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Some reverse inequalities of Hardy type on time scales

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Abstract

In this article, we obtain some new dynamic inequalities of Hardy type on time scales. The main results are derived using Fubini's theorem and the chain rule on time scales. We apply the main results to the continuous calculus, discrete calculus, and q-calculus as special cases.

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1 Introduction

In 1920, Hardy [15] proved the following result.

Theorem 1.1 *Let $\{a(n)\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers. If $p > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \left(\sum_{m=1}^{n} a(m) \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a^p(n). \tag{1.1}$$

In 1925, the continuous analogue of inequality (1.1) was given by Hardy [16] in the following form.

Theorem 1.2 *Let f be a nonnegative continuous function on $[0, \infty)$. If $p > 1$, then*

$$\int_0^{\infty} \frac{1}{x^p} \left(\int_0^x f(s) \, ds \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x) \, dx. \tag{1.2}$$

It is worthy to mention that inequality (1.2) is sharp in the sense that the constant $(\frac{p}{p-1})^p$ cannot be replaced by a smaller one.

In 1927, Littlewood and Hardy [28] established the reverse of inequality (1.2) as follows.

Theorem 1.3 *Let f be a nonnegative function on $[0, \infty)$. If $0 < p < 1$, then*

$$\int_0^{\infty} \frac{1}{x^p} \left(\int_x^{\infty} f(s) \, ds \right)^p dx \geq \left(\frac{p}{1-p} \right)^p \int_0^{\infty} f^p(x) \, dx. \tag{1.3}$$

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In 1928, Hardy [17] proved a generalization of integral inequality (1.2) in the following theorem.

Theorem 1.4 *If f is a nonnegative continuous function on $[0, \infty)$, then*

$$\int_0^\infty \frac{1}{x^\gamma} \left(\int_0^x f(s)\,ds \right)^p dx \leq \left(\frac{p}{\gamma - 1} \right)^p \int_0^\infty x^{p-\gamma} f^p(x)\,dx \quad \text{for } p \geq \gamma > 1 \qquad (1.4)$$

and

$$\int_0^\infty \frac{1}{x^\gamma} \left(\int_x^\infty f(s)\,ds \right)^p dx \leq \left(\frac{p}{1 - \gamma} \right)^p \int_0^\infty x^{p-\gamma} f^p(x)\,dx \quad \text{for } p > 1 > \gamma \geq 0. \qquad (1.5)$$

In 1928, Copson [11] generalized the discrete Hardy inequality (1.1) and obtained the next two discrete inequalities.

Theorem 1.5 *Let $\{a(n)\}_{n=1}^\infty$ and $\{\lambda(n)\}_{n=1}^\infty$ be sequences of nonnegative real numbers. Then*

$$\sum_{n=1}^\infty \frac{\lambda(n)(\sum_{m=1}^n \lambda(m)a(m))^p}{(\sum_{m=1}^n \lambda(m))^\gamma}$$
$$\leq \left(\frac{p}{\gamma - 1} \right)^p \sum_{n=1}^\infty \lambda(n)a^p(n) \left(\sum_{m=1}^n \lambda(m) \right)^{p-\gamma} \quad \text{for } p \geq \gamma > 1 \qquad (1.6)$$

and

$$\sum_{n=1}^\infty \frac{\lambda(n)(\sum_{m=n}^\infty \lambda(m)a(m))^p}{(\sum_{m=1}^n \lambda(m))^\gamma}$$
$$\leq \left(\frac{p}{1 - \gamma} \right)^p \sum_{n=1}^\infty \lambda(n)a^p(n) \left(\sum_{m=1}^n \lambda(m) \right)^{p-\gamma} \quad \text{for } p > 1 > \gamma \geq 0. \qquad (1.7)$$

In the same paper [11], Copson obtained the following discrete inequality of Hardy type.

Theorem 1.6 *Let $\{a_n\}_{n=1}^\infty$ be a sequence of nonnegative real numbers. If $p > 1$, then*

$$\sum_{n=1}^\infty \left(\sum_{k=n}^\infty a_k \right)^p \leq p^p \sum_{n=1}^\infty (na_n)^p. \qquad (1.8)$$

In 1970, Leindler [23] studied some variants of the discrete Hardy inequality (1.1) and was able to prove the following.

Theorem 1.7 *If $\{a(n)\}_{n=1}^\infty$ and $\{\lambda(n)\}_{n=1}^\infty$ are sequences of nonnegative real numbers and $p > 1$, then*

$$\sum_{n=1}^\infty \lambda(n) \left(\sum_{m=1}^n a(m) \right)^p \leq p^p \sum_{n=1}^\infty \lambda^{1-p}(n)a^p(n) \left(\sum_{m=n}^\infty \lambda(m) \right)^p \qquad (1.9)$$

and

$$\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{m=n}^{\infty} a(m) \right)^p \leq p^p \sum_{n=1}^{\infty} \lambda^{1-p}(n) a^p(n) \left(\sum_{m=1}^{n} \lambda(m) \right)^p. \tag{1.10}$$

In the same paper [23], Leindler studied the case that the summation $\sum_{n=1}^{\infty} \lambda(m) < \infty$ on the left-hand side of inequality (1.6) is replaced with the summation $\sum_{m=n}^{\infty} \lambda(m) < \infty$. His result can be written as follows.

Theorem 1.8 *Let* $\{a(n)\}_{n=1}^{\infty}$ *and* $\{\lambda(n)\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers with* $\sum_{m=n}^{\infty} \lambda(m) < \infty$. *If* $p > 1$ *and* $0 \leq \gamma < 1$, *then*

$$\sum_{n=1}^{\infty} \frac{\lambda(n)(\sum_{m=1}^{n} \lambda(m)a(m))^p}{(\sum_{m=n}^{\infty} \lambda(m))^\gamma}$$
$$\leq \left(\frac{p}{1-\gamma} \right)^p \sum_{n=1}^{\infty} \lambda(n) a^p(n) \left(\sum_{m=n}^{\infty} \lambda(m) \right)^{p-\gamma}. \tag{1.11}$$

In 1976, Copson [12] gave the continuous versions of inequalities (1.6) and (1.7). Specifically, he established the following result.

