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Existence of positive solutions for the fractional q-difference boundary value problem

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Abstract

In this paper, we investigate the existence of positive solutions for a class of fractional boundary value problems involving q-difference. By using the fixed point theorem of cone mappings, two existence results are obtained. Examples are given to illustrate the abstract results.

MSC: 34B16; 34B18

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1 Introduction

The theory of q-calculus or quantum calculus was initially developed by [6, 7] and it has many applications in the fields of hypergeometric series, particle physics, quantum mechanics and complex analysis. For a general introduction of q-calculus or quantum calculus, we refer to [1, 2, 8]. Recently, fractional boundary value problems with q-difference have been investigated by many authors; see [3, 4, 9–11] and the references therein. In [3], Ferreira considered the existence of positive solutions for the nonlinear q-fractional boundary value problem (BVP)

$$\begin{cases} D_q^\alpha u(t) = -f(t,u(t)), & t \in I := (0,1), \\ u(0) = D_q u(0) = 0, & D_q u(1) = \beta \geq 0, \end{cases} \tag{1.1}$$

where $0 < q < 1$, $2 < \alpha \leq 3$, $f : I^* \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $I^* = [0,1]$, $\mathbb{R}^+ = [0,+\infty)$. By utilizing a fixed point theorem in cones, he obtained the following existence theorem.

Theorem A *Let $\tau = q^n$ with $n \in \mathbb{N}$. Suppose that $f(t,u)$ is a nonnegative continuous function on $[0,1] \times \mathbb{R}^+$. If there exist two positive constants $r_2 > r_1 > 0$ such that the function f satisfies*

$(P1)$ $\frac{\beta}{[\alpha-1]_q} + M \max_{(t,u)\in[0,1]\times[0,r_1]} f(t,u) \leq r_1$;

($P2$) $\frac{\beta}{[\alpha-1]_q} + N \max_{(t,u)\in[\tau,1]\times[\tau^{\alpha-1}r_2,r_2]} f(t,u) \geq r_2$,
where

$$[\alpha-1]_q = \frac{1-q^{\alpha-1}}{1-q},$$

$$M = \int_0^1 G(1,qs)\, d_q s,$$

$$N = \max_{t\in[0,1]} \int_\tau^1 G(t,qs)\, d_q s,$$

$G(t,qs)$ is the Green's function which will be specified later, then the BVP (1.1) has a solution satisfying $u(t) > 0$ for $t \in (0,1]$.

Clearly, the conditions ($P1$) and ($P2$) are strong in application. In 2015, Li et al. [9] studied a class of fractional Schrödinger equations with q-difference of the form

$$D_q^\alpha u(t) + \frac{n}{\hbar}\big(\aleph - \rho(t)\big)u(t) = 0, \quad t \in I, \tag{1.2}$$

where $\rho(t)$ is the trapping potential, n is the mass of a particle, $\hbar$ is the Planck constant, $\aleph$ is the energy of a particle. Let $\lambda = \frac{n}{\hbar}$ and $h(t) = \aleph - \rho(t)$. They transformed Eq. (1.2) to

$$D_q^\alpha u(t) + \lambda h(t) f\big(u(t)\big) = 0, \quad t \in I, \tag{1.3}$$

subject to the boundary conditions

$$u(0) = D_q u(0) = D_q u(1) = 0, \tag{1.4}$$

where $0 < q < 1$, $2 < \alpha \leq 3$, $f : I^* \times \mathbb{R} \to (0,\infty)$ is continuous, $h : I \to (0,\infty)$ is continuous. By applying a fixed point theorem in cones, they proved several theorems for the existence of positive solutions of the problem (1.3)–(1.4). Here, we just list two important results of [9].

