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Further generalizations of Hadamard and Fejér–Hadamard fractional inequalities and error estimates

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Abstract
The aim of this paper is to generalize the fractional Hadamard and Fejér–Hadamard inequalities. By using a generalized fractional integral operator containing extended Mittag-Leffler function via monotone function, for convex functions we generalize well known fractional Hadamard and Fejér–Hadamard inequalities. Also we study the error bounds of these generalized Hadamard and Fejér–Hadamard inequalities. We also obtain some published results from presented inequalities.

Keywords: Convex functions; Hadamard inequality; Generalized fractional integrals; Mittag-Leffler function

1 Introduction

Fractional integral operators are useful in the generalization of classical mathematical concepts. Nowadays researchers of different fields are utilizing fractional integral operators to get amazing results, for instance, fractional differential equations and fractional order systems are used to interpret different physical and mathematical phenomena. In the near past, fractional integral operators have been used in the formation of fractional versions of many well known integral inequalities. The inequalities of Hadamard, Ostrowski, Grüss, Minkowski, and many others were studied in terms of fractional calculus operators (derivative and integral), see [1–5, 7, 9–18, 25]. Our goal in this paper is to establish Hadamard and Fejér–Hadamard inequalities for a generalized fractional integral operator containing Mittag-Leffler function for a monotone increasing function. The most classical fractional derivative and integral formulas are renowned as Riemann–Liouville fractional integral and derivative operators. The Riemann–Liouville fractional integral operators are defined as follows [24]:

Definition 1 Let $f \in L_1[a,b]$. Then Riemann–Liouville fractional integrals of order $\tau \in \mathbb{C}$ where $\Re(\tau) > 0$ are defined as follows:

$$I_{a^+}^{\tau} f(x) = \frac{1}{\Gamma(\tau)} \int_a^x (x-t)^{\tau-1} f(t)\,dt, \quad x > a, \tag{1.1}$$

$$I_{b\_}^{\tau} f(x) = \frac{1}{\Gamma(\tau)} \int_x^b (t-x)^{\tau-1} f(t)\,dt, \quad x < b. \tag{1.2}$$

After establishing the existence of Riemann–Liouville fractional integral operators, the researchers started to think in this direction and consequently they further generalized and extended these operators in different ways, for instance, see [3, 8, 19, 26] and references therein. A generalization of the Riemann–Liouville fractional integral operators by a monotone increasing function is given in [19].

Definition 2 Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\mu \in \mathbb{C}$ where $\Re(\mu) > 0$ are defined as follows:

$$ {}_g^{\mu} I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \bigl(g(x) - g(t)\bigr)^{\mu-1} g'(x) f(t)\,dt, \quad x > a,$$
$$ {}_g^{\mu} I_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \bigl(g(t) - g(x)\bigr)^{\mu-1} g'(x) f(t)\,dt, \quad x < b,$$

where $\Gamma(\cdot)$ is the gamma function.

The Riemann–Liouville fractional integral operators were also generalized by using the Mittag-Leffler function. In [24] Salim and Faraj defined the following fractional integral operators involving an extended Mittag-Leffler function in the kernel.

Definition 3 Let α, β, k, l, γ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing Mittag-Leffler function, $\epsilon_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} f$ and $\epsilon_{\alpha,\beta,l,\omega,b\_}^{\gamma,\delta,k} f$, for a real valued continuous function f are defined as follows:

$$\bigl(\epsilon_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} f\bigr)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}\bigl(\omega(x-t)^{\alpha}\bigr) f(t)\,dt, \tag{1.3}$$

$$\bigl(\epsilon_{\alpha,\beta,l,\omega,b\_}^{\gamma,\delta,k} f\bigr)(x) = \int_x^b (t-x)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}\bigl(\omega(t-x)^{\alpha}\bigr) f(t)\,dt, \tag{1.4}$$

where the function $E_{\alpha,\beta,l}^{\gamma,\delta,k}(t)$ is the Mittag-Leffler function defined as

$$E_{\alpha,\beta,l}^{\gamma,\delta,k}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} t^n}{\Gamma(\alpha n + \beta)(\delta)_{ln}}, \tag{1.5}$$

where $(\gamma)_{nk}$ is the generalized Pochhammer symbol $(\gamma)_{nk} = \frac{\Gamma(\gamma+nk)}{\Gamma(\gamma)}$.

Further, fractional integral operators containing the extended generalized Mittag-Leffler function in their kernels are defined as follows:

Definition 4 ([3]) Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional operators $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f$ and $\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f$ are defined as follows:

$$\bigl(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f\bigr)(x;p) = \int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\bigl(\omega(x-t)^{\mu};p\bigr) f(t)\,dt, \tag{1.6}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f\right)(x;p) = \int_x^b (t-x)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega(t-x)^{\mu};p\right)f(t)\,dt, \tag{1.7}$$

where

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)(c)_{nk}t^n}{\beta(\gamma,c-\gamma)\Gamma(\mu n+\alpha)(l)_{n\delta}} \tag{1.8}$$

is the extended generalized Mittag-Leffler function.

