Lee and Kim *Advances in Difference Equations* (2020) 2020:431
https://doi.org/10.1186/s13662-020-02886-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On the new type of degenerate poly-Genocchi numbers and polynomials

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Abstract

Kim and Kim (J. Math. Anal. Appl. 487:124017, 2020) introduced the degenerate logarithm function, which is the inverse of the degenerate exponential function, and defined the degenerate polylogarithm function. They also studied a new type of the degenerate Bernoulli polynomials and numbers by using the degenerate polylogarithm function. Motivated by their research, we subdivide this paper into two parts. In Sect. 2, we construct a new type of degenerate Genocchi polynomials and numbers by using the degenerate polylogarithm function, called the degenerate poly-Genocchi polynomials and numbers, deriving several combinatorial identities related to the degenerate poly-Genocchi numbers and polynomials. Then, in Sect. 3, we also consider the degenerate unipoly Genocchi polynomials attached to an arithmetic function by using the degenerate polylogarithm function. In particular, we provide some new explicit computational identities of degenerate unipoly polynomials related to special numbers and polynomials.

MSC: 11B73; 11B83; 05A19

Keywords: Degenerate poly-Genocchi numbers and polynomials; Degenerate polylogarithm functions; Degenerate Bernoulli numbers and polynomials; Degenerate Euler numbers and polynomials; Degenerate unipoly functions; Degenerate unipoly Genocchi polynomials

1 Introduction

In recent years, many mathematicians have researched various special polynomials and numbers which included the Stirling numbers, central factorial numbers, Bernoulli numbers, Euler numbers, (central) Bell numbers, Cauchy numbers, and others [2–8]. Significantly, Carlitz [9, 10] initiated a study of degenerate versions of some special polynomials and numbers, namely the degenerate Bernoulli and Euler polynomials and numbers. Since then, many mathematicians have been studying degenerate versions of special polynomials and numbers such as Bernoulli, Euler, and Genocchi polynomials and numbers, and others [1, 11–26]. Notably, Genocchi numbers have been extensively studied in many different contexts such as: elementary number theory, complex analytic number theory, differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory, and in quantum physics (quantum groups)[20–22, 27–29].

In 1997, Kaneko [2] introduced poly-Bernoulli numbers which are defined by the polylogarithm function. The polyexponential functions were first studied by Hardy [30] and reconsidered by Kim and Kim [1, 17] in view of an inverse to the polylogarithm functions which were studied by Jaonquière [31], Lewis [8], and Zagier [32]. Kim et al. [18] also studied a new type of the degenerate poly-Bernoulli polynomials by using the degenerate modified polyexponential functions.

Furthermore, Kim and Kim [15] introduced the degenerate logarithm function (the inverse of the degenerate exponential function) and studied a new type of the degenerate Bernoulli polynomials and numbers by using the degenerate polylogarithm function. Influenced by Kim et al.'s research, as well as the importance and potential for applications in number theory, combinatorics, and other fields of applied mathematics, we define a new type of the degenerate poly-Genocchi polynomials and the degenerate unipoly Genocchi polynomials, and provide several combinatorial identities related to these polynomials and numbers.

Now, as is well established in academia, the ordinary Bernoulli polynomials $B_n(x)$ and the Genocchi polynomials $G_n(x)$, $(n \in \mathbb{N} \cup \{0\})$ are respectively defined by their generating functions as follows (see[9, 13, 14, 20]):

$$\left(\frac{t}{e^t - 1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \qquad \frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{1}$$

When $x = 0$, $B_n = B_n(0)$ and $G_n = G_n(0)$ are respectively called the Bernoulli numbers and the Genocchi numbers.

We note that by (1)

$$G_{2n+1} = B_{2n+1} = 0 \quad (n \in \mathbb{N}), \qquad G_n = 2\bigl(1 - 2^n\bigr)B_n. \tag{2}$$

The Euler polynomials are given by

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \text{(see [9, 20])}. \tag{3}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

For any nonzero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}} = e_\lambda^1(t) \quad \text{(see [1, 13–25])}. \tag{4}$$

By Taylor expansion, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!} \quad \text{(see [12–15])}, \tag{5}$$

where

$$(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda)\cdots\bigl(x - (n - 1)\lambda\bigr) \quad (n \geq 1).$$

Note that

$$\lim_{\lambda\to 0} e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^{xt}. \tag{6}$$

In [9, 10], Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials, respectively given by

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}, \qquad \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $B_{n,\lambda} = B_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers, and $E_{n,\lambda} = E_{n,\lambda}(0)$ are called the degenerate Euler numbers.

Note that $\lim_{\lambda\to 0} B_{n,\lambda}(x) = B_n(x)$, $(n \geq 0)$ and $\lim_{\lambda\to 0} E_{n,\lambda}(x) = E_n(x)$, $(n \geq 0)$.

