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Stability and solvability for a class of optimal control problems described by non-instantaneous impulsive differential equations

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Abstract

In this paper, we investigate the existence and stability of solutions for a class of optimal control problems with 1-mean equicontinuous controls, and the corresponding state equation is described by non-instantaneous impulsive differential equations. The existence theorem is obtained by the method of minimizing sequence, and the stability results are established by using the related conclusions of set-valued mappings in a suitable metric space. An example with the measurable admissible control set, in which the controls are not continuous, is given in the end.

MSC: 34H05; 49J15; 34D20

Keywords: Optimal control; Non-instantaneous impulse; Stability analysis

1 Introduction

Impulsive phenomena are results of a sudden change in the state of system due to external interference; they often occur in nature and human activities. According to the duration of the change, the phenomena of this kind are divided into two categories. One is that the duration of this change is relatively short compared with the total duration of the whole process, which is called instantaneous impulse. The other is that they start from any fixed point and remain active for a limited time interval, namely the effects are continuous. We call it non-instantaneous impulse (see [1–8]).

Most of the mathematical models extracted from impulsive phenomena are characterized by impulsive differential equations, which can be classified into two categories in accordance with the type of impulse: instantaneous impulsive differential equations and non-instantaneous ones. Now, a large number of references deal with the impulsive differential equations. By the type of impulse, they include the non-instantaneous case [1–12] and the instantaneous case [13–22].

This paper is devoted to the study of the differential equation with impulse of non-instantaneous type on account of its reality and significance. For instance, the state change process of some elements during intravenous drug injection, periodic fishing, and criterion for pest management is depicted by non-instantaneous impulsive differential equations (see [20]). It is shown that the investigation of non-instantaneous impulsive differential equations is of great importance to nature and human beings themselves.

In the following, we will briefly sketch some existing results about the differential equations with non-instantaneous impulse.

In 2013, Hernández and O'Regan, depending on the background of pharmacokinetics, considered a Cauchy problem for a class of new semi-linear evolution equations with non-instantaneous impulse moments (see [1]), which is

$$\begin{cases} x'(t) = Ax(t) + f(t,x(t)), & t \in (s_i, t_{i+1}], i = 0,1,\ldots,m, \\ x(t) = g_i(t,x(t)), & t \in (t_i, s_i], i = 1,\ldots,m, \\ x(0) = x_0 \in X, \end{cases}$$

where $t_i < s_i$, $i = 0,1,\ldots,m$, $m \in \mathbb{N}$, $s_0 = 0$, $t_{m+1} = T$, $A : D(A) \subseteq X \to X$ is an infinitesimal generator of C_0 semigroup $\{T(t), t \geq 0\}$ in Banach space X, $f : [0,T] \times X \to X$, and $g_i : (t_i, s_i] \times X \to X$, $i = 1,\ldots,m$.

This paper has initiated the study for differential equations with non-instantaneous impulse. Since then, relevant research work has just been getting started.

Pierri and O'Regan dealt with the existence of solutions for a kind of semi-linear abstract differential equations with non-instantaneous impulse in 2013 (see [9]). Then, Fečkan, Wang, and Zhou paid attention to the existence of periodic solutions for the following non-instantaneous impulsive periodic systems [10, 12]:

$$\begin{cases} x'(t) = Ax(t) + f(t,x(t)), & t \in (s_i, t_{i+1}], i = 0,1,\ldots,\infty, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1,\ldots,\infty, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i], i = 1,\ldots,\infty, \end{cases}$$

and

$$\begin{cases} x'(t) + A(t)x(t) = f(t,x(t)), & t \in (s_i, t_{i+1}], i = 0,1,\ldots,\infty, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1,\ldots,\infty, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i] i = 1,\ldots,\infty. \end{cases}$$

In 2015, Hernandez et al. introduced and studied a new model of abstract impulsive differential equations [4]

$$\begin{cases} u'(t) = f(t,u(t)) + Au(t), & t \in (s_i, t_{i+1}], i = 0,1,\ldots,N, \\ u(t) = h_i(t, u_{|I_i(t)}), & t \in (t_i, s_i], i = 1,\ldots,N, \\ u(0) = x_0, \end{cases}$$

where A is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, $x_0 \in X$, $I_i(t)$ is a $2^{[0,t]}$-set valued function.

Hernandez et al. improved substantially the theory on differential equations with non-instantaneous impulses. And since then, based on their work, many scholars have been devoted to the study of abstract differential equations with non-instantaneous impulses, such as [5–7].

In 2017, Ravi Agarwal et al. considered the initial value problem with non-instantaneous impulses (see [3])

$$\begin{cases} x' = f_k(t,x), & t \in (s_k, t_{k+1}] \cap [t_0, T], k = 0, 1, \dots, p, \\ x(t) = \phi_k(t, x(t), x(t_k - 0)), & t \in (t_k, s_k] \cap [t_0, T], k = 1, \dots, p, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

and demonstrated that equation (1.1) was equivalent to the integral one as follows:

$$x(t; t_0, x_0) = \begin{cases} x_0 + \int_{t_0}^{t} f_0(s, x(s; t_0, x_0))\, ds, & t \in [t_0, t_1] \cap [t_0, T], \\ \phi_k(t, x(t; t_0, x_0), x(t_k - 0; t_0, x_0)), & t \in (t_k, s_k] \cap [t_0, T], k = 1, \dots, p, \\ \phi_k(s_k, x(s_k; t_0, x_0), x(t_k - 0; t_0, x_0)) & \\ \quad + \int_{s_k}^{t} f_k(s, x(s; t_0, x_0))\, ds, & t \in (s_k, t_{k+1}] \cap [t_0, T], k = 1, \dots, p. \end{cases} \tag{1.2}$$

In 2018, Yao et al. obtained the sufficient conditions for the existence and uniqueness of extremum solutions for a class of non-instantaneous impulsive boundary value problems (see (1.3)) by means of the method of upper and lower solutions along with monotone iteration technique (see [11]):

$$\begin{cases} x' = f_k(t, x(t)), & t \in (s_k, t_{k+1}] \cap [t_0, T], k = 0, 1, \dots, p, \\ x(t) = \varphi_k(t, x(t), x(t_k - 0)), & t \in (t_k, s_k] \cap [t_0, T], k = 1, \dots, p, \\ x(0) = lx(T). \end{cases} \tag{1.3}$$

Optimal control theory originated in the late 1950s, and the maximum principle founded by the former Soviet mathematician L.C. Pontryagin marked the beginning of a new stage in its process. Kalman put forward the concept of controllability in 1963 (see [23]), which played an important role in the field of mathematical control theory. Today, many researchers are devoted to the controllability of problems, such as those defined on abstract Banach spaces [24–26] (for more details about the study on abstract differential equations, we refer the readers to a series of Hernandez's articles, such as [27, 28]).

