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An extension of Darbo's fixed point theorem for a class of system of nonlinear integral equations

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Abstract
We introduce an extension of Darbo's fixed point theorem via a measure of noncompactness in a Banach space. By using our extension we study the existence of a solution for a system of nonlinear integral equations, which is an extended result of (Aghajani and Haghighi in Novi Sad J. Math. 44(1):59–73, 2014). We give an example to show the specified existence results.

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1 Introduction and preliminaries

The degree of noncompactness of a set is measured by means of functions called measures of noncompactness. A quantitative characteristic $\alpha(A)$ measuring the degree of noncompactness of a subset A in a metric space was first considered by Kuratowski [25] in 1930 in connection with problems of general topology. In fixed point theory, one of the most important results is due to Darbo [16], who used this measure to generalize both the classical Schauder fixed point principle and (a special variant of) Banach's contraction mapping principle for so-called condensing operators. A condensing operator is a mapping under which the image of any set is in a certain sense more compact than the set itself. Indeed, the condensing operators have properties similar to those of compact ones. There are some other definitions of measures of noncompactness the authors tried to introduce in an axiomatic way. First, it appeared in the paper of Sadovskii [33], but his axiomatics seems to be too general. In 1980, Banas [9] introduced another axiomatic measure of noncompactness, which was very useful in applications. With the establishment of these comprehensive axiomatics, measures of noncompactness are widely applied in fixed point theory and are especially useful in investigations connected with differential equations, integral equations, functional integral equations, operator equations in Banach spaces [21], fractional differential equations, fractional integral equations, and integro-differential equations [15]. Up to now, many authors have presented results on the existence of solutions for the mentioned equations with their applications by using measures of noncompactness and other techniques [1–10, 12–14, 17–20, 22–32, 34].

Here we use the methodology of MNC to enlarge the Darbo fixed point theorem [16]. Our goal is extending the results of [3] from two dimensions to three dimensions and the results of [30] on the existence of three-dimensional fixed points and tripled fixed points for a class of operators in a Banach space.

Throughout this study, we use:

- $\digamma$: a real Banach space;
- $B(z,\sigma)$: the closed ball with center z and radius σ;
- $\operatorname{con} Y$: the convex hull of a set Y;
- $\operatorname{co} \bar{Z}$: the closed convex hull of a set Z;
- $M_{\digamma}$: the set of all bounded subsets of $\digamma$;
- $N_{\digamma}$: the set of all relatively compact subsets of $\digamma$.

Definition 1.1 ([10]) . A function $\mu : M_{\digamma} \to [0,+\infty)$ is said to be MNC in $\digamma$ if it satisfies the following conditions:

(A_1) The family $\ker \mu = \{Z \in M_{\digamma} : \mu(Z) = 0\} \neq \emptyset$, and $\ker \mu \subseteq N_{\digamma}$.

(A_2) If $Z \subseteq Y$, then $\mu(Z) \leq \mu(Y)$.

(A_3) $\mu(\bar{Z}) = \mu(Z)$.

(A_4) $\mu(\operatorname{Conv} Z) = \mu(Z)$.

(A_5) $\mu(\lambda_1 Z + (1-\lambda_1)Y) \leq \lambda_1 \mu(Z) + (1-\lambda_1)\mu(Y)$ for $\lambda_1 \in [0,1]$.

(A_6) If $\digamma_r \in M_{\digamma}$ is such that $Z_{r+1} \subset Z_r$ for $r = 1,2,\dots$ and $\lim_{r\to+\infty} \mu(Z_r) = 0$, then $Z_\infty = \bigcap_{r=1}^{+\infty} Z_r \neq \emptyset$.

Theorem 1.1 (Schauder [2]) *Let Λ be a nonempty bounded closed convex subset of $\digamma$. Then every compact mapping $F : \Lambda \to \Lambda$ has at least one fixed point.*

Theorem 1.2 (Darbo [9]) *Let $F : \Lambda \to \Lambda$ be a continuous mapping, and let Λ be a bounded closed convex subset of $\digamma$. Suppose that there exists a constant $K \in [0,1)$ such that $\mu(F(Z)) \leq K\mu(Z)$ for any $Z \subseteq \Lambda$. Then F has a fixed point.*

Definition 1.2 ([11]) A point (x,y,z) is called a tripled fixed point of a mapping $F : Z^3 \to Z$ if

$$F(x,y,z) = x, \qquad F(y,x,z) = y, \qquad F(z,y,x) = z.$$

Theorem 1.3 ([10]) *Let $\mu_1, \mu_2, \dots, \mu_r$ be MNCs of $\digamma_1, \digamma_2, \dots, \digamma_r$, respectively. Moreover, assume that $B : \mathbb{R}_+^r \to \mathbb{R}_+$ is convex and that $B(x_1,x_2,\dots,x_r) = 0$ iff $x_j = 0$ for $j = 1,2,\dots,r$. Then*

$$\hat{\mu}(Z) = B\big(\mu_1(Z_1), \mu_2(Z_2), \dots, \mu_r(Z_r)\big)$$

defines an MNC in $\digamma_1 \times \digamma_2 \times \cdots \times \digamma_r$. Here Z_j denote the natural projections of Z into $\digamma_j$ for $j = 1,2,\dots,r$.

Example 1.1 ([1]) Let μ_1, μ_2, μ_3 be MNCs in $\digamma_1, \digamma_2, \digamma_3$, respectively. Moreover, suppose that $B : \mathbb{R}_+^3 \to \mathbb{R}_+$ is convex and that $B(x_1, x_2, x_3) = 0$ iff $x_j = 0$ for $j = 1, 2, 3$ Then

$$\hat{\mu}(Z) = B\big(\mu_1(Z_1), \mu_2(Z_2), \mu_3(Z_3)\big)$$

defines an MNC in $\digamma_1 \times \digamma_2 \times \digamma_3$. Here Z_j denote the natural projections of Z into $\digamma_j$ for $j = 1, 2, 3$.

Example 1.2 ([1]) Let μ be an MNC in $\digamma$, and let $B(x, y, z) = \max\{x, y, z\}$ for $(x, y, z) \in \mathbb{R}_+^3$. Then B is convex, and if $B(x, y, z) = \max\{x, y, z\} = 0$ iff $x = y = z = 0$, then clearly all the conditions of Theorem 1.3 are satisfied. Therefore $\hat{\mu}(Z) = \max(\mu_1(Z_1), \mu_2(Z_2), \mu_3(Z_3))$ is an MNC on $\digamma \times \digamma \times \digamma$. Here Z_j denote the natural projections of Z into $\digamma_j$ for $j = 1, 2, 3$.

Example 1.3 ([1]) Let μ be an MNC in $\digamma$, and let $B(x, y, z) = x + y + z$ for $(x, y, z) \in \mathbb{R}_+^3$. Then B is convex, and if $B(x, y, z) = x + y + z = 0$ iff $x = y = z = 0$, the clearly all the conditions of Theorem 1.3 are satisfied. Therefore $\hat{\mu}(Z) = \mu_1(Z_1) + \mu_2(Z_2) + \mu_3(Z_3))$ is an MNC on $\digamma \times \digamma \times \digamma$. Here Z_j denote the natural projections of Z into $\digamma_j$ for $j = 1, 2, 3$.

Lemma 1.4 (Aghajani [2]) *Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing upper semicontinuous function. Then the following two conditions are equivalent*:

(i) $\lim_{n\to+\infty} \theta^n(\zeta) = 0$, $\zeta > 0$.
(ii) $\theta(\zeta) < \zeta$, $\zeta > 0$.

2 Main results

First, we denote by $\hat{\varphi}$ the class of functions $\tilde{\varphi} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties:

(i) $\tilde{\varphi}$ is a nondecreasing continuous function on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$.
(ii) $\lim_{r\to+\infty} \hat{\theta}^r(\zeta) = 0$, $\zeta > 0$, where $\hat{\theta}(\zeta) = \tilde{\varphi}(\zeta, \zeta, \zeta)$.
(iii) $\frac{1}{3}(\tilde{\varphi}(\zeta_1, \nu_1, \varphi_1) + \tilde{\varphi}(\zeta_2, \nu_2, \varphi_2) + \tilde{\varphi}(\zeta_3, \nu_3, \varphi_3)) \le \tilde{\varphi}(\frac{\zeta_1+\zeta_2+\zeta_3}{3}, \frac{\nu_1+\nu_2+\nu_3}{3}, \frac{\varphi_1+\varphi_2+\varphi_3}{3})$

for all $\zeta_1, \nu_1, \varphi_1, \zeta_2, \nu_2, \varphi_2, \zeta_3, \nu_3, \varphi_3 \in \mathbb{R}_+$.