Theorem 1.9 *Let f and λ be nonnegative continuous functions on* $[0, \infty)$. *Then*

$$\int_0^\infty \frac{\lambda(x)(\int_0^x \lambda(s)f(s)\,ds)^p}{(\int_0^x \lambda(s)\,ds)^\gamma}\,dx$$
$$\leq \left(\frac{p}{\gamma-1} \right)^p \int_0^\infty \lambda(x) f^p(x) \left(\int_0^x \lambda(s)\,ds \right)^{p-\gamma}\,dx \quad \text{for } p \geq \gamma > 1 \tag{1.12}$$

and

$$\int_0^\infty \frac{\lambda(x)(\int_x^\infty \lambda(s)f(s)\,ds)^p}{(\int_0^x \lambda(s)\,ds)^\gamma}\,dx$$
$$\leq \left(\frac{p}{1-\gamma} \right)^p \int_0^\infty \lambda(x) f^p(x) \left(\int_0^x \lambda(s)\,ds \right)^{p-\gamma}\,dx \quad \text{for } p > 1 > \gamma \geq 0. \tag{1.13}$$

In 1982, Lyon [29] established a reverse version of the discrete Hardy inequality (1.1) for the special case when $p = 2$. His result asserts the following.

Theorem 1.10 *Let* $\{a_n\}_{n=0}^{\infty}$ *be a nonincreasing sequence of nonnegative real numbers. Then*

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right)^2 \geq \frac{\pi^2}{6} \sum_{n=0}^{\infty} a_n^2. \tag{1.14}$$

In 1986, Renaud [33] gave a generalization of Lyon's inequality (1.14) in the following form.

Theorem 1.11 *Let $\{a_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of nonnegative real numbers. If $p > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \left(\sum_{k=1}^{n} a_k \right)^p \geq \zeta(p) \sum_{n=1}^{\infty} a_n^p, \tag{1.15}$$

where $\zeta(p)$ is the Riemann zeta function.

The integral analogous of inequality (1.15), which was proved in the same paper [33], is as follows.

Theorem 1.12 *Let f be a nonincreasing nonnegative function on $[0, \infty)$. If $p > 1$, then*

$$\int_0^{\infty} \frac{1}{x^p} \left(\int_0^{x} f(s) \, ds \right)^p \geq \frac{p}{p-1} \int_0^{\infty} f^p(x) \, dx. \tag{1.16}$$

Also in [33], Renaud proved the reverse of inequality (1.8) and the integral version of this reverse inequality. In fact, he proved the following two results.

Theorem 1.13 *Let $\{a_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of nonnegative real numbers. If $p > 1$, then*

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_k \right)^p \geq \sum_{n=1}^{\infty} n^p a_n^p. \tag{1.17}$$

Theorem 1.14 *Let f be a nonincreasing nonnegative function on $[0, \infty)$. If $p > 1$, then*

$$\int_0^{\infty} \left(\int_x^{\infty} f(s) \, ds \right)^p dx \geq \int_0^{\infty} x^p f^p(x) \, dx. \tag{1.18}$$

In 1987, Bennett [5], similarly to what Leindler did in Theorem 1.8, proved the following result.

Theorem 1.15 *Let $\{a(n)\}_{n=1}^{\infty}$ and $\{\lambda(n)\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers with $\sum_{m=n}^{\infty} \lambda(m) < \infty$. If $p \geq \gamma > 1$, then*

$$\sum_{n=1}^{\infty} \frac{\lambda(n)(\sum_{m=n}^{\infty} \lambda(m)a(m))^p}{(\sum_{m=n}^{\infty} \lambda(m))^{\gamma}} \leq \left(\frac{p}{\gamma-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) a^p(n) \left(\sum_{m=n}^{\infty} \lambda(m) \right)^{p-\gamma}. \tag{1.19}$$

In 1990, the reverses of inequalities (1.9) and (1.10) were shown by Leindler in [24] as follows.

Theorem 1.16 *If $\{a(n)\}_{n=1}^{\infty}$ and $\{\lambda(n)\}_{n=1}^{\infty}$ are sequences of nonnegative real numbers and $0 < p \leq 1$, then*

$$\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{m=1}^{n} a(m) \right)^p \geq p^p \sum_{n=1}^{\infty} \lambda^{1-p}(n) a^p(n) \left(\sum_{m=n}^{\infty} \lambda(m) \right)^p \tag{1.20}$$

and

$$\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{m=n}^{\infty} a(m) \right)^p \geq p^p \sum_{n=1}^{\infty} \lambda^{1-p}(n) a^p(n) \left(\sum_{m=1}^{n} \lambda(m) \right)^p. \tag{1.21}$$

In 2002, Kaijser et al. [22], using the convex functions, established a generalization of the integral Hardy inequality (1.2) in the following form.

Theorem 1.17 *If f is a nonnegative function on $[0, \infty)$ and $\Phi \geq 0$ is a convex increasing function on $[0, \infty)$, then*

$$\int_0^{\infty} \frac{1}{x} \Phi \left(\frac{1}{x} \int_0^x f(t) \, dt \right) dx \leq \int_0^{\infty} \frac{\Phi(f(x))}{x} \, dx. \tag{1.22}$$

The theory of time scales, which has recently received a lot of attention, was initiated by Stefan Hilger in his PhD thesis [18] in order to unify discrete and continuous analysis [19]. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is defined as an arbitrary closed subset of the real numbers $\mathbb{R}$, see [9, 10]. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see [21]), i.e., when $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. The books on the subject of time scales by Bohner and Peterson [9, 10] summarize and organize much of time scale calculus. During the past two decades, a number of dynamic inequalities have been established by some authors which are motivated by some applications (see [1, 2, 7, 8, 13, 20, 25–27, 42]).

In 2005, Řehák [32] was a pioneer in extending Hardy-type inequalities to time scales. He extended the original Hardy inequalities (1.1) and (1.2) to an arbitrary time scale, and he applied his results to give an application in the oscillation theory of half-linear dynamic equations, and so, he unified them in one form as shown next.

Theorem 1.18 *Let $\mathbb{T}$ be a time scale, and $f \in C_{rd}([a, \infty)_{\mathbb{T}}, [0, \infty))$, $\Lambda(t) = \int_a^t f(s) \Delta s$ for $t \in [a, \infty)_{\mathbb{T}}$.*

$$\int_a^{\infty} \left(\frac{\Lambda^{\sigma}(t)}{\sigma(t) - a} \right)^p \Delta t < \left(\frac{p}{p-1} \right)^p \int_a^{\infty} f^p(t) \Delta t, \quad p > 1, \tag{1.23}$$

unless $f \equiv 0$.

Furthermore, if $\mu(t)/t \to 0$ as $t \to \infty$, then inequality (1.23) is sharp.

In 2008, Ozkan and Yildirim [31] established a new dynamic Hardy-type inequality with weight functions that can be considered as the time scales extension of inequality (1.22). Their result is the following theorem.