Theorem B *Suppose that* ($H1$) *and one of* ($H2$) *and* ($H3$) *hold, where*
($H1$) *$h(t)$ is continuous for $t \in (0,1)$ such that $\int_0^1 h(t)\, d_q t < +\infty$;*
($H2$) $\lim_{u\to 0} \frac{f(u)}{u} = \infty$;
($H3$) $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$.
Then the problem (1.3)–(1.4) has at least one positive solution provided that

$$0 < \lambda < \frac{\sup_{r>0} \frac{r}{\max_{0\leq u\leq r} f(u)}}{\max_{t\in[0,1]} \int_0^1 G(t,qs)h(s)\, d_q s}, \tag{1.5}$$

where $r > 0$ is constant.

Theorem C *Suppose that* ($H1$) *and one of* ($H4$) *and* ($H5$) *hold, where*
($H4$) $\lim_{u\to 0} \frac{f(u)}{u} = 0$;
($H5$) $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.

Then the problem (1.3)–(1.4) *has at least one positive solution provided that*

$$\frac{\inf_{r>0} \frac{r}{\min_{\tau r\le u\le r} f(u)}}{\min_{t\in[\tau,1]}\int_\tau^1 G(t,qs)h(s)\,d_qs} < \lambda < \infty, \tag{1.6}$$

where $r > 0$ *is constant.*

It is obvious that the conditions $(H2)$–$(H5)$ are weaker than $(P1)$–$(P2)$, but (1.5) and (1.6) are not easy to verify in application.

In the present work, we consider the fractional boundary value problem (Fr-BVP) with q-difference of the form

$$\begin{cases} D_q^\alpha u(t) + \omega(t)f(t,\delta(t)u(t)) = 0, \quad t\in I, \\ u(0) = D_qu(0) = D_qu(1) = 0, \end{cases} \tag{1.7}$$

where $0 < q < 1$, $2 < \alpha \le 3$, $\omega \in C[0,1]$, $\delta \in C(I^*,(0,+\infty))$, $f \in C(I\times\mathbb{R}^+,\mathbb{R}^+)$, f may be singular at $t = 0$ and/or 1. Here, $\delta(t)$ is a scaling function of u in the nonlinearity f.

For the sake of simplicity, denote

$$\delta_m = \min_{t\in I^*}\delta(t), \qquad \delta_M = \max_{t\in I^*}\delta(t).$$

Throughout this paper, we always assume that the functions f and ω satisfy the following conditions.

$(A1)$ $\omega \in C[0,1]$ and there exists $\xi > 0$ such that $\omega(t) \ge \xi$ for $t \in I$;

$(A2)$ $\int_0^1 G(1,qs)f(s,\delta_M)\,d_qs < +\infty$;

$(A3)$ $f(t,\delta_m) > 0$ for any $t\in I$ and there exist constants $\sigma_1 \ge \sigma_2 > 1$ such that, for every $\tau \in (0,1]$,

$$\tau^{\sigma_1}f(t,x) \le f(t,\tau x) \le \tau^{\sigma_2}f(t,x) \tag{1.8}$$

for any $t\in I$ and $x\in\mathbb{R}^+$;

$(A4)$ $f(t,\delta_m) > 0$ for any $t\in I$ and there exist constants $0 < \sigma_3 \le \sigma_4 < 1$ such that, for every $\tau\in(0,1]$,

$$\tau^{\sigma_4}f(t,x) \le f(t,\tau x) \le \tau^{\sigma_3}f(t,x) \tag{1.9}$$

for any $t\in I$ and $x\in\mathbb{R}^+$.

Remark 1.1

(1) If f satisfies the assumption $(A3)$ or $(A4)$, then $f(t,x)$ is non-decreasing with respect to $x\in\mathbb{R}^+$ for every $t\in I$.

(2) The condition (1.8) is equivalent to

$$\tau^{\sigma_2}f(t,x) \le f(t,\tau x) \le \tau^{\sigma_1}f(t,x), \quad \forall\tau\ge 1. \tag{1.10}$$

(3) The condition (1.9) is equivalent to

$$\tau^{\sigma_3}f(t,x) \le f(t,\tau x) \le \tau^{\sigma_4}f(t,x), \quad \forall\tau\ge 1.$$

Remark 1.2 The assumptions ($A3$) and ($A4$) are order conditions. They are much easier to verify in application than the conditions ($H2$)–($H5$) and (1.5), (1.6).