Remark 2.1 If $\tilde{\varphi}(\zeta, \zeta, \zeta)$ is nondecreasing and continuous, then $\hat{\theta}(\zeta)$ is also nondecreasing and continuous. Now by Lemma 1.4 the following two statements are equivalent:

(i) $\lim_{r\to+\infty} \hat{\theta}^r(\zeta) = 0$, $\zeta > 0$.
(ii) $\hat{\theta}(\zeta) < \zeta$, $\zeta > 0$.

Thus $\tilde{\varphi}(\zeta, \zeta, \zeta) < \zeta$, $\zeta > 0$.

For example, the functions $\tilde{\varphi}(\zeta, \nu, \varphi) = \ln(1 + \frac{\zeta+\nu+\varphi}{3})$ and $\tilde{\varphi}(\zeta, \nu, \varphi) = U_1\zeta + U_2\nu + U_3\varphi$, where $U_1, U_2, U_3 \in \mathbb{R}^+$ and $U_1 + U_2 + U_3 < 1$, belong to $\hat{\varphi}$.

Theorem 2.2 *Let Λ be a nonempty bounded closed convex subset of $\digamma$, and let $F : \Lambda \times \Lambda \times \Lambda \to \Lambda \times \Lambda \times \Lambda$ be a continuous function satisfying*

$$\hat{\mu}\big(F(Z)\big) \le \tilde{\varphi}\big(\hat{\mu}(Z), \hat{\mu}(Z), \hat{\mu}(Z)\big)$$

for any subset of Z of $\Lambda \times \Lambda \times \Lambda$, where $\hat{\mu}(Z)$ is defined in Example 1.1, and $\tilde{\varphi} \in \hat{\varphi}$. Then F has at least one fixed point in $\Lambda \times \Lambda \times \Lambda$.

Proof Define the sequence $\{\Lambda_r \times \Lambda_r \times \Lambda_r\}_{r=1}^{+\infty}$ by induction: $\Lambda_0 \times \Lambda_0 \times \Lambda_0 = \Lambda \times \Lambda \times \Lambda$, and $\Lambda_r \times \Lambda_r \times \Lambda_r = \mathrm{ConvF}(\Lambda_{r-1} \times \Lambda_{r-1} \times \Lambda_{r-1})$ for $r = 1, 2, 3, \dots$.

We have $F(\Lambda_0 \times \Lambda_0 \times \Lambda_0) = F(\Lambda \times \Lambda \times \Lambda) \subseteq \Lambda \times \Lambda \times \Lambda = \Lambda_0 \times \Lambda_0 \times \Lambda_0$, that is, $\Lambda_1 \times \Lambda_1 \times \Lambda_1 \subseteq \Lambda_0 \times \Lambda_0 \times \Lambda_0$.

Continuing this way, we can show that

$$\cdots \subseteq \Lambda_r \times \Lambda_r \times \Lambda_r \subseteq \cdots \subseteq \Lambda_1 \times \Lambda_1 \times \Lambda_1 \subseteq \Lambda_0 \times \Lambda_0 \times \Lambda_0.$$

If there exists an integer $P > 0$ such that $\hat{\mu}(\Lambda_P \times \Lambda_P \times \Lambda_P) = 0$, then $\Lambda_P \times \Lambda_P \times \Lambda_P$ is relatively compact, and since

$$F(\Lambda_P \times \Lambda_P \times \Lambda_P) \subseteq \operatorname{Conv} F(\Lambda_P \times \Lambda_P \times \Lambda_P) = \Lambda_{P+1} \times \Lambda_{P+1} \times \Lambda_{P+1} \subseteq \Lambda_P \times \Lambda_P \times \Lambda_P,$$

by Theorem 1.1 F has a fixed point. Thus $\hat{\mu}(\Lambda_r \times \Lambda_r \times \Lambda_r) > 0$ for all $r \geq 0$. We obtain

$$\begin{aligned}
&\hat{\mu}(\Lambda_{r+1} \times \Lambda_{r+1} \times \Lambda_{r+1}) \\
&\quad = \hat{\mu}(\operatorname{Conv} F(\Lambda_r \times \Lambda_r \times \Lambda_r) \\
&\quad = \hat{\mu}\big(F(\Lambda_r \times \Lambda_r \times \Lambda_r)\big) \\
&\quad \leq \tilde{\varphi}\big(\hat{\mu}(\Lambda_r \times \Lambda_r \times \Lambda_r), \hat{\mu}(\Lambda_r \times \Lambda_r \times \Lambda_r), \hat{\mu}(\Lambda_r \times \Lambda_r \times \Lambda_r)\big) \\
&\quad = \hat{\theta}\big(\hat{\mu}(\Lambda_r \times \Lambda_r \times \Lambda_r)\big) \\
&\quad = \hat{\theta}\big(\hat{\mu}\big(\operatorname{Conv} F(\Lambda_{r-1} \times \Lambda_{r-1} \times \Lambda_{r-1})\big)\big) \\
&\quad = \hat{\theta}\big(\hat{\mu}\big(F(\Lambda_{r-1} \times \Lambda_{r-1} \times \Lambda_{r-1})\big)\big) \\
&\quad \leq \hat{\theta}\big(\tilde{\varphi}\big(\hat{\mu}(\Lambda_{r-1} \times \Lambda_{r-1} \times \Lambda_{r-1}), \hat{\mu}(\Lambda_{r-1} \times \Lambda_{r-1} \times \Lambda_{r-1}), \hat{\mu}(\Lambda_{r-1} \times \Lambda_{r-1} \times \Lambda_{r-1})\big)\big) \\
&\quad = \hat{\theta}^2\big(\hat{\mu}(\Lambda_{r-1} \times \Lambda_{r-1} \times \Lambda_{r-1})\big) \\
&\quad \leq \cdots \leq \hat{\theta}^r\big(\hat{\mu}(\Lambda_1 \times \Lambda_1 \times \Lambda_1)\big).
\end{aligned}$$

Therefore $\hat{\mu}(\Lambda_{r+1} \times \Lambda_{r+1} \times \Lambda_{r+1}) \to 0$ as $r \to +\infty$.

Since $\Lambda_{r+1} \times \Lambda_{r+1} \times \Lambda_{r+1} \subseteq \Lambda_r \times \Lambda_r \times \Lambda_r$ for $r = 0, 1, 2, \dots$, in view of (A_6), the set $\Lambda_\infty \times \Lambda_\infty \times \Lambda_\infty = \bigcap_{r=1}^{+\infty} \Lambda_r \times \Lambda_r \times \Lambda_r$ is a closed convex subset of $\Lambda \times \Lambda \times \Lambda$ invariant under the operator F and belongs to $\ker \hat{\mu}$, that is, F maps $\Lambda_\infty \times \Lambda_\infty \times \Lambda_\infty$ into itself and thus by Theorem 1.1 F has at least one fixed point in $\Lambda_\infty \times \Lambda_\infty \times \Lambda_\infty$ and thus in $\Lambda \times \Lambda \times \Lambda$. □

Theorem 2.3 *Let Λ be a nonempty bounded closed convex subset of $\digamma$, let $\hat{\mu}$ be an arbitrary MNC, and let $F : \Lambda \times \Lambda \times \Lambda \to \Lambda$ be a continuous function satisfying*

$$\mu\big(F(Z_1 \times Z_2 \times Z_3)\big) \leq \tilde{\varphi}\big(\mu(Z_1), \mu(Z_2), \mu(Z_3)\big)$$

for all $Z_1, Z_2, Z_3 \subseteq \Lambda$, where $\tilde{\varphi} \in \hat{\varphi}$. Then F has a tripled fixed point.

