\infty)},\tag{5.2}$$

and let $\Omega^{\circ}=\{f\in C_B[0,\infty):f',f''\in C_B[0,\infty)\}$. For $h\in C_B[0,\infty)$, Peetre's K-functional is defined by

$$K_2^{\circ}\big(h,\delta^{\circ}\big)=\inf_{f\in\Omega^{\circ}}\left\{\left(\|h-f\|_{C_B[0,\infty)}+\delta^{\circ}\left\|f''\right\|_{C_B^2[0,\infty)}\right):f\in\Omega^{\circ}\right\}.\tag{5.3}$$

For a positive constant M, one has $K_2^{\circ}(h,\delta^{\circ})\leq M\omega_2^{\circ}(h,\sqrt{\delta^{\circ}})$, where $\delta^{\circ}>0$ and the second order modulus of continuity ω_2° is defined by

$$\omega_2^{\circ}\big(h,\sqrt{\delta^{\circ}}\big)=\sup_{0<u<\sqrt{\delta^{\circ}}}\sup_{y\in[0,\infty)}\left|h(y+2u)-2h(y+u)+h(y)\right|.\tag{5.4}$$

Theorem 5.1 *Let $m > 2$, $m \in \mathbb{N}$. Then for all $f \in C_B^{\kappa}[0,\infty)$ we have*

$$\left|L_m^{u_m,v_m}(f;y) - f(y)\right| \le \left(\Psi_m + \frac{(\Psi_m)^2}{2}\right)\|f\|_{C_B^{\kappa}[0,\infty)},$$

where Ψ_m is defined by Theorem 4.1.

Proof By Taylor's formula, one has

$$f(t) = f(y) + f'(y)(t-y) + f''(\chi)\frac{(t-y)^2}{2}, \quad \chi \in (y,t),$$
$$\left|f(t) - f(y)\right| \le W_1|t-y| + \frac{1}{2}W_2(t-y)^2,$$

where

$$W_1 = \sup_{y\in[0,\infty)}\left|f'(y)\right| = \left\|f'\right\|_{C_B[0,\infty)} \le \|f\|_{C_B^{\kappa}[0,\infty)},$$
$$W_2 = \sup_{y\in[0,\infty)}\left|f''(y)\right| = \left\|f''\right\|_{C_B[0,\infty)} \le \|f\|_{C_B^{\kappa}[0,\infty)}.$$

Therefore,

$$\left|f(t) - f(y)\right| \le \left(|t-y| + \frac{(t-y)^2}{2}\right)\|f\|_{C_B^{\kappa}[0,\infty)}$$

and

$$\left|L_m^{u_m,v_m}(f,y) - f(y)\right| = \left|L_m^{u_m,v_m}\left(f(t) - f(y);y\right)\right| \le L_m^{u_m,v_m}\left(\left|f(t) - f(y)\right|;y\right).$$

Thus, we get

$$\left|L_m^{u_m,v_m}(f;y) - f(y)\right| \le \left(L_m^{u_m,v_m}\left(|t-y|;y\right) + \frac{L_m^{u_m,v_m}((t-y)^2;y)}{2}\right)\|f\|_{C_B^{\kappa}[0,\infty)}$$
$$\le \left(\Psi_m + \frac{(\Psi_m)^2}{2}\right)\|f\|_{C_B^{\kappa}[0,\infty)},$$

where

$$L_m^{u_m,v_m}\left(|t-y|;y\right) \le \sqrt{L_m^{u_m,v_m}\left((t-y)^2;y\right)} = \sqrt{L_m^{u_m,v_m}(\Upsilon_2;y)}.$$

Hence the result. □

Theorem 5.2 *For every $h \in C_B[0,\infty)$ and $m > 2$, $m \in \mathbb{N}$, we have*

$$\left|L_m^{u_m,v_m}(h;y) - h(y)\right|$$
$$\le 2M\left\{\omega_2^{\circ}\left(h;\sqrt{2\Psi_m + \frac{(\Psi_m)^2}{4}}\right) + \min\left(1, 2\Psi_m + \frac{(\Psi_m)^2}{4}\right)\|h\|_{C_B[0,\infty)}\right\}.$$