Theorem 1.19 *Assume that $f \in C_{rd}([a, b]_{\mathbb{T}}, (c, d))$ and Φ is a convex function on (c, d). Further, let $u \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R}_+)$ such that the delta integral $\int_t^b \frac{u(s)}{(s-a)(\sigma(s)-a)} \Delta s$ exists as a finite number. Then*

$$\int_a^b \frac{u(t)}{t-a} \Phi \left(\frac{1}{\sigma(t) - a} \int_a^{\sigma(t)} f(s) \Delta s \right) \Delta t \leq \int_0^{\infty} \Phi(f(t)) \left(\int_t^b \frac{u(s)}{(s-a)(\sigma(s)-a)} \Delta s \right) \Delta t.$$

In 2014, Saker et al. [39] established a generalization of Řehák's result in the following form.

Theorem 1.20 *Let $a \in [0, \infty)_{\mathbb{T}}$ and define, for $t \in [0, \infty)_{\mathbb{T}}$,*

$$\Phi(t) := \int_a^t \lambda(s) g(s) \Delta s \quad and \quad \Lambda(t) := \int_a^t \lambda(s) \Delta s.$$

If $p \geq q > 1$, then

$$\int_a^\infty \lambda(t) \frac{(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^q} \Delta t \leq \left(\frac{p}{q-1} \right)^p \int_a^\infty \lambda(t) \frac{(\Lambda^\sigma(t))^{q(p-1)}}{(\Lambda(t))^{p(q-1)}} g^p(t) \Delta t. \tag{1.24}$$

Recently, in 2017, Agarwal et al. [3] gave the time scales version of inequality (1.16) as follows.

Theorem 1.21 *Suppose that $\mathbb{T}$ is a time scale such that $0 \in \mathbb{T}$. Further, assume that f is a nonincreasing nonnegative function on $[0, \infty)_{\mathbb{T}}$. If $p > 1$, then*

$$\int_0^\infty \frac{1}{t^p} \left(\int_0^t f(s) \Delta s \right)^p \Delta t \geq \frac{p}{p-1} \int_0^\infty f^p(t) \Delta t. \tag{1.25}$$

After these initial results, many generalizations, extensions, and refinements of a dynamic Hardy inequality were made by various authors. For a comprehensive survey on the dynamic inequalities of Hardy type on time scales, one can refer to the papers [14, 30, 31, 34–38, 40, 41] and the book [4].

In this article, we state and prove some reverse Hardy-type dynamic inequalities on time scales. The obtained Hardy-type dynamic inequalities are completely original, and thus, we get some new integral and discrete inequalities of Hardy type. In addition to that, some of our results generalize inequality (1.25) and give the time scales version of inequalities (1.17) and (1.18).

We will need the following important relations between calculus on time scales $\mathbb{T}$ and continuous calculus on $\mathbb{R}$, discrete calculus on $\mathbb{Z}$. Note that:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \mu(t) = 0, \qquad f^\Delta(t) = f'(t), \qquad \int_a^b f(t) \Delta t = \int_a^b f(t)\, dt. \tag{1.26}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \qquad \mu(t) = 1, \qquad f^\Delta(t) = \Delta f(t), \qquad \int_a^b f(t) \Delta t = \sum_{t=a}^{b-1} f(t). \tag{1.27}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$, then

$$\sigma(t) = t + h, \qquad \mu(t) = h, \qquad f^\Delta(t) = \frac{f(t+h) - f(t)}{h},$$

$$\int_a^b f(t) \Delta t = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} h f(ht). \tag{1.28}$$

(iv) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then

$$\sigma(t) = qt, \qquad \mu(t) = (q-1)t, \qquad f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t},$$

$$\int_a^b f(t)\Delta t = (q-1) \sum_{t=\log_q a}^{\log_q b - 1} q^t f\left(q^t\right). \tag{1.29}$$

One of the forms of the chain rule on time scales is the following form.

Lemma 1.22 (Chain rule, see [9]) *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, $g : \mathbb{T} \to \mathbb{R}$ be a delta differentiable function on $\mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then there is c in the interval $[t, \sigma(t)]$ such that*

$$(f \circ g)^\Delta(t) = f'\left(g(c)\right)g^\Delta(t). \tag{1.30}$$

The following lemma due to Keller is known as Keller's chain rule on time scales.

Lemma 1.23 (Chain rule, see [9]) *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and $g : \mathbb{T} \to \mathbb{R}$ is a delta differentiable function. Then*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)g^\Delta(t)\right) dh \right\} g^\Delta(t). \tag{1.31}$$

Next, we write Fubini's theorem on time scales.

Lemma 1.24 (Fubini's thoerem, see [6]) *Let f be bounded and Δ-integrable over $R = [a, b) \times [c, d)$ and suppose that the single integrals*

$$I(t) = \int_c^d f(t,s)\Delta s \quad and \quad K(s) = \int_a^b f(t,s)\Delta t$$

exist for each $t \in [a, b)$ and for each $s \in [c, d)$, respectively. Then the iterated integrals

$$\int_a^b \Delta t \int_c^d f(t,s)\Delta s \quad and \quad \int_c^d \Delta s \int_a^b f(t,s)\Delta t$$

exist and the equality

$$\int_a^b \Delta t \int_c^d f(t,s)\Delta s = \int_c^d \Delta s \int_a^b f(t,s)\Delta t$$

holds.

Now we are ready to state and prove our main results.

2 Main results

Throughout this section, any time scale $\mathbb{T}$ is unbounded above, and we will assume that the right-hand sides of the inequalities converge if the left-hand sides converge.

The following result will establish a new weighted dynamic Hardy inequality and, as special cases of it, we will be able to obtain two original integral and discrete inequalities. Inequalities (1.17) and (1.18) can be recaptured as special cases of these obtained integral and discrete inequalities.