Proof First, $\hat{\mu}(Z) = \mu(Z_1) + \mu(Z_2) + \mu(Z_3)$ is an MNC in $\digamma \times \digamma \times \digamma$, where Z_1, Z_2, and Z_3 denote the natural projections of $Z \subseteq \Lambda \times \Lambda \times \Lambda$ into $\digamma$. Let $\hat{F} : \Lambda \times \Lambda \times \Lambda \to \Lambda \times \Lambda \times \Lambda$

be the mapping defined as $\hat{F}(x,y,z) = (F(x,y,z),F(y,x,z),F(z,y,x)$ for $(x,y,z) \in \Lambda \times \Lambda \times \Lambda$. Since F is continuous, $\hat{F}$ is also continuous. Then by Theorem 2.2 we have

$$\begin{aligned}
&\hat{\mu}\big(\hat{F}(Z)\big) \\
&\quad \leq \hat{\mu}\big(F(Z_1 \times Z_2 \times Z_3), F(Z_2 \times Z_1 \times Z_3), F(Z_3 \times Z_2 \times Z_1)\big) \\
&\quad = \mu(F(Z_1 \times Z_2 \times Z_3) + \mu(F(Z_2 \times Z_1 \times Z_3) + \mu(F(Z_3 \times Z_2 \times Z_1) \\
&\quad \leq \tilde{\varphi}\big(\mu(Z_1),\mu(Z_2),\mu(Z_3)\big) + \tilde{\varphi}\big(\mu(Z_2),\mu(Z_1),\mu(Z_3)\big) + \tilde{\varphi}\big(\mu(Z_3),\mu(Z_2),\mu(Z_1)\big) \\
&\quad \leq 3\tilde{\varphi}\bigg(\frac{\mu(Z_1)+\mu(Z_2)+\mu(Z_3)}{3}, \frac{\mu(Z_2)+\mu(Z_1)+\mu(Z_3)}{3}, \frac{\mu(Z_3)+\mu(Z_2)+\mu(Z_1)}{3}\bigg) \\
&\quad \leq 3\tilde{\varphi}\bigg(\frac{\hat{\mu}(Z)}{3}, \frac{\hat{\mu}(Z)}{3}, \frac{\hat{\mu}(Z)}{3}\bigg).
\end{aligned}$$

Hence

$$\frac{1}{3}\hat{\mu}\big(\hat{F}(Z)\big) \leq \tilde{\varphi}\bigg(\frac{\hat{\mu}(Z)}{3}, \frac{\hat{\mu}(Z)}{3}, \frac{\hat{\mu}(Z)}{3}\bigg).$$

Putting $\hat{\mu}_1 = \frac{1}{3}\hat{\mu}$, we have

$$\hat{\mu}\big(\hat{F}(Z)\big) \leq \tilde{\varphi}\big(\hat{\mu}(Z), \hat{\mu}(Z), \hat{\mu}(Z)\big).$$

Also, $\hat{\mu}_1$ is an MNC. So by Theorem 2.2 F has a tripled fixed point. □

Corollary 2.4 *Let Λ be a nonempty bounded closed convex subset of $\digamma$, let μ be an arbitrary MNC, and let $F : \Lambda \times \Lambda \times \Lambda \to \Lambda$ be a continuous mapping such that for some nonnegative constants U_1, U_2, U_3 with $U_1 + U_2 + U_3 < 1$,*

$$\mu(F(Z_1 \times Z_2 \times Z_3) \leq U_1\mu(Z_1) + U_2\mu(Z_2) + U_3\mu(Z_3)$$

for all $Z_1, Z_2, Z_3 \subseteq \Lambda$. Then F has a tripled fixed point.

Proof Setting $\tilde{\varphi}(\zeta,\nu,\varphi) = U_1\zeta + U_2\nu + U_3\varphi$ in Theorem 2.3, we obtain the result. □

Corollary 2.5 *Let Λ be a nonempty closed bounded convex subset of $\digamma$, let μ be an arbitrary MNC, and let $F : \Lambda \times \Lambda \times \Lambda \to \Lambda$ be a continuous mapping such that*

$$\mu(F(Z_1 \times Z_2 \times Z_3) \leq \ln\bigg(1 + \frac{\mu(Z_1)+\mu(Z_2)+\mu(Z_3)}{3}\bigg)$$

for all $Z_1, Z_2, Z_3 \subseteq \Lambda$. Then F has at least one tripled fixed point.

Proof Putting $\tilde{\varphi}(\zeta,\nu,\varphi) = \ln(1 + \frac{\zeta+\nu+\varphi}{3})$ in Theorem 2.3, we obtain the result. □

3 Applications

Recall that $\digamma = BC(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$ is the Banach space of all real-valued continuous bounded functions defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ with the standard norm

$$\|x\| = \sup\big\{\big|x(\zeta,\nu,\varphi)\big| : \zeta,\nu,\varphi \geq 0\big\}.$$

Let Z be a fixed nonempty bounded subset of $\digamma$ and fix $\tilde{\epsilon} > 0$, $G > 0$, and $x \in Z$. The modulus of continuity of x on $[0, G]$ is defined as

$$\begin{aligned}\omega^G(x, \tilde{\epsilon}) = \sup\bigl\{\bigl|x(\zeta, \nu, \varphi) - x(\bar{\zeta}, \bar{\nu}, \bar{\varphi}\bigr| : \zeta, \nu, \varphi, \bar{\zeta}, \bar{\nu}, \bar{\varphi} \in [0, G], |\zeta - \bar{\zeta}| \le \epsilon,\\ |\nu - \bar{\nu}| \le \tilde{\epsilon}, |\varphi - \bar{\varphi}| \le \tilde{\epsilon}\bigr\}.\end{aligned}$$

Further, let

$$\begin{aligned}&\omega^G(Z, \tilde{\epsilon}) = \sup\bigl\{\omega^G(x, \tilde{\epsilon}) : x \in Z\bigr\},\\ &\omega_0^G(Z) = \lim_{\tilde{\epsilon}\to 0} \omega^G(Z, \tilde{\epsilon}),\end{aligned}$$

and

$$\omega_0(Z) = \lim_{G\to+\infty} \omega_0^G(Z).$$

Besides, for three fixed numbers $\zeta, \nu, \varphi \in \mathbb{R}^+$, we define rhe function $\ddot{\mu}$ on the family $M_\digamma$ as

$$\mu(Z) = \omega_0(Z) + \lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam} Z(\zeta, \nu, \varphi),$$

where $Z(\zeta, \nu, \varphi) = \{z(\zeta, \nu, \varphi) : \zeta, \nu, \varphi \in \mathbb{R}^+\}$, and

$$\operatorname{diam} Z(\zeta, \nu, \varphi) = \sup\bigl\{\bigl|x(\zeta, \nu, \varphi) - y(\zeta, \nu, \varphi)\bigr| : x, y \in Z\bigr\}.$$

Finally, we are going to prove the existence results for the system

$$\begin{aligned}
&x(\zeta, \nu, \varphi)\\
&\quad= g(\zeta, \nu, \varphi, x(\zeta, \nu, \varphi), y(\zeta, \nu, \varphi), z(\zeta, \nu, \varphi),\\
&\qquad \int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\bigl(\zeta, \nu, \varphi, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\bigr),\\
&y(\zeta, \nu, \varphi)\\
&\quad= g(\zeta, \nu, \varphi, y(\zeta, \nu, \varphi), x(\zeta, \nu, \varphi), z(\zeta, \nu, \varphi),\\
&\qquad \int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\bigl(\zeta, \nu, \varphi, \xi, \rho, \psi, y(\xi, \rho, \psi), x(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\bigr),\\
&z(\zeta, \nu, \varphi)\\
&\quad= g(\zeta, \nu, \varphi, z(\zeta, \nu, \varphi), y(\zeta, \nu, \varphi), x(\zeta, \nu, \varphi),\\
&\qquad \int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\bigl(\zeta, \nu, \varphi, \xi, \rho, \psi, z(\xi, \rho, \psi), y(\xi, \rho, \psi), x(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\bigr),
\end{aligned} \tag{1}$$

where $\zeta, \nu, \varphi \in \mathbb{R}^+$.

Consider the following assumptions:

(i) $d, e, k : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous.