Proof As previously, we easily conclude that

$$\begin{aligned}\left|L_m^{u_m,v_m}(h;y)-h(y)\right| &\leq \left|L_m^{u_m,v_m}(h-f;y)\right|+\left|L_m^{u_m,v_m}(f;y)-f(y)\right|+\left|h(y)-f(y)\right|\\ &\leq 2\|h-f\|_{C_B(\mathbb{R}^+)}+\left(\Psi_m+\frac{(\Psi_m)^2}{2}\right)\|f\|_{C_B^{\kappa}[0,\infty)}\\ &\leq 2\left(\|h-f\|_{C_B[0,\infty)}+\left(2\Psi_m+\frac{(\Psi_m)^2}{4}\right)\|f\|_{C_B^{\kappa}[0,\infty)}\right).\end{aligned}$$

By taking infimum and using (5.3), we get

$$\left|L_m^{u_m,v_m}(h;y)-h(y)\right|\leq 2K_2\left(h;2\Psi_m+\frac{(\Psi_m)^2}{4}\right).$$

Now, for an absolute constant $M>0$ [10], we use the relation

$$K_2^{\circ}\left(h;\delta^{\circ}\right)\leq M\left\{\omega_2^{\circ}\left(h;\sqrt{\delta^{\circ}}\right)+\min\left(1,\delta^{\circ}\right)\|h\|\right\}.$$

This completes the proof. □

6 Conclusion

We have constructed an integral type modification of Jakimovski–Leviatan operators by using beta function and two sequences of unbounded and increasing functions $\{u_m\}$, $\{v_m\}$ such that $\lim_{m\to\infty}\frac{u_m}{v_m}=1+O(\frac{1}{v_m})$ and $\lim_{m\to\infty}\frac{1}{v_m}=0$. We derived some uniform convergence results of these operators via Korovkin's theorem and obtained the rate of convergence by using the modulus of continuity and Lipschitz class. Furthermore, we obtained some direct theorems with the help of Peetre's K-functional.

Acknowledgements
Not applicable.

Funding
No funding available.

Availability of data and materials
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1] Department of Mathematics, Faculty of Science, University of Tabuk, PO Box 4279, Tabuk, 71491, Saudi Arabia. [2] Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan. [3] Department of Mathematics, Aligarh Muslim University, Aligarh, 202002, India. [4] Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

Publisher's Note

Received: 13 January 2020 Accepted: 21 July 2020 Published online: 03 August 2020