Theorem 2.1 *Assume that $\mathbb{T}$ is a time scale with $0 \leq a \in \mathbb{T}$. Moreover, suppose that f and λ are nonnegative rd-continuous functions on $[a, \infty)_{\mathbb{T}}$ with f nonincreasing. If $p \geq 1$ and $\gamma \geq 0$, then*

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{(\Lambda^\sigma(t))^\gamma} \Delta t \geq \int_a^\infty \frac{\lambda(t)\Lambda^p(t)f^p(t)}{(\Lambda^\sigma(t))^\gamma} \Delta t, \tag{2.1}$$

where

$$\Phi(t) = \int_t^\infty \lambda(s)f(s)\Delta s \quad and \quad \Lambda(t) = \int_a^t \lambda(s)\Delta s.$$

Proof As f is nonincreasing, we have for $x \geq t \geq a$

$$F(x,t) = \int_t^x \lambda(s)f(s)\Delta s \geq f(x)\int_t^x \lambda(s)\Delta s.$$

Then

$$f(x)F^{p-1}(x,t) \geq \left[\int_t^x \lambda(s)\Delta s\right]^{p-1} f^p(x). \tag{2.2}$$

Applying the chain rule (1.31) and using $F^{\Delta_x}(x,t) = \lambda(x)f(x) \geq 0$, where Δ_x denotes the delta derivative with respect to x, we get

$$\begin{aligned}
\left(F^p(x,t)\right)^{\Delta_x} &= pF^{\Delta_x}(x,t)\int_0^1 \left[hF(\sigma(x),t) + (1-h)F(x,t)\right]^{p-1} dh \\
&\geq p\lambda(x)f(x)\int_0^1 \left[hF(x,t) + (1-h)F(x,t)\right]^{p-1} dh \\
&= p\lambda(x)f(x)F^{p-1}(x,t).
\end{aligned} \tag{2.3}$$

Combining (2.2) with (2.3) gives

$$\left(F^p(x,t)\right)^{\Delta_x} \geq p\lambda(x)\left[\int_t^x \lambda(s)\Delta s\right]^{p-1} f^p(x),$$

and so (note that $x \geq t \geq a$ and hence, because Λ is nondecreasing, $\Lambda^\sigma(x) \geq \Lambda^\sigma(t) \geq 0$)

$$\frac{\lambda(t)(F^p(x,t))^{\Delta_x}}{(\Lambda^\sigma(t))^\gamma} \geq \frac{p\lambda(t)\lambda(x)}{(\Lambda^\sigma(t))^\gamma}\left[\int_t^x \lambda(s)\Delta s\right]^{p-1} f^p(x) \geq \frac{p\lambda(t)\lambda(x)}{(\Lambda^\sigma(x))^\gamma}\left[\int_t^x \lambda(s)\Delta s\right]^{p-1} f^p(x).$$

Integrating both sides with respect to x over $[t, \infty)_{\mathbb{T}}$ gives

$$\frac{\lambda(t)\Phi^p(t)}{(\Lambda^\sigma(t))^\gamma} = \int_t^\infty \frac{\lambda(t)(F^p(x,t))^{\Delta_x}}{(\Lambda^\sigma(t))^\gamma}\Delta x \geq p\int_t^\infty \frac{\lambda(t)\lambda(x)}{(\Lambda^\sigma(x))^\gamma}\left[\int_t^x \lambda(s)\Delta s\right]^{p-1} f^p(x)\Delta x.$$

Integrating both sides again, but this time with respect to t over $[a, \infty)_{\mathbb{T}}$, produces

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{(\Lambda^\sigma(t))^\gamma}\Delta t \geq p \int_a^\infty \left[\int_t^\infty \frac{\lambda(t)\lambda(x)}{(\Lambda^\sigma(x))^\gamma}\left[\int_t^x \lambda(s)\Delta s\right]^{p-1}f^p(x)\Delta x\right]\Delta t. \quad (2.4)$$

Using Fubini's theorem on time scales, inequality (2.4) can be rewritten as

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{(\Lambda^\sigma(t))^\gamma}\Delta t \geq p \int_a^\infty \lambda(x)\left(\Lambda^\sigma(x)\right)^{-\gamma}f^p(x)\left[\int_a^x \lambda(t)\left[\int_t^x \lambda(s)\Delta s\right]^{p-1}\Delta t\right]\Delta x. \quad (2.5)$$

Now, from the chain rule (1.30), there exists $c \in [t, \sigma(t)]$ such that (here Δ_t denotes the delta derivative with respect to t)

$$-\left[\left(\int_t^x \lambda(s)\Delta s\right)^p\right]^{\Delta_t} = p\lambda(t)\left[\int_c^x \lambda(s)\Delta s\right]^{p-1} \leq p\lambda(t)\left[\int_t^x \lambda(s)\Delta s\right]^{p-1}. \quad (2.6)$$

Substituting (2.6) into (2.5) leads to

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{(\Lambda^\sigma(t))^\gamma}\Delta t \geq \int_a^\infty \lambda(x)\left(\Lambda^\sigma(x)\right)^{-\gamma}f^p(x)\left[\int_a^x -\left[\left(\int_t^x \lambda(s)\Delta s\right)^p\right]^{\Delta_t}\Delta t\right]\Delta x$$

$$= \int_a^\infty \frac{\lambda(x)\Lambda^p(x)f^p(x)}{(\Lambda^\sigma(x))^\gamma}\Delta x.$$

This shows the validity of inequality (2.1). $\qquad\square$

Corollary 2.2 *In Theorem* 2.1, *if we take $\lambda(t) = 1$ and $\gamma = 0$, then inequality (2.1) reduces to*

$$\int_a^\infty \left(\int_t^\infty f(s)\Delta s\right)^p \Delta t \geq \int_a^\infty t^p f^p(t)\Delta t,$$

which is the time scales version of inequalities (1.17) and (1.18).

Corollary 2.3 *If $\mathbb{T} = \mathbb{R}$ in Theorem* 2.1, *then, using relations (1.26), inequality (2.1) reduces to*

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{\Lambda^\gamma(t)}\,dt \geq \int_a^\infty \lambda(t)\Lambda^{p-\gamma}(t)f^p(t)\,dt,$$

where

$$\Phi(t) = \int_t^\infty \lambda(s)f(s)\,ds \quad and \quad \Lambda(t) = \int_a^t \lambda(s)\,ds.$$

Remark 2.4 In Corollary 2.3, if we take $\lambda(t) = 1$, $a = 1$, and $\gamma = 0$, then we recapture inequality (1.18).