In [20], Kim et al. considered the degenerate Genocchi polynomials given by

$$\frac{2t}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}. \tag{8}$$

When $x = 0$, $G_{n,\lambda} = G_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

As is well known, for $s \in \mathbb{C}$, the polylogarithm function is defined by a power series in z, which is also a Dirichlet series in s

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots \quad \bigl(\text{see [8, 14]}\bigr). \tag{9}$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$: it can be extended to $|z| \geq 1$ by analytic continuation.

From (9), we note that

$$\mathrm{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z). \tag{10}$$

Recently, Kim and Kim [15] introduced the degenerate logarithm function $\log_\lambda(1+t)$, which is the inverse of the degenerate exponential function $e_\lambda(t)$ and the motivation for the definition of degenerate polylogarithm function as follows:

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!} = \frac{1}{\lambda}\sum_{n=1}^{\infty} (\lambda)_n \frac{t^n}{n!} = \frac{1}{\lambda}\bigl((1+t)^\lambda - 1\bigr). \tag{11}$$

Here, $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t) = e_\lambda(\log_\lambda(t)) = t$. We note that

$$\lim_{\lambda\to 0} \log_\lambda(1+t) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{t^n}{n} = \log(1+t). \tag{12}$$

Thus, the degenerate polylogarithm function is defined by

$$l_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}x^n,\quad k\in\mathbb{Z}\bigl(|x|<1\bigr),\bigl(\text{see [15]}\bigr).\tag{13}$$

We note that

$$\lim_{\lambda\to 0}l_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\mathrm{Li}_k(x).\tag{14}$$

By (11) and (13), we see that

$$l_{1,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n!}x^n=-\log_{\lambda}(1-x).\tag{15}$$

In [15], they also studied a new type of degenerate poly-Bernoulli polynomials and numbers by using the degenerate polylogarithm function as follows:

$$\frac{l_{k,\lambda}(1-e_{\lambda}(-t))}{1-e_{\lambda}(-t)}e_{\lambda}^{x}(-t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.\tag{16}$$

When $x=0$, $\beta_{n,\lambda}^{(k)}=\beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

Moreover, they observed that

$$\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(1)}\frac{t^n}{n!}=\frac{1}{1-e_{\lambda}(-t)}l_{1,\lambda}\bigl(1-e_{\lambda}(-t)\bigr)=\frac{-t}{e_{\lambda}(-t)-1}=\sum_{n=0}^{\infty}(-1)^nB_{n,\lambda}\frac{t^n}{n!}.\tag{17}$$

Kim [15] introduced the degenerate Stirling numbers of the second kind as follows:

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l\quad(n\geq 0).\tag{18}$$

As an inversion formula of (18), the degenerate Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda}\quad(n\geq 0),\bigl(\text{see [8, 18]}\bigr).\tag{19}$$

From (18) and (19), it is well known that

$$\frac{1}{k!}\bigl(e_{\lambda}(t)-1\bigr)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\quad(k\geq 0),\bigl(\text{see [13, 15]}\bigr)\tag{20}$$

and

$$\frac{1}{k!}\bigl(\log_{\lambda}(1+t)\bigr)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\quad(k\geq 0),\bigl(\text{see [15]}\bigr).\tag{21}$$

This paper is subdivided into two parts. In Sect. 2, we construct a new type of degenerate Genocchi polynomials and numbers, called the degenerate poly-Genocchi polynomials and numbers, by using the degenerate polylogarithm function, deriving several combinatorial identities related to the degenerate poly-Genocchi numbers and polynomials. In Sect. 3, we also consider the degenerate unipoly Genocchi polynomials attached to an arithmetic function by using the degenerate polylogarithm function. In particular, we provide some new explicit computational identities of degenerate unipoly polynomials related to special numbers and polynomials.

2 A new type degenerate poly-Genocchi numbers and polynomials

In this section, we define the new type degenerate poly-Genocchi polynomials by using the degenerate polylogarithm function which are called the degenerate poly-Genocchi polynomials as follows:

$$\frac{l_{k,\lambda}(1-e_\lambda(-2t))}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^\infty g_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{22}$$

When $x=0$, $g_{n,\lambda}^{(k)}=g_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.