Recently, Farid defined a unified integral operator in [8] (see also [20]) as follows:

Definition 5 Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$ be functions such that f is positive and $f\in L_1[a,b]$ and g are differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$ and $\omega,\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\mu,\delta>0$ and $0<\nu\leq\delta+\mu$. Then for $x\in[a,b]$ the left and right integral operators are defined as follows:

$$\left({}_gF^{\phi,\gamma,\delta,\nu,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p) = \int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(t)\right)^{\mu};p\right)f(t)d\left(g(t)\right), \tag{1.9}$$

$$\left({}_gF^{\phi,\gamma,\delta,\nu,c}_{\mu,\alpha,l,\omega,b^-}f\right)(x;p) = \int_x^b \frac{\phi(g(t)-g(x))}{g(t)-g(x)}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\omega\left(g(t)-g(x)\right)^{\mu};p\right)f(t)d\left(g(t)\right). \tag{1.10}$$

The following definition of a generalized fractional integral operator containing extended Mittag-Leffler function in the kernel for a monotone increasing function g can be extracted by setting $\phi(x)=x^{\tau}$ in Definition 5.

Definition 6 Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$ be functions such that f is positive and $f\in L_1[a,b]$ and g are differentiable and strictly increasing. Also let $\omega,\tau,\delta,\rho,c\in\mathbb{C}$, $\Re(\tau),\Re(\delta)>0$, $\Re(c)>\Re(\rho)>0$ with $p\geq 0$, $\sigma,r>0$ and $0<k\leq r+\sigma$. Then for $x\in[a,b]$ the left and right integral operators are defined as follows:

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\right)(x;p) = \int_a^x \left(g(x)-g(t)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(x)-g(t)\right)^{\sigma};p\right)f(t)d\left(g(t)\right), \tag{1.11}$$

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}f\right)(x;p) = \int_x^b \left(g(t)-g(x)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(t)-g(x)\right)^{\sigma};p\right)f(t)d\left(g(t)\right). \tag{1.12}$$

The following remark provides a connection of Definition 6 with already known operators:

Remark 1

(i) If we take $p=0$ and $g(x)=x$ in equation (1.11), then it reduces to the fractional integral operator defined by Salim and Faraj in [24].

(ii) If we take $\delta=r=1$ and $g(x)=x$ in (1.11), then it reduces to the fractional integral operator defined by Rahman *et al.* in [23].

(iii) If we set $p=0$, $\delta=r=1$ and $g(x)=x$ in (1.11), then it reduces to the integral operator introduced by Srivastava and Tomovski in [26].

(iv) If we take $p=0$, $\delta=r=k=1$ and $g(x)=x$ in (1.11), then it reduces to the integral operator defined by Prabhaker in [22].

(v) If we take $p = \omega = 0$ and $g(x) = x$ in (1.11), then it reduces to the Riemann–Liouville fractional integral operator.

2 Preliminary results

The aim of this paper is to generalize the Hadamard and the Fejér–Hadamard-type inequalities for fractional integral operators containing extended generalized Mittag-Leffler function given in [1, 11, 15]. The Hadamard inequality is an equivalent presentation of convex function which has a fascinating graphical interpretation. Convex functions play an important role in the formation of new functions and inequalities. A lot of mathematicians have considered their analytical and geometrical properties to develop the theory of inequalities.

Definition 7 A function $f : [a,b] \to \mathbb{R}$ is said to be convex if

$$f\bigl(tx + (1-t)y\bigr) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in [a,b]$ and $t \in [0,1]$.

The Hadamard inequality is stated in the following theorem:

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then the following inequality holds*:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

The very first generalization of Hadamard inequality is the Fejér–Hadamard inequality which is its weighted version stated as follows:

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ be a convex function with $a < b$. Then the following inequality holds*:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx,$$

where $g : [a,b] \to \mathbb{R}$ is a nonnegative, integrable and symmetric function about $\frac{a+b}{2}$.

Clearly, for $g(x) = 1$, $x \in [a,b]$, the Hadamard inequality can be obtained. In recent past decades, by using fractional calculus operators, the Hadamard inequality has been studied extensively, see [1–5, 10–12, 15, 16, 18, 25]. For example, in [25] Sarikaya et al. gave the fractional version of the Hadamard inequality by using Riemann–Liouville fractional integral operators.

Theorem 3 *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is convex on $[a,b]$, then following inequality for the fractional integral operator holds*:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(b-a)^\beta}\bigl[I_{a^+}^\beta f(b) + I_{b^-}^\beta f(a)\bigr] \leq \frac{f(a)+f(b)}{2}, \quad \beta > 0. \tag{2.1}$$

In [25] the authors also studied the error bounds of inequality (2.1).

Farid in [6] proved the following version of Hadamard inequality using fractional integral operators given in (1.3) and (1.4).

Theorem 4 *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is convex on $[a,b]$, then following inequality for the fractional integral operator holds*:

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}1\right)(b) &\le \frac{1}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b_-}f\right)(a)\right] \\ &\le \frac{f(a)+f(b)}{2}\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b_-}1\right)(a), \quad \omega' = \frac{\omega}{(b-a)^\alpha}. \end{aligned} \tag{2.2}$$

Abbas and Farid in [1] studied the error bounds of inequality (2.2). In [16] Kang et al. proved the following version of Hadamard inequality using fractional integral operators given in (1.6) and (1.7).