In recent years, with the further development of computer science and mathematics, the optimal control problems have achieved great progress, and the applications in real life are becoming more and more extensive. A number of scholars have been committed to the study on the optimal control problems (see [8, 29–36]).

In [30], Yu investigated the existence and stability of solutions for the problem

$$J_f\big(u^*\big) = \min_{u \in U} J_f(u),$$

where

$$J_f(u) \triangleq h\big(x(T)\big) + \int_{t_0}^{T} g\big(t, x(t), u(t)\big)\, dt,$$

h, g are continuous and $x(t)$ satisfies the following differential equation:

$$\begin{cases} \dot{x} = f(t, x(t), u(t)), & t \in [t_0, T], \\ x(t_0) = x^0. \end{cases}$$

In [37], Deng and Wei considered the existence and stability analysis for the above nonlinear optimal control problems with 1-mean equicontinuous controls. Different from [30], Deng and Wei got the existence and stability results by weakening the condition of the control. And the control may be not continuous in their results, this is more in line with the general situation.

In [31], Liu et al. studied the optimal control problem for a new class of non-instantaneous impulsive differential equations, and the controllability was proved by constructing a suitable control function, namely finding $\overline{u} \in U_{ad}$ such that $J(\overline{x}, \overline{u}) \leq J(x, u)$ for all $u \in U_{ad}$, $x \in PC([0, T], \mathbb{R}^{n_1})$, where

$$J(x,u) = \int_0^T \|y(t) - y_d(t)\|^2 \, dt = \int_0^T \|C(t)x(t) + D(t)u(t) - y_d(t)\|^2 \, dt.$$

$y_d \in PC([0, T], \mathbb{R}^{n_1})$ is a given piecewise continuous function and $x(t)$ satisfies the following differential equation:

$$\begin{cases} \dot{x} = A(t)x(t) + f(t, x(t)) + B(t)u(t), & t \in \bigcup_{j=0}^{N} [s_i, t_{i+1}], \\ x(t) = B_i(t)x(t_i^-), & t \in (t_i, s_i], i = 1, 2, \dots, N, \\ x(t_i^+) = x(t_i^-), & i = 1, 2, \dots, N, \\ x(0) = x_0. \end{cases}$$

In [32], Achim Ilchmann et al. were concerned with the optimal control problem for regular linear differential-algebraic systems. In their paper, they derived an augmented system as the key to analyzing the optimal control problem with tools well known for the optimal control of ordinary differential equations.

So far, we have found that the research findings on non-instantaneous impulsive differential equations are still few, and the studies on optimal control problems with non-instantaneous impulse are also scarce. In view of the widespread use of optimal control problem in industrial and mining enterprizes, transportation, power industry, and national economic management (see [38, 39]), inspired by [3, 9, 11, 31], we mainly concentrate on the existence and stability of optimal control problem with nonlinear non-instantaneous impulsive differential equations. The problem is as follows.

Problem (P) Looking for $u^* \in \mathcal{U}[t_0, T]$ satisfying the equation

$$J_{f_0, f_1, \dots, f_p}(u^*) = \min_{u \in \mathcal{U}[t_0, T]} J_{f_0, f_1, \dots, f_p}(u), \tag{1.4}$$

where

$$J_{f_0, f_1, \dots, f_p}(u) = g(x(T)) + \int_{t_0}^T h(t, x(t), u(t)) \, dt, \tag{1.5}$$

$p > 0$ is a natural number, $\mathcal{U}[t_0, T] = \{u | u \in L^1([t_0, T]; \mathbb{R}^n), u(t) \in U\}$, $U \subset \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}$, $h : [t_0, T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, and x satisfies the equation

$$\begin{cases} \dot{x} = f_k(t, x, u), & t \in (s_k, t_{k+1}] \cap [t_0, T], k = 0, 1, \dots, p, \\ x(t) = \phi_k(t, x(t), x(t_k - 0)), & t \in (t_k, s_k] \cap [t_0, T], k = 1, \dots, p, \\ x(t_0) = x_0, \end{cases} \tag{1.6}$$

where $f_k : (s_k, t_{k+1}] \cap [t_0, T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $\phi_k : (t_k, s_k] \cap [t_0, T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$.

In the paper, the existence result is established by the method of minimizing sequence, and the stability analysis is carried out in a suitable metric space by the related conclusions of set-valued mappings.

The layout of the rest is listed as follows. In Sect. 2, we review some standard facts that are necessary for the paper, such as some important definitions and lemmas. In Sect. 3, it is shown that there exists a unique solution for non-instantaneous impulsive differential equation (1.6). Then, in Sect. 4, it is shown that the optimal control problem **(P)** is solvable in the defined space with the help of minimizing sequence method. Finally, in Sect. 5, we get the results of stability for the optimal control problem by applying related conclusions on set-valued mappings, and in the end, an example is given to illustrate these results.

2 Preliminaries

In order to proceed smoothly, we do some necessary preparations in this section, and the first one is to construct topological spaces.

Let $\{t_i\}_{i=1}^{p+1}$ and $\{s_i\}_{i=0}^{p}$ be numbers such that $0 = s_0 < t_i < s_i < t_{i+1}$, $i = 1, 2, \dots p$, $0 = s_0 < t_0 < t_1$, $t_p < T \leq t_{p+1}$ and p is a natural number.

For the sake of convenience, the norms of all function spaces in the following are uniformly written as the symbol $\|\cdot\|$ without confusion, and the readers can identify the meaning of the norms from the context easily.

Set

$$PC[t_0,T] = \left\{ \begin{array}{r} x|x:[t_0,T]\to\mathbb{R}, x\in C\big((t_k,s_k]\cap[t_0,T]\big), x(t_k+0)<\infty, k=1,2,\dots,p, \\ x\in C\big((s_k,t_{k+1}]\cap[t_0,T]\big), x(s_k+0)<\infty, k=0,1,\dots,p, \end{array} \right\},$$

in which the norm is defined by $\|x\| = \max_{t_0\leq t\leq T}\|x(t)\|$, and

$$PC_m[t_0,T] = \big\{x = (x_1,\dots,x_m)^{\mathrm{T}} | x_i \in PC[t_0,T], i = 1,2,\dots m\big\}$$

in which the norm is defined as $\|x\| = \max_{t_0\leq t\leq T}\|x(t)\|$, where $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^{\mathrm{T}} \in PC_m[t_0,T]$ and

$$\big\|x(t)\big\| = \sqrt{\big(x_1(t)\big)^2 + \big(x_2(t)\big)^2 + \cdots + \big(x_m(t)\big)^2}.$$

It is easy to prove that $(PC_m[t_0,T], \|\cdot\|)$ is a Banach space.

