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Explicit criteria for the qualitative properties of differential equations with p -Laplacian-like operator

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Abstract

The aim of this work is to study qualitative properties of solutions for a fourth-order neutral nonlinear differential equation, driven by a p -Laplace differential operator. Some oscillation criteria for the equation under study have been obtained by comparison theory. The obtained results improve the well-known oscillation results present in the literature. Some examples are provided to show the applicability of the obtained results.

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1 Introduction

Differential equations of fourth-order appear in models concerning biological, physical, and chemical phenomena, optimization, mathematics of networks, dynamical systems, see [1].

We study the oscillatory behavior of the fourth-order neutral nonlinear differential equation of the form

$$\begin{cases} (r(x)|w'''(x)|^{p_1-2}w'''(x))' + \sum_{i=1}^j q_i(x)|u'''(\vartheta_i(x))|^{p_2-2}u'''(\vartheta_i(x)) = 0, \\ j \geq 1, \quad p_2 \geq p_1, \quad r(x) > 0, \quad r'(x) \geq 0, \quad x \geq x_0 > 0, \end{cases} \quad (1)$$

where $w(x) := u(x) + a(x)u(\tau(x))$ and the first term means the p -Laplace-type operator ($1 < p_i < \infty$, $i = 1, 2$). The main results are obtained under the following conditions: $r \in C[x_0, \infty)$, $a, q_i \in C[x_0, \infty)$, $q_i(x) > 0$, $0 \leq a(x) < a_0 < 1$, $\tau, \vartheta_i \in C[x_0, \infty)$, $\tau(x) \leq x$, $\lim_{x \rightarrow \infty} \tau(x) = \lim_{x \rightarrow \infty} \vartheta_i(x) = \infty$, $i = 1, 2, \dots, j$, and under the condition

$$\int_{x_0}^{\infty} \frac{1}{r^{1/(p_1-1)}(s)} ds = \infty. \quad (2)$$

The p -Laplace equations have some significant applications in elasticity theory and continuum mechanics, see [2] (power-law fluids), and in general in nonlinear phenomena, see

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[3] (capillary phenomena). For some results concerning the oscillatory behavior of equations driven by a p -Laplace differential operator, we mention the papers [4–6].

In [7], the authors used a classical variational approach based on the critical points theory to prove the existence of at least one nontrivial weak solution of a double-phase Dirichlet problem. Here the differential operator of the problem is the sum of two p -Laplacian-type operators with variable exponents. This fact could provide new ideas for further investigations. The authors characterized the continuous spectrum of double-phase equations (to improve the regularity theory for such a kind of operators and classify solutions).

Nastasi [8] established an existence result of a nontrivial weak solution to (p, q) -Laplacian problem on a noncompact Riemannian manifold. The special setting led the author to develop the Maz'ya's approach, by working with isocapacitary inequalities to characterize the compact embeddings.

2 Mathematical background—hypotheses

In this section we collect some relevant facts and auxiliary results from the existing literature. Also, we fix the notation.

Currently, researchers have become more concerned with the topic of oscillation of differential equations in [9–27]. Li et al. [4], using the Riccati transformation together with integral averaging technique, focused on the oscillations of the equation

$$\begin{cases} (r(x)|w'''(x)|^{p-2}w'''(x))' + \sum_{i=1}^j q_i(x)|y(\vartheta_i(x))|^{p-2}y(\vartheta_i(x)) = 0, \\ 1 < p < \infty, \quad x \geq x_0 > 0. \end{cases}$$

In [28, 29], the comparison method with first and second order equations was used to investigate every solution u of

$$\begin{cases} (r(x)|u^{(n-1)}(x)|^{p-2}u^{(n-1)}(x))' + \sum_{i=1}^j q_i(x)g(u(\vartheta_i(x))) = 0, \\ j \geq 1, \quad x \geq x_0 > 0, \end{cases}$$

where n is even and $p > 1$ is a real number, in the case where $\vartheta_i(x) \geq \nu$ (with $r \in C^1((0, \infty), \mathbb{R}), q_i \in C([0, \infty), \mathbb{R}), i = 1, 2, \dots, j$).