Theorem 5 *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is convex on $[a,b]$, then following inequality for the extended generalized fractional integral holds*:

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left(\epsilon^{\omega',\delta,q,r,c}_{a^+,\alpha,\beta,\tau}1\right)(b;p) &\le \frac{1}{2}\left[\left(\epsilon^{\omega',\delta,q,r,c}_{a^+,\alpha,\beta,\tau}f\right)(b;p) + \left(\epsilon^{\omega',\delta,q,r,c}_{b^-,\alpha,\beta,\tau}f\right)(a;p)\right] \\ &\le \frac{f(a)+f(b)}{2}\left(\epsilon^{\omega',\delta,q,r,c}_{b^-,\alpha,\beta,\tau}1\right)(a;p), \quad \omega' = \frac{\omega}{(b-a)^\alpha}. \end{aligned} \tag{2.3}$$

In [11] Farid et al. studied the error bounds of (2.3). Many authors have analyzed the fractional versions of the Hadamard inequality and further produced a plenty of such versions for other fractional integral operators (see [1, 4, 5, 10–12, 16, 21, 23, 25]). In Sect. 3 we will derive the generalized Hadamard and Fejér–Hadamard fractional integral inequalities for fractional integral operators given in (1.11) and (1.12). In Sect. 4 we will study the error estimates of these inequalities by proving two identities. The connection with already known results is described by considering particular functions and parameters of Mittag-Leffler function.

3 Hadamard and Fejér–Hadamard inequalities

Theorem 6 *Let $f,g : [a,b] \to \mathbb{R}$, $0 < a < b$, $\mathrm{Range}(g) \subset [a,b]$, be functions such that f is positive, $f \in L_1[a,b]$ and convex on $[a,b]$, and g is differentiable and strictly increasing. Then the following inequalities for the extended generalized fractional integral operators defined in (1.11) and (1.12) hold*:

$$\begin{aligned} &f\left(\frac{g(a)+g(b)}{2}\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}1\right)(b;p) \\ &\quad\le \frac{1}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}f\circ g\right)(b;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b^-}f\circ g\right)(a;p)\right] \\ &\quad\le \frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b^-}1\right)(a;p); \quad \omega' = \frac{\omega}{(g(b)-g(a))^\sigma}. \end{aligned}$$

Proof For the convex function f, we have

$$2f\left(\frac{g(a)+g(b)}{2}\right) \le f\bigl(tg(a)+(1-t)g(b)\bigr) + f\bigl((1-t)g(a)+tg(b)\bigr). \tag{3.1}$$

Further, from (3.1), one can obtain the following inequality:

$$\begin{aligned}
&2f\left(\frac{g(a)+g(b)}{2}\right)\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}\left(\omega t^{\sigma};p\right)dt\\
&\quad\le\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}\left(\omega t^{\sigma};p\right)f\left(tg(a)+(1-t)g(b)\right)dt\\
&\qquad+\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}\left(\omega t^{\sigma};p\right)f\left((1-t)g(a)+tg(b)\right)dt.
\end{aligned}\tag{3.2}$$

Setting $tg(a)+(1-t)g(b)=g(x)$, that is, $t=\frac{g(b)-g(x)}{g(b)-g(a)}$ and $(1-t)g(a)+tg(b)=g(y)$, i.e., $t=\frac{g(y)-g(a)}{g(b)-g(a)}$ in (3.2), we get the following inequality:

$$\begin{aligned}
&2f\left(\frac{g(a)+g(b)}{2}\right)\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',a^+}^{\rho,r,k,c}1\right)(b;p)\\
&\quad\le\left[\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',a^+}^{\rho,r,k,c}f\circ g\right)(b;p)+\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',b^-}^{\rho,r,k,c}f\circ g\right)(a;p)\right].
\end{aligned}\tag{3.3}$$

Further, by using the convexity of f, one can obtain

$$f\left(tg(a)+(1-t)g(b)\right)+f\left((1-t)g(a)+tg(b)\right)\le f\left(g(a)\right)+f\left(g(b)\right).\tag{3.4}$$

This leads to the following integral inequality:

$$\begin{aligned}
&\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}\left(\omega t^{\alpha};p\right)f\left(tg(a)+(1-t)g(b)\right)dt\\
&\qquad+\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}\left(\omega t^{\alpha};p\right)f\left((1-t)g(a)+tg(b)\right)dt\\
&\quad\le\left(f\left(g(a)\right)+f\left(g(b)\right)\right)\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}\left(\omega t^{\alpha};p\right)dt.
\end{aligned}\tag{3.5}$$