(ii) $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist a function $\varphi_3 \in \hat{\varphi}$ and a nondecreasing continuous function $\varphi_4 : \mathbb{R}^+ \to \mathbb{R}$ with $\varphi_4(0) = 0$ such that

$$\left|g(\zeta,\nu,\varphi,x,y,z,w) - g(\zeta,\nu,\varphi,\tilde{x},\tilde{y},\tilde{z},\tilde{w})\right| \leq \varphi_3\big(|x-\tilde{x}|,|y-\tilde{y}|,|z-\tilde{z}|\big) + \varphi_4\big(|w-\tilde{w}|\big)$$

for all $\zeta,\nu,\varphi \geq 0$ and $x,y,z,\tilde{x},\tilde{y},\tilde{z} \in \mathbb{R}$.

(iii) $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\begin{aligned} Q = \sup\bigg\{\bigg|&\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),\\ &z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\bigg|\\ &: \zeta,\nu,\varphi,\xi,\rho,\psi \in \mathbb{R}^+, x,y,z \in \mathcal{F}\bigg\} \end{aligned}$$

is finite. Further,

$$\begin{aligned} \lim_{\zeta,\nu,\varphi\to+\infty} &\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} \big|h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\big)\\ &- h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,\tilde{x}(\xi,\rho,\psi),\tilde{y}(\xi,\rho,\psi),\tilde{z}(\xi,\rho,\psi)\big)\big|\,d\xi\,d\rho\,d\psi = 0 \end{aligned}$$

for all $x,y,z,\tilde{x},\tilde{y},\tilde{z} \in \mathcal{F}$.

(iv) $\hat{Q} = \sup\{|g(\zeta,\nu,\varphi,0,0,0,0)| : \zeta,\nu,\varphi \in \mathbb{R}^+\} < +\infty$.

(v) There exists a positive solution σ of the inequality

$$\hat{Q} + \varphi_3(\hat{r},\hat{r},\hat{r}) + \varphi_4(Q) \leq \hat{r}.$$

Theorem 3.1 *Under hypotheses* (i)–(v), *equation* (1) *has at least one solution in the space* $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$.

Proof Define the operator $F : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ by

$$\begin{aligned} &F(x,y,z)(\zeta,\nu,\varphi)\\ &\quad= g\bigg(\zeta,\nu,\varphi,x(\zeta,\nu,\varphi),y(\zeta,\nu,\varphi),z(\zeta,\nu,\varphi),\\ &\qquad\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\,d\xi\,d\rho\,d\psi\bigg). \end{aligned}$$

Clearly, $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$ is a Banach space with sup norm

$$\big\|(x,y,z)\big\|_{\mathcal{F}\times\mathcal{F}\times\mathcal{F}}\| = \|x\|_{\mathcal{F}} + \|y\|_{\mathcal{F}} + \|z\|_{\mathcal{F}},$$

where $\|x\|_{\mathcal{F}} = \sup\{|x(\zeta,\nu,\varphi)| : \zeta,\nu,\varphi \geq 0\}$, $\|y\|_{\mathcal{F}} = \sup\{|y(\zeta,\nu,\varphi)| : \zeta,\nu,\varphi \geq 0\}$, and $\|z\|_{\mathcal{F}} = \sup\{|z(\zeta,\nu,\varphi)| : \zeta,\nu,\varphi \geq 0\}$ for $x,y,z \in \mathcal{F}$. Then the operator $F(x,y,z)(\zeta,\nu,\varphi)$ is continuous at any $(x,y,z) \in \mathcal{F}$. Let $\hat{B}_\sigma = \{x \in \mathcal{F} : \|x\|_{\mathcal{F}} \le \sigma\}$. Now we have

$$\begin{aligned}
&\big|(F(x,y,z)(\zeta,\nu,\varphi)\big| \\
&\quad = \biggl(\biggl|g(\zeta,\nu,\varphi,x(\zeta,\nu,\varphi),y(\zeta,\nu,\varphi),z(\zeta,\nu,\varphi), \\
&\qquad \int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi), \\
&\qquad y(\xi,\rho,\psi),z(\xi,\rho,\psi)\,d\xi\,d\rho\,d\psi\bigr)\biggr|\biggr) \\
&\quad \le \biggl(\biggl|g(\zeta,\nu,\varphi,x(\zeta,\nu,\varphi),y(\zeta,\nu,\varphi),z(\zeta,\nu,\varphi), \\
&\qquad \int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi), \\
&\qquad y(\xi,\rho,\psi),z(\xi,\rho,\psi)\,d\xi\,d\rho\,d\psi\bigr) - g(\zeta,\nu,\varphi,0,0,0,0)\biggr|\biggr) + \bigl|g(\zeta,\nu,\varphi,0,0,0,0)\bigr| \\
&\quad \le \varphi_3\bigl(\|x\|,\|y\|,\|z\|\bigr) + \varphi_4(Q) + \hat{Q} \\
&\quad \le \sigma.
\end{aligned}$$

Hence $F(\hat{B}_\sigma \times \hat{B}_\sigma \times \hat{B}_\sigma) \subseteq \hat{B}_\sigma$, which means that F is well defined.

Now we prove that F is continuous on $\hat{B}_\sigma \times \hat{B}_\sigma \times \hat{B}_\sigma$. Let $(x,y,z),(\tilde{x},\tilde{y},\tilde{z}) \in \hat{B}_\sigma \times \hat{B}_\sigma \times \hat{B}_\sigma$ and $\epsilon > 0$ with

$$\bigl\|(x,y,z) - (\alpha,\beta,\gamma)\bigr\|_{\mathcal{F}\times\mathcal{F}\times\mathcal{F}} < \frac{\tilde{\epsilon}}{3}.$$

Now we have

$$\begin{aligned}
&\big|(F(x,y,z)(\zeta,\nu,\varphi) - F(\alpha,\beta,\gamma)(\zeta,\nu,\varphi)\big| \\
&\quad = \biggl|g(\zeta,\nu,\varphi,x(\zeta,\nu,\varphi),y(\zeta,\nu,\varphi),z(\zeta,\nu,\varphi), \\
&\qquad \int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi), \\
&\qquad y(\xi,\rho,\psi),z(\xi,\rho,\psi)\,d\xi\,d\rho\,d\psi\bigr) - g(\zeta,\nu,\varphi,\alpha(\zeta,\nu,\varphi),\beta(\zeta,\nu,\varphi),\gamma(\zeta,\nu,\varphi), \\
&\qquad \int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)} h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha(\xi,\rho,\psi),\beta(\xi,\rho,\psi),\gamma(\xi,\rho,\psi)\,d\xi\,d\rho\,d\psi\bigr)\biggr| \\
&\quad \le \varphi_3\bigl(|x-\alpha|,|y-\beta|,|z-\gamma|\bigr) + \varphi_4\biggl(\biggl|\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)}\bigl\{h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi), \\
&\qquad y(\xi,\rho,\psi),z(\xi,\rho,\psi)\bigr) - h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha(\xi,\rho,\psi), \\
&\qquad \beta(\xi,\rho,\psi),\gamma(\xi,\rho,\psi)\bigr)\bigr\}\,d\xi\,d\rho\,d\psi\biggr|\biggr) \\
&\quad \le \varphi_3\bigl(\|x-\alpha\|,\|y-\beta\|,\|z-\gamma\|\bigr) \\
&\qquad + \varphi_4\biggl(\biggl|\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)}\bigl\{h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),
\end{aligned}$$

$$y(\xi,\rho,\psi),z(\xi,\rho,\psi)\big) - h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha(\xi,\rho,\psi),$$
$$\beta(\xi,\rho,\psi),\gamma(\xi,\rho,\psi)\big)\big\}\,d\xi\,d\rho\,d\psi\bigg|\bigg).$$

From (ii) and (iii) it follows that there exists $G > 0$ such that for $\zeta,\nu,\varphi > G$,

$$\varphi_4\bigg(\int_0^{d(\zeta)}\int_0^{e(\nu)}\int_0^{k(\varphi)}\big|h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\big)$$
$$- h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha(\xi,\rho,\psi),\beta(\xi,\rho,\psi),\gamma(\xi,\rho,\psi)\big)\big|\,d\xi\,d\rho\,d\psi\bigg) \le \frac{\tilde{\epsilon}}{2}$$

for all $x,y,z,\alpha,\beta,\gamma \in F$.

Consider two cases.