Suppose that $\mathcal{U}[t_0,T]$ meets the following condition (which is set forth in [37]):

(H_u): U is compact in $\mathbb{R}^n$, and $\mathcal{U}[t_0,T]$ is 1-mean equicontinuous, that is, $\forall\epsilon > 0$, there exists a constant $\delta(\epsilon) > 0$ such that for $|h| < \delta(\epsilon)$ one has

$$\sup_{u\in\mathcal{U}[t_0,T]} \int_{t_0}^{T} \big\|u(t+h) - u(t)\big\|\,dt < \epsilon,$$

where $u(t) = 0$ for $t \notin [t_0,T]$.

For U is compact in $\mathbb{R}^n$, U is a nonempty and closed subset of $\mathbb{R}^n$. Then there exists a constant $M > 0$ such that $\|z\| \leq M$, $\forall z \in U$. According to Proposition 2.1 of [37], $\mathcal{U}[t_0,T]$ is bounded and closed in $L^1([t_0,T];\mathbb{R}^n)$. Besides, in view of Lemma 3.2 of [37], $\mathcal{U}[t_0,T]$ is compact in $L^1([t_0,T];\mathbb{R}^n)$.

Let

$$B = \{u \in \mathbb{R}^n : \|u\| \le M\}$$

and

$$\mathcal{B}_k = ([s_k, t_{k+1}] \cap [t_0, T]) \times \mathbb{R}^m \times B, \quad k = 0, 1, \dots, p.$$

Consider the following conditions ($k = 0, 1, \dots, p$).

($H_{w,k}$): $\forall x^1, x^2 \in \mathbb{R}^m$, $\forall u \in B$, $\forall t \in [s_k, t_{k+1}] \cap [t_0, T]$, $f_k : \mathcal{B}_k \to \mathbb{R}^m$, then

$$\|f_k(t, x^1, u) - f_k(t, x^2, u)\| \le L_k \|x^1 - x^2\|,$$

and $\sup_{(t,x,u)\in\mathcal{B}_k} \|f_k(t, x, u)\| \le C_k$, where $L_k, C_k > 0$ are constants.

Under the above condition, we define the metric space as follows:

$$F_k = \{f_k | f_k \text{ is continuous in } \mathcal{B}_k, f_k \text{ satisfies condition } (H_{w,k})\}, k = 1, 2, \dots, m,$$

with the metric ρ_k defined as

$$\rho_k(w^1, w^2) = \sup_{(t,x,u)\in\mathcal{B}_k} \|w^1(t, x, u) - w^2(t, x, u)\|, \quad w^1, w^2 \in F_k.$$

One can demonstrate easily that (F_k, ρ_k) is a complete metric space for each $k = 0, 1, \dots, p$.

Let $\phi_k : ([t_k, s_k] \cap [t_0, T]) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $k = 1, 2, \dots, p$, and ϕ_k satisfies condition ($H_{\phi,k}$), $k = 1, 2, \dots, p$.

($H_{\phi,k}$): ϕ_k is continuous, in addition, $\forall x^1, x^2, y^1, y^2 \in \mathbb{R}^m$,

$$\|\phi_k(t, x^1, y^1) - \phi_k(t, x^2, y^2)\| \le Q_k \|x^1 - x^2\| + \overline{Q}_k \|y^1 - y^2\|,$$

and $\sup_{(t,x,y)\in([t_k,s_k]\cap[t_0,T])\times\mathbb{R}^m\times\mathbb{R}^m} \|\phi_k(t, x, y)\| \le D_k$, where $Q_k, \overline{Q}_k, D_k > 0$ ($k = 1, 2, \dots, p$) are constants and $Q_k + \overline{Q}_k < 1$ ($k = 1, 2, \dots, p$).

Next, one necessary lemma is given.

Lemma 2.1 *Assuming that conditions* (H_u), ($H_{\phi,0}$), ($H_{\phi,k}$) *and* ($H_{w,k}$) ($k = 1, \dots, p$) *are satisfied, then,* $\forall u \in \mathcal{U}[t_0, T]$, $\forall f_k \in F_k$ ($k = 0, 1, \dots, p$), *the differential equation* (1.6) *is equivalent to the following integral-algebraic one*:

$$x(t) = \begin{cases} x_0 + \int_{t_0}^t f_0(s, x(s), u(s))\, ds, & t \in [t_0, t_1] \cap [t_0, T], \\ \phi_k(t, x(t), x(t_k - 0)), & t \in (t_k, s_k] \cap [t_0, T], k = 1, \dots, p, \\ \phi_k(s_k, x(s_k), x(t_k - 0)) & \\ \quad + \int_{s_k}^t f_k(s, x(s), u(s))\, ds, & t \in (s_k, t_{k+1}] \cap [t_0, T], k = 1, \dots, p. \end{cases}$$

Proof Similar to the proof in Sect. 2 of the reference [3], it is easy to obtain this result. □

What follows are the concepts and some important conclusions related to set-valued mappings. For more details, readers can refer to [40, 41].

Definition 2.1 ([40]) Let U and F be metric spaces, a set-valued mapping $I : F \rightrightarrows U$ is called

(1) upper semi-continuous at $f \in F$ if, for each open set $G \subset U$ with $G \supset I(f)$, there exists $\delta > 0$ such that $G \supset I(f')$ for any $f' \in F$ with $\rho(f',f) < \delta$;

(2) lower semi-continuous at $f \in F$ if, for each open set $G \subset U$ with $G \cap I(f) \neq \emptyset$, there exists $\delta > 0$ such that $G \cap I(f') \neq \emptyset$ for any $f' \in F$ with $\rho(f',f) < \delta$;

(3) continuous at $f \in F$ if I is both upper semi-continuous and lower semi-continuous at f.

Definition 2.2 ([40]) Let U and F be metric spaces, a set-valued mapping $I : F \rightrightarrows U$ is called an usco mapping if I is upper semi-continuous and $I(f)$ is nonempty compact for each $f \in F$.