We point out that Bazighifan [30] gave us some results providing information on the oscillation of equations

$$\begin{cases} (r(x)|u'''(x)|^{p_1-2}u'''(x))' + q(x)|u'''(g_1(x))|^{p_2-2}u'''(g_1(x)) = 0, \\ r(x) > 0, \quad r'(x) \geq 0, \quad p_2 \geq p_1, \quad x \geq x_0 > 0, \end{cases}$$

where n is even. This time, the author used the comparison method with second order equations.

The authors of [31], using the Riccati technique, derived oscillation conditions of

$$\begin{cases} (r(x)|u^{(n-1)}(x)|^{p-2}u^{(n-1)}(x))' + q(x)g(u(\vartheta(x))) = 0, \\ 1 < p < \infty, \end{cases}$$

where n is even.

As we already mentioned in the Introduction, our aim here is to provide complementary results to [28, 29, 31]. For this purpose we briefly discuss these results.

Definition 2.1 Define sequences of functions $\{\delta_n(x)\}_{n=0}^\infty$ and $\{\sigma_n(x)\}_{n=0}^\infty$ as

$$\begin{aligned} \delta_0(x) &= \xi_*(x) \quad \text{and} \quad \sigma_0(x) = \eta_*(x), \\ \delta_n(x) &= \delta_0(x) + \int_x^\infty R_1(s) \delta_{n-1}^{p_1/(p_1-1)}(s) \, ds, \quad n \geq 1 \\ \sigma_n(x) &= \sigma_0(x) + \int_x^\infty \sigma_{n-1}^{p_1/(p_1-1)}(s) \, ds, \quad n \geq 1. \end{aligned} \tag{3}$$

We see by induction that $\delta_n(x) \leq \delta_{n+1}(x)$ and $\sigma_n(x) \leq \sigma_{n+1}(x)$ for $x \geq x_0, n \geq 1$.

Now, we are ready to introduce the precise hypotheses on the data of (1):

(H1) u is an eventually positive solution of (1).

(H2) Let $B(x) = (p_1 - 1) \varepsilon \frac{\vartheta_i^2(x) \xi \vartheta_i'(x)}{r^{1/(p_1-1)}(x)}$ and $\phi_1(x) = \int_x^\infty A(s) \, ds$ be such that

$$\liminf_{x \rightarrow \infty} \frac{1}{\phi_1(x)} \int_x^\infty B(s) \phi_1^{\frac{p_1}{p_1-1}}(s) \, ds > \frac{p_1 - 1}{p_1} \tag{4}$$

where

$$A(x) = \sum_{i=1}^j q_i(x) (1 - a_0)^{p_2-1} M^{p_1-p_2} (\vartheta_i(x)).$$

(H3) For some $\mu \in (0, 1)$, there are positive constants M_1, M_2 such that

$$\liminf_{x \rightarrow \infty} \frac{1}{\xi_*(x)} \int_x^\infty R_1(s) \xi_*^{p_1/(p_1-1)}(s) \, ds > \frac{(p_1 - 1)}{p_1} \tag{5}$$

and

$$\liminf_{x \rightarrow \infty} \frac{1}{\eta_*(x)} \int_{x_0}^\infty \eta_*^2(s) \, ds > \frac{1}{4} \tag{6}$$

where

$$\begin{aligned} R_1(x) &:= (p_1 - 1) \mu \frac{x^2}{2r^{1/(p_1-1)}(x)}, \\ \xi(x) &:= \sum_{i=1}^j q_i(x) (1 - a_0)^{p_2-1} M_1^{p_2-p_1} \varepsilon_1 \left(\frac{\vartheta_i(x)}{x} \right)^{3(p_2-1)}, \\ \eta(x) &:= (1 - a_0)^{p_2/p_1} M_2^{p_2/(p_1-2)} \int_x^\infty \left(\frac{1}{r(\delta)} \int_\delta^\infty \sum_{i=1}^j q_i(s) \frac{\vartheta_i^{p_2-1}(s)}{s^{p_2-1}} \, ds \right)^{1/(p_1-1)} \, d\delta, \\ \xi_*(x) &= \int_x^\infty \xi(s) \, ds \quad \text{and} \quad \eta_*(x) = \int_x^\infty \eta(s) \, ds. \end{aligned}$$