Case 1: If $\zeta,\nu,\varphi > G$, then

$$\big|F(x,y,z)(\zeta,\nu,\varphi) - F(\alpha,\beta,\gamma)(\zeta,\nu,\varphi)\big| \le \varphi_3\bigg(\frac{\tilde{\epsilon}}{3},\frac{\tilde{\epsilon}}{3},\frac{\tilde{\epsilon}}{3}\bigg) + \frac{\tilde{\epsilon}}{2} < \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2} = \tilde{\epsilon}.$$

Case 2: If $\zeta,\nu,\varphi \in [0,G]$, then

$$\big|F(x,y,z)(\zeta,\nu,\varphi) - F(\alpha,\beta,\gamma)(\zeta,\nu,\varphi)\big| \le \varphi_3\bigg(\frac{\tilde{\epsilon}}{3},\frac{\tilde{\epsilon}}{3},\frac{\tilde{\epsilon}}{3}\bigg) + \varphi_4(\hat{d}\hat{e}\hat{k}\omega) < \frac{\tilde{\epsilon}}{3} + \varphi_4(\hat{d}\hat{e}\hat{k}\omega),$$

where

$$\omega(\tilde{\epsilon}) = \sup\bigg\{\big|h(\zeta,\nu,\varphi,\xi,\rho,\psi,x,y,z) - h(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha,\beta,\gamma)\big| : \zeta,\nu,\varphi \in [0,G],$$
$$\xi \in [0,\hat{d}], \rho \in [0,\hat{e}],$$
$$\psi \in [0,\hat{k}], x,y,z,\alpha,\beta,\gamma \in [-\sigma,\sigma], \big\|(x,y,z) - (\alpha,\beta,\gamma)\big\|_{F\times F\times F} < \frac{\tilde{\epsilon}}{2}.\bigg\},$$

and

$$\hat{d} = \sup\big\{d(\varphi) : \varphi \in [0,G]\big\},$$
$$\hat{e} = \sup\big\{e(\nu) : \nu \in [0,G]\big\},$$
$$\hat{k} = \sup\big\{k(\zeta) : \zeta \in [0,G]\big\}.$$

From the continuity of h on $[0,G] \times [0,G] \times [0,G] \times [0,\hat{k}] \times [0,\hat{e}] \times [0,\hat{d}] \times [-\sigma,\sigma] \times [-\sigma,\sigma] \times [-\sigma,\sigma]$ we infer that $\omega(\tilde{\epsilon}) \to 0$ as $\tilde{\epsilon} \to 0$, and from the continuity of φ_4 we obtain $\varphi_4(\hat{d}\hat{e}\hat{k}\omega) \to 0$ as $\tilde{\epsilon} \to 0$.

Clearly, F is a continuous mapping from $\hat{B}_\sigma \times \hat{B}_\sigma \times \hat{B}_\sigma$ into $\hat{B}_\sigma$. Fix $G > 0$ and $\tilde{\epsilon} > 0$. Choose $\zeta_1,\zeta_2,,\nu_1,\nu_2,\varphi_1,\varphi_2 \in [0,G]$ such that $|\zeta_1 - \zeta_2| \le \tilde{\epsilon}$, $|\nu_1 - \nu_2| \le \tilde{\epsilon}$, $|\varphi_1 - \varphi_2| \le \tilde{\epsilon}$. Assume that $\zeta_1 \le \zeta_2$, $\nu_1 \le \nu_2$, $\varphi_1 \le \varphi_2$, and $(x,y,z) \in (Z_1 \times Z_2 \times Z_3)$. We get

$$\big|F(x,y,z)(\zeta_2,\nu_2,\varphi_2) - F(x,y,z)(\zeta_1,\nu_1,\varphi_1)\big|$$
$$\le \bigg|g(\zeta_2,\nu_2,\varphi_2,x(\zeta_2,\nu_2,\varphi_2),y(\zeta_2,\nu_2,\varphi_2),z(\zeta_2,\nu_2,\varphi_2),$$
$$\int_0^{d(\varphi_2)}\int_0^{e(\nu_2)}\int_0^{k(\zeta_2)} h\big(\zeta_2,\nu_2,\varphi_2,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\,d\xi\,d\rho\,d\psi\big)$$

$$
\begin{aligned}
&- g(\zeta_2, \nu_2, \varphi_2, x(\zeta_1, \nu_1, \varphi_1), y(\zeta_1, \nu_1, \varphi_1), z(\zeta_1, \nu_1, \varphi_1),\\
&\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_2, \nu_2, \varphi_2, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\big)\Big|\\
&+ \Big| g(\zeta_2, \nu_2, \varphi_2, x(\zeta_1, \nu_1, \varphi_1), y(\zeta_1, \nu_1, \varphi_1), z(\zeta_1, \nu_1, \varphi_1),\\
&\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_2, \nu_2, \varphi_2, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\big)\\
&- g(\zeta_1, \nu_1, \varphi_1, x(\zeta_1, \nu_1, \varphi_1), y(\zeta_1, \nu_1, \varphi_1), z(\zeta_1, \nu_1, \varphi_1),\\
&\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_2, \nu_2, \varphi_2, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\big)\Big|\\
&+ \Big| g(\zeta_1, \nu_1, \varphi_1, x(\zeta_1, \nu_1, \varphi_1), y(\zeta_1, \nu_1, \varphi_1), z(\zeta_1, \nu_1, \varphi_1),\\
&\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_2, \nu_2, \varphi_2, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\big)\\
&- g\Bigg(\zeta_1, \nu_1, \varphi_1, x(\zeta_1, \nu_1, \varphi_1), y(\zeta_1, \nu_1, \varphi_1), z(\zeta_1, \nu_1, \varphi_1),\\
&\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_1, \nu_1, \varphi_1, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi),\\
&z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\big)\, d\delta\, d\phi\, d\psi \Bigg)\Bigg|\\
&+ \Big| g(\zeta_1, \nu_1, \varphi_1, x(\zeta_1, \nu_1, \varphi_1), y(\zeta_1, \nu_1, \varphi_1), z(\zeta_1, \nu_1, \varphi_1),\\
&\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_1, \nu_1, \varphi_1, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\big)\\
&- g\Bigg(\zeta_1, \nu_1, \varphi_1, x(\zeta_1, \nu_1, \varphi_1), y(\zeta_1, \nu_1, \varphi_1), z(\zeta_1, \nu_1, \varphi_1),\\
&\int_0^{d(\varphi_1)} \int_0^{e(\nu_1)} \int_0^{k(\zeta_1)} h\big(\zeta_1, \nu_1, \varphi_1, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\, d\xi\, d\rho\, d\psi\big)\Big|\\
\leq{}& \varphi_3\big(\big|x(\zeta_2, \nu_2, \varphi_2) - z(\zeta_1, \nu_1, \varphi_1)\big|, \big|y(\zeta_2, \nu_2, \varphi_2) - x(\zeta_1, \nu_1, \varphi_1)\big|,\\
&\big|z(\zeta_2, \nu_2, \varphi_2) - y(\zeta_1, \nu_1, \varphi_1)\big|\big)\\
&+ \omega_\sigma^G(g, \epsilon)\\
&+ \varphi_4\bigg(\bigg|\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_2, \nu_2, \varphi_2, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\big)\\
&- h\big(\zeta_1, \nu_1, \varphi_1, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi), z(\xi, \rho, \psi)\big)\, d\xi\, d\rho\, d\psi\bigg|\bigg)\\
&+ \varphi_4\bigg(\bigg|\int_0^{d(\varphi_2)} \int_0^{e(\nu_2)} \int_0^{k(\zeta_2)} h\big(\zeta_1, \nu_1, \varphi_1, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi),\\
&z(\xi, \rho, \psi)\big)\, d\xi\, d\rho\, d\psi\\
&- \int_0^{d(\varphi_1)} \int_0^{e(\nu_1)} \int_0^{k(\zeta_1)} h\big(\zeta_1, \nu_1, \varphi_1, \xi, \rho, \psi, x(\xi, \rho, \psi), y(\xi, \rho, \psi),
\end{aligned}
$$