Setting $tg(a)+(1-t)g(b)=g(x)$, that is, $t=\frac{g(b)-g(x)}{g(b)-g(a)}$ and $(1-t)g(a)+tg(b)=g(y)$, i.e., $t=\frac{g(y)-g(a)}{g(b)-g(a)}$ in (3.5), and after some calculations, we get

$$\begin{aligned}
&\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',a^+}^{\rho,r,k,c}f\circ g\right)(b;p)+\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',b^-}^{\rho,r,k,c}f\circ g\right)(a;p)\\
&\quad\le\left(f\left(g(a)\right)+f\left(g(b)\right)\right)\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',b^-}^{\rho,r,k,c}1\right)(a;p).
\end{aligned}\tag{3.6}$$

Combining (3.3) and (3.6), we get the required result. □

Theorem 7 *Let* $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g),\mathrm{Range}(h)\subset[a,b]$, *be functions such that* f, h *are positive*, $f,h\in L_1[a,b]$ *and* f *convex on* $[a,b]$, *where* g *is differentiable and strictly increasing. If* $f(g(a)+g(b)-g(x))=f(g(x))$, *then the following inequalities for the extended generalized fractional integral operators defined in* (1.11) *and* (1.12) *hold*:

$$f\left(\frac{g(a)+g(b)}{2}\right)\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',a^+}^{\rho,r,k,c}h\circ g\right)(b;p)$$

$$\leq \frac{1}{2}\Big[\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}(h\circ g)(f\circ g)\big)(b;p) + \big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b^-}(h\circ g)(f\circ g)\big)(a;p)\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2}\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b^-}h\circ g\big)(a;p), \quad \omega' = \frac{\omega}{(g(b)-g(a))^\sigma}.$$

Proof Multiplying both sides of (3.1) by $t^{\tau-1}h(tg(a)+(1-t)g(b))E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;p)$ and integrating on $[0,1]$, we get

$$\begin{aligned}
&2f\left(\frac{g(a)+g(b)}{2}\right)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;p)h\big(tg(a)+(1-t)g(b)\big)\,dt\\
&\quad\leq \int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;p)h\big(tg(a)+(1-t)g(b)\big)f\big(tg(a)+(1-t)g(b)\big)\,dt\\
&\qquad+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;p)h\big(tg(a)+(1-t)g(b)\big)f\big((1-t)g(a)+tg(b)\big)\,dt.
\end{aligned}\tag{3.7}$$

Setting $tg(a)+(1-t)g(b)=g(x)$, that is, $t=\frac{g(b)-g(x)}{g(b)-g(a)}$ and $(1-t)g(a)+tg(b)=g(a)+g(b)-g(x)$, in (3.7), also using $f(g(a)+g(b)-g(x))=f(g(x))$, the following inequality is obtained:

$$\begin{aligned}
&2f\left(\frac{g(a)+g(b)}{2}\right)\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}h\circ g\big)(b;p)\\
&\quad\leq \Big[\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}(h\circ g)(f\circ g)\big)(b;p) + \big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b^-}(h\circ g)(f\circ g)\big)(a;p)\Big].
\end{aligned}\tag{3.8}$$

Multiplying by $t^{\tau-1}h(tg(a)+(1-t)g(b))E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\alpha;p)$ both sides of (3.4) and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\alpha;p)h(\big(tg(a)+(1-t)g(b)\big)f\big(tg(a)+(1-t)g(b)\big)\,dt\\
&\qquad+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\alpha;p)h(\big(tg(a)+(1-t)g(b)\big)f\big((1-t)g(a)+tg(b)\big)\,dt\\
&\quad\leq \big(f\big(g(a)\big)+f\big(g(b)\big)\big)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\alpha;p)h(\big(tg(a)+(1-t)g(b)\big)\,dt.
\end{aligned}\tag{3.9}$$

Setting $tg(a)+(1-t)g(b)=g(x)$ and using $f(g(a)+g(b)-g(x))=f(g(x))$, we get

$$\begin{aligned}
&\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}(h\circ g)(f\circ g)\big)(b;p) + \big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b^-}(h\circ g)(f\circ g)\big)(a;p)\\
&\quad\leq \big(f\big(g(a)\big)+f\big(g(b)\big)\big)\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b^-}(h\circ g)\big)(a;p).
\end{aligned}\tag{3.10}$$

Combining (3.8) and (3.10), we get the required result. □

Remark 2 The Hadamard and Fejér–Hadamard inequalities given in Theorems 2–5 are special cases of theorems of this section.

4 Estimates and error bounds of Hadamard and Fejér–Hadamard inequalities

To find error estimates of inequalities proved in Sect. 3, first we prove the following lemmas.

Lemma 1 *Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$, be functions such that f is positive and $f\in L_1[a,b]$, and g is differentiable and strictly increasing. If $f(g(t))=f(g(a)+g(b)-g(t))$, then we have*

$$\begin{aligned}\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\big)(b;p)&=\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}f\circ g\big)(a;p)\\&=\frac{1}{2}\big[\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\big)(b;p)+\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}f\circ g\big)(a;p)\big].\end{aligned}\tag{4.1}$$

Proof By Definition 6 of the extended generalized fractional integral operator, we have

$$\begin{aligned}&\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\big)(b;p)\\&\quad=\int_a^b\big(g(b)-g(t)\big)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\big(\omega\big(g(b)-g(t)\big)^{\sigma};p\big)f\circ g(t)d\big(g(t)\big).\end{aligned}\tag{4.2}$$