Remark 1.3 If $\delta(t) \equiv 1$ for $t \in [0,1]$ and $f(t,u) = f(u)$, then the Fr-BVP (1.7) becomes to the problem (1.3)–(1.4) with $\omega(t) = \lambda h(t)$. Therefore, the Fr-BVP (1.7) is more general than the problem (1.3)–(1.4).

By using the fixed point theorem of cone mappings, we obtain the following theorems.

Theorem 1.1 *Let the assumptions* ($A1$)–($A3$) *hold. Then the Fr-BVP* (1.7) *has at least one positive solution* $u \in C[0,1]$.

Theorem 1.2 *Let the assumptions* ($A1$), ($A2$) *and* ($A4$) *hold. Then the Fr-BVP* (1.7) *has at least one positive solution* $u \in C[0,1]$.

The rest of this paper is organized as follows. In Sect. 2 we introduce some preliminaries and notations which are useful in our proof. In Sect. 3, we will prove Theorems 1.1 and 1.2. Examples are given in Sect. 4 to illustrate the abstract results.

2 Preliminaries

In this section, we introduce some definitions and notations on fractional q-difference equations. Some related lemmas are also given in this section. For $q \in (0,1)$ and $a, b, \alpha \in \mathbb{R}$, we denote

$$[\alpha]_q := \frac{1-q^\alpha}{1-q}$$

and

$$(a-b)^{(\alpha)} := a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{n+\alpha}}.$$

The q-analogue of the power function $(a-b)^n$ is defined by

$$(a-b)^0 = 1$$

and

$$(a-b)^n = \prod_{k=1}^{\infty} \left(a-bq^k\right), \quad n \in \mathbb{N}.$$

The q-gamma function is given by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R} \setminus \{0,-1,-2,\ldots\},$$

and it satisfies $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$.

Let ℓ be a function defined on $[0,1]$. The q-derivative of ℓ is

$$(D_q\ell)(t) = \frac{\ell(t) - \ell(qt)}{(1-q)t}, \quad t > 0,$$

and

$$(D_q\ell)(0) = \lim_{t\to 0}(D_q\ell)(t).$$

The q-derivative of ℓ of high order is given by

$$\left(D_q^0\ell\right)(t) = \ell(t), \quad t \in [0,1],$$

and

$$\left(D_q^n\ell\right)(t) = D_q\left(D_q^{n-1}\ell\right)(t), \quad t \in [0,1], n \in \mathbb{N}.$$

The following definitions of fractional q-calculus are cited from [3].

Definition 2.1 The fractional q-integral of the Riemann–Liouville type of order $\alpha \geq 0$ for the function ℓ is defined by

$$\left(I_q^0\ell\right)(t) = \ell(t), \quad t \in [0,1],$$

and

$$\left(I_q^\alpha\ell\right)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\ell(s)\,d_qs, \quad \alpha > 0, t \in [0,1].$$

Definition 2.2 The fractional q-derivative of the Riemann–Liouville type of order $\alpha \geq 0$ for the function ℓ is defined by

$$\left(D_q^0\ell\right)(t) = \ell(t), \quad t \in [0,1],$$

and

$$\left(D_q^\alpha\ell\right)(t) = \left(D_q^m I_q^{m-\alpha}\ell\right)(t)\alpha > 0, \quad t \in [0,1],$$

where $m := \lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

We refer the reader to the papers [3, 10] and the monographs [1, 2] for more details on the definitions of fractional q-calculus.