When $k=1$, from (15), we see that $g_{n,\lambda}^{(1)}(x)=G_{n,\lambda}(x)$ $(n\geq 0)$ are the degenerate Genocchi polynomials because of

$$l_{1,\lambda}\bigl(1-e_\lambda(-2t)\bigr)=-\log_\lambda\bigl(1-1+e_\lambda(-2t)\bigr)=2t. \tag{23}$$

From (5), we observe that

$$\frac{d}{dx}e_\lambda(x)=e_\lambda^{1-\lambda}(x),\qquad \frac{d}{dx}l_{k,\lambda}(x)=\frac{1}{x}l_{k-1,\lambda}(x). \tag{24}$$

Theorem 1 *For $n\geq 2$, $k\in\mathbb{Z}$, we have*

$$\begin{aligned}&l_{k,\lambda}\bigl(1-e_\lambda(-2x)\bigr)\\&\quad=2\int_0^x\frac{-2e_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}\underbrace{\int_0^t\frac{-2e_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}\cdots\cdots\int_0^t\frac{-2te_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}}_{(k-2)\text{-times}}\,dt\,dt\cdots dt.\end{aligned} \tag{25}$$

Proof By using (25), we have

$$\frac{d}{dx}\bigl(1-e_\lambda(-2x)\bigr)=2e_\lambda^{1-\lambda}(-2x) \tag{26}$$

and

$$\frac{d}{dx}l_{k,\lambda}\bigl(1-e_\lambda(-2x)\bigr)=\frac{2e_\lambda^{1-\lambda}(-2t)}{(1-e_\lambda(-2x))}l_{k-1,\lambda}\bigl(1-e_\lambda(-2x)\bigr). \tag{27}$$

Thus, from (15), (26), and (27), we get the desired result as follows:

$$\begin{aligned} &l_{k,\lambda}\bigl(1-e_\lambda(-2x)\bigr)\\ &\quad = 2\int_0^x \frac{-2e_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}\underbrace{\int_0^t \frac{-2e_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}\cdots\cdots\int_0^t \frac{-2te_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}}_{(k-2)\text{-times}}\,dt\,dt\cdots dt. \end{aligned} \tag{28}$$

□

Theorem 2 *For $n \geq 0$ and $k = 2$, we have*

$$g_{n,\lambda}^{(2)} = \sum_{l=0}^{n}\binom{n}{l}(-2)^l\frac{B_{l,\lambda}(1-\lambda)}{l+1}G_{n-l,\lambda}. \tag{29}$$

Proof By using (16), (17), and Theorem 1, we get

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}^{(2)}\frac{x^n}{n!} &= \frac{2}{e_\lambda(x)+1}\int_0^x \frac{-2t}{e_\lambda(-2t)-1}e_\lambda^{1-\lambda}(-2t)\,dt\\ &= \frac{2}{e_\lambda(x)+1}\int_0^x \sum_{l=0}^{\infty} B_{l,\lambda}(1-\lambda)\frac{(-2t)^l}{l!}dt\\ &= \frac{2x}{e_\lambda(x)+1}\sum_{l=0}^{\infty}(-2)^l\frac{B_{l,\lambda}(1-\lambda)}{l+1}\frac{x^l}{l!}\\ &= \left(\sum_{m=0}^{\infty} G_{m,\lambda}\frac{x^m}{m!}\right)\left(\sum_{l=0}^{\infty}(-2)^l\frac{B_{l,\lambda}(1-\lambda)}{l+1}\frac{x^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-2)^l\frac{B_{l,\lambda}(1-\lambda)}{l+1}G_{n-l,\lambda}\right)\frac{x^n}{n!}. \end{aligned} \tag{30}$$

Therefore, by comparing the coefficients on both sides of (30), we get what we wanted. □

Theorem 3 *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\begin{aligned} g_{n,\lambda}^{(k)} = {}&\sum_{n_1+n_2+\cdots+n_{k-1}=m}\binom{n}{m}(-2)^m\binom{n}{n_1,n_2,\ldots,n_{k-1}}\\ &\times\frac{B_{n_1,\lambda}(1-\lambda)}{n_1+1}\cdots\cdots\frac{B_{n_{k-1},\lambda}(1-\lambda)}{n_1+n_2+\cdots+n_{k-1}+1}G_{n-m,\lambda}. \end{aligned} \tag{31}$$

Proof By using (8), Theorem 1, and Theorem 2, we have

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}^{(k)}\frac{x^n}{n!} &= \frac{l_{k,\lambda}(1-e_\lambda(-2x))}{e_\lambda(x)+1}\\ &= \left(\frac{2x}{e_\lambda(x)+1}\right)\sum_{n_1+n_2+\cdots+n_{k-1}=m}(-2)^m\binom{n}{n_1,n_2,\ldots,n_{k-1}}\\ &\quad\times\frac{B_{n_1,\lambda}(1-\lambda)}{n_1+1}\cdots\cdots\frac{B_{n_{k-1},\lambda}(1-\lambda)}{n_1+n_2+\cdots+n_{k-1}+1} \end{aligned}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_{k-1}=m}\binom{n}{m}(-2)^m\binom{n}{n_1,n_2,\ldots,n_{k-1}}\right.$$
$$\left.\times \frac{B_{n_1,\lambda}(1-\lambda)}{n_1+1}\cdot\cdots\cdot\frac{B_{n_{k-1},\lambda}(1-\lambda)}{n_1+n_2+\cdots+n_{k-1}+1}G_{n-m,\lambda}\right)\frac{x^m}{m!}. \tag{32}$$

Therefore, by comparing the coefficients on both sides of (32), we get what we desired. □

The following lemma is easily obtained by (5) and (22).