If we replace $g(t)$ by $g(a)+g(b)-g(t)$ in (4.2), then we get

$$\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\big)(b;p)=\int_a^b\big(g(t)-g(a)\big)^{\tau-1}E^{\rho,r,q,c}_{\sigma,\tau,\delta}\big(\omega\big(g(t)-g(a)\big)^{\sigma};p\big)f\circ g(t)d\big(g(t)\big).$$

This implies

$$\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\big)(b;p)=\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}f\circ g\big)(a;p).\tag{4.3}$$

By adding equations (4.2) and (4.3), we get (4.1). □

Lemma 2 *Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$, be functions such that f is positive and $f\circ g\in L_1[a,b]$, where g is differentiable, strictly increasing and h is continuous. If $f'\circ g\in L_1[a,b]$ and $h(g(t))=h(g(a)+g(b)-g(t))$, then the following equality for the extended generalized fractional integral operators (1.11) and (1.12) holds:*

$$\begin{aligned}&\left(\frac{f(g(a))+f(g(b))}{2}\right)\big[\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\big)(b;p)+\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}h\circ g\big)(a;p)\big]\\&\qquad-\big[\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(h\circ g)(f\circ g)\big)(b;p)+\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}(h\circ g)(f\circ g)\big)(a;p)\big]\\&\quad=\int_a^b\left[\int_a^t\big(g(b)-g(s)\big)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\big(\omega\big(g(b)-g(s)\big)^{\sigma};p\big)h\circ g(s)d\big(g(s)\big)\right.\\&\qquad\left.-\int_t^b\big(g(s)-g(a)\big)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\big(\omega\big(g(s)-g(a)\big)^{\sigma};p\big)h\circ g(s)d\big(g(s)\big)\right]\\&\qquad\times f'\big(g(t)\big)d\big(g(t)\big).\end{aligned}\tag{4.4}$$

Proof To prove this lemma, we consider its right-hand side. Upon integrating by parts and after simplification, we have

$$\begin{aligned}&\int_a^b\left[\int_a^t\big(g(b)-g(s)\big)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\big(\omega\big(g(b)-g(s)\big)^{\sigma};p\big)h\big(g(s)\big)d\big(g(s)\big)\right]f'\big(g(t)\big)d\big(g(t)\big)\\&\quad=f\big(g(b)\big)\left(\int_a^b\big(g(b)-g(s)\big)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\big(\omega\big(g(b)-g(s)\big)^{\sigma};p\big)h\big(g(s)\big)d\big(g(s)\big)\right)\end{aligned}$$

$$- \int_a^b \left(\left(g(b) - g(t) \right)^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c} \left(\omega \left(g(b) - g(t) \right)^{\sigma}; p \right) \right) h\left(g(t)\right) f\left(g(t)\right) d\left(g(t)\right)$$
$$= f\left(g(b)\right) \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} h \circ g \right)(b;p) - \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} (h \circ g)(f \circ g) \right)(b;p).$$

By using Lemma 1, we have

$$\begin{aligned} &\int_a^b \left[\int_a^t \left(g(b) - g(s) \right)^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c} \left(\omega \left(g(b) - g(s) \right)^{\sigma}; p \right) h\left(g(s)\right) d\left(g(s)\right) \right] f'\left(g(t)\right) d\left(g(t)\right) \\ &= \frac{f(g(b))}{2} \left[\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} h \circ g \right)(b;p) + \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,b^-}^{\rho,r,k,c} h \circ g \right)(a;p) \right] \\ &\quad - \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} (h \circ g)(f \circ g) \right)(b;p). \end{aligned} \tag{4.5}$$

Similarly,

$$\begin{aligned} &\int_a^b \left[- \int_t^b \left(g(s) - g(a) \right)^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c} \left(\omega \left(g(s) - g(a) \right)^{\sigma}; p \right) h\left(g(s)\right) d\left(g(s)\right) \right] \\ &\quad \times f'\left(g(t)\right) d\left(g(t)\right) \\ &= \frac{f(g(a))}{2} \left[\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} h \circ g \right)(b;p) + \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,b^-}^{\rho,r,k,c} h \circ g \right)(a;p) \right] \\ &\quad - \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,b^-}^{\rho,r,k,c} (h \circ g)(f \circ g) \right)(a;p). \end{aligned} \tag{4.6}$$

Adding (4.5) and (4.6), we get (4.4). □

In [3], Andric et al. proved the absolute convergence of the function $E_{\sigma,\tau,\delta}^{\rho,r,k,c}(t;p)$ for $k < r + \Re(\sigma)$. If we let $\sum_{n=0}^{\infty} |\frac{\beta_p(\rho+nk,c-\rho)(c)_{nk} t^n}{\beta(\rho,c-\rho)\Gamma(\sigma n+\tau)(\delta)_{nr}}| = M$, then $|E_{\sigma,\tau,\delta}^{\rho,r,k,c}(t;p)| \leq M$, which we will use to prove the next results.