Corollary 2.5 *If $\mathbb{T} = h\mathbb{Z}$ in Theorem* 2.1, *then, using relations (1.28), inequality (2.1) reduces to*

$$\sum_{t=\frac{a}{h}}^\infty \frac{\lambda(ht)\Phi^p(ht)}{\Lambda^\gamma(ht+h)} \geq \sum_{t=\frac{a}{h}}^\infty \frac{\lambda(ht)\Lambda^p(ht)f^p(ht)}{\Lambda^\gamma(ht+h)},$$

where

$$\Phi(t) = h \sum_{s=\frac{t}{h}}^{\infty} \lambda(hs)f(hs) \quad and \quad \Lambda(t) = h \sum_{s=\frac{a}{h}}^{\frac{t}{h}-1} \lambda(hs).$$

Corollary 2.6 *For* $\mathbb{T} = \mathbb{Z}$, *we simply take* $h = 1$ *in Corollary* 2.5. *In this case, inequality* (2.1) *reduces to*

$$\sum_{t=a}^{\infty} \frac{\lambda(t)\Phi^p(t)}{\Lambda^\gamma(t+1)} \geq \sum_{t=a}^{\infty} \frac{\lambda(t)\Lambda^p(t)f^p(t)}{\Lambda^\gamma(t+1)},$$

where

$$\Phi(t) = \sum_{s=t}^{\infty} \lambda(s)f(s) \quad and \quad \Lambda(t) = \sum_{s=a}^{t-1} \lambda(s).$$

Remark 2.7 In Corollary 2.6, if we take $\lambda(t) = 1$, $a = 1$, and $\gamma = 0$, then we recapture inequality (1.17).

Corollary 2.8 *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem* 2.1, *then, using relations* (1.29), *inequality* (2.1) *reduces to*

$$\sum_{t=(\log_q a)}^{\infty} \frac{q^t\lambda(q^t)\Phi^p(q^t)}{\Lambda^\gamma(q^{t+1})} \geq \sum_{t=(\log_q a)}^{\infty} \frac{q^t\lambda(q^t)\Lambda^p(q^t)f^p(q^t)}{\Lambda^\gamma(q^{t+1})},$$

where

$$\Phi(t) = (q-1) \sum_{s=(\log_q t)}^{\infty} q^s\lambda(q^s)f(q^s) \quad and \quad \Lambda(t) = (q-1) \sum_{s=(\log_q a)}^{(\log_q t)-1} q^s\lambda(q^s).$$

Now, it seems interesting to study inequality (2.1) in the case of the integral $\int_a^t \lambda(s)\Delta s$ on the left-hand side being replaced by the integral $\int_t^\infty \lambda(s)\Delta s$. In fact, that is what we are going to do in the next theorem.

Theorem 2.9 *Let* $\mathbb{T}$ *be a time scale with* $0 \leq a \in \mathbb{T}$. *Furthermore, let* f *and* λ *be nonnegative rd-continuous functions on* $[a, \infty)_{\mathbb{T}}$ *with* f *nonincreasing. If* $p \geq 1$ *and* $\gamma > 1$, *then*

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{\Omega^\gamma(t)} \Delta t \geq \frac{p}{\gamma-1} \int_a^\infty \lambda(t)\Omega^{p-\gamma}(t)f^p(t)\Delta t, \tag{2.7}$$

where

$$\Phi(t) = \int_t^\infty \lambda(s)f(s)\Delta s \quad and \quad \Omega(t) = \int_t^\infty \lambda(s)\Delta s.$$

Proof Since f is nonincreasing, we have for $t \geq x \geq a$

$$\Phi(x) = \int_x^\infty \lambda(s)f(s)\Delta s \leq f(x)\Omega(x).$$

So,

$$f(x)\Phi^{p-1}(x) \leq \Omega^{p-1}(x)f^p(x). \tag{2.8}$$

By utilizing the chain rule (1.31) and using $\Phi^\Delta(x) = -\lambda(x)f(x) \leq 0$, we get

$$\begin{aligned}
\left(\Phi^p(x)\right)^\Delta &= p\Phi^\Delta(x) \int_0^1 \left[h\Phi^\sigma(x) + (1-h)\Phi(x)\right]^{p-1} dh \\
&\geq -p\lambda(x)f(x) \int_0^1 \left[h\Phi(x) + (1-h)\Phi(x)\right]^{p-1} dh \\
&= -p\lambda(x)f(x)\Phi^{p-1}(x). \tag{2.9}
\end{aligned}$$

From (2.8) and (2.9) we get

$$\left(\Phi^p(x)\right)^\Delta \geq -p\lambda(x)\Omega^{p-1}(x)f^p(x).$$

Thus,

$$\frac{\lambda(t)(\Phi^p(x))^\Delta}{\Omega^\gamma(t)} \geq \frac{-p\lambda(t)\lambda(x)\Omega^{p-1}(x)f^p(x)}{\Omega^\gamma(t)}.$$

Therefore, upon integrating both sides with respect to x over $[a,t]_{\mathbb{T}}$,

$$\frac{\lambda(t)[\Phi^p(t) - \Phi^p(a)]}{\Omega^\gamma(t)} = \int_a^t \frac{\lambda(t)(\Phi^p(x))^\Delta}{\Omega^\gamma(t)}\Delta x \geq -p\int_a^t \frac{\lambda(t)\lambda(x)\Omega^{p-1}(x)f^p(x)}{\Omega^\gamma(t)}\Delta x.$$

Since $\Phi^p(t) \geq \Phi^p(t) - \Phi^p(a)$, we have

$$\frac{\lambda(t)\Phi^p(t)}{\Omega^\gamma(t)} \geq -p\int_a^t \frac{\lambda(t)\lambda(x)\Omega^{p-1}(x)f^p(x)}{\Omega^\gamma(t)}\Delta x.$$

Then, by integrating both sides with respect to t over $[a,\infty)_{\mathbb{T}}$, we get

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{\Omega^\gamma(t)}\Delta t \geq -p\int_a^\infty \left[\int_a^t \frac{\lambda(t)\lambda(x)\Omega^{p-1}(x)f^p(x)}{\Omega^\gamma(t)}\Delta x\right]\Delta t. \tag{2.10}$$

With the help of Fubini's theorem on time scales, inequality (2.10) can be rewritten as

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{\Omega^\gamma(t)}\Delta t \geq p\int_a^\infty \lambda(x)\Omega^{p-1}(x)f^p(x)\left[\int_x^\infty -\lambda(t)\Omega^{-\gamma}(t)\Delta t\right]\Delta x. \tag{2.11}$$

Now, from the chain rule (1.30), there exists $c \in [t,\sigma(t)]$ with

$$-\left(\Omega^{1-\gamma}(t)\right)^\Delta = -(\gamma-1)\lambda(t)\Omega^{-\gamma}(c) \leq -(\gamma-1)\lambda(t)\Omega^{-\gamma}(t). \tag{2.12}$$