In order to prove the existence of positive solutions of the Fr-BVP (1.7), for any $h \in C[0,1]$, we first consider the linear Fr-BVP

$$\begin{cases} D_q^\alpha u(t) + h(t) = 0, & t \in I, \\ u(0) = D_q u(0) = D_q u(1) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1 ([3]) *Let $0 < q < 1$ and $2 < \alpha \le 3$. For any $h \in C[0,1]$, the linear Fr-BVP* (2.1) *has a unique solution expressed by*

$$u(t) = \int_0^1 G(t,qs)h(s)\,d_qs,$$

where

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-qs)^{(\alpha-2)}t^{\alpha-1} - (t-qs)^{(\alpha-1)}, & 0 \le qs \le t < 1, \\ (1-qs)^{(\alpha-2)}t^{\alpha-1}, & 0 \le t \le qs < 1, \end{cases} \tag{2.2}$$

is the Green's function of the linear Fr-BVP (2.1).

Lemma 2.2 ([3]) *The Green's function $G(t,qs)$ has the following properties*:
(i) $G(t,qs) \ge 0$ *for all* $t,s \in I^*$;
(ii) $t^{\alpha-1}G(1,qs) \le G(t,qs) \le G(1,qs)$ *for all* $t,s \in I^*$.

By Lemma 2.1, we can define the solution of the Fr-BVP (1.7) as follows.

Definition 2.3 A function $u \in C[0,1]$ is called a solution of the Fr-BVP (1.7) if it satisfies the integral equation

$$u(t) = \int_0^1 G(t,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs, \quad t \in I^*.$$

If $u(t) > 0$ for $t \in I$, then it is called a positive solution of the Fr-BVP (1.7).

Let $E := C[0,1]$. Then E is a Banach space endowed with the norm

$$\|u\| = \max_{t\in I^*}\bigl|u(t)\bigr|, \quad \forall u \in E.$$

Let $\eta \in (0,1)$. Define a cone K in E by

$$K = \Bigl\{u \in E : u(t) \ge 0, t \in I^*, \min_{t\in[\eta,1]} u(t) \ge \eta^{\alpha-1}\|u\|\Bigr\}.$$

Then K is a nonempty closed convex cone of E.

Define an operator $Q : K \to E$ by

$$(Qu)(t) = \int_0^1 G(t,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs, \quad t \in I^*. \tag{2.3}$$

Lemma 2.3 *Let the assumptions* (*A*1)–(*A*3) *hold. Then $Q : K \to E$ is well defined, and $u \in E$ is a positive solution of the Fr-BVP* (1.7) *if and only if u is a positive fixed point of Q.*

Proof For fixed $u \in E$ with $u(t) \geq 0$ for all $t \in I^*$, choosing a constant $a \in (0,1)$ such that $a\|u\| < 1$. Then, for any $t \in I^*$, by (1.8) and (1.10), we have

$$\begin{aligned} f\bigl(t,\delta(t)u(t)\bigr) &\leq \left(\frac{1}{a}\right)^{\sigma_1} f\bigl(t,a\delta(t)u(t)\bigr) \\ &\leq \left(\frac{1}{a}\right)^{\sigma_1} \bigl[au(t)\bigr]^{\sigma_2} f\bigl(t,\delta(t)\bigr) \\ &\leq a^{\sigma_2-\sigma_1}\|u\|^{\sigma_2} f(t,\delta_M). \end{aligned}$$

So, for any $t \in I^*$, by (2.2), we have

$$\begin{aligned} 0 &< \int_0^1 G(t,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &\leq \int_0^1 G(1,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &\leq a^{\sigma_2-\sigma_1}\|u\|^{\sigma_2}\|\omega\| \int_0^1 G(1,qs)f(s,\delta_M)\,d_qs \\ &< +\infty. \end{aligned}$$

This implies that the operator $Q: K \to E$ is well defined. By Definition 2.3, $u \in E$ is a positive solution of the Fr-BVP (1.7) if and only if u is a positive fixed point of Q. □

Lemma 2.4 *If the assumptions* (*A*1), (*A*2) *and* (*A*4) *hold, then* $Q: K \to E$ *is well defined, and* $u \in E$ *is a positive solution of the Fr-BVP* (1.7) *if and only if* u *is a positive fixed point of* Q.