Theorem 8 *Let* $f, g, h : [a,b] \to \mathbb{R}$, $0 < a < b$, $\mathrm{Range}(g) \subset [a,b]$, *be functions such that* f *is positive and* $(f \circ g)' \in L_1[a,b]$, *where* g *is differentiable and strictly increasing, and* h *is continuous. Also let* $h(g(t)) = h(g(a) + g(b) - g(t))$ *and* $|(f \circ g)'|$ *be convex. Then for* $k < r + \Re(\sigma)$, *the following inequality holds*:

$$\begin{aligned} &\left| \left(\frac{f(g(a)) + f(g(b))}{2} \right) \left[\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} h \right)(b;p) + \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,b^-}^{\rho,r,k,c} h \right)(a;p) \right] \right. \\ &\quad \left. - \left[\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} (f \circ g)(h \circ g) \right)(b;p) + \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega,b^-}^{\rho,r,k,c} (f \circ g)(h \circ g) \right)(a;p) \right] \right| \\ &\leq \frac{\|h\|_\infty M (g(b) - g(a))^{\tau+1}}{\tau(\tau+1)} (1 - \Phi) \left[\left| f'\left(g(a)\right) \right| + \left| f'\left(g(b)\right) \right| \right], \end{aligned}$$

where $\|h\|_\infty = \sup_{t \in [a,b]} |h(t)|$ *and*

$$\begin{aligned} \Phi &= \frac{1}{\tau+2} \left[\left(\frac{g(\frac{a+b}{2}) - g(a)}{g(b) - g(a)} \right)^{\tau+2} + \left(\frac{g(b) - g(\frac{a+b}{2})}{g(b) - g(a)} \right)^{\tau+2} \right] \\ &\quad - \frac{\tau+1}{\tau+2} \left[\left(\frac{g(\frac{a+b}{2}) - g(a)}{g(b) - g(a)} \right)^{\tau+2} + \left(\frac{g(b) - g(\frac{a+b}{2})}{g(b) - g(a)} \right)^{\tau+2} \right] \\ &\quad - \left(\frac{g(\frac{a+b}{2}) - g(a)}{g(b) - g(a)} \right)^{\tau+1} \left(\frac{g(b) - g(\frac{a+b}{2})}{g(b) - g(a)} \right) + \left(\frac{g(\frac{a+b}{2}) - g(a)}{g(b) - g(a)} \right) \left(\frac{g(b) - g(\frac{a+b}{2})}{g(b) - g(a)} \right)^{\tau+1}. \end{aligned}$$

Proof By using Lemma 2, we have

$$\begin{aligned}
&\left|\frac{f(g(a))+f(g(b))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}h\circ g\right)(a;p)\right]\right.\\
&\quad\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|\\
&\le\int_a^b\left|\left[\int_a^t\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)h\left(g(s)\right)d\left(g(s)\right)\right.\right.\\
&\quad\left.\left.-\int_t^b\left(g(s)-g(a)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(s)-g(a)\right)^{\sigma};p\right)h\left(g(s)\right)d\left(g(s)\right)\right]\right|\\
&\quad\times\left|f'\left(g(t)\right)\right|d\left(g(t)\right).
\end{aligned}\tag{4.7}$$

Using the convexity of $|f'(g)|$ on $[a,b]$, we have

$$\left|f'\left(g(t)\right)\right|\le\frac{g(b)-g(t)}{g(b)-g(a)}\left|f'\left(g(a)\right)\right|+\frac{g(t)-g(a)}{g(b)-g(a)}\left|f'\left(g(b)\right)\right|,\quad t\in[a,b].\tag{4.8}$$

If we replace $g(s)$ by $g(a)+g(b)-g(s)$ and use $h(g(s))=h(g(a)+g(b)-g(s))$, $t'=g^{-1}(g(a)+g(b)-g(t))$ in the following second integral, we get

$$\begin{aligned}
&\left|\int_a^t\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)h\left(g(s)\right)d\left(g(s)\right)\right.\\
&\quad\left.-\int_t^b\left(g(s)-g(a)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(s)-g(a)\right)^{\sigma};p\right)h\left(g(s)\right)d\left(g(s)\right)\right|\\
&=\left|-\int_t^a\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)h\left(g(s)\right)d\left(g(s)\right)\right.\\
&\quad\left.-\int_a^{t'}\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)h\left(g(s)\right)d\left(g(s)\right)\right|\\
&=\left|\int_t^{t'}\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)h\left(g(s)\right)d\left(g(s)\right)\right|\\
&\le\begin{cases}\int_t^{t'}|(g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(b)-g(s))^{\sigma};p)h(g(s))|d(g(s)), & t\in[a,\frac{a+b}{2}],\\ \int_{t'}^{t}|(g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(b)-g(s))^{\sigma};p)h(g(s))|d(g(s)), & t\in[\frac{a+b}{2},b].\end{cases}
\end{aligned}\tag{4.9}$$