Lemma 4 *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$g_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n}\binom{n}{m}g_{m,\lambda}^{(k)}(x)_{n-m,\lambda}. \tag{33}$$

Theorem 5 *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$g_{n,\lambda}^{(k)}(1) + g_{n,\lambda}^{(k)} = \sum_{m=1}^{n}\frac{(1)_{m,1/\lambda}(-1)^{n-1}}{m^{k-1}}\lambda^{m-1}2^n S_{2,\lambda}(n,m). \tag{34}$$

Proof By using Lemma 4, (5), and (22), we observe that

$$\begin{aligned} l_{k,\lambda}\big(1-e_\lambda(-2t)\big) &= \big(e_\lambda(t)+1\big)\sum_{l=0}^{\infty}g_{l,\lambda}^{(k)}\frac{t^l}{l!} \\ &= \left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}+1\right)\left(\sum_{l=0}^{\infty}g_{l,\lambda}^{(k)}\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}g_{l,\lambda}^{(k)}+g_{n,\lambda}^{(k)}\right)\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\big(g_{n,\lambda}^{(k)}(1)+g_{n,\lambda}^{(k)}\big)\frac{t^n}{n!}. \end{aligned} \tag{35}$$

On the other hand, from (13) and (20), we have

$$\begin{aligned} l_{k,\lambda}\big(1-e_\lambda(-2t)\big) &= \sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-\lambda)^{m-1}}{(m-1)!m^k}\big(1-e_\lambda(-2t)\big)^m \\ &= \sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-1)^{m-1}\lambda^{m-1}}{m^{k-1}}\frac{(-1)^m(e_\lambda(-2t)-1)^m}{m!} \\ &= \sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-1)^{-1}\lambda^{m-1}}{m^{k-1}}\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)(-1)^n\frac{2^n t^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,1/\lambda}(-1)^{n-1}}{m^{k-1}}\lambda^{m-1}2^n S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{36}$$

Therefore, by comparing the coefficients of (35) and (36), we get what we wanted. □

Theorem 6 *For $n \geq 0$, $k = 1$, we have*

$$\sum_{m=1}^{n}(1)_{m,1/\lambda}\lambda^{m-1}2^{n}S_{2,\lambda}(n,m) = \begin{cases} 2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Proof From Theorem 5 and (23), we have

$$\begin{aligned} 2t &= l_{1,\lambda}\bigl(1 - e_{\lambda}(-2t)\bigr) \\ &= \sum_{n=1}^{\infty}\Biggl(\sum_{m=1}^{\infty}(1)_{m,1/\lambda}(-1)^{n-1}\lambda^{m-1}2^{n}S_{2,\lambda}(n,m)\Biggr)\frac{t^{n}}{n!}. \end{aligned} \tag{38}$$

Therefore, by comparing the coefficients on both sides of (38), we have the desired result. □

Theorem 7 *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$g_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_{m}S_{2,\lambda}(m,l)g_{n-l,\lambda}^{(k)}. \tag{39}$$

Proof From (20) and (22), we get

$$\begin{aligned} \sum_{n=0}^{\infty}g_{n,\lambda}^{(k)}(x)\frac{t^{n}}{n!} &= \frac{l_{k,\lambda}(1 - e_{\lambda}(-2t))}{e_{\lambda}(t) + 1}\bigl(e_{\lambda}(t) - 1 + 1\bigr)^{x} \\ &= \sum_{i=0}^{\infty}g_{i,\lambda}^{(k)}\frac{t^{i}}{i!}\sum_{m=0}^{\infty}(x)_{m}\frac{(e_{\lambda}(t) - 1)^{m}}{m!} \\ &= \sum_{i=0}^{\infty}g_{i,\lambda}^{(k)}\frac{t^{i}}{i!}\sum_{l=0}^{\infty}\sum_{m=0}^{l}(x)_{m}S_{2,\lambda}(m,l)\frac{t^{l}}{l!} \\ &= \sum_{n=0}^{\infty}\Biggl(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_{m}S_{2,\lambda}(m,l)g_{n-l,\lambda}^{(k)}\Biggr)\frac{t^{n}}{n!}. \end{aligned} \tag{40}$$

Therefore, by comparing the coefficients on both sides of (40), we have the desired result. □

Theorem 8 *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$g_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(1)_{m,1/\lambda}(-1)^{l}}{(l+1)m^{k-1}}\lambda^{m-1}2^{l}S_{2,\lambda}(l+1,m)G_{n-l,\lambda}(x). \tag{41}$$