Lemma 2.2 ([40]) *Let U and F be metric spaces, a set-valued mapping $I : F \rightrightarrows U$ is closed if* $\mathrm{Graph}(I)$ *is closed, where* $\mathrm{Graph}(I) := \{(f,u) \in F \times U : u \in I(f)\}$ *is the graph of I.*

Definition 2.3 ([40]) Let U and F be metric spaces, $I : F \rightrightarrows U$ is a set-valued mapping.

(1) For each $f \in F$, $u \in I(f)$ is called an essential solution if $\forall \varepsilon > 0$, there exists $\delta > 0$ such that, for any $f' \in F$ with $\rho(f',f) < \delta$, it has $\|u - u'\| < \varepsilon$ for some $u' \in I(f')$.

(2) The optimal control problem associated with $f \in F$ is called essential if each $u \in I(f)$ is essential.

Lemma 2.3 ([41]) *Let U and F be metric spaces and $I : F \rightrightarrows U$ be a set-valued mapping. If I is closed and U is compact, then I is upper semi-continuous.*

Lemma 2.4 ([42, 43]) *Let U be a metric space, F be a complete metric space, and $I : F \rightrightarrows U$ be an usco mapping. There exists a dense residual subset E of F such that I is lower semi-continuous at each $f \in E$.*

Definition 2.4 ([40]) Let (X,d) be a metric space, $\mathcal{A}$, $\mathcal{B}$ are any two nonempty bounded sets in X. We call

$$H(\mathcal{A},\mathcal{B}) = \inf\bigl\{\varepsilon > 0 : \mathcal{A} \subset U(\varepsilon,\mathcal{B}), \mathcal{B} \subset U(\varepsilon,\mathcal{A})\bigr\}$$

the Hausdorff metric between $\mathcal{A}$ and $\mathcal{B}$, where

$$U(\varepsilon,\mathcal{A}) = \bigl\{x \in X : \exists a \in \mathcal{A},\ \text{such that } d(a,x) < \varepsilon\bigr\},$$
$$U(\varepsilon,\mathcal{B}) = \bigl\{x \in X : \exists b \in \mathcal{B},\ \text{such that } d(b,x) < \varepsilon\bigr\}.$$

3 Existence and uniqueness of solutions for the differential equation with non-instantaneous impulse

In this section, it is demonstrated that there exists a unique solution for the non-instantaneous impulsive differential equation (1.6).

Theorem 3.1 *Supposing that conditions (H_u), $(H_{w,0})$, $(H_{w,k})$, and $(H_{\phi,k})$ $(k = 1,\dots,p)$ are all satisfied, then for each $u \in \mathcal{U}[t_0,T]$, the differential equation (1.6) has a unique solution $x \in PC_m[t_0,T]$. Since the unique solution x depends on u, we denote it as x_u. Furthermore, the map $u \to x_u$ is continuous from $L^1([t_0,T];\mathbb{R}^n)$ into $PC_m[t_0,T]$.*

Proof Similar to the proof of Theorem 3.2.1 in [44], by constructing an equivalent norm in $PC_m[t_0, T]$

$$\|x\|_* = \max_{t_0 \leq t \leq T} e^{-\chi t}\|x\|,$$

where $\chi > 0$ is a constant and satisfies that

$$\frac{L_0}{\chi}\left(1 - e^{\chi(t_0 - t_1)}\right) < 1$$

and

$$\frac{L_k}{\chi}\left(1 - e^{\chi(s_k - t_{k+1})}\right) < 1 - Q_k - \overline{Q}_k \quad (k = 1, 2, \ldots, p),$$

we can come to the conclusion according to the contraction mapping principle in a Banach space. By the same discussion of the proof of Theorem 3.2.1 in [44], we can also get the results that the map $u \to x_u$ is continuous from $L^1([t_0, T]; \mathbb{R}^n)$ into $PC_m[t_0, T]$. □

4 Solvability for the optimal control problem

In this section, we will deal with the existence of solutions for the optimal control problem **(P)** by the method of minimization.

Lemma 4.1 *Let $f_k^q \in F_k$ $(k = 0, \ldots, p)$, $\{u^q\} \subset \mathcal{U}[t_0, T]$, and ϕ_k^q be functions satisfying condition $(H_{\phi,k})$ $(k = 1, \ldots, p)$. Supposed that conditions (H_u), $(H_{w,0})$, and $(H_{w,k})$ all hold, if $f_k^q \to f_k$ $(q \to +\infty, k = 0, \ldots, p)$, $u^q \to u$ $(q \to +\infty)$ in $L^1([t_0, T]; \mathbb{R}^n)$ and $\phi_k^q \to \phi_k$ $(q \to +\infty, k = 1, \ldots, p)$ in $C(((t_k, s_k] \cap [t_0, T]) \times \mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$, then $x^q \to x$ $(q \to +\infty)$ in $PC_m[t_0, T]$, where*

$$x(t) = \begin{cases} x_0 + \int_{t_0}^t f_0(s, x(s), u(s))\,ds, & t \in [t_0, t_1] \cap [t_0, T], \\ \phi_k(t, x(t), x(t_k - 0)), & t \in (t_k, s_k] \cap [t_0, T], k = 1, \ldots, p, \\ \phi_k(s_k, x(s_k), x(t_k - 0)) & \\ \quad + \int_{s_k}^t f_k(s, x(s), u(s))\,ds, & t \in (s_k, t_{k+1}] \cap [t_0, T], k = 1, \ldots, p, \end{cases}$$

and

$$x^q(t) = \begin{cases} x_0 + \int_{t_0}^t f_0^q(s, x^q(s), u^q(s))\,ds, & t \in [t_0, t_1] \cap [t_0, T], \\ \phi_k^q(t, x^q(t), x^q(t_k - 0)), & t \in (t_k, s_k] \cap [t_0, T], k = 1, \ldots, p, \\ \phi_k^q(s_k, x^q(s_k), x^q(t_k - 0)) & \\ \quad + \int_{s_k}^t f_k^q(s, x^q(s), u^q(s))\,ds, & t \in (s_k, t_{k+1}] \cap [t_0, T], k = 1, \ldots, p. \end{cases}$$

Proof By using the Gronwall inequality and Theorem 3.1, similar to the arguments of Theorem 3.2 in [37], we can get this result easily. □

The following corollary is a direct consequence of the above lemma.