Lemma 2.5 $Q: K \to K$ *is a completely continuous operator.*

Proof For any $u \in K$ and $t \in I^*$, by Lemma 2.2 and (1.10), we have $(Qu)(t) \geq 0$ on I^* and

$$\begin{aligned} \min_{t\in[\eta,1]}(Qu)(t) &\geq \min_{t\in[\eta,1]} \int_0^1 t^{\alpha-1}G(1,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &= \eta^{\alpha-1}\int_0^1 G(1,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &\geq \eta^{\alpha-1}\|Qu\|. \end{aligned}$$

Hence, $Q: K \to K$. By the Ascoli–Arzela theorem, one can prove that $Q: K \to K$ is completely continuous. □

At last, we state a fixed point theorem of cone mapping to end this section, which is useful in the proof of our main results.

Lemma 2.6 ([5]) *Let* E *be a Banach space,* $P \subset E$ *a cone in* E. *Assume that* Ω_1 *and* Ω_2 *are two bounded and open subset of* E *with* $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. *If*

$$Q: P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$$

is a completely continuous operator such that either

(i) $\|Qu\| \leq \|u\|$, $\forall u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \geq \|u\|$, $\forall u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Qu\| \geq \|u\|$, $\forall u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \leq \|u\|$, $\forall u \in P \cap \partial\Omega_2$,

Then Q *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Proof of the main results

In this section, we will apply Lemma 2.6 to prove the existence of positive solutions of the Fr-BVP (1.7). For any $0 < r < R$, let

$$\Omega_r = \{u \in E : \|u\| < r\}, \qquad \Omega_R = \{u \in E : \|u\| < R\}.$$

Then $\partial\Omega_r = \{u \in E : \|u\| = r\}$, $\partial\Omega_R = \{u \in E : \|u\| = R\}$.

Proof of Theorem 1.1 On the one hand, defining an operator $Q : K \to E$ as in (2.3), we prove that there exists a constant $r \in (0, 1]$ such that

$$\|Qu\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_r.$$

In fact, for $u \in K$ with $\|u\| \leq 1$, we have

$$f\bigl(t, \delta(t)u(t)\bigr) \leq u^{\sigma_2}(t) f\bigl(t, \delta(t)\bigr) \leq \|u\|^{\sigma_2} f(t, \delta_M), \quad \forall t \in I^*.$$

So, by Lemma 2.2, we have

$$\begin{aligned} \|Qu\| &= \max_{t \in I^*} \left| \int_0^1 G(t, qs)\omega(s) f\bigl(s, \delta(s)u(s)\bigr)\, d_q s \right| \\ &\leq \|u\|^{\sigma_2} \|\omega\| \int_0^1 G(1, qs) f(s, \delta_M)\, d_q s \\ &= \beta_1 \|u\|^{\sigma_2}, \end{aligned}$$

where $\beta_1 = \|\omega\| \int_0^1 G(1, qs) f(s, \delta_M)\, d_q s$.

If $\beta_1 > 1$, choosing $r = (\frac{1}{\beta_1})^{\frac{1}{\sigma_2 - 1}}$, then $r \in (0, 1)$. For any $u \in K \cap \partial\Omega_r$, we have

$$\|Qu\| \leq \beta_1 \|u\|^{\sigma_2} = \beta_1^{1 - \frac{\sigma_2}{\sigma_2 - 1}} = r = \|u\|.$$

If $\beta_1 \leq 1$, choosing $r = 1$, then, for any $u \in K \cap \partial\Omega_r$, we have

$$\|Qu\| \leq \beta_1 \|u\|^{\sigma_2} = \beta_1 \leq 1 = r = \|u\|.$$

On the other hand, we prove that there exists a constant $R > 1$ such that

$$\|Qu\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_R.$$

In fact, for $u \in K$ with $u(t) \geq 1$ for $t \in I^*$, we have

$$f\bigl(t, \delta(t)u(t)\bigr) \geq u^{\sigma_2}(t) f\bigl(t, \delta(t)\bigr) \geq u^{\sigma_2}(t) f(t, \delta_m), \quad \forall t \in I^*.$$