(H4) For some $\mu_1 \in (0, 1)$, we have

$$\limsup_{x \rightarrow \infty} \left(\frac{\mu_1 x^3}{6r^{1/(p_1-1)}(x)} \right)^{p_1-1} \delta_n(x) > 1 \tag{7}$$

and

$$\limsup_{x \rightarrow \infty} \lambda x \sigma_n(x) > 1, \tag{8}$$

for some n .

(H5) For some n , we have

$$\int_{x_0}^{\infty} \xi(x) \exp\left(\int_{x_0}^x R_1(s) \delta_n^{1/(p_1-1)}(s) ds\right) dx = \infty \tag{9}$$

and

$$\int_{x_0}^{\infty} \eta(x) \exp\left(\int_{x_0}^x \sigma_n^{1/(p_1-1)}(s) ds\right) dx = \infty. \tag{10}$$

3 Main results

Next, we mention some important lemmas:

Lemma 3.1 ([32]) *Let w satisfy $w^{(i)}(x) > 0, i = 0, 1, \dots, n$, and $w^{(n+1)}(x) < 0$ eventually. Then, for every $\varepsilon_1 \in (0, 1)$, $w(x)/w'(x) \geq \varepsilon_1 x/n$ eventually.*

Lemma 3.2 ([10]) *Let w satisfy $w(x) > 0$ and $w^{(n-1)}(x)w^{(n)}(x) \leq 0, x \geq x_w$, then there exist constants $\theta, 0 < \theta < 1$ and $\varepsilon > 0$ such that*

$$w'(\theta x) \geq \varepsilon x^{n-2} w^{(n-1)}(x),$$

for all sufficiently large x .

Lemma 3.3 ([33]) *Let w satisfy $w^{(n-1)}(x)w^{(n)}(x) \leq 0$ and $\lim_{x \rightarrow \infty} w(x) \neq 0$, then*

$$w(x) \geq \frac{\mu}{(n-1)!} x^{n-1} |w^{(n-1)}(x)| \quad \text{for } \mu \in (0, 1).$$

Lemma 3.4 ([34]) *If (H1) holds, then we can distinguish the following situations:*

- (G₁) $w^{(k)}(x) > 0, \quad k = 1, 2, 3,$
- (G₂) $w^{(k)}(x) > 0, \quad k = 1, 3,$ and $w''(x) < 0,$

for $x \geq x_1$, where $x_1 \geq x_0$ is sufficiently large.

Theorem 3.1 *If (H2) holds, then (1) is oscillatory.*

Proof Let (H1) hold, then there exists an $x_1 \geq x_0$ such that $u(x) > 0$, $u(\tau(x)) > 0$ and $u(\vartheta_i(x)) > 0$ for $x \geq x_1$. Since $r'(x) > 0$, we have

$$\begin{aligned} w(x) > 0, \quad w'(x) > 0, \quad w'''(x) > 0, \\ w^{(4)}(x) < 0 \quad \text{and} \quad (r(x)(w'''(x))^{p_1-1})' \leq 0, \end{aligned} \tag{11}$$

for $x \geq x_1$. From the definition of w , we get

$$u(x) \geq w(x) - a_0 u(\tau(x)) \geq w(x) - a_0 w(\tau(x)) \geq (1 - a_0)w(x),$$

which together with (1) gives

$$(r(x)(w'''(x))^{p_1-1})' \leq - \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} w^{p_2-1}(\vartheta_i(x)). \tag{12}$$