Combining (2.12) and (2.11) yields

$$\int_a^\infty \frac{\lambda(t)\Phi^p(t)}{\Omega^\gamma(t)}\Delta t \geq \frac{p}{\gamma-1}\int_a^\infty \lambda(x)\Omega^{p-1}(x)f^p(x)\left[\int_x^\infty -\left(\Omega^{1-\gamma}(t)\right)^\Delta \Delta t\right]\Delta x$$

$$= \frac{p}{\gamma - 1} \int_a^\infty \lambda(x) \Omega^{p-\gamma}(x) f^p(x) \Delta x,$$

from which inequality (2.7) follows. □

Corollary 2.10 *If* $\mathbb{T} = \mathbb{R}$ *in Theorem* 2.9, *then, using relations* (1.26), *inequality* (2.7) *boils down to*

$$\int_a^\infty \frac{\lambda(t) \Phi^p(t)}{\Omega^\gamma(t)} \, dt \geq \frac{p}{\gamma - 1} \int_a^\infty \lambda(t) \Omega^{p-\gamma}(t) f^p(t) \, dt,$$

where

$$\Phi(t) = \int_t^\infty \lambda(s) f(s) \, ds \quad and \quad \Omega(t) = \int_t^\infty \lambda(s) \, ds.$$

Corollary 2.11 *If* $\mathbb{T} = h\mathbb{Z}$ *in Theorem* 2.9, *then, using relations* (1.28), *inequality* (2.7) *boils down to*

$$\sum_{t=\frac{a}{h}}^\infty \frac{\lambda(ht) \Phi^p(ht)}{\Omega^\gamma(ht)} \geq \frac{p}{\gamma - 1} \sum_{t=\frac{a}{h}}^\infty \lambda(ht) \Omega^{p-\gamma}(ht) f^p(ht),$$

where

$$\Phi(t) = h \sum_{s=\frac{t}{h}}^\infty \lambda(hs) f(hs) \quad and \quad \Omega(t) = h \sum_{s=\frac{t}{h}}^\infty \lambda(hs).$$

Corollary 2.12 *For* $\mathbb{T} = \mathbb{Z}$, *we simply take* $h = 1$ *in Corollary* 2.11. *In this case, inequality* (2.7) *boils down to*

$$\sum_{t=a}^\infty \frac{\lambda(t) \Phi^p(t)}{\Omega^\gamma(t)} \geq \frac{p}{\gamma - 1} \sum_{t=a}^\infty \lambda(t) \Omega^{p-\gamma}(t) f^p(t),$$

where

$$\Phi(t) = \sum_{s=t}^\infty \lambda(s) f(s) \quad and \quad \Omega(t) = \sum_{s=t}^\infty \lambda(s).$$

Corollary 2.13 *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem* 2.9, *then, using relations* (1.29), *inequality* (2.7) *boils down to*

$$\sum_{t=(\log_q a)}^\infty \frac{q^t \lambda(q^t) \Phi^p(q^t)}{\Omega^\gamma(q^t)} \geq \frac{p}{\gamma - 1} \sum_{t=(\log_q a)}^\infty q^t \lambda(q^t) \Omega^{p-\gamma}(q^t) f^p(q^t),$$

where

$$\Phi(t) = (q-1) \sum_{s=(\log_q t)}^\infty q^s \lambda(q^s) f(q^s) \quad and \quad \Omega(t) = (q-1) \sum_{s=(\log_q t)}^\infty q^s \lambda(q^s).$$

The following theorem gives a generalization of Theorem 1.21.

Theorem 2.14 *Suppose that $\mathbb{T}$ is a time scale with $0 \leq a \in \mathbb{T}$. Moreover, assume that f and λ are nonnegative rd-continuous functions on $[a, \infty)_{\mathbb{T}}$ with f nonincreasing. If $p \geq 1$ and $\gamma > 1$, then*

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{\Lambda^\gamma(t)} \Delta t \geq \frac{p}{\gamma - 1} \int_a^\infty \lambda(t)\Lambda^{p-\gamma}(t)f^p(t)\Delta t, \tag{2.13}$$

where

$$\Psi(t) = \int_a^t \lambda(s)f(s)\Delta s \quad and \quad \Lambda(t) = \int_a^t \lambda(s)\Delta s.$$

Proof As f is nonincreasing, we have for $x \geq a$

$$\Psi(x) = \int_a^x \lambda(s)f(s)\Delta s \geq f(x)\int_a^x \lambda(s)\Delta s = f(x)\Lambda(x),$$

then

$$f(x)\Psi^{p-1}(x) \geq f^p(x)\Lambda^{p-1}(x). \tag{2.14}$$

Using the chain rule (1.31) and the fact that $\Psi^\Delta(x) = \lambda(x)f(x) \geq 0$, we get

$$\begin{aligned}
\left(\Psi^p(x)\right)^\Delta &= p\Psi^\Delta(x)\int_0^1 \left[h\Psi\left(\sigma(x)\right) + (1-h)\Psi(x)\right]^{p-1} dh \\
&\geq p\lambda(x)f(x)\int_0^1 \left[h\Psi(x) + (1-h)\Psi(x)\right]^{p-1} dh \\
&= p\lambda(x)f(x)\Psi^{p-1}(x).
\end{aligned} \tag{2.15}$$

Combining (2.14) with (2.15) gives

$$\left(\Psi^p(x)\right)^\Delta \geq p\lambda(x)\Lambda^{p-1}(x)f^p(x),$$

and thus

$$\frac{\lambda(t)(\Psi^p(x))^\Delta}{\Lambda^\gamma(t)} \geq \frac{p\lambda(t)\lambda(x)\Lambda^{p-1}(x)f^p(x)}{\Lambda^\gamma(t)}.$$

Therefore,

$$\frac{\lambda(t)\Psi^p(t)}{\Lambda^\gamma(t)} = \int_a^t \frac{\lambda(t)(\Psi^p(x))^\Delta}{\Lambda(t)} \Delta x \geq p\int_a^t \frac{\lambda(t)\lambda(x)\Lambda^{p-1}(x)f^p(x)}{\Lambda^\gamma(t)} \Delta x,$$

and hence

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{\Lambda^\gamma(t)} \Delta t \geq p\int_a^\infty \left[\int_a^t \frac{\lambda(t)\lambda(x)\Lambda^{p-1}(x)f^p(x)}{\Lambda^\gamma(t)} \Delta x\right]\Delta t. \tag{2.16}$$

By making use of Fubini's theorem on time scales, inequality (2.16) can be rewritten as

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{\Lambda^\gamma(t)}\Delta t \geq p \int_a^\infty \lambda(x)\Lambda^{p-1}(x)f^p(x)\left[\int_x^\infty \lambda(t)\Lambda^{-\gamma}(t)\Delta t\right]\Delta x. \tag{2.17}$$