Thus, for every $u \in K \cap \partial\Omega_R$, by Lemma 2.5, we have $Qu \in K$ and, for any $\eta \in (0,1)$,

$$\begin{aligned}
\|Qu\| &\geq \min_{t\in[\eta,1]} (Qu)(t) \\
&= \min_{t\in[\eta,1]} \int_0^1 G(t,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\
&\geq \min_{t\in[\eta,1]} \int_0^1 G(t,qs)\omega(s)u^{\sigma_2}(s)f(s,\delta_m)\,d_qs \\
&\geq \min_{t\in[\eta,1]} t^{\alpha-1}\xi \int_\eta^1 G(1,qs)u^{\sigma_2}(s)f(s,\delta_m)\,d_qs \\
&\geq \eta^{(\sigma_2+1)(\alpha-1)}\xi\|u\|^{\sigma_2} \int_\eta^1 G(1,qs)f(s,\delta_m)\,d_qs \\
&= \beta_2\|u\|^{\sigma_2},
\end{aligned}$$

where $\beta_2 = \eta^{(\sigma_2+1)(\alpha-1)}\xi \int_\eta^1 G(1,qs)f(s,\delta_m)\,d_qs$.

If $\beta_2 < 1$, choosing $R = (\frac{1}{\beta_2})^{\frac{1}{\sigma_2-1}}$, then $R > 1 \geq r$. For any $u \in K \cap \partial\Omega_R$, we have

$$\|Qu\| \geq \beta_2\|u\|^{\sigma_2} = \beta_2^{1-\frac{\sigma_2}{\sigma_2-1}} = R = \|u\|.$$

If $\beta_2 \geq 1$, choosing $R = \beta_2 + 1$, then $R > 1 \geq r$. For $u \in K \cap \partial\Omega_R$, we have

$$\|Qu\| \geq \beta_2\|u\|^{\sigma_2} \geq \beta_2\|u\| \geq \|u\|.$$

Hence, by Lemma 2.6, Q has at least one fixed point $u^* \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$ satisfying $0 < r \leq \|u^*\| \leq R$. Hence for $\eta \in (0,1)$, $\min_{t\in[\eta,1]} u^*(t) \geq \eta^{\alpha-1}\|u^*\| > 0$ and it is a positive solution of the Fr-BVP (1.7). □

Proof of Theorem 1.2 Similar to the proof of Theorem 1.1, we can prove this theorem. So we omit the details here. □

Remark 3.1 If $\omega \in L^\infty[0,T]$, the results in Theorem 1.1 and 1.2 are still true.

4 Examples

Example 4.1 Consider the following BVP:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{5}{2}}u(t) + \frac{5-\sin\pi t^2}{t(1-t)}(e^{3t}u^3(t) + e^{2t}u^2(t)) = 0, & t \in (0,1), \\ u(0) = 0, \quad D_{\frac{1}{2}}u(0) = D_{\frac{1}{2}}u(1) = 0. \end{cases} \tag{4.1}$$

Let $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $f(t,\delta(t)u(t)) = \frac{1}{t(1-t)}(e^{3t}u^3(t) + e^{2t}u^2(t))$ and $\omega(t) = 5 - \sin\pi t^2$, where $\delta(t) = e^t > 0$. Then $\xi = 4$, $\delta_m = 1$ and $\delta_M = e$. Clearly, $f(t,\delta_m) = \frac{2}{t(1-t)} > 0$ and

$$\int_0^1 G\biggl(1,\frac{1}{2}s\biggr)f(s,\delta_M)\,d_{\frac{1}{2}}s \leq \int_0^1 \frac{(1-\frac{1}{2}s)^{(\frac{1}{2})}(e^2+e^3)}{\Gamma_{\frac{1}{2}}(\frac{5}{2})s(1-s)}\,d_{\frac{1}{2}}s < +\infty,$$

where $\Gamma_{\frac{1}{2}}(\frac{5}{2}) = \frac{(\frac{1}{2})^{(\frac{3}{2})}}{(\frac{1}{2})^{\frac{3}{2}}}$. Hence the conditions $(A1)$ and $(A2)$ hold.

