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# Computation of semi-analytical solutions of fuzzy nonlinear integral equations

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## Abstract

In this article, we use a fuzzy number in its parametric form to solve a fuzzy nonlinear integral equation of the second kind in the crisp case. The main theme of this article is to find a semi-analytical solution of fuzzy nonlinear integral equations. A hybrid method of Laplace transform coupled with Adomian decomposition method is used to find the solution of the fuzzy nonlinear integral equations including fuzzy nonlinear Fredholm integral equation, fuzzy nonlinear Volterra integral equation, and fuzzy nonlinear singular integral equation of Abel type kernel. We also provide some suitable examples to better understand the proposed method.

**Keywords:** Semi-analytical solution; Fuzzy nonlinear Fredholm integral equation; Fuzzy nonlinear Volterra integral equation; Fuzzy singular integral equation; Fuzzy number

## 1 Introduction

Recently, the notions of fuzzy differential equations (FDEs) and fuzzy integral equations (FIEs) with fuzzy control have attracted the researchers. Different definitions of the fuzzy derivative and fuzzy integral have been established and extended to fuzzy calculus. For the existence and uniqueness of the solution of these equations, sufficient conditions have been provided, and for the approximate solution different numerical algorithms have been designed. Before going into detail, some of the basic topics need to be presented here. Zadeh introduced the concept of fuzzy set [1], followed by that of fuzzy number, and implemented it in fuzzy control [2] and approximate reasoning [3, 4]. The researchers provided that a fuzzy number is a collection of  $\alpha$ -cut. They also provided various arithmetic operations on fuzzy numbers [5–12] which are a base for fuzzy calculus. Various researchers implanted a fuzzy number in Banach and topological spaces [13–20] and it has many applications in fuzzy linear systems [21, 22], fuzzy linear least squares problems [23–27], fuzzy eigenvalues and eigenvectors [21, 28], and to characterize a compact set [17, 29]. Chang and Zadeh [30] introduced fuzzy mapping, and based on Zadeh's extension principle [1] Dubois and Prade [31] developed elementary fuzzy calculus. Different definitions for fuzzy derivative and fuzzy integration were provided in [1, 12, 13, 15, 31–33].

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It is mentioned that integral equations have many applications in various scientific and mathematical disciplines. To model any physical problem by using integral equations with exact parameters is not an easy task, in the real world problems it is almost impossible. Therefore, various techniques are used to measure uncertainty in the data. To handle such problems, Zadeh introduced fuzzy set in 1965 [1]. Over time, many researchers, scientists, and mathematicians extended fuzzy set in various other structures and solved problems having unprecise and uncertain data. Recently, fuzzy set and system and fuzzy logic are used in the fields of fuzzy topological spaces [34], fuzzy metric spaces [35], fuzzy differential equations [36, 37], to control chaotic systems [38, 39], and in applied physics [40, 41]. The interest in fuzzy integral equations is growing very fast in many problems. So, deterministic models of integral equations are used instead of applying the fuzzy integral equation. Hence, a dire need is to develop a numerical procedure and mathematical models that would deal with the general fuzzy integral equations and solve them appropriately. Various applications and methods for solving linear and nonlinear integral equations are given in [42]. The fuzzy integral equation is one of the most dominant fields of fuzzy set theory [43, 44]. In the last few decades, the concepts of fuzzy integral equation and fuzzy integro-differential equation have stimulated researchers. Wu and Ma [19] provided the first application of fuzzy integral and investigated fuzzy Fredholm integral equation of the second kind (FF-2), while Park et al. [45] showed the existence of a solution of FIEs. A large amount of research work has been done to investigate FIEs [46–57]. This is due to various applications of FIEs in the scientific field. Therefore, finding an efficient and accurate algorithm for investigating FIE has been one the hot areas of research in recent time. To achieve these goals, various methods and procedures were used to handle integral equations, for details, see [58–60].