From the chain rule (1.30), there is $c \in [t, \sigma(t)]$ such that

$$\left(\Lambda^{1-\gamma}(t)\right)^\Delta = (1-\gamma)\Lambda^{-\gamma}(c)\Lambda^\Delta(t) \geq (1-\gamma)\lambda(t)\Lambda^{-\gamma}(t). \tag{2.18}$$

Substituting (2.18) into (2.17) gives

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{\Lambda^\gamma(t)}\Delta t \geq \frac{p}{1-\gamma}\int_a^\infty \lambda(x)\Lambda^{p-1}(x)f^p(x)\left[\int_x^\infty \left(\Lambda^{1-\gamma}(t)\right)^\Delta \Delta t\right]\Delta x$$

$$= \frac{p}{\gamma-1}\int_a^\infty \lambda(x)\Lambda^{p-\gamma}(x)f^p(x)\Delta x.$$

This completes the proof. □

Remark 2.15 In Theorem 2.14, if we take $\lambda(t) = 1$, $a = 0$, and $p = \gamma$, then we recapture Theorem 1.21.

Corollary 2.16 *If* $\mathbb{T} = \mathbb{R}$ *in Theorem* 2.14, *then, using relation* (1.26), *inequality* (2.13) *reduces to*

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{\Lambda^\gamma(t)}\,dt \geq \frac{p}{\gamma-1}\int_a^\infty \lambda(t)\Lambda^{p-\gamma}(t)f^p(t)\,dt,$$

where

$$\Psi(t) = \int_a^t \lambda(s)f(s)\,ds \quad and \quad \Lambda(t) = \int_a^t \lambda(s)\,ds.$$

Corollary 2.17 *If* $\mathbb{T} = h\mathbb{Z}$ *in Theorem* 2.14, *then, using relations* (1.28), *inequality* (2.13) *reduces to*

$$\sum_{t=\frac{a}{h}}^\infty \frac{\lambda(ht)\Psi^p(ht)}{\Lambda^\gamma(ht)} \geq \frac{p}{\gamma-1}\sum_{t=\frac{a}{h}}^\infty \lambda(ht)\Lambda^{p-\gamma}(ht)f^p(ht),$$

where

$$\Psi(t) = h\sum_{s=\frac{a}{h}}^{\frac{t}{h}-1}\lambda(hs)f(hs) \quad and \quad \Lambda(t) = h\sum_{s=\frac{a}{h}}^{\frac{t}{h}-1}\lambda(hs).$$

Corollary 2.18 *For* $\mathbb{T} = \mathbb{Z}$, *we simply take* $h = 1$ *in Corollary* 2.17. *In this case, inequality* (2.13) *reduces to*

$$\sum_{t=a}^\infty \frac{\lambda(t)\Psi^p(t)}{\Lambda^\gamma(t)} \geq \frac{p}{\gamma-1}\sum_{t=a}^\infty \lambda(t)\Lambda^{p-\gamma}(t)f^p(t),$$

where

$$\Psi(t) = \sum_{s=a}^{t-1} \lambda(s)f(s) \quad and \quad \Lambda(t) = \sum_{s=a}^{t-1} \lambda(s).$$

Corollary 2.19 *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem* 2.14*, then, using relations* (1.29)*, inequality* (2.13) *reduces to*

$$\sum_{t=(\log_q a)}^{\infty} \frac{q^t \lambda(q^t)\Psi^p(q^t)}{\Lambda^\gamma(q^t)} \geq \frac{p}{\gamma-1} \sum_{t=(\log_q a)}^{\infty} q^t \lambda(q^t)\Lambda^{p-\gamma}(q^t)f^p(q^t),$$

where

$$\Psi(t) = (q-1)\sum_{s=(\log_q a)}^{(\log_q t)-1} q^s \lambda(q^s)f(q^s) \quad and \quad \Lambda(t) = (q-1)\sum_{s=(\log_q a)}^{(\log_q t)-1} q^s \lambda(q^s).$$

Next, we will study inequality (2.13) in the case of the integral $\int_a^t \lambda(s)\Delta s$ on the left-hand side being replaced by the integral $\int_t^\infty \lambda(s)\Delta s$.

Theorem 2.20 *Let* $\mathbb{T}$ *be a time scale with* $0 \leq a \in \mathbb{T}$*. Furthermore, assume that* f *and* λ *are nonnegative rd-continuous functions on* $[a,\infty)_{\mathbb{T}}$ *with* f *nonincreasing. If* $p \geq 1$ *and* $0 \leq \gamma < 1$*, then*

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{(\Omega^\sigma(t))^\gamma}\Delta t \geq \frac{p}{1-\gamma}\int_a^\infty \lambda(t)\Lambda^{p-1}(t)\Omega^{1-\gamma}(t)f^p(t)\Delta t, \tag{2.19}$$

where

$$\Psi(t) = \int_a^t \lambda(s)f(s)\Delta s, \qquad \Lambda(t) = \int_a^t \lambda(s)\Delta s \quad and \quad \Omega(t) = \int_t^\infty \lambda(s)\Delta s.$$

Proof Since f is nonincreasing, we have for $x \geq a$

$$\Psi(x) = \int_a^x \lambda(s)f(s)\Delta s \geq f(x)\int_a^x \lambda(s)\Delta s = f(x)\Lambda(x),$$

then

$$f(x)\Psi^{p-1}(x) \geq f^p(x)\Lambda^{p-1}(x). \tag{2.20}$$

Employing the chain rule (1.31) and using $\Psi^\Delta(x) = \lambda(x)f(x) \geq 0$, we get

$$\begin{aligned}
\left(\Psi^p(x)\right)^\Delta &= p\Psi^\Delta(x)\int_0^1 \left[h\Psi(\sigma(x)) + (1-h)\Psi(x)\right]^{p-1}dh \\
&\geq p\lambda(x)f(x)\int_0^1 \left[h\Psi(x) + (1-h)\Psi(x)\right]^{p-1}dh \\
&= p\lambda(x)f(x)\Psi^{p-1}(x). \tag{2.21}
\end{aligned}$$

Combining (2.20) with (2.21) leads to

$$\left(\Psi^p(x)\right)^\Delta \geq p\lambda(x)\Lambda^{p-1}(x)f^p(x),$$

and so

$$\frac{\lambda(t)(\Psi^p(x))^\Delta}{(\Omega^\sigma(t))^\gamma} \geq \frac{p\lambda(t)\lambda(x)\Lambda^{p-1}(x)f^p(x)}{(\Omega^\sigma(t))^\gamma}.$$