Proof From (8), (13), and (36), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}g_{n,\lambda}^{(k)}(x)\frac{t^{n}}{n!} \\ &\quad = \frac{2t}{e_{\lambda}(t) + 1}e_{\lambda}^{x}(t)\frac{1}{2t}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^{k}}\bigl(1 - e_{\lambda}(-2t)\bigr)^{m} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2t}{e_\lambda(t)+1} e_\lambda^x(t) \frac{1}{2t} \sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \frac{(1)_{m,1/\lambda}(-1)^{l-1}}{m^{k-1}} \lambda^{m-1} 2^l S_{2,\lambda}(l,m) \right) \frac{t^l}{l!} \\
&= \sum_{i=0}^{\infty} G_{i,\lambda}(x) \frac{t^i}{i!} \frac{1}{2t} \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{(1)_{m,1/\lambda}(-1)^{l}}{m^{k-1}} \lambda^{m-1} 2^{l+1} S_{2,\lambda}(l+1,m) \right) \frac{t^{l+1}}{(l+1)!} \\
&= \sum_{i=0}^{\infty} G_{i,\lambda}(x) \frac{t^i}{i!} \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{(1)_{m,1/\lambda}(-1)^{l}}{(l+1)m^{k-1}} \lambda^{m-1} 2^{l} S_{2,\lambda}(l+1,m) \right) \frac{t^{l}}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=1}^{l+1} \binom{n}{l} \frac{(1)_{m,1/\lambda}(-1)^{l}}{(l+1)m^{k-1}} \lambda^{m-1} 2^{l} S_{2,\lambda}(l+1,m) G_{n-l,\lambda}(x) \right) \frac{t^n}{n!}.
\end{aligned} \tag{42}
$$

Therefore, by comparing the coefficients on both sides of (42), we have the desired result. □

Theorem 9 *For $n \geq 1$, $k \in \mathbb{Z}$, we have*

$$
\begin{aligned}
g_{n,\lambda}^{(k)}(x) = {} & \sum_{l=1}^{n} \sum_{m=1}^{l} \sum_{j=1}^{m} \binom{n}{l} \binom{l}{m} \frac{(1)_{l-m+1,\lambda}(1)_{j,1/\lambda}(-1)^{m-1}\lambda^{j-1}2^{n-l+m-1}}{(l-m+1)j^{k-1}} \\
& \times S_{2,\lambda}(m,j) B_{n-l,\frac{\lambda}{2}}\left(\frac{x}{2}\right).
\end{aligned} \tag{43}
$$

Proof From (5), (7), and (36), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} g_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} \\
&= \frac{l_{k,\lambda}(1-e_\lambda(-2t))}{e_\lambda(t)+1} e_\lambda^x(t) \\
&= \frac{1}{e_\lambda^2(t)-1} e_\lambda^x(t) (e_\lambda^t - 1) \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} \frac{(1)_{j,1/\lambda}(-1)^{m-1}}{j^{k-1}} \lambda^{j-1} 2^m S_{2,\lambda}(m,j) \right) \frac{t^m}{m!} \\
&= \frac{1}{e_{\frac{\lambda}{2}}(2t)-1} e_{\frac{\lambda}{2}}^{\frac{x}{2}}(2t) \sum_{i=1}^{\infty} (1)_{i,\lambda} \frac{t^i}{i!} \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} \frac{(1)_{j,1/\lambda}(-1)^{m-1}}{j^{k-1}} \lambda^{j-1} 2^m S_{2,\lambda}(m,j) \right) \frac{t^m}{m!} \\
&= \frac{t}{e_{\frac{\lambda}{2}}(2t)-1} e_{\frac{\lambda}{2}}^{\frac{x}{2}}(2t) \sum_{i=0}^{\infty} \frac{(1)_{i+1,\lambda}}{i+1} \frac{t^i}{i!} \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} \frac{(1)_{j,1/\lambda}(-1)^{m-1}}{j^{k-1}} \lambda^{j-1} 2^m S_{2,\lambda}(m,j) \right) \frac{t^m}{m!} \\
&= \sum_{\alpha=0}^{\infty} \frac{1}{2} B_{\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right) \frac{2^\alpha t^\alpha}{\alpha!} \sum_{i=0}^{\infty} \frac{(1)_{i+1,\lambda}}{i+1} \frac{t^i}{i!} \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} \frac{(1)_{j,1/\lambda}(-1)^{m-1}}{j^{k-1}} \lambda^{j-1} 2^m S_{2,\lambda}(m,j) \right) \frac{t^m}{m!} \\
&= \sum_{\alpha=0}^{\infty} B_{\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right) \frac{2^{\alpha-1} t^\alpha}{\alpha!} \\
&\quad \times \sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \sum_{j=1}^{m} \binom{l}{m} \frac{(1)_{l-m+1,\lambda}(1)_{j,1/\lambda}(-1)^{m-1}\lambda^{j-1}2^m}{(l-m+1)j^{k-1}} S_{2,\lambda}(m,j) \right) \frac{t^l}{l!}
\end{aligned}
$$