From (4.7), (4.8), (4.9), and using the absolute convergence of Mittag-Leffler function, we have

$$\begin{aligned}
&\left|\frac{f(g(a))+f(g(b))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}h\circ g\right)(a;p)\right]\right.\\
&\quad\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|\\
&\le\int_a^{\frac{a+b}{2}}\left(\int_t^{t'}\left|\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)h\left(g(s)\right)\right|d\left(g(s)\right)\right)\\
&\quad\times\left(\frac{g(b)-g(t)}{g(b)-g(a)}\left|f'\left(g(a)\right)\right|+\frac{g(t)-g(a)}{g(b)-g(a)}\left|f'\left(g(b)\right)\right|\right)d\left(g(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{a+b}{2}}^{b}\left(\int_{t'}^{t}\left|\left(g(b)-g(s)\right)^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)h\left(g(s)\right)\right|d\left(g(s)\right)\right)\\
&\times\left(\frac{g(b)-g(t)}{g(b)-g(a)}\left|f'\left(g(a)\right)\right|+\frac{g(t)-g(a)}{g(b)-g(a)}\left|f'\left(g(b)\right)\right|\right)d\left(g(t)\right)\\
\le{}&\frac{\|h\|_{\infty}M}{\tau(g(b)-g(a))}\\
&\times\left[\int_{a}^{\frac{a+b}{2}}\left(\left(g(b)-g(t)\right)^{\tau}-\left(g(t)-g(a)\right)^{\tau}\left(g(b)-g(t)\right)\left|f'\left(g(a)\right)\right|\right)d\left(g(t)\right)\right.\\
&+\int_{a}^{\frac{a+b}{2}}\left(\left(g(b)-g(t)\right)^{\tau}-\left(g(t)-g(a)\right)^{\tau}\left(g(t)-g(a)\right)\left|f'\left(g(b)\right)\right|\right)d\left(g(t)\right)\\
&+\int_{\frac{a+b}{2}}^{b}\left(\left(g(t)-g(a)\right)^{\tau}-\left(g(b)-g(t)\right)^{\tau}\left(g(b)-g(t)\right)\left|f'\left(g(a)\right)\right|\right)d\left(g(t)\right)\\
&\left.+\int_{\frac{a+b}{2}}^{b}\left(\left(g(t)-g(a)\right)^{\tau}-\left(g(b)-g(t)\right)^{\tau}\left(g(t)-g(a)\right)\left|f'\left(g(b)\right)\right|\right)d\left(g(t)\right)\right].
\end{aligned}
\tag{4.10}
$$

After combining the terms of the above inequality, we have the following values:

$$
\begin{aligned}
&\int_{a}^{\frac{a+b}{2}}\left(\left(g(b)-g(t)\right)^{\tau}-\left(g(t)-g(a)\right)^{\tau}\right)\left(g(b)-g(t)\right)d\left(g(t)\right)\\
&=\int_{\frac{a+b}{2}}^{b}\left(\left(g(t)-g(a)\right)^{\tau}-\left(g(b)-g(t)\right)^{\tau}\right)\left(g(t)-g(a)\right)d\left(g(t)\right)\\
&=\frac{(g(b)-g(a))^{\tau+2}}{\tau+2}-\frac{(g(b)-g(\frac{a+b}{2}))^{\tau+2}}{\tau+2}\\
&\quad-\frac{(g(\frac{a+b}{2})-g(a))^{\tau+1}}{\tau+1}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)-\frac{(g(\frac{a+b}{2})-g(a))^{\tau+2}}{(\tau+1)(\tau+2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{a}^{\frac{a+b}{2}}\left(\left(g(b)-g(t)\right)^{\tau}-\left(g(t)-g(a)\right)^{\tau}\right)\left(g(t)-g(a)\right)d\left(g(t)\right)\\
&=\int_{\frac{a+b}{2}}^{b}\left(\left(g(t)-g(a)\right)^{\tau}-\left(g(b)-g(t)\right)^{\tau}\right)\left(g(b)-g(t)\right)d\left(g(t)\right)\\
&=-\frac{(g(\frac{a+b}{2})-g(a))^{\tau+1}}{\tau+1}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\\
&\quad+\frac{(g(b)-g(a))^{\tau+2}}{(\tau+1)(\tau+2)}-\frac{(g(\frac{a+b}{2})-g(a))^{\tau+2}}{(\tau+1)(\tau+2)}-\frac{(g(b)-g(\frac{a+b}{2}))^{\tau+2}}{\tau+2}.
\end{aligned}
$$

Using the above calculations of integrals in (4.10), we get the required inequality. □

Remark 3

(i) In Theorem 8, if we put $g=I$, we get [11, Theorem 2.3],

(ii) In Theorem 8, if we put $p=0$ and $g=I$, we get [1, Theorem 2.3].

(iii) In Theorem 8, if we put $\omega=p=0$ and $g=I$, we get [15, Theorem 2.36].