Corollary 4.1 *Let $f_k \in F_k$ $(k = 0, \ldots, p)$, $\{u^q\} \subset \mathcal{U}[t_0, T]$, and ϕ_k^q be functions meeting hypothesis $(H_{\phi,k})$ $(k = 1, \ldots, p)$. Assume that conditions (H_u), $(H_{w,0})$, and $(H_{w,k})$ $(k = 1, \ldots, p)$*

all hold, if $u^q \to u$ $(q \to +\infty)$ *in* $L^1([t_0,T];\mathbb{R}^n)$ *and* $\phi_k^q \to \phi_k$ $(q \to +\infty,\ k = 1,\dots,p)$ *in* $C(((t_k,s_k] \cap [t_0,T]) \times \mathbb{R}^m \times \mathbb{R}^m;\mathbb{R}^m)$, *then*

$$x^q \to x \quad (q \to +\infty) \text{ in } PC_m[t_0,T].$$

Now, in order to accomplish the objective of our study, we need to do some proper restrictions on the functions g and h in equation (1.5), which are set forth below.

(H_{gh}): $g : \mathbb{R}^m \to \mathbb{R}$ and $h : [t_0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ are both continuous functions.

For simplicity, in the latter part of this section, we suppose that conditions (H_u), $(H_{w,0})$, $(H_{w,k})$, $(H_{\phi,k})$ $(k = 1,\dots,p)$, and (H_{gh}) are all satisfied.

Lemma 4.2 *Let* $f_k^q \in F_k$ *and* $\{u^q\} \subset \mathcal{U}[t_0,T]$. *If* $f_k^q \to f_k$ $(q \to +\infty,\ k = 0,1,\dots,p)$ *and* $u^q \to u$ $(q \to +\infty)$ *in* $L^1([t_0,T];\mathbb{R}^n)$, *then* $J_{f_0^q,f_1^q,\dots,f_p^q}(u^q) \to J_{f_0,f_1,\dots,f_p}(u) q \to +\infty$.

Proof By the definition of J, we know that

$$J_{f_0^q,f_1^q,\dots,f_p^q}(u^q) = g\bigl(x^q(T)\bigr) + \int_{t_0}^{T} h\bigl(t,x^q(t),u^q(t)\bigr)\,dt,$$
$$J_{f_0,f_1,\dots,f_p}(u) = g\bigl(x(T)\bigr) + \int_{t_0}^{T} h\bigl(t,x(t),u(t)\bigr)\,dt,$$

where x^q and x are the solutions of equation (1.6) corresponding to f_k^q, u^q and f, u, respectively.

From the hypothesis of this lemma and Lemma 4.1, we immediately obtain that $x^q \to x$.

Since g and h are continuous, it follows that

$$g\bigl(x^q(T)\bigr) \to g\bigl(x(T)\bigr), \quad q \to +\infty \tag{4.1}$$

and

$$h\bigl(t,x^q(t),u^q(t)\bigr) \to h\bigl(t,x(t),u(t)\bigr), \quad q \to +\infty. \tag{4.2}$$

Next, we will show that the solution of equation (1.6) is bounded, which is independent with regard to f_k and u. The demonstration of the boundedness of x is presented in the following three steps.

(1) For $t \in [t_0,t_1] \cap [t_0,T]$,

$$\begin{aligned}\bigl\|x(t)\bigr\| &= \biggl\|x_0 + \int_{t_0}^{t} f_0\bigl(s,x(s),u(s)\bigr)\,ds\biggr\| \\ &\le \|x_0\| + C_0T;\end{aligned}$$

(2) For $t \in (t_k,s_k] \cap [t_0,T]$, $k = 1,\dots,p$, $\|x(t)\| = \|\phi_k(t,x(t),x(t_k-0))\| \le D_k$;

(3) For $t \in (s_k,t_{k+1}] \cap [t_0,T]$, $k = 1,\dots,p$,

$$\begin{aligned}\bigl\|x(t)\bigr\| &\le \bigl\|\phi_k\bigl(s_k,x(s_k),x(t_k-0)\bigr)\bigr\| + \int_{s_k}^{t} \bigl\|f_k\bigl(s,x(s),u(s)\bigr)\bigr\|\,ds \\ &\le D_k + C_kT,\end{aligned}$$

where C_0, C_k, D_k $(k = 1,\dots,p)$ are constants in conditions $(H_{w,0})$, $(H_{w,k})$, and $(H_{\phi,k})$ $(k = 1,\dots,p)$.

It is followed up by the step (1) to (3) that $\|x\| \le \overline{A} < +\infty$, where $\overline{A} = \max_{k=1,\dots,p}\{\|x_0\| + C_0T, D_k + C_kT\}$.

It is noticeable that $\|u\| \le M$ by virtue of condition (H_u), which implies that $h(t,x(t),u(t))$ is bounded for $t \in [t_0,T]$ in cooperation with the continuity of h. With the aid of the dominated convergence theorem, equation (4.2) leads to the fact that

$$\int_{t_0}^{T} h\bigl(t,x^q(t),u^q(t)\bigr)\,dt \to \int_{t_0}^{T} h\bigl(t,x(t),u(t)\bigr)\,dt. \tag{4.3}$$

Combining (4.1) with (4.3), it holds that

$$J_{f_0^q,f_1^q,\dots,f_p^q}\bigl(u^q\bigr) \to J_{f_0,f_1,\dots,f_p}(u), \quad q \to +\infty.$$

Then the proof is finished. □

Then, two corollaries are given in the following as a result of Lemma 4.2.

Corollary 4.2 *Let $f_k \in F_k$ $(k = 0,1,\dots,p)$ and $\{u^q\} \subset \mathcal{U}[t_0,T]$ with $u^q \to u$ $(q \to +\infty)$, then*

$$J_{f_0,f_1,\dots,f_p}\bigl(u^q\bigr) \to J_{f_0,f_1,\dots,f_p}(u), \quad q \to +\infty.$$

Corollary 4.3 *Let $f_k^q \in F_k$ with $f_k^q \to f_k$ $(q \to +\infty, k = 0,1,\dots,p)$ and $u \in \mathcal{U}[t_0,T]$, then*

$$J_{f_0^q,f_1^q,\dots,f_p^q}(u) \to J_{f_0,f_1,\dots,f_p}(u), \quad q \to +\infty.$$

Now, let us demonstrate one of the main results in the paper, that is, the existence of solutions for the optimal control problem **(P)**.