$$
\begin{aligned}
& z(\xi,\rho,\psi)\big)\Big)\,d\xi\,d\rho\,d\psi\Big|\Big) \\
& \le \varphi_3\big(\omega^G(x,\tilde{\epsilon}),\omega^G(y,\tilde{\epsilon}),\omega^G(z,\tilde{\epsilon})\big)+\omega_\sigma^G(g,\tilde{\epsilon})+\varphi_4\big(\hat{d}\hat{e}\hat{k}\omega_\sigma^G(h,\tilde{\epsilon})\big) \\
& \quad + \varphi_4\Big(\Big|\int_0^{d(\varphi_2)}\int_0^{e(\nu_2)}\int_0^{k(\zeta_2)} h\big(\zeta_1,\nu_1,\varphi_1,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi), \\
& z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi \\
& \quad - \int_0^{d(\varphi_1)}\int_0^{e(\nu_1)}\int_0^{k(\zeta_1)} h\big(\zeta_1,\nu_1,\varphi_1,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi), \\
& z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\Big|\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\omega^G(x,\tilde{\epsilon}) &= \sup\big\{\big|x(\zeta_2,\nu_2,\varphi_2)-x(\zeta_1,\nu_1,\varphi_1)\big| : \zeta_1,\zeta_2,\nu_1,\nu_2,\varphi_1,\varphi_2\in[0,G], |\zeta_1-\zeta_2|\le\tilde{\epsilon}, \\
&\qquad |\nu_1-\nu_2|\le\tilde{\epsilon}, |\varphi_1-\varphi_2|\le\tilde{\epsilon}\big\}, \\
\omega^G(y,\tilde{\epsilon}) &= \sup\big\{\big|y(\zeta_2,\nu_2,\varphi_2)-y(\zeta_1,\nu_1,\varphi_1)\big| : \zeta_1,\zeta_2,\nu_1,\nu_2,\varphi_1,\varphi_2\in[0,G], |\zeta_1-\zeta_2|\le\tilde{\epsilon}, \\
&\qquad |\nu_1-\nu_2|\le\tilde{\epsilon}, |\varphi_1-\varphi_2|\le\tilde{\epsilon}\big\}, \\
\omega^G(z,\tilde{\epsilon}) &= \sup\big\{\big|z(\zeta_2,\nu_2,\varphi_2)-z(\zeta_1,\nu_1,\varphi_1)\big| : \zeta_1,\zeta_2,\nu_1,\nu_2,\varphi_1,\varphi_2\in[0,G], |\zeta_1-\zeta_2|\le\tilde{\epsilon}, \\
&\qquad |\nu_1-\nu_2|\le\tilde{\epsilon}, |\varphi_1-\varphi_2|\le\tilde{\epsilon}\big\}, \\
\omega_\sigma^G(g,\tilde{\epsilon}) &= \sup\big\{\big|g(\zeta_2,\nu_2,\varphi_2,x,y,z,w)-g(\zeta_1,\nu_1,\varphi_1,x,y,z,w)\big| : \\
&\qquad \zeta_1,\zeta_2,\nu_1,\nu_2,\varphi_1,\varphi_2\in[0,G], \\
&\qquad |\zeta_1-\zeta_2|\le\tilde{\epsilon}, |\nu_1-\nu_2|\le\tilde{\epsilon}, |\varphi_1-\varphi_2|\le\tilde{\epsilon}, x,y,z\in[-\sigma,\sigma], w\in[-L,L]\big\}, \\
\omega_\sigma^G(h,\tilde{\epsilon}) &= \sup\big\{\big|h(\zeta_2,\nu_2,\varphi_2,\xi,\rho,\psi,x,y,z)-h(\zeta_1,\nu_1,\varphi_1,\xi,\rho,\psi,x,y,z)\big| : \\
&\qquad \zeta_1,\zeta_2,\nu_1,\nu_2,\varphi_1,\varphi_2,\in[0,G], \\
&\qquad |\zeta_1-\zeta_2|\le\tilde{\epsilon}, |\nu_1-\nu_2|\le\tilde{\epsilon}, |\varphi_1-\varphi_2|\le\tilde{\epsilon}, x,y,z\in[-\sigma,\sigma], \xi\in[0,\hat{k}], \\
&\qquad \rho\in[0,\hat{e}], \psi\in[0,\hat{d}]\big\}, \\
L &= \hat{k}\hat{e}\hat{d}\sup\big\{|h(\zeta,\nu,\varphi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)| : \zeta,\nu,\varphi\in[0,G], \\
&\quad \xi\in[0,\hat{k}], \rho\in[0,\hat{e}], \psi\in[0,\hat{d}], x,y,z\in[-\sigma,\sigma]\big\}.
\end{aligned}
$$

As h and g are uniformly continuous on $[0,G]\times[0,G]\times[0,G]\times[-\sigma,\sigma]\times[-\sigma,\sigma]\times[-\sigma,\sigma]\times[-L,L]$ and $[0,G]\times[0,G]\times[0,G]\times[0,\hat{k}]\times[0,\hat{e}]\times[0,\hat{d}]\times[-\sigma,\sigma]\times[-\sigma,\sigma]\times[-\sigma,\sigma]$, respectively, we infer that $\omega_\sigma^G(g,\tilde{\epsilon})$, $\omega_\sigma^G(h,\tilde{\epsilon})$ as $\tilde{\epsilon}\to 0$.

Again, from the uniform continuity of k, e, and d on $[0,L]$ we get that $k(\zeta_2)\to k(\zeta_1)$, $e(\nu_2)\to e(\nu_1)$ and $d(\varphi_2)\to d(\varphi_1)$ as $\tilde{\epsilon}\to 0$, So,

$$
\begin{aligned}
& \Big|\int_0^{d(\varphi_2)}\int_0^{e(\nu_2)}\int_0^{k(\zeta_2)} h\big(\zeta_1,\nu_1,\varphi_1,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi \\
& \quad - \int_0^{d(\varphi_1)}\int_0^{e(\nu_1)}\int_0^{k(\zeta_1)} h\big(\zeta_1,\nu_1,\varphi_1,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi), \\
& \quad z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\Big| \to 0,
\end{aligned}
$$

which gives

$$\varphi_2\bigg(\bigg|\int_0^{d(\varphi_2)}\int_0^{e(\nu_2)}\int_0^{k(\zeta_2)} h\big(\zeta_1,\nu_1,\varphi_1,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi$$
$$-\int_0^{d(\varphi_1)}\int_0^{e(\nu_1)}\int_0^{k(\zeta_1)} h\big(\zeta_1,\nu_1,\varphi_1,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),$$
$$z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\bigg|\bigg)\to 0$$

as $\tilde{\epsilon}\to 0$. We have

$$\omega_0^G\big(F(Z_1\times Z_2\times Z_3)\big)\leq\varphi_3\big(\omega_0^G(Z_1),\omega_0^G(Z_2),\omega_0^G(Z_3)\big).$$

Taking $G\to+\infty$, we get

$$\omega_0\big(F(Z_1\times Z_2\times Z_3)\big)\leq\varphi_3\big(\omega_0(Z_1),\omega_0(Z_2),\omega_0(Z_3)\big). \tag{2}$$

For arbitrary $(x,y,z),(\beta,\gamma,\alpha)\in Z_1\times Z_2\times Z_3$ and $\zeta,\nu,\varphi\in\mathbb{R}_+$, we have

$$\begin{aligned}
&\big|F(x,y,z)(\zeta,\nu,\varphi)-F(\alpha,\beta,\gamma)(\zeta,\nu,\varphi)\big|\\
&\quad\leq\varphi_3\big(\big|x(\zeta,\nu,\varphi)-\alpha(\zeta,\nu,\varphi)\big|,\big|y(\zeta,\nu,\varphi)-\beta(\zeta,\nu,\varphi)\big|,\big|z(\zeta,\nu,\varphi)-\gamma(\zeta,\nu,\varphi)\big|\big)\\
&\qquad+\varphi_4\bigg(\bigg|\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)}\big(h(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\\
&\qquad-h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha(\xi,\rho,\psi),\beta(\xi,\rho,\psi),\gamma(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\bigg|\bigg),\\
&\quad\leq\varphi_3\big(\operatorname{diam}\big(Z_1(\zeta,\nu,\varphi),\operatorname{diam}\big(Z_2(\zeta,\nu,\varphi),\operatorname{diam}\big(Z_3(\zeta,\nu,\varphi)\big)\big)\big)\\
&\qquad+\varphi_4\bigg(\bigg|\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)}\big(h(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\\
&\qquad-h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha(\xi,\rho,\psi),\beta(\xi,\rho,\psi),\gamma(\xi,\rho,\psi)\big)\,d\delta\,d\phi\,d\psi\bigg|\bigg).
\end{aligned}$$