For $\tau \in (0,1]$, since

$$\frac{\tau^3}{t(1-t)}\left(x^3+x^2\right) \le f(t,\tau x) = \frac{1}{t(1-t)}\left(\tau^3x^3+\tau^2x^2\right) \le \frac{\tau^2}{t(1-t)}\left(x^3+x^2\right),$$

then the condition $(A3)$ is satisfied with $\sigma_1 = 3$, $\sigma_2 = 2$. Hence, by Theorem 1.1, the BVP (4.1) has at least one positive solution $u \in C[0,1]$.

Example 4.2 Consider the following BVP:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{5}{2}}u(t) + \frac{\cos 3t^2+2}{t(1-t)}(e^{\frac{t}{3}}u^{\frac{1}{3}}(t) + e^{\frac{t}{4}}u^{\frac{1}{4}}(t)) = 0, & t \in (0,1), \\ u(0) = 0, \qquad D_{\frac{1}{2}}u(0) = D_{\frac{1}{2}}u(1) = 0. \end{cases} \tag{4.2}$$

Let $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $f(t,\delta(t)u(t)) = \frac{1}{t(1-t)}(e^{\frac{t}{3}}u^{\frac{1}{3}}(t) + e^{\frac{t}{4}}u^{\frac{1}{4}}(t))$ and $\omega(t) = \cos 3t^2 + 2$, where $\delta(t) = e^t > 0$. Then $\xi = 1$, $\delta_m = 1$ and $\delta_M = e$. Only we verify $(A4)$. For $\tau \in (0,1]$, since

$$\frac{\tau^{\frac{1}{3}}}{t(1-t)}\left(x^{\frac{1}{3}}+x^{\frac{1}{4}}\right) \le f(t,\tau x) = \frac{1}{t(1-t)}\left(\tau^{\frac{1}{3}}x^{\frac{1}{3}}+\tau^{\frac{1}{4}}x^{\frac{1}{3}}\right) \le \frac{\tau^{\frac{1}{4}}}{t(1-t)}\left(x^{\frac{1}{3}}+x^{\frac{1}{4}}\right),$$

then the condition $(A4)$ is satisfied with $\sigma_3 = \frac{1}{4}$, $\sigma_4 = \frac{1}{3}$. Hence, by Theorem 1.2, the BVP (4.2) has at least one positive solution $u \in C[0,1]$.

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None of the authors have any competing interests in the manuscript.

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References

1. Ahmad, B., Ntouyas, S., Tariboon, J.: Quantum Calculus: New Concepts, Impulsive IVPs and BVPs, Inequalities. World Scientific, Singapore (2016)
2. Ernst, T.: A Comprehensive Treatment of q-Calculus. Springer, Basel (2012)
3. Ferreira, R.: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. **61**, 367–373 (2011)
4. Gao, C., Lv, L., Wang, Y.: Spectra of a discrete Sturm–Liouville problem with eigenparameter-dependent boundary conditions in Pontryagin space. Quaest. Math. (2019). https://doi.org/10.2989/16073606.2019.1680456
5. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cone. Academic Press, New York (1988)
6. Jackson, D., Fukuda, T., Dunn, O., Majors, E.: On q-definite integrals. J. Pure Appl. Math. **41**, 193–203 (1910)

7. Jackson, F.: On q-functions and a certain difference operator. Trans. R. Soc. Edinb. **46**, 253–281 (1908)
8. Kac, V., Cheung, P.: Quantum Calculus. Springer, New York (2001)
9. Li, X., Han, Z., Li, X.: Boundary value problems of fractional q-difference Schrödinger equations. Appl. Math. Lett. **46**, 100–105 (2015)
10. Mao, J., Zhao, Z., Wang, C.: The unique iterative positive solution of fractional boundary value problem with q-difference. Appl. Math. Lett. **10**, 106002 (2020)
11. Wang, G.: Twin iterative positive solutions of fractional q-difference Schrödinger equations. Appl. Math. Lett. **76**, 103–109 (2018)