Theorem 4.1 *Suppose that conditions (H_u), $(H_{w,0})$, $(H_{w,k})$, $(H_{\phi,k})$ $(k = 1,\dots,p)$, and (H_{gh}) are all satisfied, problem* (**P**) *has at least one solution, that is, there exists $u^* \in \mathcal{U}[t_0,T]$ satisfying the equation $J_{f_0,f_1,\dots,f_p}(u^*) = \min_{u\in\mathcal{U}[t_0,T]} J_{f_0,f_1,\dots,f_p}(u)$.*

Proof The proof of Lemma 4.2 shows that the solution of equation (1.6) is bounded. Given the continuity of g and h, by condition (H_u), it implies that

$$\begin{aligned} J_{f_0,f_1,\dots,f_p}(u) &\ge -\bigl|J_{f_0,f_1,\dots,f_p}(u)\bigr| \\ &= -\left|g\bigl(x(T)\bigr) + \int_{t_0}^{T} h\bigl(t,x(t),u(t)\bigr)\,dt\right| \\ &\ge -\bigl|g\bigl(x(T)\bigr)\bigr| - \int_{t_0}^{T} \bigl|h\bigl(t,x(t),u(t)\bigr)\bigr|\,dt \\ &> -\infty, \end{aligned}$$

namely $J_{f_0,f_1,\dots,f_p}(u)$ is bounded below.

Take a minimizing sequence $\{u^j\}_{j=1}^{+\infty} \subset \mathcal{U}[t_0, T]$ such that

$$J_{f_0,f_1,\dots,f_p}(u^j) \to \inf_{u\in\mathcal{U}[t_0,T]} J_{f_0,f_1,\dots,f_p}(u), \quad j \to +\infty. \tag{4.4}$$

In the light of the compactness of $\mathcal{U}[t_0, T]$, it has a convergent subsequence $\{u^{j'}\}_{j=1}^{+\infty} \subset \{u^j\}_{j=1}^{+\infty} \subset \mathcal{U}[t_0, T]$ such that

$$u^{j'} \to u^* \in \mathcal{U}[t_0, T], \quad j' \to +\infty.$$

Therefore,

$$J_{f_0,f_1,\dots,f_p}(u^{j'}) \to \inf_{u\in\mathcal{U}[t_0,T]} J_{f_0,f_1,\dots,f_p}(u), \quad j' \to +\infty. \tag{4.5}$$

In accordance with Corollary 4.2, we have that

$$J_{f_0,f_1,\dots,f_p}(u^{j'}) \to J_{f_0,f_1,\dots,f_p}(u^*), \quad j' \to +\infty. \tag{4.6}$$

Combining (4.5) and (4.6), it leads to that

$$J_{f_0,f_1,\dots,f_p}(u^*) = \min_{u\in\mathcal{U}[t_0,T]} J_{f_0,f_1,\dots,f_p}(u).$$

This finishes the proof. □

5 Stability of optimal control problem

In this section, the main task is to discuss the stability of optimal solutions for the optimal control problem **(P)**. We will characterize them by the stability of the set of all solutions for problem **(P)**, which is denoted by $I(f_0, f_1, \dots, f_p)$(see below). In other words, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $H(I(f_0', f_1', \dots, f_p'), I(f_0, f_1, \dots, f_p)) < \varepsilon$ with $\widetilde{\rho}(f', f) < \delta$, where H is the Hausdorff metric induced by the metric defined on $\mathcal{F}$(see below).

Set a metric space

$$\mathcal{F} \triangleq F_0 \times F_1 \times \cdots \times F_p,$$

on which the metric is defined as

$$\widetilde{\rho}(f^1, f^2) = \max_{0\le j\le p} \|f_j^1 - f_j^2\|,$$

where $f^1 = (f_0^1, \dots, f_p^1) \in \mathcal{F}$ and $f^2 = (f_0^2, \dots, f_p^2) \in \mathcal{F}$.

Obviously, $(\mathcal{F}, \widetilde{\rho})$ is a complete metric space.

Consider the set-valued mapping $I : \mathcal{F} \rightrightarrows \mathcal{U}[t_0, T]$,

$$I(f_0, f_1, \dots, f_p) = \{u \in \mathcal{U}[t_0, T] | u \text{ is the solution of problem } (\mathbf{P}) \text{ corresponding to } f = (f_0, f_1, \dots, f_p) \in \mathcal{F}\}.$$

Whereafter, some meaningful conclusions are approached by similar proofs in [30] in the following.

Theorem 5.1 *$I(f_0,f_1,\ldots,f_p) \neq \emptyset$ for each $f = (f_0,f_1,\ldots,f_p) \in \mathcal{F}$.*

Proof The result can be obtained directly by Theorem 4.1. □

Theorem 5.2 *$I : \mathcal{F} \rightrightarrows \mathcal{U}[t_0,T]$ is a usco mapping.*

Proof In view of Lemma 2.3 and the compactness of $\mathcal{U}[t_0,T]$, it is sufficient to show that Graph(I) is closed, where

$$\mathrm{Graph}(I) := \bigl\{(f,u) \in \mathcal{F} \times \mathcal{U}[t_0,T] : u \in I(f_0,f_1,\ldots,f_p)\bigr\}.$$

Let $\{f^q\} \subset \mathcal{F}$ with $f^q \to f = (f_0,f_1,\ldots,f_p) \in \mathcal{F}$ and $\{u^q\} \subset I(f_0,f_1,\ldots,f_p)$ with $u^q \to u^* \in \mathcal{U}[t_0,T]$. In order to prove the closeness of Graph(I), we shall just conclude that $u^* \in I(f_0,f_1,\ldots,f_p)$.

In fact, in consideration of $u^q \in I(f^q)$ for each $q \in \mathbb{N}$, we have

$$J_{f_0^q,f_1^q,\ldots,f_p^q}\bigl(u^q\bigr) \le J_{f_0^q,f_1^q,\ldots,f_p^q}(u), \quad \forall u \in \mathcal{U}[t_0,T],$$

where $f^q = (f_0^q,\ldots,f_p^q) \in \mathcal{F}$.

Since $f^q \to f$ $(q \to +\infty)$ and $u^q \to u^*$ $(q \to +\infty)$, according to Lemma 4.2 and Corollary 4.3, one can get that

$$J_{f_0^q,f_1^q,\ldots,f_p^q}\bigl(u^q\bigr) \to J_{f_0,f_1,\ldots,f_p}\bigl(u^*\bigr), \quad q \to +\infty,$$

and

$$J_{f_0^q,f_1^q,\ldots,f_p^q}(u) \to J_{f_0,f_1,\ldots,f_p}(u), \quad q \to +\infty, \forall u \in \mathcal{U}[t_0,T],$$

respectively. Thus, it has

$$J_{f_0,f_1,\ldots,f_p}\bigl(u^*\bigr) \le J_{f_0,f_1,\ldots,f_p}(u), \quad \forall u \in \mathcal{U}[t_0,T],$$

which indicates that $u^* \in I(f_0,f_1,\ldots,f_p)$. Then the proof is completed. □

Theorem 5.3 *$I : \mathcal{F} \rightrightarrows U$ is lower semi-continuous at $f = (f_0,f_1,\ldots,f_p) \in \mathcal{F}$ if and only if problem* (**P**) *associated with f is essential.*

Proof First of all, we show that the lower semi-continuity of $I : \mathcal{F} \rightrightarrows \mathcal{U}[t_0,T]$ at $f = (f_0,f_1,\ldots,f_p) \in \mathcal{F}$ results in the fact that problem **(P)** associated with f is essential.