Thus,

$$\frac{\lambda(t)\Psi^p(t)}{(\Omega^\sigma(t))^\gamma} = \int_a^t \frac{\lambda(t)(\Psi^p(x))^\Delta}{(\Omega^\sigma(t))^\gamma}\Delta x \geq p\int_a^t \frac{\lambda(t)\lambda(x)\Lambda^{p-1}(x)f^p(x)}{(\Omega^\sigma(t))^\gamma}\Delta x,$$

and hence

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{(\Omega^\sigma(t))^\gamma}\Delta t \geq p\int_a^\infty \left[\int_a^t \frac{\lambda(t)\lambda(x)\Lambda^{p-1}(x)f^p(x)}{(\Omega^\sigma(t))^\gamma}\Delta x\right]\Delta t. \tag{2.22}$$

Employing Fubini's theorem on time scales, inequality (2.22) can be rewritten as

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{(\Omega^\sigma(t))^\gamma}\Delta t \geq p\int_a^\infty \lambda(x)\Lambda^{p-1}(x)f^p(x)\left[\int_x^\infty \lambda(t)\left(\Omega^\sigma(t)\right)^{-\gamma}\Delta t\right]\Delta x. \tag{2.23}$$

Now, from the chain rule (1.30), there exists $c \in [t,\sigma(t)]$ such that

$$\left(\Omega^{1-\gamma}(t)\right)^\Delta = (1-\gamma)\Omega^{-\gamma}(c)\Omega^\Delta(t) \geq (\gamma-1)\lambda(t)\left(\Omega^\sigma(t)\right)^{-\gamma}. \tag{2.24}$$

Substituting (2.24) into (2.23) yields

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{(\Omega^\sigma(t))^\gamma}\Delta t \geq \frac{p}{\gamma-1}\int_a^\infty \lambda(x)\Lambda^{p-1}(x)f^p(x)\left[\int_x^\infty \left(\Omega^{1-\gamma}(t)\right)^\Delta\Delta t\right]\Delta x$$

$$= \frac{p}{1-\gamma}\int_a^\infty \lambda(x)\Lambda^{p-1}(x)\Omega^{1-\gamma}(x)f^p(x)\Delta x,$$

which is our desired inequality (2.19). □

Corollary 2.21 *If* $\mathbb{T} = \mathbb{R}$ *in Theorem* 2.20*, then, using relations* (1.26)*, inequality* (2.19) *boils down to*

$$\int_a^\infty \frac{\lambda(t)\Psi^p(t)}{\Omega^\gamma(t)}\,dt \geq \frac{p}{1-\gamma}\int_a^\infty \lambda(t)\Lambda^{p-1}(t)\Omega^{1-\gamma}(t)f^p(t)\,dt,$$

where

$$\Psi(t) = \int_a^t \lambda(s)f(s)\,ds, \qquad \Lambda(t) = \int_a^t \lambda(s)\,ds \quad and \quad \Omega(t) = \int_t^\infty \lambda(s)\,ds.$$

Corollary 2.22 *If* $\mathbb{T} = h\mathbb{Z}$ *in Theorem* 2.20*, then, using relations* (1.28)*, inequality* (2.19) *boils down to*

$$\sum_{t=\frac{a}{h}}^\infty \frac{\lambda(ht)\Psi^p(ht)}{\Omega^\gamma(ht+h)} \geq \frac{p}{1-\gamma}\sum_{t=\frac{a}{h}}^\infty \lambda(ht)\Lambda^{p-1}(ht)\Omega^{1-\gamma}(ht)f^p(ht),$$

where

$$\Psi(t) = h \sum_{s=\frac{a}{h}}^{\frac{t}{h}-1} \lambda(hs)f(hs), \qquad \Lambda(t) = h \sum_{s=\frac{a}{h}}^{\frac{t}{h}-1} \lambda(hs) \quad and \quad \Omega(t) = h \sum_{s=\frac{t}{h}}^{\infty} \lambda(hs).$$

Corollary 2.23 *For* $\mathbb{T} = \mathbb{Z}$, *we simply take* $h = 1$ *in Corollary* 2.22. *In this case, inequality* (2.19) *boils down to*

$$\sum_{t=a}^{\infty} \frac{\lambda(t)\Psi^p(t)}{\Omega^\gamma(t+1)} \geq \frac{p}{1-\gamma} \sum_{t=a}^{\infty} \lambda(t)\Lambda^{p-1}(t)\Omega^{1-\gamma}(t)f^p(t),$$

where

$$\Psi(t) = \sum_{s=a}^{t-1} \lambda(s)f(s), \qquad \Lambda(t) = \sum_{s=a}^{t-1} \lambda(s) \quad and \quad \Omega(t) = \sum_{s=t}^{\infty} \lambda(s).$$

Corollary 2.24 *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem* 2.20, *then, using relations* (1.29), *inequality* (2.19) *boils down to*

$$\sum_{t=(\log_q a)}^{\infty} \frac{q^t\lambda(q^t)\Psi^p(q^t)}{\Omega^\gamma(q^{t+1})} \geq \frac{p}{1-\gamma} \sum_{t=(\log_q a)}^{\infty} q^t\lambda(q^t)\Lambda^{p-1}(q^t)\Omega^{1-\gamma}(q^t)f^p(q^t),$$

where

$$\Psi(t) = (q-1)\sum_{s=(\log_q a)}^{(\log_q t)-1} q^s\lambda(q^s)f(q^s), \qquad \Lambda(t) = (q-1)\sum_{s=(\log_q a)}^{(\log_q t)-1} q^s\lambda(q^s) \quad and$$

$$\Omega(t) = (q-1)\sum_{s=(\log_q t)}^{\infty} q^s\lambda(q^s).$$

3 Conclusion

In the present article, by making use of the time scales version of Fubini's theorem and the chain rule, we have successfully obtained some new reverse dynamic Hardy-type inequalities. The obtained inequalities generalize some dynamic inequalities known in the literature. In order to illustrate the theorems for each type of inequality applied to various time scales such as $\mathbb{R}$, $h\mathbb{Z}$, $\overline{q^{\mathbb{Z}}}$, and $\mathbb{Z}$ as a sub case of $h\mathbb{Z}$. Possible future work includes studying different generalizations and variants of the dynamic Hardy inequality using the results presented in this article.

Competing interests
The authors announce that there are not any competing interests.

Authors' contributions
All authors have read and finalized the manuscript with equal contribution.

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