Since (x,y,z), (α,β,γ), and ζ, ν, φ are arbitrary, we get

$$\begin{aligned}
&\operatorname{diam}F(Z_1\times Z_2\times Z_3)(\zeta,\nu,\varphi)\\
&\quad\leq\varphi_3\big(\operatorname{diam}\big(Z_1(\zeta,\nu,\varphi),\operatorname{diam}\big(Z_2(\zeta,\nu,\varphi)\big),\operatorname{diam}\big(Z_3(\zeta,\nu,\varphi)\big)\big)\big)\\
&\qquad+\varphi_4\bigg(\bigg|\int_0^{d(\varphi)}\int_0^{e(\nu)}\int_0^{k(\zeta)}\big(h(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\\
&\qquad-h\big(\zeta,\nu,\varphi,\xi,\rho,\psi,\alpha(\xi,\rho,\psi),\beta(\xi,\rho,\psi),\gamma(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\bigg|\bigg).
\end{aligned}$$

As $\zeta,\nu,\varphi\to+\infty$,

$$\begin{aligned}
&\lim_{\zeta,\nu,\varphi\to+\infty}\sup\operatorname{diam}F(Z_1\times Z_2\times Z_3)(\zeta,\nu,\varphi)\\
&\quad\leq\varphi_3\Big(\lim_{\zeta,\nu,\varphi\to+\infty}\sup\operatorname{diam}\big(Z_1(\zeta,\nu,\varphi)\big),\lim_{\zeta,\nu,\varphi\to+\infty}\sup\operatorname{diam}\big(Z_2(\zeta,\nu,\varphi)\big),\\
&\qquad\lim_{\zeta,\nu,\varphi\to+\infty}\sup\operatorname{diam}\big(Z_3(\zeta,\nu,\varphi)\big)\Big).
\end{aligned}\tag{3}$$

From (2) and (3) we have

$$\begin{aligned}
&\omega\big(F(Z_1 \times Z_2 \times Z_3)\big) + \lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam} F(Z_1 \times Z_2 \times Z_3)(\zeta,\nu,\varphi) \\
&\quad\le \varphi_3(\omega_0(Z_1),\omega_0(Z_2),\omega_0(Z_3) + \varphi_3\Big(\lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam}\big(Z_1(\zeta,\nu,\varphi)\big), \\
&\qquad \lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam}\big(Z_2(\zeta,\nu,\varphi)\big), \lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam}\big(Z_3(\zeta,\nu,\varphi)\big)\Big) \\
&\quad\le 3\varphi_3\bigg(\frac{\omega_0(Z_1) + \lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam}(Z_1(\zeta,\nu,\varphi))}{3}, \\
&\qquad \frac{\omega_0(Z_2)) + \lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam}(Z_2(\zeta,\nu,\varphi))}{3}, \\
&\qquad \frac{\omega_0(Y_3) + \lim_{\zeta,\nu,\varphi\to+\infty} \sup \operatorname{diam}(Z_3(\zeta,\nu,\varphi))}{3}\bigg).
\end{aligned}$$

Therefore

$$\frac{1}{3}\mu\big(F(Z_1 \times Z_2 \times Z_3)\big) \le \varphi_3\left(\frac{\mu}{3},\frac{\mu}{3},\frac{\mu}{3}\right).$$

Putting $\frac{1}{3}\mu = \hat{\mu}$, we get

$$\hat{\mu}\big(F(Z_1 \times Z_2 \times Z_3)\big) \le \varphi_3\big(\hat{\mu}(Z_1),\hat{\mu}(Z_2),\hat{\mu}(Z_3)\big).$$

Hence equation (1) has a tripled fixed point in the space $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$, and thus the system has a solution in $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$. □

Example 3.1 Consider the system of NIEs

$$\begin{aligned}
&x(\zeta,\nu,\varphi) \\
&\quad= \frac{1}{11}e^{-(\zeta^3+\nu^3+\varphi^3)} + \frac{\zeta^3 \ln(1 + x(\zeta,\nu,\varphi))}{5(1+\nu^3)} + \frac{e^{-\nu^2} \ln(1 + y(\zeta,\nu,\varphi)}{9} + \frac{\varphi^2 \ln(1 + z(\zeta,\nu,\varphi)}{4(1+\varphi^2)} \\
&\qquad + \ln\bigg(1 + \frac{1}{3}\int_0^\varphi \int_0^\nu \int_0^\zeta \big(\cos^2\big(1 + 2\xi^3 x(\xi,\rho,\psi)\big) + \sin\big(\psi^3 y(\xi,\rho,\psi)\big) \\
&\qquad + \cos\big(\rho^5 z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\big)/e^{\zeta^2\nu^2\varphi^2}\bigg), \\
&y(\zeta,\nu,\varphi) \\
&\quad= \frac{1}{11}e^{-(\zeta^3+\nu^3+\varphi^3)} + \frac{\zeta^3 \ln(1 + y(\zeta,\nu,\varphi))}{5(1+\nu^3)} + \frac{e^{-\nu^2} \ln(1 + x(\zeta,\nu,\varphi)}{9} \\
&\qquad + \frac{\varphi^2 \ln(1 + z(\zeta,\nu,\varphi)}{4(1+\varphi^2)} \qquad\qquad (4) \\
&\qquad + \ln\bigg(1 + \frac{1}{3}\int_0^\varphi \int_0^\nu \int_0^\zeta \big(\cos^2\big(1 + 2\xi^3 y(\xi,\rho,\psi)\big) + \sin\big(\psi^3 x(\xi,\rho,\psi)\big) \\
&\qquad + \cos\big(\rho^5 z(\xi,\rho,\psi)\big)\,d\xi\,d\rho\,d\psi\big)/e^{\zeta^2\nu^2\varphi^2}\bigg),
\end{aligned}$$

$$
\begin{aligned}
&z(\zeta,\nu,\varphi)\\
&\quad= \frac{1}{11}e^{-(\zeta^3+\nu^3+\varphi^3)} + \frac{\zeta^3\ln(1+z(\zeta,\nu,\varphi))}{5(1+\nu^3)} + \frac{e^{-\nu^2}\ln(1+y(\zeta,\nu,\varphi)}{9} + \frac{\varphi^2\ln(1+x(\zeta,\nu,\varphi)}{4(1+\varphi^2)}\\
&\qquad + \ln\biggl(1+\frac{1}{3}\int_0^{\varphi}\int_0^{\nu}\int_0^{\zeta}\bigl(\cos^2\bigl(1+2\xi^3 z(\xi,\rho,\psi)\bigr)+\sin\bigl(\psi^3 y(\xi,\rho,\psi)\bigr)\\
&\qquad + \cos\bigl(\rho^5 x(\xi,\rho,\psi)\bigr)\,d\xi\,d\rho\,d\psi\bigr)/e^{\zeta^2\nu^2\varphi^2}\biggr).
\end{aligned}
$$

This system is a particular form of (1) with

$$
\begin{aligned}
g(\zeta,\nu,\varphi,x,y,z,w) &= \frac{1}{11}e^{-(\zeta^3+\nu^3+\varphi^3)} + \frac{\zeta^3\ln(1+x(\zeta,\nu,\varphi))}{5(1+\zeta^3)} + \frac{e^{-\nu^2}\ln(1+y(\zeta,\nu,\varphi)}{9}\\
&\quad + \frac{\varphi^2\ln(1+z(\zeta,\nu,\varphi)}{4(1+\varphi^2)} + \ln\biggl(1+\frac{|w|}{3}\biggr),\\
h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,&x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\bigr)\\
&= \frac{\cos^2(1+2\delta^3x(\xi,\rho,\psi))+\sin(\phi^3y(\xi,\rho,\psi))+\cos(\psi^5z(\xi,\rho,\psi))\,d\xi\,d\rho\,d\psi}{e^{\zeta^2\nu^2\varphi^2}},
\end{aligned}
$$

$\varphi_3(\zeta,\nu,\varphi)=\ln(1+\frac{\zeta+\nu+\varphi}{3})$, $d(\varphi)=\varphi$, $e(\nu)=\nu$, $k(\zeta)=\zeta$, $\varphi_4(\zeta)=\frac{\zeta}{3}$.