Motivated by the aforesaid work, in this article, we use a hybrid method of Laplace transform coupled with Adomian decomposition method to solve the fuzzy nonlinear integral equation of the type

$$\omega(h, \gamma) = f(h, \gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u)\omega(u, \gamma) du, \tag{1}$$

where  $f(h, \gamma)$  and  $G(h, u)$  are the fuzzy functions, and  $\omega(h, \gamma)$  is the unrevealed function, a nonlinear term that appeared under the integral, i.e.,  $\ln(\omega(u, \gamma))$ ,  $\exp(\omega(u, \gamma))$ , and  $\omega^2(u, \gamma)$  etc.,  $\gamma \in (0, 1)$  is a fuzzy parameter, and  $\mu$  is a constant parameter. The two-variable function  $G(h, u)$  is called kernel of fuzzy integral equations,  $\alpha_1(h)$  and  $\beta_1(h)$  are limits of integration. If both the limits of integration are real numbers, then Eq. (1) is called fuzzy nonlinear Fredholm integral equation. If one of its limits, say  $\alpha_1(h)$ , is constant and the other limit, say  $\beta_1(h)$ , is variable, then Eq. (1) is called fuzzy nonlinear Volterra integral equation; and if the kernel has a singularity in the domain of its integration, then Eq. (1) is called fuzzy nonlinear singular integral equation of Abel type kernel.

The parametric case of Eq. (1) is

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} \underline{G}(h, u)\underline{\omega}(u, \gamma) du, \\ \overline{\omega}(h, \gamma) = \overline{f}(h, \gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} \overline{G}(h, u)\overline{\omega}(u, \gamma) du, \end{cases} \tag{2}$$

where  $\omega(h, \gamma) = (\underline{\omega}(h, \gamma), \overline{\omega}(h, \gamma))$ ,  $f(h, \gamma) = (\underline{f}(h, \gamma), \overline{f}(h, \gamma))$

and

$$\begin{aligned} \underline{G(h, u)\omega(u, \gamma)} &= \begin{cases} G(h, u)\underline{\omega}(h, \gamma) & G(h, u) \geq 0, \\ G(h, u)\overline{\omega}(u, \gamma) & G(h, u) < 0, \end{cases} \\ \overline{G(h, u)\omega(u, \gamma)} &= \begin{cases} G(h, u)\overline{\omega}(h, \gamma) & G(h, u) \geq 0, \\ G(h, u)\underline{\omega}(u, \gamma) & G(h, u) < 0. \end{cases} \end{aligned}$$

By this method, we compute fuzzy solutions of the above integral equation (2) and provide various examples to demonstrate the procedure.

We organize the article as follows:

Section 2 presents the general procedure for nonlinear FIE. In Sect. 3, some test problems [61] are solved by the proposed method. In Sect. 4, the comparison and discussion of the proposed method with the existing techniques [62–64] are given, while Sect. 5 is devoted to the conclusion of the article.

### 2 General procedure to handle fuzzy nonlinear integral equations

To solve the fuzzy nonlinear integral Eq. (1) in a fuzzy sense, the parametric form of Eq. (1) can be written into two integral equations as follows:

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u)\underline{\omega}^n(u, \gamma) du, \\ \overline{\omega}(h, \gamma) = \overline{f}(h, \gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u)\overline{\omega}^n(u, \gamma) du, \end{cases} \tag{3}$$

applying Laplace transform on Eq. (3)

$$\begin{cases} L[\underline{\omega}(h, \gamma)] = L[\underline{f}(h, \gamma)] + L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u)\underline{\omega}^n(u, \gamma) du], \\ L[\overline{\omega}(h, \gamma)] = L[\overline{f}(h, \gamma)] + L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u)\overline{\omega}^n(u, \gamma) du], \end{cases} \tag{4}$$

operating inverse Laplace transform on Eq. (4), we obtain

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u)\underline{\omega}^n(u, \gamma) du]], \\ \overline{\omega}(h, \gamma) = \overline{f}(h, \gamma) + L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u)\overline{\omega}^n(u, \gamma) du]]. \end{cases} \tag{5}$$