$$= \sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\sum_{m=1}^{l}\sum_{j=1}^{m}\binom{n}{l}\binom{l}{m}\frac{(1)_{l-m+1,\lambda}(1)_{j,1/\lambda}(-1)^{m-1}\lambda^{j-1}2^{n-l+m-1}}{(l-m+1)j^{k-1}}\right.$$
$$\left.\times S_{2,\lambda}(m,j)B_{n-l,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\right)\frac{t^n}{n!}. \tag{44}$$

Therefore, by comparing the coefficients on both sides of (44), we have the desired result. □

Theorem 10 *For $n \geq 1$ and $k \in \mathbb{Z}$, we get*

$$\sum_{m=1}^{n}\sum_{l=0}^{n-m}\binom{n}{m}(-1)^{l+n-m}2^{-l-1}\frac{m(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k}S_{1,\lambda}(n-m,l)E_{l,\lambda}$$
$$= \sum_{m=0}^{n}(-1)^{m+n}2^{-m}S_{1,\lambda}(n,m)g_{l,\lambda}^{(k)}. \tag{45}$$

Proof Replace t by $-\frac{1}{2}\log_{\lambda}(1-t)$ in (22). From (7), (13), and (21), the left-hand side of (22) is

$$\begin{aligned}
&\frac{l_{k,\lambda}(t)}{(e_{\lambda}(-\frac{1}{2}\log_{\lambda}(1-t))+1)} \\
&\quad= \frac{1}{2}\frac{2}{e_{\lambda}(-\frac{1}{2}\log_{\lambda}(1-t))+1}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}t^m \\
&\quad= \frac{1}{2}\sum_{l=0}^{\infty}E_{l,\lambda}\frac{(-\frac{1}{2}\log_{\lambda}(1-t))^l}{l!}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}\frac{t^m}{m!} \\
&\quad= \frac{1}{2}\sum_{l=0}^{\infty}E_{l,\lambda}\left(-\frac{1}{2}\right)^l\sum_{j=l}^{\infty}S_{1,\lambda}(j,l)\frac{(-t)^j}{j!}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}\frac{t^m}{m!} \\
&\quad= \frac{1}{2}\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}E_{l,\lambda}(-1)^{l+j}2^{-l}S_{1,\lambda}(j,l)\right)\frac{t^j}{j!}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}\frac{t^m}{m!} \\
&\quad= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{l=0}^{n-m}\binom{n}{m}(-1)^{l+n-m}2^{-l-1}\frac{m(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k}S_{1,\lambda}(n-m,l)E_{l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{46}$$

On the other hand, the right-hand side of (22) is

$$\begin{aligned}
\sum_{m=0}^{\infty}g_{m,\lambda}^{(k)}\frac{(-\frac{1}{2}\log_{\lambda}(1-t))^m}{m!} &= \sum_{m=0}^{\infty}g_{m,\lambda}^{(k)}\left(-\frac{1}{2}\right)^m\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{(-1)^n t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(-1)^{m+n}2^{-m}S_{1,\lambda}(n,m)g_{m,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{47}$$

Therefore, by comparing the coefficients of (46) and (47), we get what we wanted. □

3 The degenerate unipoly Genocchi polynomials and numbers

Let p be any arithmetic function which is real- or complex-valued function defined on the set of positive integers $\mathbb{N}$. Kim and Kim [5] defined the unipoly function attached to $p(x)$

by

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)x^n}{n^k} \quad (k \in \mathbb{Z}). \tag{48}$$

Moreover,

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x) \tag{49}$$

is the ordinary polylogarithm function.

In this paper, we define the degenerate unipoly function attached to $p(x)$ as follows:

$$u_{k,\lambda}(x|p) = \sum_{n=1}^{\infty} p(n)\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n^k}x^n. \tag{50}$$

We note that

$$u_{k,\lambda}\left(x|\frac{1}{\Gamma}\right) = l_{k,\lambda}(x) \tag{51}$$

is the degenerate polylogarithm function.

We also define the degenerate unipoly Genocchi polynomials by

$$\frac{u_{k,\lambda}(1-e_\lambda(-2t)|p)}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} g_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!}. \tag{52}$$

When $x = 0$, $g_{n,\lambda,p}^{(k)} = g_{n,\lambda,p}^{(k)}(0)$ is the degenerate unipoly Genocchi numbers.