Theorem 9 *Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$, be functions such that f is positive and $(f\circ g)'\in L_1[a,b]$, where g is differentiable and strictly increasing, and h is continuous. Also let $h(g(t))=h(g(a)+g(b)-g(t))$ and $|(f\circ g)'|^q$, $q>1$ be convex. Then for $k<r+\Re(\sigma)$, the following inequality holds*:

$$\begin{aligned}
&\left|\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}h\circ g\right)(a;p)\right]\right.\\
&\quad\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|\\
&\quad\le\frac{2\|h\|_\infty M(g(b)-g(a))^{\tau+1}}{\tau(\tau+1)}\left[(1-\Psi)^{1-\frac{1}{q}}(1-\Phi)^{\frac{1}{q}}\right]\\
&\quad\quad\times\left(\frac{|f'(g(a))|^q+|f'(g(b))|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}\tag{4.11}$$

where $\|h\|_\infty=\sup_{t\in[a,b]}|h(t)|$, $\Psi=(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)})^{\tau+1}+(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)})^{\tau+1}$, *and*

$$\begin{aligned}
\Phi={}&\frac{1}{\tau+2}\left[\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+2}+\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+2}\right]\\
&-\frac{\tau+1}{\tau+2}\left[\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+2}+\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+2}\right]\\
&-\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+1}\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)+\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+1}.
\end{aligned}$$

Proof Using Lemma 2, power mean inequality, (4.9), and convexity of $|f'(g)|^q$, we get

$$\begin{aligned}
&\left|\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}h\circ g\right)(a;p)\right]\right.\\
&\quad\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|\\
&\quad\le\left[\int_a^b\left|\left(\int_t^{a+b-t}\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)\right)\right.\right.\\
&\quad\quad\left.\left.\times h\left(g(s)\right)d\left(g(s)\right)\right|d\left(g(t)\right)\right]^{1-\frac{1}{q}}\\
&\quad\quad\times\left[\int_a^b\left|\left(\int_t^{a+b-t}\left(g(b)-g(s)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(g(b)-g(s)\right)^{\sigma};p\right)\right)h\left(g(s)\right)d\left(g(s)\right)\right|\right.\\
&\quad\quad\left.\times\left|f'\left(g(t)\right)\right|^q d\left(g(t)\right)\right]^{\frac{1}{q}}.
\end{aligned}\tag{4.12}$$

Since $|f'(g)|^q$ is convex on $[a,b]$, we have

$$\left|f'\left(g(t)\right)\right|^q\le\frac{g(b)-g(t)}{g(b)-g(a)}\left|f'\left(g(a)\right)\right|^q+\frac{g(t)-g(a)}{g(b)-g(a)}\left|f'\left(g(b)\right)\right|^q.\tag{4.13}$$

Using (4.13), $\|h\|_\infty = \sup_{t\in[a,b]} |h(t)|$, and the absolute convergence of Mittag-Leffler function, inequality (4.12) becomes

$$\begin{aligned}
&\bigg| \frac{f(g(a))+f(g(b))}{2}\big[\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+} h\circ g\big)(b;p) + \big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-} h\circ g\big)(a;p)\big] \\
&\quad - \big[\big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\big)(b;p) + \big({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}(f\circ g)(h\circ g)\big)(a;p)\big]\bigg| \\
&\le \frac{\|h\|_\infty M}{\tau}\bigg[\int_a^{\frac{a+b}{2}} \big\{\big(g(b)-g(t)\big)^\tau - \big(g(t)-g(b)\big)^\tau\big\} d\big(g(t)\big) \\
&\quad + \int_{\frac{a+b}{2}}^b \big\{\big(g(t)-g(a)\big)^\tau - \big(g(b)-g(t)\big)^\tau\big\} d\big(g(t)\big)\bigg]^{1-\frac{1}{q}} \\
&\quad \times \bigg[\int_a^{\frac{a+b}{2}} \big\{\big(g(b)-g(t)\big)^\tau - \big(g(t)-g(b)\big)^\tau\big\} \\
&\quad \times \bigg(\frac{g(b)-g(t)}{g(b)-g(a)}\big|f'\big(g(a)\big)\big|^q + \frac{g(t)-g(a)}{g(b)-g(a)}\big|f'\big(g(b)\big)\big|^q\bigg) d\big(g(t)\big) \\
&\quad + \int_{\frac{a+b}{2}}^b \big\{\big(g(t)-g(a)\big)^\tau - \big(g(b)-g(t)\big)^\tau\big\} \\
&\quad \times \bigg(\frac{g(b)-g(t)}{g(b)-g(a)}\big|f'\big(g(a)\big)\big|^q + \frac{g(t)-g(a)}{g(b)-g(a)}\big|f'\big(g(b)\big)\big|^q\bigg) d\big(g(t)\big)\bigg]^{\frac{1}{q}}.
\end{aligned}$$

After integrating and simplifying the above inequality, we get (4.11). □

Remark 4

(i) In Theorem 9, if we put $g = I$, we get [11, Theorem 2.5].

(ii) In Theorem 9, if we put $p = 0$ and $g = I$, we get [1, Theorem 2.6].

(iii) In Theorem 9, if we put $\omega = p = 0$ and $g = I$, we get [15, Theorem 2.8].

5 Concluding remarks

The results of this paper provide the fractional Hadamard and Fejér–Hadamard inequalities in a generalized form. By proving two identities, the error estimates of these inequalities are established. Furthermore, the results deducible from the proved inequalities are published in [1, 11, 15]. Also in special cases the reader can obtain results for fractional integral operators described in Remark 1.

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