Consider that the lower and upper fuzzy limit solutions of Eq. (5) can be extended by the Laplace decomposition algorithm into an infinite series as follows:

$$\begin{cases} \underline{\omega}(h, \gamma) = \sum_{n=0}^{\infty} \underline{\omega}_n(h, \gamma), \\ \overline{\omega}(h, \gamma) = \sum_{n=0}^{\infty} \overline{\omega}_n(h, \gamma), \end{cases} \tag{6}$$

and nonlinear lower and upper limit terms ( $\underline{\omega}^n(h, \gamma), \overline{\omega}^n(h, \gamma)$ ) can be written as

$$\begin{cases} \underline{\omega}^n(u, \gamma) = \sum_{n=0}^{\infty} B_n(u, \gamma), \\ \overline{\omega}^n(u, \gamma) = \sum_{n=0}^{\infty} \hat{B}_n(u, \gamma), \end{cases} \tag{7}$$

where  $(B_n(u, \gamma), \hat{B}_n(u, \gamma))$  are the Adomian polynomials [65]. By putting Eq. (6) and Eq. (7) in Eq. (5), we get

$$\begin{cases} \sum_{n=0}^{\infty} \underline{\omega}_n(h, \gamma) = \underline{f}(h, \gamma) + L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) [\sum_{n=0}^{\infty} B_n(u, \gamma)] du]], \\ \sum_{n=0}^{\infty} \overline{\omega}_n(h, \gamma) = \overline{f}(h, \gamma) + L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) [\sum_{n=0}^{\infty} \hat{B}_n(u, \gamma)] du]]. \end{cases} \tag{8}$$

The Adomian polynomials in Eq. (7) can be generated by several means, and we use the following recursive formulation:

$$\begin{cases} B_n(u, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum_{k=0}^{\infty} \lambda^k \underline{\omega}_k)]_{\lambda=0}, \\ \hat{B}_n(u, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum_{k=0}^{\infty} \lambda^k \overline{\omega}_k)]_{\lambda=0}, \end{cases} \tag{9}$$

for  $n = 0, 1, 2, 3, \dots$ . Thus, in general, comparing the recursive relation of Eq. (8) termwise, we get for  $n = 0$

$$\begin{cases} \underline{\omega}_0(h, \gamma) = \underline{f}(h, \gamma), \\ \overline{\omega}_0(h, \gamma) = \overline{f}(h, \gamma), \end{cases}$$

at  $n = 1$

$$\begin{cases} \underline{\omega}_1(h, \gamma) = L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) B_0(u, \gamma) du]], \\ \overline{\omega}_1(h, \gamma) = L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) \hat{B}_0(u, \gamma) du]], \end{cases}$$

at  $n = 2$

$$\begin{cases} \underline{\omega}_2(h, \gamma) = L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) B_1(u, \gamma) du]], \\ \overline{\omega}_2(h, \gamma) = L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) \hat{B}_1(u, \gamma) du]], \end{cases}$$

and so on up to  $n + 1$ , we get

$$\begin{cases} \underline{\omega}_{n+1}(h, \gamma) = L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) B_n(u, \gamma) du]], \\ \overline{\omega}_{n+1}(h, \gamma) = L^{-1}[L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) \hat{B}_n(u, \gamma) du]], \quad n \geq 0, \end{cases} \tag{10}$$

where  $(\underline{f}(h, \gamma), \overline{f}(h, \gamma))$  represents the source term.

### 3 Test problems of nonlinear fuzzy integral equations

We solve a few examples by considering the proposed method to obtain an analytical approximate solution of fuzzy nonlinear integral equations of different types. Applications of the proposed method are more simple and upgrade its order.