Define

$$\varpi(x) := \frac{r(x)(w'''(x))^{p_1-1}}{w^{p_1-1}(\zeta \vartheta_i(x))}, \tag{13}$$

for some a constant $\zeta \in (0, 1)$. By differentiating the above and using (12), we get

$$\begin{aligned} \varpi'(x) \leq & \frac{- \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} w^{p_2-1}(\vartheta_i(x))}{w^{p_1-1}(\zeta \vartheta_i(x))} \\ & - (p_1 - 1) \frac{r(x)(w'''(x))^{p_1-1} w'(\zeta \vartheta_i(x)) \zeta \vartheta_i'(x)}{w^{p_1}(\zeta \vartheta_i(x))}. \end{aligned}$$

From Lemma 3.2, there exists a constant $\varepsilon > 0$ such that

$$\begin{aligned} \varpi'(x) \leq & - \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} w^{p_2-p_1}(\vartheta_i(x)) \\ & - (p_1 - 1) \frac{r(x)(w'''(x))^{p_1-1} \varepsilon \vartheta_i^2(x) w'''(\vartheta_i(x)) \zeta \vartheta_i'(x)}{w^{p_1}(\zeta \vartheta_i(x))}, \end{aligned}$$

which implies

$$\varpi'(x) \leq - \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} w^{p_2-p_1}(\vartheta_i(x)) - (p_1 - 1) \varepsilon \frac{r(x) \vartheta_i^2(x) \zeta \vartheta_i'(x) (w'''(x))^{p_1}}{w^{p_1}(\zeta \vartheta_i(x))}.$$

Using (13), we find

$$\varpi'(x) \leq - \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} w^{p_2-p_1}(\vartheta_i(x)) - (p_1 - 1) \varepsilon \frac{\vartheta_i^2(x) \zeta \vartheta_i'(x)}{r^{1/(p_1-1)}(x)} \varpi^{p_1/(p_1-1)}(x). \tag{14}$$

Since $w'(x) > 0$, there exist an $x_2 \geq x_1$ and a constant $M > 0$ such that

$$w(x) > M.$$

Then, (14) turns into

$$\varpi'(x) \leq - \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} M^{p_2-p_1} (\vartheta_i(x)) - (p_1 - 1)\varepsilon \frac{\vartheta_i^2(x)\zeta \vartheta_i'(x)}{r^{1/(p_1-1)}(x)} \varpi^{p_1/(p_1-1)}(x),$$

that is,

$$\varpi'(x) + A(x) + B(x)\varpi^{p_1/(p_1-1)}(x) \leq 0. \tag{15}$$

Integrating (15) from x to l , we obtain

$$\varpi(l) - \varpi(x) + \int_x^l A(s) ds + \int_x^l B(s)\varpi^{p_1/(p_1-1)}(s) ds \leq 0.$$

Letting $l \rightarrow \infty$ and using $\varpi > 0$ and $\varpi' < 0$, we have

$$\varpi(x) \geq \phi_1(x) + \int_x^\infty B(s)\varpi^{p_1/(p_1-1)}(s) ds.$$

This implies

$$\frac{\varpi(x)}{\phi_1(x)} \geq 1 + \frac{1}{\phi_1(x)} \int_x^\infty B(s)\phi_1^{p_1/(p_1-1)}(s) \left(\frac{\varpi(s)}{\phi_1(s)}\right)^{p_1/(p_1-1)} ds. \tag{16}$$

Let $\lambda = \inf_{x \geq x} \varpi(x)/\phi_1(x)$, then obviously $\lambda \geq 1$. So, from (4) and (16), we find

$$\lambda \geq 1 + (p_1 - 1) \left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)},$$

or

$$\frac{\lambda}{p_1} \geq \frac{1}{p_1} + \frac{(p_1 - 1)}{p_1} \left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)},$$

which contradicts $\lambda \geq 1$ and $(p_1 - 1) > 0$.