We note that

$$g_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x) = \sum_{n=0}^{\infty} g_{n,\lambda}^{(k)}(x). \tag{53}$$

The next lemma is intended to be used conveniently to prove some of the theorems below.

Lemma 11 *For $k \in \mathbb{Z}$, we have*

$$u_{k,\lambda}\big(1-e_\lambda(-2t)|p\big) = \sum_{l=1}^{\infty}\left(\sum_{m=1}^{l} \frac{p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{m^k}S_{2,\lambda}(l,m)\right)\frac{t^l}{l!}. \tag{54}$$

Proof From (20) and (50), we have

$$\begin{aligned}
u_{k,\lambda}\big(1-e_\lambda(-2t)|p\big) &= \sum_{m=1}^{\infty} \frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k}\big(1-e_\lambda(-2t)\big)^m\frac{m!}{m!} \\
&= \sum_{m=1}^{\infty} \frac{p(m)(-1)^{-1}\lambda^{m-1}(1)_{m,1/\lambda}m!}{m^k}\frac{(e_\lambda(-2t)-1)^m}{m!}
\end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{p(m)(-1)^{-1}\lambda^{m-1}(1)_{m,1/\lambda} m!}{m^k} \sum_{l=m}^{\infty} S_{2,\lambda}(l,m) \frac{(-2t)^l}{l!}$$
$$= \sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \frac{p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda} m! 2^l}{m^k} S_{2,\lambda}(l,m) \right) \frac{t^l}{l!}. \tag{55}$$

Thus, we have what we wanted. □

Theorem 12 *For $n \geq 1$, $k \in \mathbb{Z}$, we get*

$$g_{n,\lambda,p}^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{l} \binom{n}{l} \frac{m! p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda} 2^{l-1} S_{2,\lambda}(l,m)}{m^k} E_{n-l,\lambda}. \tag{56}$$

Proof From (7) and Lemma 11, we have

$$\sum_{n=0}^{\infty} g_{n,\lambda,p}^{(k)} \frac{t^n}{n!}$$
$$= \frac{u_{k,\lambda}(1 - e_\lambda(-2t)|p)}{e_\lambda(t) + 1}$$
$$= \frac{2}{e_\lambda(t) + 1} \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{m! p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda} 2^{l-1}}{m^k} S_{2,\lambda}(l,m) \right) \frac{t^l}{l!}$$
$$= \sum_{i=0}^{\infty} E_{i,\lambda} \frac{t^i}{i!} \sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \frac{m! p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda} 2^{l-1} S_2(l,m)}{m^k} \right) \frac{t^l}{l!}$$
$$= \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \sum_{m=1}^{l} \binom{n}{l} \frac{m! p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda} 2^{l-1} S_{2,\lambda}(l,m)}{m^k} E_{n-l,\lambda} \right) \frac{t^n}{n!}. \tag{57}$$

Thus, by comparing the coefficients on both sides of (57), we have the desired result. □

Theorem 13 *For $n \geq 0$, $k \in \mathbb{Z}$, we get*

$$g_{n,\lambda,p}^{(k)} = \sum_{l=0}^{n} \sum_{m=1}^{l+1} \binom{n}{l} \frac{p(m)(-1)^l \lambda^{m-1}(1)_{m,1/\lambda} m! 2^l}{m^k (l+1)} S_{2,\lambda}(l+1,m) G_{n-l,\lambda}. \tag{58}$$

Proof From (8) and Lemma 11, we have

$$\sum_{n=0}^{\infty} g_{n,\lambda,p}^{(k)} \frac{t^n}{n!}$$
$$= \frac{u_{k,\lambda}(1 - e_\lambda(-2t)|p)}{e_\lambda(t) + 1} e_\lambda^x(t)$$
$$= \frac{1}{e_\lambda(t) + 1} \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{p(m)(-1)^l \lambda^{m-1}(1)_{m,1/\lambda} m! 2^{l+1}}{m^k} S_{2,\lambda}(l+1,m) \right) \frac{t^{l+1}}{(l+1)!}$$
$$= \frac{2t}{e_\lambda(t) + 1} \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{p(m)(-1)^l \lambda^{m-1}(1)_{m,1/\lambda} m! 2^l}{m^k (l+1)} S_{2,\lambda}(l+1,m) \right) \frac{t^l}{l!}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} G_{i,\lambda}\frac{t^i}{i!}\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{m^k(l+1)}S_{2,\lambda}(l+1,m)\right)\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{m^k(l+1)}S_{2,\lambda}(l+1,m)G_{n-l,\lambda}\right)\frac{t^n}{n!}. \qquad (59)
\end{aligned}
$$

Thus, by comparing the coefficients on both sides of (59), we have the desired result. □

Theorem 14 *For $k \in \mathbb{Z}$, we have*

$$
\begin{aligned}
&g_{n,\lambda,p}^{(k)}(x) \\
&= \sum_{\alpha=1}^{n}\sum_{l=0}^{\alpha-1}\sum_{m=1}^{l+1}\binom{n}{\alpha}\binom{\alpha}{l}(1)_{\alpha-l,\lambda}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{(l+1)m^k}S_{2,\lambda}(l+1,m)B_{n-\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right)
\end{aligned}
$$

if $n \geq 1$ and $g_{0,\lambda,p}^{(k)}(x) = 0$.