*Example 1* Consider the second kind fuzzy nonlinear Volterra integral equation in parametric form:

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + \int_0^h \omega^2(u, \gamma) du, \\ \overline{\omega}(h, \gamma) = \overline{f}(h, \gamma) + \int_0^h \omega^2(u, \gamma) du, \end{cases} \tag{11}$$

where  $\mu = 1, 0 \leq h \leq 1, 0 \leq u \leq h, G(h, u) = 1, 0 \leq \gamma \leq 1$  and nonhomogenous term is  $(\underline{f}(h, \gamma), \bar{f}(h, \gamma)) = ((\gamma^2 + \gamma)h, (7 - \gamma)h)$ .

To solve Eq. (11) using the proposed method, first taking Laplace transform on Eq. (11), we have

$$\begin{cases} L[\underline{\omega}(h, \gamma)] = L[\underline{f}(h, \gamma)] + L[\int_0^h \omega^2(u, \gamma) du], \\ L[\bar{\omega}(h, \gamma)] = L[\bar{f}(h, \gamma)] + L[\int_0^h \omega^2(u, \gamma) du]. \end{cases} \tag{12}$$

Operating the inverse Laplace transform on Eq. (12), we get

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + L^{-1}[L[\int_0^h \omega^2(u, \gamma) du]], \\ \bar{\omega}(h, \gamma) = \bar{f}(h, \gamma) + L^{-1}[L[\int_0^h \omega^2(u, \gamma) du]], \end{cases} \tag{13}$$

$$\begin{cases} \underline{\omega}(h, \gamma) = (\gamma^2 + \gamma)h + L^{-1}[L[\int_0^h \omega^2(u, \gamma) du]], \\ \bar{\omega}(h, \gamma) = (7 - \gamma)h + L^{-1}[L[\int_0^h \omega^2(u, \gamma) du]]. \end{cases} \tag{14}$$

Assume that the lower and upper limit fuzzy solutions of Eq. (14) are an infinite series as follows:

$$\begin{cases} \underline{\omega}(h, \gamma) = \sum_{n=0}^{\infty} \underline{\omega}_n(h, \gamma), \\ \bar{\omega}(h, \gamma) = \sum_{n=0}^{\infty} \bar{\omega}_n(h, \gamma). \end{cases} \tag{15}$$

Putting Eq. (15) in Eq. (14), we get

$$\begin{cases} \sum_{n=0}^{\infty} \underline{\omega}_n(h, \gamma) = (\gamma^2 + \gamma)h + L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_n(u, \gamma)] du]], \\ \sum_{n=0}^{\infty} \bar{\omega}_n(h, \gamma) = (7 - \gamma)h + L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \hat{B}_n(u, \gamma)] du]], \end{cases} \tag{16}$$

where  $(B_n, \hat{B}_n)$  are Adomian polynomials which represent the nonlinear term and can be decomposed as follows:

$$\begin{cases} B_n(u, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [(\sum_{k=0}^{\infty} \lambda^k \underline{\omega}_k)]_{\lambda=0}^2, \\ \hat{B}_n(u, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [(\sum_{k=0}^{\infty} \lambda^k \bar{\omega}_k)]_{\lambda=0}^2. \end{cases} \tag{17}$$

Now, we first solve the lower limit fuzzy solution of Eq. (11). So comparing the lower limit terms of Eq. (16) and solving, we obtain

$$\begin{cases} \underline{\omega}_0(h, \gamma) = (\gamma^2 + \gamma)h, \\ \underline{\omega}_1(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_0(u, \gamma)] du]] = \frac{1}{3}(\gamma^2 + \gamma)^2 h^3, \\ \underline{\omega}_2(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_1(u, \gamma)] du]] = \frac{2}{15}(\gamma^2 + \gamma)^3 h^5, \\ \underline{\omega}_3(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_2(u, \gamma)] du]] = \frac{17}{315}(\gamma^2 + \gamma)^4 h^7, \\ \vdots \end{cases} \tag{18}$$