Let $u \in I(f_0,f_1,\ldots,f_p)$, $\forall \varepsilon > 0$, then $V(u,\varepsilon) \cap I(f_0,f_1,\ldots,f_p) \neq \emptyset$, where $V(u,\varepsilon)$ is the open neighborhood of u. For the set-valued mapping I is lower semi-continuous at f, there exists $\delta > 0$ such that $V(u,\varepsilon) \cap I(f_0',f_1',\ldots,f_p') \neq \emptyset$ for any $f' \in \mathcal{F}$ subject to $\widetilde{\rho}(f',f) < \delta$. Take $u' \in V(u,\varepsilon) \cap I(f_0',f_1',\ldots,f_p')$, then $u' \in I(f_0',f_1',\ldots,f_p')$ and $\|u - u'\| < \varepsilon$. Hence, u is essential, which implies that problem **(P)** associated with f is essential.

Conversely, if problem **(P)** associated with f is essential, any element of $I(f_0,f_1,\ldots,f_p)$ is essential in accordance with Definition 2.3. For any open set G with $G \cap I(f_0,f_1,\ldots,f_p) \neq \emptyset$, there exists $\varepsilon > 0$ such that the open neighborhood $V(u,\varepsilon)$ of u satisfies $V(u,\varepsilon) \subset G$ for each $u \in G \cap I(f_0,f_1,\dots,f_p)$. Because u is an essential solution, there exists $\delta > 0$ such that, for any $f' \in \mathcal{F}$ satisfying $\widetilde{\rho}(f',f) < \delta$, we have that $\|u - u'\| < \varepsilon$ for some $u' \in I(f_0',f_1',\dots,f_p')$. It shows that $u' \in (V(u,\varepsilon) \cap I(f_0',f_1',\dots,f_p'))$. Consequently, $G \cap I(f_0',f_1',\dots,f_p') \neq \emptyset$ for any $f' \in \mathcal{F}$ with $\widetilde{\rho}(f',f) < \delta$, that is, I is lower semi-continuous at f. Then the proof is ended. □

Remark 5.1 Theorem 5.2 and Theorem 5.3 indicate that if problem **(P)** associated with $f = (f_0,f_1,\dots,f_p) \in \mathcal{F}$ is essential, then the set-valued mapping $I : \mathcal{F} \rightrightarrows \mathcal{U}[t_0,T]$ is continuous at f. What is more, since $I : \mathcal{F} \rightrightarrows \mathcal{U}[t_0,T]$ is compact-valued, then by Theorem 17.15 of [41], the solution set $I(f_0,f_1,\dots,f_p)$ is stable.

Remark 5.2 Considering Lemma 2.4, there exists a dense residual subset $\mathcal{E}$ of $\mathcal{F}$ such that problem **(P)** associated with $f = (f_0,f_1,\dots,f_p) \in \mathcal{E}$ is essential, the solution set $I(f_0,f_1,\dots,f_p)$ is stable for most $f = (f_0,f_1,\dots,f_p) \in \mathcal{F}$. Moreover, it suggests that each optimal control problem associated with $f = (f_0,f_1,\dots,f_p) \in \mathcal{F}$ can be closely approximated arbitrarily by an essential optimal control problem.

In the end, an example is presented for illustrations.

Example 5.1 Let

$$\mathcal{U}[1,6] = \left\{ u^\alpha : u^\alpha(t) = \begin{cases} -\frac{t}{\alpha} + \frac{1}{t}, & t \in [1,2], \\ -\frac{t}{2\alpha} + \frac{1}{t}, & t \in [2,6], \end{cases} \alpha = 18,19,\dots \right\}.$$

Obviously, $\mathcal{U}[1,6]$ satisfies condition (H_u). In fact, $\forall u \in \mathcal{U}[1,6]$, $\forall \epsilon > 0$, there exists $\delta = \min\{1, \frac{\epsilon}{5}\}$ such that when $|h| < \delta$,

$$\begin{aligned}
&\int_1^6 \bigl|u(t+h) - u(t)\bigr|\,dt \\
&= \int_1^{2-h} \bigl|u(t+h) - u(t)\bigr|\,dt + \int_{2-h}^2 \bigl|u(t+h) - u(t)\bigr|\,dt \\
&\quad + \int_2^{6-h} \bigl|u(t+h) - u(t)\bigr|\,dt + \int_{6-h}^6 \bigl|u(t+h) - u(t)\bigr|\,dt \\
&= \int_1^{2-h} \left|-\frac{t+h}{\alpha} + \frac{1}{t+h} + \frac{t}{\alpha} - \frac{1}{t}\right| dt + \int_{2-h}^2 \left|-\frac{t+h}{2\alpha} + \frac{1}{t+h} + \frac{t}{\alpha} - \frac{1}{t}\right| dt \\
&\quad + \int_2^{6-h} \left|-\frac{t+h}{2\alpha} + \frac{1}{t+h} + \frac{t}{2\alpha} - \frac{1}{t}\right| dt + \int_{6-h}^6 \left|\frac{t}{2\alpha} - \frac{1}{t}\right| dt \\
&\le \int_1^{2-h} \frac{h}{\alpha}\,dt + \int_1^{2-h} \left(\frac{1}{t} - \frac{1}{t+h}\right) dt + \int_{2-h}^2 \frac{t-h}{2\alpha}\,dt + \int_{2-h}^2 \left(\frac{1}{t} - \frac{1}{t+h}\right) dt \\
&\quad + \int_2^{6-h} \frac{h}{2\alpha}\,dt + \int_2^{6-h} \left(\frac{1}{t} - \frac{1}{t+h}\right) dt + \int_{6-h}^6 \left(\frac{1}{t} - \frac{t}{2\alpha}\right) dt \\
&\le \int_1^2 \frac{h}{\alpha}\,dt + \int_1^6 \left(\frac{1}{t} - \frac{1}{t+h}\right) dt + \int_{2-h}^2 \frac{2}{2\alpha}\,dt + \int_2^6 \frac{h}{2\alpha}\,dt + \int_{6-h}^6 \left(1 - \frac{1}{2\alpha}\right) dt \\
&= \frac{h}{\alpha} + \ln\left(1 + \frac{5h}{6+h}\right) + \frac{h}{\alpha} + \frac{2h}{\alpha} + \left(1 - \frac{1}{2\alpha}\right) h
\end{aligned}$$

$$\begin{aligned} &\leq \frac{2h}{\alpha} + \frac{2h}{3+h} + h \\ &\leq 4h < \epsilon. \end{aligned}$$

So, it is a nonempty compact subset of $L^1([1,6])$.