References

1. Acar, E., İzgi, A., Serenbay, S.K.: Note on Jakimovski–Leviatan operators preserving e^{-x}. Appl. Math. Nonlinear Sci. **4**(2), 543–549 (2019)
2. Al-Salam, W.A.: *q*-Appell polynomials. Ann. Mat. Pura Appl. **4**, 31–45 (1967)
3. Alotaibi, A., Mursaleen, M.: Approximation of Jakimovski–Leviatan-beta type integral operators via q-calculus. AIMS Math. **5**(4), 3019–3034 (2020)
4. Ansari, K.J., Mursaleen, M., Rahman, S.: Approximation by Jakimovski–Leviatan operators of Durrmeyer type involving multiple Appell polynomials. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(2), 1007–1024 (2019)
5. Appell, P.: Une classe de polynômes. Ann. Sci. Éc. Norm. Supér. **9**, 119–144 (1880)
6. Ari, D.A., Serenbay, S.K.: Approximation by a generalization of the Jakimovski–Leviatan operators. Filomat **33**(8), 2345–2353 (2019)
7. Atakut, C., Büyükyazici, I.: Approximation by modified integral type Jakimovski–Leviatan operators. Filomat **30**, 29–39 (2016)
8. Bernstein, S.N.: Démonstation du théorème de Weierstrass fondée sur le calcul de probabilités. Commun. Soc. Math. Kharkow **13**, 1–2 (1912–1913)
9. Büyükyazıcı, İ., Tanberkan, H., Serenbay, S., Atakut, C.: Approximation by Chlodowsky type Jakimovski–Leviatan operators. J. Comput. Appl. Math. **259**, 153–163 (2014)
10. Ciupa, A.: A class of integral Favard–Szász type operators. Stud. Univ. Babeş–Bolyai, Math. **40**, 39–47 (1995)
11. Gadžiev, A.D.: A problem on the convergence of a sequence of positive linear operators on unbounded sets, and theorems that are analogous to P. P. Korovkin's theorem. Dokl. Akad. Nauk SSSR **218**, 1001–1004 (1974) (Russian)
12. Gadžiev, A.D.: Weighted approximation of continuous functions by positive linear operators on the whole real axis. Izv. Akad. Nauk Azerb. SSR, Ser. Fiz.-Teh. Mat. Nauk **5**, 41–45 (1975) (Russian)
13. Jakimovski, A., Leviatan, D.: Generalized Szász operators for the approximation in the infinite interval. Mathematica **11(34)**, 97–103 (1969)
14. Kac, V., De Sole, A.: On integral representations of *q*-gamma and *q*-beta functions. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. **9**, 11–29 (2005)
15. Keleshteri, M.E., Mahmudov, N.I.: A study on *q*-Appell polynomials from determinantal point of view. Appl. Math. Comput. **260**, 351–369 (2015)
16. Korovkin, P.P.: Linear Operators and Approximation Theory. Hindustan Publ. Co., Delhi (1960)
17. Milovanović, G.V., Mursaleen, M., Nasiruzzaman, M.: Modified Stancu type Dunkl generalization of Szász–Kantorovich operators. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**(1), 135–151 (2018)
18. Mohiuddine, S.A., Acar, T., Alghamdi, M.A.: Genuine modified Bernstein–Durrmeyer operators. J. Inequal. Appl. **2018**, Article ID 104 (2018)
19. Mursaleen, M., Ansari, K.J., Nasiruzzaman, M.: Approximation by *q*-analogue of Jakimovski–Leviatan operators involving *q*-Appell polynomials. Iran. J. Sci. Technol. Trans. A, Sci. **41**, 891–900 (2017)
20. Mursaleen, M., Karakaya, V., Ertürk, M., Gürsoy, F.: Weighted statistical convergence and its application to Korovkin type approximation theorem. Appl. Math. Comput. **218**, 9132–9137 (2012)
21. Mursaleen, M., Khan, T.: On approximation by Stancu type Jakimovski–Leviatan–Durrmeyer operators. Azerb. J. Math. **7**(1), 16–26 (2017)
22. Mursaleen, M., Nasiruzzaman, M.: Approximation of modified Jakimovski–Leviatan-beta type operators. Constr. Math. Anal. **1**(2), 88–98 (2018)
23. Mursaleen, M., Nasiruzzaman, M., Alotaibi, A.: On modified Dunkl generalization of Szász operators via *q*-calculus. J. Inequal. Appl. **2017**, Article ID 38 (2017)
24. Mursaleen, M., Nasiruzzaman, M., Srivastava, H.M.: Approximation by bicomplex beta operators in compact BC-disks. Math. Methods Appl. Sci. **39**, 2916–2929 (2016)
25. Nasiruzzaman, Md., Mukheimer, A., Mursaleen, M.: Approximation results on Dunkl generalization of Phillips operators via q-calculus. Adv. Differ. Equ. **2019**, Article ID 244 (2019)
26. Srivastava, H.M., Mursaleen, M., Nasiruzzaman, M.: Approximation by a class of q-beta operators of the second kind via the Dunkl-type generalization on weighted spaces. Complex Anal. Oper. Theory **13**(3), 1537–1556 (2019)