and for the upper limit fuzzy solution of Eq. (11), comparing the upper limit terms of Eq. (16) and solving

$$\begin{cases} \bar{\omega}_0(h, \gamma) = (7 - \gamma)h, \\ \bar{\omega}_1(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^\infty \hat{B}_0(u, \gamma)] du]] = \frac{1}{3}(7 - \gamma)^2 h^3, \\ \bar{\omega}_2(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^\infty \hat{B}_1(u, \gamma)] du]] = \frac{2}{15}(7 - \gamma)^3 h^5, \\ \bar{\omega}_3(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^\infty \hat{B}_2(u, \gamma)] du]] = \frac{17}{315}(7 - \gamma)^4 h^7, \\ \vdots \end{cases} \tag{19}$$

putting Eq. (18) and Eq. (19) in Eq. (15), we get the approximate lower and upper limit fuzzy solutions as follows:

$$\begin{cases} \underline{\omega}(h, \gamma) = (\gamma^2 + \gamma)h + \frac{1}{3}(\gamma^2 + \gamma)^2 h^3 + \frac{2}{15}(\gamma^2 + \gamma)^3 h^5 + \frac{17}{315}(\gamma^2 + \gamma)^4 h^7 + \dots, \\ \bar{\omega}(h, \gamma) = (7 - \gamma)h + \frac{1}{3}(7 - \gamma)^2 h^3 + \frac{2}{15}(7 - \gamma)^3 h^5 + \frac{17}{315}(7 - \gamma)^4 h^7 + \dots \end{cases} \tag{20}$$

*Example 2* Consider the second kind fuzzy nonlinear Fredholm integral equation in a parametric form:

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + \int_0^1 \underline{\omega}^2(u, \gamma) du, \\ \bar{\omega}(h, \gamma) = \bar{f}(h, \gamma) + \int_0^1 \bar{\omega}^2(u, \gamma) du, \end{cases} \tag{21}$$

where  $0 \leq h, u \leq 1, 0 \leq \gamma \leq 1, G(h, u) = 1$  and  $(\underline{f}(h, \gamma), \bar{f}(h, \gamma)) = (\gamma, 2 - \gamma)$ .

To solve Eq. (21) by the proposed method, taking Laplace transform, we have

$$\begin{cases} L[\underline{\omega}(h, \gamma)] = L[\underline{f}(h, \gamma)] + L[\int_0^1 \underline{\omega}^2(u, \gamma) du], \\ L[\bar{\omega}(h, \gamma)] = L[\bar{f}(h, \gamma)] + L[\int_0^1 \bar{\omega}^2(u, \gamma) du]. \end{cases} \tag{22}$$

Now, applying the inverse Laplace transform on Eq. (22), we get

$$\begin{cases} \underline{\omega}(h, \gamma) = \gamma + L^{-1}[L[\int_0^1 \underline{\omega}^2(u, \gamma) du]], \\ \bar{\omega}(h, \gamma) = (2 - \gamma) + L^{-1}[L[\int_0^1 \bar{\omega}^2(u, \gamma) du]]. \end{cases} \tag{23}$$

Assume that the lower and upper limit fuzzy solutions of Eq. (23) are an infinite series as follows:

$$\begin{cases} \underline{\omega}(h, \gamma) = \sum_{n=0}^\infty \underline{\omega}_n(h, \gamma), \\ \bar{\omega}(h, \gamma) = \sum_{n=0}^\infty \bar{\omega}_n(h, \gamma). \end{cases} \tag{24}$$