The proof is complete. □

Theorem 3.2 *If (H3) holds, then (1) is oscillatory.*

Proof Let (1) have a nonoscillatory solution in $[x_0, \infty)$. Without loss of generality, we let $u(x) > 0$. Then, there exists an $x_1 \geq x_0$ such that $u(\tau(x)) > 0$ and $u(\vartheta_i(x)) > 0$ for $x \geq x_1$. From Lemma 3.4, there are two cases (G_1) and (G_2).

For case (G_1), define

$$\omega(x) := \frac{r(x)(w'''(x))^{p_1-1}}{w^{p_1-1}(x)}. \tag{17}$$

From (12), we obtain

$$\omega'(x) \leq - \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} \frac{w^{p_2-1}(\vartheta_i(x))}{w^{p_1-1}(x)} - (p_1 - 1) \frac{r(x)(w'''(x))^{p_1-1}}{w^{p_1}(x)} w'(x). \tag{18}$$

From Lemma 3.1, we find

$$\frac{w'(x)}{w(x)} \leq \frac{3}{\varepsilon_1 x}.$$

Integrating again from $\vartheta_i(x)$ to x , we find

$$\frac{w(\vartheta_i(x))}{w(x)} \geq \varepsilon_1 \frac{\vartheta_i^3(x)}{x^3}. \tag{19}$$

It follows from Lemma 3.3 that

$$w'(x) \geq \frac{\mu_1}{2} x^2 w'''(x), \tag{20}$$

for all $\mu_1 \in (0, 1)$. Since $w'(x) > 0$, there exists an $x_2 \geq x_1$ such that

$$w(x) > M. \tag{21}$$

From (18), (19), (20), and (21), we obtain

$$\omega'(x) + \sum_{i=1}^j q_i(x)(1 - a_0)^{p_2-1} M_1^{p_2-p_1} \varepsilon_1 \left(\frac{\vartheta_i(x)}{x}\right)^{3(p_2-1)} + \frac{(p_1 - 1)\mu x^2}{2r^{1/(p_1-1)}(x)} \omega^{p_1/(p_1-1)}(x) \leq 0,$$

that is,

$$\omega'(x) + \xi(x) + R_1(x)\omega^{p_1/(p_1-1)}(x) \leq 0. \tag{22}$$

Integrating (22) from x to l , we find

$$\omega(l) - \omega(x) + \int_x^l \xi(s) ds + \int_x^l R_1(s)\omega^{p_1/(p_1-1)}(s) ds \leq 0.$$

Letting $l \rightarrow \infty$ and using $\omega > 0, \omega' < 0$, we get

$$\omega(x) \geq \xi_*(x) + \int_x^\infty R_1(s)\omega^{p_1/(p_1-1)}(s) ds. \tag{23}$$

This implies

$$\frac{\omega(x)}{\xi_*(x)} \geq 1 + \frac{1}{\xi_*(x)} \int_x^\infty R_1(s)\xi_*^{p_1/(p_1-1)}(s) \left(\frac{\omega(s)}{\xi_*(s)}\right)^{p_1/(p_1-1)} ds. \tag{24}$$

Let $\lambda = \inf_{x \geq x} \omega(x)/\xi_*(x)$, then $\lambda \geq 1$. So, from (5) and (24), we obtain

$$\lambda \geq 1 + (p_1 - 1) \left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)},$$

or

$$\frac{\lambda}{p_1} \geq \frac{1}{p_1} + \frac{(p_1 - 1)}{p_1} \left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)},$$

which contradicts $\lambda \geq 1$ and $(p_1 - 1) > 0$.