Proof From (5), (7), and Lemma 11, we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} g_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!} \\
&= \frac{u_{k,\lambda}(1-e_\lambda(-2t)|p)}{e_\lambda(t)+1}e_\lambda^x(t) \\
&= \frac{1}{e_\lambda(t)+1}\cdot\frac{e_\lambda(t)-1}{e_\lambda(t)-1}e_\lambda^x(t) \\
&\quad\times\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^{l+1}}{m^k}S_{2,\lambda}(l+1,m)\right)\frac{t^{l+1}}{(l+1)!} \\
&= \frac{2te_\lambda^x(t)}{e_{\frac{\lambda}{2}}(2t)-1}(e_\lambda(t)-1)\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{m^k}S_{2,\lambda}(l+1,m)\right)\frac{t^l}{(l+1)!} \\
&= \sum_{i=0}^{\infty}B_{i,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\frac{2^it^i}{i!}\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{(l+1)m^k}S_{2,\lambda}(l+1,m)\right)\frac{t^l}{l!} \\
&= \sum_{i=0}^{\infty}B_{i,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\frac{2^it^i}{i!} \\
&\quad\times\sum_{\alpha=1}^{\infty}\left(\sum_{l=0}^{\alpha-1}\sum_{m=1}^{l+1}\binom{\alpha}{l}(1)_{\alpha-l,\lambda}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{(l+1)m^k}S_{2,\lambda}(l+1,m)\right)\frac{t^\alpha}{\alpha!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{\alpha=1}^{n}\sum_{l=0}^{\alpha-1}\sum_{m=1}^{l+1}\binom{n}{\alpha}\binom{\alpha}{l}(1)_{\alpha-l,\lambda}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{(l+1)m^k}\right. \\
&\quad\left.\times S_{2,\lambda}(l+1,m)B_{n-\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\right)\frac{t^n}{n!}. \qquad (60)
\end{aligned}
$$

Thus, by comparing the coefficients on both sides of (60), we obtain the desired theorem. □

4 Conclusion

In this paper, we constructed a new type degenerate Genocchi polynomials and numbers by using the degenerate polylogarithm function, called degenerate poly-Genocchi polynomials and numbers. We represented the following: the generating function of the degenerate poly-Genocchi numbers by iterated integrals in Theorem 1; the explicit degenerate poly-Genocchi numbers in terms of the degenerate Bernoulli polynomials and the degenerate Genocchi numbers in Theorem 3. Not to mention, we obtained in Theorems 5 and 7 that the degenerated poly-Genocchi polynomials are represented by the degenerated poly-Genocchi numbers and the degenerate Stirling numbers of the second kind. We also demonstrated in Theorem 10 that the degenerate poly-Genocchi polynomials are represented by the degenerated poly-Genocchi numbers and the degenerate Stirling numbers of the first kind. We expressed those polynomials in terms of the degenerate Genocchi polynomials and the degenerate Stirling numbers of the second kind in Theorem 8 and the degenerate poly-Bernoulli polynomials in Theorem 9.

On the other hand, in Sect. 3, we defined the degenerate unipoly Genocchi polynomials by using the degenerate polylogarithm function and obtained: the identity degenerate unipoly Genocchi polynomials in terms of the degenerate Euler numbers and the degenerate Stirling numbers of the second kind in Theorem 12; the degenerate Genocchi numbers and the degenerate Stirling numbers of the second kind in Theorem 13; the degenerate Bernoulli polynomials and the degenerate Stirling numbers of the second kind in Theorem 14.

The field of degenerate versions is widely applied not only to number theory and combinatorics but also to symmetric identities, differential equations, and probability theory. As one of our future projects, we would like to continue to study degenerate versions of certain special polynomials and numbers and their applications to physics, economics, and engineering as well as mathematics.

Acknowledgements
The authors would like to thank the referees for the detailed and valuable comments that helped improve the original manuscript in its present form. Also, the authors thank Jangjeon Institute for Mathematical Science for the support of this research.

Funding
This research was supported by the Daegu University Research Grant, 2019.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
All authors reveal that there is no ethical problem in the production of this paper.

Competing interests
The authors declare no conflict of interests.

Consent for publication
All authors want to publish this paper in this journal.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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Publisher's Note

Received: 28 May 2020 Accepted: 9 August 2020 Published online: 18 August 2020

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