Putting Eq. (24) in Eq. (23), we get

$$\begin{cases} \sum_{n=0}^\infty \underline{\omega}_n(h, \gamma) = \gamma + L^{-1}[L[\int_0^1 [\sum_{n=0}^\infty B_n(u, \gamma)] du]], \\ \sum_{n=0}^\infty \bar{\omega}_n(h, \gamma) = (2 - \gamma) + L^{-1}[L[\int_0^1 [\sum_{n=0}^\infty \hat{B}_n(u, \gamma)] du]], \end{cases} \tag{25}$$

where  $(B_n, \hat{B}_n)$  are Adomian polynomials which represent nonlinear term and can be solved as follows:

$$\begin{cases} B_n(u, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [(\sum_{k=0}^{\infty} \lambda^k \underline{\omega}_k)]_{\lambda=0}^2, \\ \hat{B}_n(u, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [(\sum_{k=0}^{\infty} \lambda^k \bar{\omega}_k)]_{\lambda=0}^2. \end{cases} \tag{26}$$

Now first we solve the lower limit fuzzy solution of Eq. (21). So, comparing the lower limit case of Eq. (25) termwise and solving, we obtain

$$\begin{cases} \underline{\omega}_0(h, \gamma) = \gamma, \\ \underline{\omega}_1(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_0(u, \gamma)] du]] = \gamma^2, \\ \underline{\omega}_2(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_1(u, \gamma)] du]] = 2\gamma^3, \\ \underline{\omega}_3(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_2(u, \gamma)] du]] = 5\gamma^4, \\ \vdots \end{cases} \tag{27}$$

and for the upper limit fuzzy solutions of Eq. (21), comparing the upper limit case of Eq. (25) termwise and solving, we obtain

$$\begin{cases} \bar{\omega}_0(h, \gamma) = 2 - \gamma, \\ \bar{\omega}_1(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \hat{B}_0(u, \gamma)] du]] = (2 - \gamma)^2, \\ \bar{\omega}_2(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \hat{B}_1(u, \gamma)] du]] = 2(2 - \gamma)^3, \\ \bar{\omega}_3(h, \gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \hat{B}_2(u, \gamma)] du]] = 5(2 - \gamma)^4, \\ \vdots \end{cases} \tag{28}$$

Put Eq. (27) and Eq. (28) in Eq. (25) to get the approximate lower and upper limit fuzzy solutions:

$$\begin{cases} \underline{\omega}(h, \gamma) = \gamma + \gamma^2 + 2\gamma^3 + 5\gamma^4 + \dots, \\ \bar{\omega}(h, \gamma) = (2 - \gamma) + (2 - \gamma)^2 + 2(2 - \gamma)^3 + 5(2 - \gamma)^4 + \dots. \end{cases} \tag{29}$$

*Example 3* Consider the second kind fuzzy nonlinear singular integral equation of Abel type kernel of parametric form:

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + \int_0^h \frac{\underline{\omega}^2(u, \gamma)}{\sqrt{h-u}} du, \\ \bar{\omega}(h, \gamma) = \bar{f}(h, \gamma) + \int_0^h \frac{\bar{\omega}^2(u, \gamma)}{\sqrt{h-u}} du, \end{cases} \tag{30}$$

where  $(\underline{f}(h, \gamma), \bar{f}(h, \gamma)) = (h\gamma - \frac{16}{15}\gamma^2 h^{5/2}, h(3 - \gamma) - \frac{16}{15}(3 - \gamma)^2 h^{5/2})$  and  $0 \leq h \leq 1, 0 \leq u \leq h, 0 \leq \gamma \leq 1$ .