For case (G_2) , integrating (12) from x to m , we obtain

$$r(m)(w'''(m))^{p_1-1} - r(x)(w'''(x))^{p_1-1} \leq - \int_x^m \sum_{i=1}^j q_i(s)(1 - a_0)^{p_2-1} w^{p_2-1}(\vartheta_i(s)) \, ds. \tag{25}$$

From Lemma 3.1, we find

$$w(x) \geq \varepsilon_1 x w'(x) \quad \text{and hence} \quad w(\vartheta_i(x)) \geq \varepsilon_1 \frac{\vartheta_i(x)}{x} w(x). \tag{26}$$

For (25), letting $m \rightarrow \infty$ and using (26), we see that

$$r(x)(w'''(x))^{p_1-1} \geq \varepsilon_1 (1 - a_0)^{p_2-1} w^{p_2-1}(x) \int_x^\infty \sum_{i=1}^j q_i(s) \frac{\vartheta_i^{p_2-1}(s)}{s^{p_2-1}} \, ds. \tag{27}$$

Integrating (27) from x to ∞ , we obtain

$$w''(x) \leq -\varepsilon_1 (1 - a_0)^{p_2/p_1} w^{p_2/p_1}(x) \int_x^\infty \left(\frac{1}{r(\delta)} \int_\delta^\infty \sum_{i=1}^j q_i(s) \frac{\vartheta_i^{p_2-1}(s)}{s^{p_2-1}} \, ds \right)^{1/(p_1-1)} \, d\delta, \tag{28}$$

for all $\varepsilon_1 \in (0, 1)$. Define

$$y(x) = \frac{w'(x)}{w(x)}.$$

By differentiating y and from (21) and (28), we see that

$$y'(x) \leq -y^2(x) - (1 - a_0)^{p_2/p_1} M^{(p_2/p_1)-1} \int_x^\infty \left(\frac{1}{r(\delta)} \int_\delta^\infty \sum_{i=1}^j q_i(s) \frac{\vartheta_i^{p_2-1}(s)}{s^{p_2-1}} \, ds \right)^{1/(p_1-1)} \, d\delta, \tag{29}$$

hence

$$y'(x) + \eta(x) + y^2(x) \leq 0. \tag{30}$$

The rest of the proof of the case where (G_2) holds is the same as that of case (G_1) . Thus, the proof is complete. \square

Theorem 3.3 *If (H4) holds, then (1) is oscillatory.*

Proof Proceeding as in the proof of Theorem 3.2, in the case (G_1) , we see that (20) holds. By Lemma 3.3, we find

$$w(x) \geq \frac{\mu_1}{6} x^3 w'''(x). \tag{31}$$

From (17) and (31), we get

$$\frac{1}{\omega(x)} = \frac{1}{r(x)} \left(\frac{w(x)}{w'''(x)} \right)^{p_1-1} \geq \frac{1}{r(x)} \left(\frac{\mu_1}{6} x^3 \right)^{p_1-1}.$$

Thus,

$$\omega(x) \left(\frac{\mu_1 x^3}{6r^{1/(p_1-1)}(x)} \right)^{p_1-1} \leq 1.$$

Therefore,

$$\limsup_{x \rightarrow \infty} \omega(x) \left(\frac{\mu_1 x^3}{6r^{1/(p_1-1)}(x)} \right)^{p_1-1} \leq 1,$$

which contradicts (7).

The rest of the proof is the same as that for the case (G₂). Theorem 3.3 is proved. □

Corollary 3.1 *If (H5) holds, then (1) is oscillatory.*

Proof Proceeding as in the proof of Theorem 3.2, in the case (G₁), from (23) we obtain $\omega(x) \geq \delta_0(x)$.

By induction we can also see that $\omega(x) \geq \delta_n(x)$ for $x \geq x_0, n > 1$. Since the sequence $\{\delta_n(x)\}_{n=0}^\infty$ is monotone increasing and bounded above, it converges to $\delta(x)$. Using Lebesgue’s monotone convergence theorem, we find

$$\delta(x) = \lim_{n \rightarrow \infty} \delta_n(x) = \int_x^\infty R_1(x) \delta_n^{p_1/(p_1-1)}(s) ds + \delta_0(x)$$

and

$$\delta'(x) = -R_1(x) \delta_n^{p_1/(p_1-1)}(x) - \xi(x). \tag{32}$$

Since $\delta_n(x) \leq \delta(x)$, it follows from (32) that

$$\delta'(x) \leq -R_1(x) \delta_n^{1/(p_1-1)}(x) \delta(x) - \xi(x).$$

Hence, we get

$$\delta(x) \leq \exp\left(-\int_x^x R_1(s) \delta_n^{1/(p_1-1)}(s) ds\right) \left(\delta(x) - \int_x^x \xi(s) \exp\left(\int_x^s R_1(\delta) \delta_n^{1/(p_1-1)}(\delta) d\delta\right) ds \right).$$