Now we show that all the properties of Theorem 3.1 are satisfied for system (4).

(i) d, e, k, g are continuous, $|g(\zeta,\nu,\varphi,0,0,0,0)| = \frac{1}{11}e^{-(\zeta^3+\nu^3+\varphi^3)}$ is bounded for $\zeta,\nu,\varphi\in\mathbb{R}^+$, and $\hat{Q}=\frac{1}{11}$.

(ii) Let $\zeta,\nu,\varphi\in\mathbb{R}^+$ and $x,y,z,w,\tilde{x},\tilde{y},\tilde{z},\tilde{w}\in\mathbb{R}^+$ with $|x|\ge|\tilde{x}|$, $|y|\ge|\tilde{y}|$, $|z|\ge|\tilde{z}|$, $|w|\ge|\tilde{w}|$. By the mean value theorem, for $\ln(1+\frac{|w|}{3}))$ and $\ln(1+\frac{\zeta+\nu+\varphi}{3})\in\hat{\varphi}$, we have

$$
\begin{aligned}
&\bigl|g(\zeta,\nu,\varphi,x,y,z,w)-g(\zeta,\nu,\varphi,\tilde{x},\tilde{y},\tilde{z},\tilde{w})\bigr|\\
&\quad\le \frac{\zeta^3}{5(1+\zeta^3)}\bigl|\ln\bigl(1+|x|\bigr)-\ln\bigl(1+|\tilde{x}|\bigr)\bigr| + \frac{e^{-\nu^2}}{9}\bigl|\ln\bigl(1+|y|\bigr)-\ln\bigl(1+|\tilde{y}|\bigr)\bigr|\\
&\qquad + \frac{\varphi^2}{4(1+\varphi^2)}\bigl|\ln\bigl(1+|z|\bigr)-\ln\bigl(1+|\tilde{z}|\bigr)\bigr| + \biggl|\ln\biggl(1+\frac{|w|}{3}\biggr)-\ln\biggl(1+\frac{|\tilde{w}|}{3}\biggr)\biggr|\\
&\quad\le \frac{\zeta^3}{5(1+\zeta^3)}\biggl|\ln\biggl(\frac{1+|x|}{1+|\tilde{x}|}\biggr)\biggr| + \frac{e^{-\nu^2}}{9}\biggl|\ln\biggl(\frac{1+|y|}{1+|\tilde{y}|}\biggr)\biggr|\\
&\qquad + \frac{\varphi^2}{4(1+\varphi^2)}\biggl|\ln\biggl(\frac{1+|z|}{1+|\tilde{z}|}\biggr)\biggr| + \frac{1}{3}\biggl|\ln\biggl(\frac{1+\frac{|w|}{3}}{1+\frac{|\tilde{w}|}{3}}\biggr)\biggr|\\
&\quad\le \frac{1}{5}\biggl|\ln\biggl(\frac{1+|x|}{1+|\tilde{x}|}\biggr)\biggr| + \frac{1}{9}\biggl|\ln\biggl(\frac{1+|y|}{1+|\tilde{y}|}\biggr)\biggr|\\
&\qquad + \frac{1}{4}\biggl|\ln\biggl(\frac{1+|z|}{1+|\tilde{z}|}\biggr)\biggr| + \frac{1}{3}|w-\tilde{w}|\\
&\quad\le \frac{1}{4}\biggl|\ln\biggl(1+\frac{|x|-|\tilde{x}|}{1+|\tilde{x}|}\biggr)\biggr| + \frac{1}{4}\biggl|\ln\biggl(1+\frac{|y|-|\tilde{y}|}{1+|\tilde{y}|}\biggr)\biggr|\\
&\qquad + \frac{1}{4}\biggl|\ln\biggl(1+\frac{|z|-|\tilde{z}|}{1+|\tilde{z}|}\biggr)\biggr| + \frac{1}{3}|w-\tilde{w}|
\end{aligned}
$$

$$\begin{aligned}&\leq \frac{1}{4}\ln\bigl(1+|x-\tilde{x}|\bigr)+\frac{1}{4}\ln\bigl(1+|y-\tilde{y}|\bigr)+\frac{1}{4}\ln\bigl(1+|z-\tilde{z}|\bigr)+\frac{1}{3}|w-\tilde{w}|\\&\leq \ln\left(1+\frac{|x-\tilde{x}|+|y-\tilde{y}|+|z-\tilde{z}|}{3}\right)+\frac{1}{3}|w-\tilde{w}|,\\&=\varphi_3\bigl(|x-\tilde{x}|,|y-\tilde{y}|,|z-\tilde{z}|\bigr)+\varphi_4\bigl(|w-\tilde{w}|\bigr).\end{aligned}$$

(iii) Since h is continuous, for each $\zeta,\nu,\varphi,\xi,\rho,\psi\in\mathbb{R}^+$ and $x,y,z,\tilde{x},\tilde{y},\tilde{z}\in\mathbb{R}^+$, we have

$$\begin{aligned}&\bigl|h(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\\&\qquad-\bigl|h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,\tilde{x}(\xi,\rho,\psi),\tilde{y}(\xi,\rho,\psi),\tilde{z}(\xi,\rho,\psi)\bigr)\bigr|\\&\quad\leq\frac{6}{e^{\zeta^2\nu^2\varphi^2}},\\&\lim_{\zeta,\nu,\varphi\to+\infty}\int_0^{\varphi}\int_0^{\nu}\int_0^{\zeta}\bigl|h(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\\&\quad-\bigl|h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,\tilde{x}(\xi,\rho,\psi),\tilde{y}(\xi,\rho,\psi),\tilde{z}(\xi,\rho,\psi)\bigr)\bigr|\leq\lim_{\zeta,\nu,\varphi\to+\infty}\frac{6\zeta\nu\varphi}{e^{\zeta^2\nu^2\varphi^2}}=0\end{aligned}$$

for all $x,y,z,\tilde{x},\tilde{y},\tilde{z}\in\mathcal{F}$.

Moreover,

$$\bigl|h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\bigr)\bigr|\leq\frac{3}{e^{\zeta^2\nu^2\varphi^2}}.$$

Also,

$$\int_0^{\varphi}\int_0^{\nu}\int_0^{\zeta}\bigl|h\bigl(\zeta,\nu,\varphi,\xi,\rho,\psi,x(\xi,\rho,\psi),y(\xi,\rho,\psi),z(\xi,\rho,\psi)\bigr)\,d\xi\,d\rho\,d\psi\bigr|\leq\frac{3\zeta\nu\varphi}{e^{\zeta^2\nu^2\varphi^2}}$$

for all $\zeta,\nu,\varphi,\xi,\rho,\psi\in\mathbb{R}^+$ and $x,y,z\in\mathbb{R}$. Thus

$$Q=\sup\left\{\frac{3\zeta\nu\varphi}{e^{\zeta^2\nu^2\varphi^2}}:\zeta,\nu,\varphi\geq 0\right\}=\frac{3}{\sqrt{2e}}=1.2866.$$

(iv) Putting all values Q, $\hat{Q}$, φ_3, and φ_4 in the inequality.

$$\frac{1}{11}+\ln(1+\hat{r})+0.4286<\hat{r}.$$

For $\hat{r}\geq 2$, we obtain

$$\hat{r}-\frac{1}{11}-\ln(1+\hat{r})-0.4286>0.$$

Choosing $\sigma=2$, all the conditions of Theorem 3.1 are satisfied, and the system of NIEs (4) has a solution in the space $\mathcal{F}\times\mathcal{F}\times\mathcal{F}$.

4 Conclusions

There are different generalizations of Darbo's fixed point theorem. Some authors have created generalizations via measures of noncompactness. On the other hand, several authors have extended Darbo's fixed point theorem by changing the domain of mappings that possess a fixed point. In this paper, we used contractions to verify that a mapping defined on a nonempty convex bounded closed subset of a given Banach space has at least one fixed point. We prove the existence of solutions for a system of functional nonlinear integral equations in three dimensions.

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