To solve Eq. (30) by the proposed method, taking the Laplace transform on Eq. (30), we have

$$\begin{cases} L[\underline{\omega}(h, \gamma)] = L[\gamma h - \frac{16}{15}\gamma^2 h^{5/2}] + L[\int_0^h \frac{\underline{\omega}^2(u, \gamma)}{\sqrt{h-u}} du], \\ L[\bar{\omega}(h, \gamma)] = L[(3 - \gamma)h - \frac{16}{15}(3 - \gamma)^2 h^{5/2}] + L[\int_0^h \frac{\bar{\omega}^2(u, \gamma)}{\sqrt{h-u}} du]. \end{cases} \tag{31}$$

Applying the fuzzy convolution theorem on Eq. (31), we obtain

$$\begin{cases} L[\underline{\omega}(h, \gamma)] = L[\gamma h - \frac{16}{15}\gamma^2 h^{5/2}] + L[(h)^{-1/2}] \cdot L[\underline{\omega}^2(u, \gamma)], \\ L[\overline{\omega}(h, \gamma)] = L[(3 - \gamma)h - \frac{16}{15}(3 - \gamma)^2 h^{5/2}] + L[(h)^{-1/2}] \cdot L[\overline{\omega}^2(u, \gamma)], \\ L[\underline{\omega}(h, \gamma)] = L[\gamma h - \frac{16}{15}\gamma^2 h^{5/2}] + \sqrt{\frac{\pi}{u}}L[\underline{\omega}^2(u, \gamma)], \\ L[\overline{\omega}(h, \gamma)] = L[(3 - \gamma)h - \frac{16}{15}(3 - \gamma)^2 h^{5/2}] + \sqrt{\frac{\pi}{u}}L[\overline{\omega}^2(u, \gamma)]. \end{cases} \tag{32}$$

Applying the inverse Laplace transform on Eq. (32), we obtain

$$\begin{cases} \underline{\omega}(h, \gamma) = \gamma h - \frac{16}{15}\gamma^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\underline{\omega}^2(u, \gamma)]], \\ \overline{\omega}(h, \gamma) = (3 - \gamma)h - \frac{16}{15}(3 - \gamma)^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\overline{\omega}^2(u, \gamma)]]. \end{cases} \tag{33}$$

Assume that the lower and upper limit fuzzy solutions of Eq. (33) are an infinite series as follows:

$$\begin{cases} \underline{\omega}(h, \gamma) = \sum_{n=0}^{\infty} \underline{\omega}_n(h, \gamma), \\ \overline{\omega}(h, \gamma) = \sum_{n=0}^{\infty} \overline{\omega}_n(h, \gamma). \end{cases} \tag{34}$$

Putting Eq. (34) in Eq. (33), we get

$$\begin{cases} \sum_{n=0}^{\infty} \underline{\omega}_n(h, \gamma) = \gamma h - \frac{16}{15}\gamma^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\underline{\omega}^2(h, \gamma)]], \\ \sum_{n=0}^{\infty} \overline{\omega}_n(h, \gamma) = (3 - \gamma)h - \frac{16}{15}(3 - \gamma)^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\overline{\omega}^2(h, \gamma)]], \end{cases} \tag{35}$$

$$\begin{cases} \sum_{n=0}^{\infty} \underline{\omega}_n(h, \gamma) = \gamma h - \frac{16}{15}\gamma^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\sum_{n=0}^{\infty} B_n(h, \gamma)]], \\ \sum_{n=0}^{\infty} \overline{\omega}_n(h, \gamma) = (3 - \gamma)h - \frac{16}{15}(3 - \gamma)^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\sum_{n=0}^{\infty} \hat{B}_n(h, \gamma)]], \end{cases} \tag{36}$$

where  $(B_n, \hat{B}_n)$  are Adomian polynomials which represent the nonlinear terms and can be found as follows:

$$\begin{cases} B_n(h, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [(\sum_{k=0}^{\infty} \lambda^k \underline{\omega}_k)]^2_{\lambda=0}, \\ \hat{B}_n(h, \gamma) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [(\sum_{k=0}^{\infty} \lambda^k \overline{\omega}_k)]^2_{\lambda=0}. \end{cases} \tag{37}$$