This implies

$$\int_x^x \xi(s) \exp\left(\int_x^s R_1(\delta) \delta_n^{1/(p_1-1)}(\delta) d\delta\right) ds \leq \delta(x) < \infty,$$

which contradicts (9). The proof of the case where (G₂) holds is the same as that of (G₁). Corollary 3.1 is proved. □

Example 3.1 Consider the differential equation

$$\left(u(x) + \frac{1}{2} u\left(\frac{x}{2}\right) \right)^{(4)} + \frac{q_0}{x^4} u\left(\frac{x}{3}\right) = 0, \tag{33}$$

where $q_0 > 0$. Let $p_1 = p_2 = 2$, $r(x) = 1$, $a(x) = 1/2$, $\tau(x) = x/2$, $\vartheta(x) = x/3$, and $q(x) = q_0/x^4$. Then

$$A(x) = \sum_{i=1}^j q_i(x)(1 - a_0)^{(p_2-1)} M^{p_2-p_1}(\vartheta_i(x)) = \frac{q_0}{2x^4},$$

$$B(x) = (p_1 - 1)\varepsilon \frac{\vartheta_i^2(x)\zeta\vartheta_i'(x)}{r^{1/(p_1-1)}(x)} = \frac{\varepsilon x^2}{27},$$

$$\phi_1(x) = \frac{q_0}{6x^3}$$

and

$$\liminf_{x \rightarrow \infty} \frac{1}{\phi_1(x)} \int_x^\infty B(s)\phi_1^{p_1/(p_1-1)}(s) ds > \frac{(p_1 - 1)}{p_1^{p_1/(p_1-1)}},$$

$$\liminf_{x \rightarrow \infty} \frac{6\varepsilon q_0 x^3}{972} \int_x^\infty \frac{ds}{s^4} > \frac{1}{4},$$

$$q_0 > 121.5\varepsilon,$$

for some $\varepsilon > 0$. Thus, by Theorem 3.1, every solution of equation (33) is oscillatory if $q_0 > 121.5\varepsilon$.

Example 3.2 Consider a differential equation

$$(u(x) + a_0u(\tau_0x))^{(n)} + \frac{q_0}{x^n}u(\vartheta_0x) = 0, \quad q_0 > 0. \tag{34}$$

Let $p = 2$, $x_0 = 1$, $r(x) = 1$, $a(x) = a_0$, $\tau(x) = \tau_0x$, $\vartheta(x) = \vartheta_0x$, and $q(x) = q_0/x^n$. Then we easily see that condition (5) holds and condition (6) is satisfied. Hence, by Theorem 3.2, every solution of equation (34) is oscillatory.

4 Conclusions

Our aim of this article was to study the qualitative behavior of a fourth-order neutral non-linear differential equation, driven by a p -Laplace differential operator. The obtained oscillation theorems complement the well-known oscillation results present in the literature. In this line of work, one can investigate oscillatory conditions for a fourth-order equation of the type:

$$\begin{cases} (r(x)|y'''(x)|^{p_1-2}y'''(x))' + a(x)f(y'''(x)) + \sum_{i=1}^j q_i(x)|y(\sigma_i(x))|^{p_2-2}y(\sigma_i(x)) = 0, \\ x \geq x_0, \quad \sigma_i(x) \leq x, \quad j \geq 1, \quad 1 < p_2 \leq p_1 < \infty. \end{cases}$$

which is of interest to the authors, in particular, the case of $p_2 > p_1$.

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