Now, consider the following optimal control problems: Looking for $u^* \in \mathcal{U}[1,6]$ satisfying

$$J_{f_0,f_1,f_2}\bigl(u^*\bigr) = \min_{u\in\mathcal{U}} J_{f_0,f_1,f_2}(u),$$

where

$$\begin{aligned} &J_{f_0,f_1,f_2}(u) = g\bigl(x(6)\bigr) + \int_1^6 h\bigl(t,x(t),u(t)\bigr)\,dt, \\ &g(x) = x, \qquad h(t,x,u) = t^2, \end{aligned}$$

and

$$f_0(t,x,u) = u, \qquad f_1(t,x,u) = u + 1, \qquad f_2(t,x,u) = u + 2.$$

In addition, $x(t)$ is the solution of the non-instantaneous impulsive differential equation

$$\begin{cases} \dot{x} = f_k(t,x,u^*), & t \in (s_k, t_{k+1}], k = 0,1,2, \\ x = \sin t + x(t_k - 0), & t \in (t_k, s_k], k = 1,2, \\ x(1) = 0, \end{cases}$$

where $1 = s_0 < t_1 = 2 < s_1 = 3 < t_2 = 4 < s_2 = 5 < t_3 = 6$.

Through simple calculation, $x(t)$ can be written as

$$x(t) = \begin{cases} -\frac{1}{2\alpha}t^2 + \ln t + \frac{1}{2\alpha}, & t \in (1,2], \\ \sin t + c_1, & t \in (2,3], \\ -\frac{1}{4\alpha}t^2 + \ln t + t + c_2, & t \in (3,4], \\ \sin t + c_3, & t \in (4,5], \\ -\frac{1}{4\alpha}t^2 + \ln t + 2t + c_4, & t \in (5,6], \end{cases}$$

where

$$\begin{aligned} &c_1 = -\frac{3}{2\alpha} + \ln 2, \qquad c_2 = \frac{3}{4\alpha} + \ln\frac{2}{3} + \sin 3 - 3, \\ &c_3 = -\frac{13}{4\alpha} + \ln\frac{8}{3} + 1 + \sin 3, \qquad c_4 = \frac{3}{\alpha} + \ln\frac{8}{15} + \sin 3 + \sin 5 - 9. \end{aligned}$$

Then

$$J_{f_0,f_1,f_2}(u) = -\frac{6}{\alpha} + \ln\frac{16}{5} + \sin 3 + \sin 5 + 3 + \frac{215}{3}.$$

It shows that $J_{f_0,f_1,f_2}(u)$ reaches the minimum when $\alpha = 18$, i.e., $I(f_0,f_1,f_2) = \{u^{18}\}$.

Take

$$f_0^m(t,x,u) = u + \frac{1}{m}, \qquad f_1^m(t,x,u) = u + 1 + \frac{1}{m},$$
$$f_2^m(t,x,u) = u + 2 + \frac{1}{m}, \quad m = 1,2,\ldots;$$

therefore, the solution $x^m(t)$ of the above differential equation with non-instantaneous impulse corresponding to $f^m = (f_0^m, f_1^m, f_2^m)$ can be represented as

$$x^m(t) = \begin{cases} -\frac{1}{2\alpha}t^2 + \ln t + \frac{1}{m}t + \frac{1}{2\alpha} - \frac{1}{m}, & t \in (1,2], \\ \sin t + c_1', & t \in (2,3], \\ -\frac{1}{4\alpha}t^2 + \ln t + \frac{1}{m}t + t + c_2', & t \in (3,4], \\ \sin t + c_3', & t \in (4,5], \\ -\frac{1}{4\alpha}t^2 + \ln t + \frac{1}{m}t + 2t + c_4', & t \in (5,6], \end{cases}$$

where

$$c_1' = -\frac{3}{2\alpha} + \ln 2 + \frac{1}{m}, \qquad c_2' = \frac{3}{4\alpha} + \ln\frac{2}{3} + \sin 3 - 3 - \frac{2}{m},$$
$$c_3' = -\frac{13}{4\alpha} + \ln\frac{8}{3} + 1 + \sin 3 + \frac{2}{m}, \qquad c_4' = \frac{3}{\alpha} + \ln\frac{8}{15} + \sin 3 + \sin 5 - 9 - \frac{3}{m}.$$

Then

$$J_{f_0^m f_1^m f_2^m}(u) = -\frac{6}{\alpha} + \ln\frac{16}{5} + \sin 3 + \sin 5 + 3 + \frac{215}{3} + \frac{3}{m}.$$

Similarly, $J_{f_0,f_1,f_2}(u)$ attains the minimum when $\alpha = 18$, i.e., $I(f_0^m, f_1^m, f_2^m) = \{u^{18}\}$.

From the previous discussion, we can see that

$$\widetilde{\rho}\big(f^m, f\big) = \max_{0\le j\le 2}\big\|f_j^m - f_j\big\| = \frac{1}{m} \to 0, \quad m \to +\infty$$

and for any $m > 0$,

$$\big\|(u^m)^* - u^*\big\| = \big\|u^{18} - u^{18}\big\| = 0.$$

Consequently, u^{18} is essential. What is more, $H(I(f_0^m, f_1^m, f_2^m), I(f_0, f_1, \ldots, f_p)) = H(u^{18}, u^{18}) = 0$, so the optimal controller u^{18} is stable.

6 Conclusions

In this paper, we obtain some important conclusions about the existence and stability of solutions for the optimal control problem described by nonlinear non-instantaneous impulsive ordinary differential equations. It is concluded that most of the optimal control problems are stable in the complete metric space $\mathcal{F}$, that is, the optimal solution set $I(f_0, f_1, \ldots, f_p)$ will not perturb largely when some disturbances occur in the function $f = (f_0, f_1, \ldots, f_p) \in \mathcal{F}$.

In the future, we will deal with the optimal control problem with nonlinear differential equations with non-instantaneous impulses of fractional order.

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