Now, first we solve the lower limit fuzzy solution of Eq. (30). So, comparing the lower limit terms of Eq. (36) and solving, we obtain

$$\begin{cases} \underline{\omega}_0(h, \gamma) = \gamma h - \frac{16}{15}\gamma^2 h^{5/2}, \\ \underline{\omega}_1(h, \gamma) = L^{-1}[\sqrt{\frac{\pi}{u}}L[B_0(u, \gamma)]] = \frac{16}{15}\gamma^2 h^{5/2} + \frac{16}{45}\gamma^4 h^3 - \frac{7}{12}\gamma^3 \pi h^4, \\ \vdots \end{cases} \tag{38}$$



and for the upper limit solution of Eq. (30), comparing the upper limit terms of Eq. (36) and solving, we obtain

$$\begin{cases} \bar{\omega}_0(h, \gamma) = (3 - \gamma)h - \frac{16}{15}(3 - \gamma)^2 h^{5/2}, \\ \bar{\omega}_1(h, \gamma) = L^{-1}[\sqrt{\frac{\pi}{u}}[L[\sum_{n=0}^{\infty} \hat{B}_0(u, \gamma)]]], \\ \quad = \frac{16}{15}(3 - \gamma)^2 h^{5/2} + \frac{112}{90}(3 - \gamma)^2 \pi h^3 - \frac{7}{12}(3 - \gamma)^3 \pi h^4, \\ \vdots \end{cases} \tag{39}$$

and so on. Compute the various terms of the series fuzzy solutions like we computed in Eq. (39) and in Eq. (34). After computation of the terms and putting them in Eq. (38), we cancel the noise terms and get the lower and upper limit solution as follows:

$$\begin{cases} \underline{\omega}(h, \gamma) = \gamma h, \\ \bar{\omega}(h, \gamma) = (3 - \gamma)h. \end{cases} \tag{40}$$

**4 Comparison with HAM and discussion**

In HAM the convergence of solution in the form of series depends upon four factors, i.e., the initial guess, the auxiliary linear operator, the auxiliary function which we define for homotopy, and the auxiliary parameter  $\hbar$ . Further, if we select  $\hbar = -1$ , and the auxiliary function also equals 1, HPM is obtained. Hence, HPM is a special case of HAM whose convergence only depends upon two factors: the auxiliary linear operator and the initial guess. So, given the initial guess and the auxiliary linear operator, the HPM approach cannot provide other ways to ensure that the solution is convergent. On the other hand, the proposed method solution for both linear and nonlinear problems is obtained in the form of a series showing higher convergence of the method. Among all other analytical methods, the proposed method is an efficient analytical method to solve nonlinear problems of differential or integral equations. Therefore, a hybrid method is the combination of two powerful methods: Laplace transform and Adomian decomposition method. The mentioned method does not need any kind of discretization or linearization. It also does not need a predefined parameter which controls these schemes as needed in HAM. Therefore, our proposed method is an efficient analytical technique for treating those equations that represent nonlinear models. If we consider our proposed model without initial condition, it will converge to a particular solution.

**5 Conclusion**

In this article, different types of fuzzy nonlinear integral equations of Abel type kernel are solved by using a hybrid method of integral transform and decomposition technique. Further, it is shown that the solution of fuzzy nonlinear integral equations by the newly proposed algorithm is easily, effectively and accurately convergent. To show the efficiency of the proposed algorithm, some numerical examples are given, which give the exactness of the method. Shortly, some modification is required to extend the method for the system of fuzzy nonlinear integral equations, in which the coefficients are mixed.

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**Abbreviations**

Fuzzy differential equation (FDE); Fuzzy integral equation (FIE); Fuzzy Fredholm integral equation of the second kind (FF-2); Adomian decomposition method (ADM); Laplace Adomian decomposition method (LADM); Homotopy perturbation method (HPM).

**Availability of data and materials**

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

**Ethics approval and consent to participate**

Not applicable.

**Competing interests**

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**Authors' contributions**

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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